Lesson 2

The Nature of Infinity

定义无极限
Dìngyì wú jí xiàn
Define infinity

During the week following my last meeting with Dr. Wu, I found myself pondering a number of questions. The work of Cantor had established the need for the concept of Infinity in mathematics, but this fact did not imply that Infinity exists in the real world. Yet, Dr. Wu had suggested that in some way, experienceable by us as humans, Infinity really does exist. He had said that if my individuality could somehow "melt away" even for a moment, I could get a glimpse of Infinity. This was fascinating to me, but every effort I made to "get beyond" my individuality resulted in frustration. I really didn't have any idea what to do. One thing I did notice, though, was that *trying* to reach some non-individual state locked me into my individuality—I saw that I was *an individual trying to be non-individual!* So putting a lot of effort into it wasn't going to work.

Spinoza and Plotinus on the Infinite

Dr. Wu had mentioned Spinoza and Plotinus as two philosophers who seemed to have experienced this non-individual state in their lives. So I went to the library to see what they had to say.

I spent a long time trying to make sense of their philosophies. At no point did either philosopher reveal exactly how he came to experience the Infinite. But I did find a few nuggets that I jotted down for future reference.

Baruch de Spinoza was a 17th century "rationalist" philosopher.[1] Spinoza's name for the Infinite was *Substance* (also *God*), which "exists by virtue of itself alone." According to Spinoza, Substance is the essence of man's soul, and could be "known" by way of a higher kind of knowing—*intuitive knowledge*—which was beyond sensory or rational forms of knowing. His "proof" of the existence of the Infinite makes sense only if one has had this kind of higher kind of cognition. His argument went something like this: "I can imagine a being having all possible perfections. Existence is a perfection. Therefore, God, the Absolute Infinite, must exist." The reason that my "imaginings" don't lead me astray here, according to Spinoza, is that they are to be understood as "intuitive cognitions" or "clear and distinct ideas"—in other words, they are based on a direct and undeniable experience of the Infinite.

[1] A good resource on Spinoza that I found was Wolfson, H.A. (1958).

Plotinus, on the other hand, was a Greek philosopher who lived in the 3rd century A.D.[2] He revived the philosophy of Plato, though most historians view Plotinus' version of Platonism as being colored with a mystical flavor (yet Plotinus himself seems to have believed that he was a faithful exponent of Plato's original philosophy). He is considered to be the first *Neoplatonist*. Plotinus referred to the Infinite[3] as *the One*; it was to be understood as "beyond Being," "ineffable and indescribable," "simple and formless," the "not this," the "first cause," the "ultimate object of all desire," and identical with Plato's *Good*.[4] Yet, however *ultimate* he took It to be, Plotinus viewed the Infinite as directly accessible to human experience. In one passage, he explains that souls, drawn to experience the Infinite, in time become "divinely inspired" and "inflamed with love" leading to an extraordinary state of bliss, steadily "being raised by that which imparts to it [the soul] its love." Ultimately, the soul "transcends . . . the intellect, but is not able to run beyond the Good because there is nothing beyond It." Porphyry, Plotinus' foremost disciple, says that Plotinus often had such experiences himself—on several occasions, Porphyry was present to witness Plotinus immersed in these exalted states. One tool for ascending to the One that Plotinus mentions is *the dialectic*[5] which takes the mind to the first principles underlying existence; having arrived at the first cause of all, Plotinus explains, the intellect becomes quiet and absorbed in contemplation of its own Infinite nature.

Delving into these philosophers' worlds was inspiring, but ultimately unsatisfying. I didn't find myself any closer to the Infinite after studying Spinoza's "proof"; and I knew for a fact that I was not, so far at least, one of the fortunate recipients of Plotinus' "divine inspiration" who were led to a transcendental encounter with the One. How is a 20th century college freshman supposed to meet up with the Infinite? It seemed that I had a long way to go.

As I was leaving the library, the "New Arrivals" section caught my eye, and I became absorbed in a book in which, in an interview format, a variety of accomplished individuals talked about the path their lives had taken. One of these was a modern-day philosopher—Dr. Jonathan Shear, a university professor of philosophy—who had had his own experience of the Infinite. His

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[2] A good reference is O'Brien, E. (1964).

[3] I learned later that the notion of "infinity" (*apeiron*) among Greek philosophers tended to connote chaos and disorder—"unboundedness" in the sense of "lacking orderly boundaries." This was not Plotinus' view of it, however. One passage from his *Enneads* makes this point clear (Plotinus, 1992):

> This All is universal power, of infinite extent and infinite in potency, a god so great that all his parts are infinite. (5.8.9)

[4] In Plato's philosophy, the *Good* or the *One* represents the ultimate, the source of all things. He used the sun as an analogy for the Good: Just as the sun is, in a sense, responsible for the growth of living things in the physical world, but is itself beyond earthly evolution, so likewise is the Good the source of knowledge, truth, and even being itself, in the *intelligible world*, but is beyond all of these. Plato talks about the Good in his dialogues, particularly in *The Republic*. The following is one such passage (Plato, 1972b, p. 346, S.N. 509):

> You will say, I think, that the sun imparts to things which are seen not only their visibility, but likewise their generation, growth, and nourishment, not being itself generation....We may say therefore that things which are known have not only this from the Good, that they are known, but likewise that their being and essence are thence derived, while the Good itself is not essence, but beyond essence, transcending it in both dignity and power.

[5] As I would learn later from Dr. Wu, this is the same "dialectic" method that Plato discusses in *The Republic*. See Lesson 5 for more on this topic.

account provided exactly the kind of encouragement I was needing. In the quote that I jotted down, he describes one of his earliest meditative experiences in his days as an undergraduate mathematics major.

> I had an experience of unboundedness, infinity, in meditation. I had been a serious mathematics student and thought that infinity only meant you can always add one, without ending—in other words, that there is no real infinity. However, after this expansion of consciousness, I saw that this was wrong. I realized infinity can be experienced.[6]

After reading this, I felt reassured that it was indeed possible to have this experience, and I started to relax more about finding my own way.

Set Notation

It also seemed that I had a long way to go regarding Dr. Wu's new Infinite Hotel problem. In this problem, I had to find a way to make room for *infinitely many* new guests in the already-full Infinite Hotel. I could use my earlier insights to accommodate a thousand, a million, a billion new guests, but, as Dr. Wu had warned, this technique would not solve the Infinite Superbowl problem since I couldn't expect every current guest to move up *infinitely many* rooms to make room for the new guests. I finally had to give up.

By the time Monday rolled around, it hit me that I must have been a fool to think I would benefit from this course of study with Dr. Wu. I was thinking of pulling out while I still could. After arriving at Dr. Wu's office, but before I had a chance to broach the subject, Dr. Wu asked,

"Have you had any luck with that Infinite Superbowl problem?"

"No Dr. Wu," I began. "I'm starting to think—"

"It is a more difficult problem," he said, interrupting, "but you will find that an answer will come soon. A new skill you will develop in these studies is to allow answers and new perspectives to come to you in their own time, and to be at peace with the process. As we proceed, you will have many questions for which you may not be able to find satisfactory answers, at least not right away. Gradually you will find that your way of looking at things shifts; after one shift, you may find that a host of questions that stubbornly refused to be answered are all of a sudden cleared up in an instant. You must be patient and allow this process to happen. Accept the fact that, for as long as you are working with me, there will be a significant number of unresolved issues floating around. Once you accept this as a way of life, it will stop bothering you and will actually become a source of continual motivation."

He certainly seemed to know where I was coming from.

"I guess that means you aren't going to tell me the solution?" I asked.

"Right," he smiled. "But let's take a closer look today at Infinity in mathematics. This should be a big help.

"The first basic point to remember is that Infinity is studied in mathematics by studying *infinite sets*. The most fundamental infinite set is the set of natural numbers, which we write like this:"

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[6] Ellis, G.A. (2012, p. 70).

He wrote:

$$\{1, 2, 3, 4, 5, \ldots\}$$

"In this notation, called *elliptical notation*, the three dots $(\ldots)$ indicate that the numbers keep going according to the pattern that is indicated by the first few elements listed in the set. The curly braces signify that we are talking about a *set*. Each of the numbers between these two braces is understood to be an *element* of the set. Is this clear?"

"I think so," I said. "So, the three dots mean that all the natural numbers, like 6, 200, 5 million—all of them—lie between the two curly braces?"

"Right."

"So you have written the *set of natural numbers* on the board."

"Exactly. Unfortunately, this notation is not entirely precise. The difficulty is that we are assuming that anyone looking at $\{1, 2, 3, 4, 5, \ldots\}$ will see the same 'pattern' as we do. There are in fact many different ways to continue the sequence $1, 2, 3, 4, 5, \ldots$. We might have in mind any of the following:

$$1, 2, 3, 4, 5, 10, 15, 20, 25, 50, 75, 100, 125, \ldots \ \textit{or}$$
$$1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 21, 22, 23, 24, 25, \ldots$$

and there are many other possibilities.

"When we need to be precise and remove any possibility of ambiguous interpretation, there is another notation that is used to specify a set; it is called *set-builder notation*. As an example, we can specify the set of natural numbers as follows:

$$\{x \mid x \text{ is a natural number}\}$$

"The notation is read in the following way: 'The set of all x such that x is a natural number.' As long as we know the meaning of 'is a natural number', the set has been precisely specified.

"More formally still, one may insist that 'is a natural number' is not entirely clear. It turns out that, using tools from the theory of sets, one can specify all natural numbers without assuming knowledge of the natural numbers in advance. We will take up this point in later lessons.

"For now, consider one more example: Suppose we wish to specify the set of even natural numbers. In set-builder notation, we could write it like this:

$$\{x \mid x \text{ is an even natural number}\},$$

or, to provide more detail, we could write:

$$\{x \mid \text{for some natural number } y,\ x = 2y\}.$$

"This second way of expressing the set of even numbers relies on the mathematical definition of 'even': A natural number n is said to be *even* if it is divisble by 2; that is, if it is '2 times' another number.

"In general, set-builder notation takes the following form:

$$\{x \mid \phi(x)\}$$

where $\phi(x)$ represents some property (such as 'is a natural number') that x could have.

"For the purpose of our discussions, I will use the more informal elliptical notation whenever possible since, at first anyway, it is more intuitively clear."

After a short pause, Dr. Wu continued, "Continuing our main thread of ideas, then, this set $\{1, 2, 3, 4, 5, \ldots\}$ of natural numbers is the fundamental infinite set that Cantor's work introduced into the mainstream of mathematics. Can you think of any other sets that are also infinite?"

This didn't seem too hard. To be safe, I said,

"Well, what about the set consisting of the elements $0, 1, 2, 3, 4, \ldots$?"

"So you want also to include 0? Certainly the resulting set is also infinite. We call this the set of *whole numbers*. Here are some other infinite sets we will need, along with their names:"

He wrote:

$$\begin{aligned}
\text{the set of natural numbers:}\quad & \mathbb{N} &= \{1, 2, 3, 4, \ldots\}\\
\text{the set of whole numbers:}\quad & \mathbb{W} &= \{0, 1, 2, 3, 4, \ldots\}\\
\text{the set of even numbers:}\quad & \mathbf{E} &= \{2, 4, 6, 8, \ldots\}\\
\text{the set of odd numbers:}\quad & \mathbf{O} &= \{1, 3, 5, 7, \ldots\}\\
\text{the set of integers:}\quad & \mathbb{Z} &= \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}
\end{aligned}$$

When I saw Dr. Wu writing these sets on the board—particularly the sets of even and odd numbers—I all of a sudden saw how to solve the Infinite Superbowl problem. I could see that Dr. Wu wouldn't want me to take him on a tangent right then, so I jotted a note to myself so I wouldn't forget my insight.

First Attempts at a Definition of *Infinity*

"These are important examples of infinite sets. But we need to understand what *infinite sets* really are a bit more deeply. You will find out fairly soon that not all infinite sets can be written in this convenient notation— by listing a few elements of the sets and following them with three dots. For this reason, we will need to come up with a *definition* of 'infinite set' so that you can recognize an infinite set when you encounter one.

"To give a definition, though, we need to know what a *set* is. The easiest thing to do at this stage is to simply assert that a *set* is any collection of things assembled together. For our purposes, as we are just beginning, this definition will be good enough. But as I will show you later, this definition has an inherent flaw; the flaw is so serious that, if it is not corrected, mathematics would end up being an inconsistent mess. The difficulty was first spotted by Bertrand Russell in 1900. At the time, the only official 'definition' of a set was given by Cantor, who did not make his concept of a set precise. He declared that a set is '. . . any collection into a whole of definite, distinct object of our intuition or thought.' This definition left considerable room for interpretation, and the common interpretation of the time was to think of a set as being simply a collection of objects. Russell's work showed that this loose interpretation of Cantor's notion of set led to serious problems, which were later resolved by introducing considerably more rigor in the formulation. For the time being, though, we will use this loose definition as our provisional definition of 'set' and revise it later on.

"This leaves us with the task of defining 'infinite' in a mathematical way. Historically, this has been done in a number of ways, all of which have turned out to be equivalent. The definition I wish to use is based on the basic observation that $\mathbb{N}$, the set of natural numbers, is for us 'the' fundamental infinite set. To tell you the definition I have in mind, I will need to introduce one other concept. First, though, let me ask—do you have an idea for how we might define an 'infinite set'?"

I really had no idea. I looked up at the board at the infinite sets he had written and one thing that hit me right away was that each of the sets began with a few numbers and were followed by three dots (...). This gave me an idea.

"What if we say that a set is infinite if it goes on forever?" I suggested.

"Like the sets on the board?" he asked.

"Right."

"We will need to make your idea more precise if we wish to use it as a definition. But first I want you to consider one other set: the set of all real numbers between 0 and 1, inclusive. Does this seem to be an infinite set to you?"

I thought for a moment. I had to remind myself that real numbers included all the fractions. And I certainly could think of infinitely many fractions between 0 and 1:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$$

"Yes, there must be infinitely many such numbers," I answered.

"Good. But do the reals between 0 and 1 inclusive 'go on forever' in the sense that you mean?"

"I guess not," I replied. "I mean, there are numbers that are bigger than all of these, like 2 and 3. This isn't true with $\mathbb{N}$, **E** or any of the other sets you mentioned."

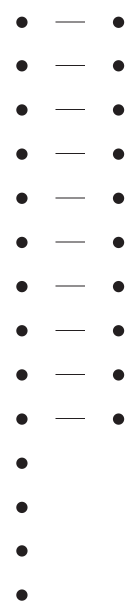
"Right," Dr. Wu began. "As you will discover soon, it's actually not possible to arrange the real numbers in this set in a sequence at all. So we need a definition of 'infinite' that will allow us to recognize that sets like those on the board *as well as* other sets that may not be expressible in this way are in fact infinite.

Cantor's Approach: 1-1 Correspondences

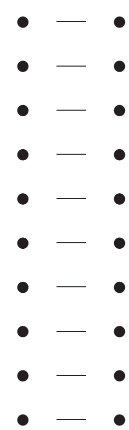
"To address this problem, Cantor began with the following basic question about sets in general: What does it mean to say that one set has *the same size* as another set? Likewise, what does it mean to say that one set is bigger than another? For small finite sets, a reasonable answer is that two sets have the same size if they have the same number of elements; and the set A is bigger than the set B if A has more elements than B. This definition, however, stops making sense (at least at our present stage) when we wish to compare two *infinite* sets since we have no way of talking about 'how many' elements such sets contain.

"The alternative that Cantor devised is actually familiar to anyone who has had to compare the sizes of relatively large collections. An example from everyday life is the phenomenon of picking which line to stand in when you want to buy a ticket to get into a movie theater. Certainly, you want to stand in the shortest line—the shortest line has the fewest number of people; picking this line means you minimize your time in line. But how do you decide which line is shortest? One way to tell is to count all the people in each line; the line with the smallest number of people must be shortest. But if the lines are long, this approach takes too long. The approach that we all take is to pick the line that is literally the shortest. Consider the following schematic diagram of the situation:"

Dr. Wu drew two vertical columns of dots, representing two lines of people at a movie theater:



"In the diagram, the vertical column of dots (representing a line of people) on the right is shorter. Notice that we can match up, one for one, the people in the line on the left with the people in the line on the right all the way to the end of the right line. Here is what this match-up looks like, up to the end of the right line:



"This matching process guarantees that the set of people in the left line who have been matched in this way with the set of people in the right line must have the same size. We don't need to know how many people are in each of these lines; the fact that people in the right line have been matched one for one with an initial portion of the line of people on the left clearly means that the line on the left is longer.

"This idea, when applied to sets, leads to the conclusion that we can tell that two sets have the same size if we can find a way to match the elements of one set one-for-one with the elements of the other set. Cantor formalized this idea of one-for-one match-ups using the concept of a *1-1 correspondence*. A 1-1 correspondence between two sets A and B is a one-to-one matchup of the elements of A with the elements of B, so that every element of A is matched with just one element of B, and every element of B is matched with just one element of A. For example, if $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, we can obtain a 1-1 correspondence between A and B by matching 1 with a, 2 with b, and 3 with c. Notice that in this example, every element of A and also every element of B has been accounted for; no elements have been left out of the matching.

"Two sets A and B are said to *have the same size* if there is a 1-1 correspondence between them. For example the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers has the same size as the set $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$ of whole numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers because we can find a

1-1 correspondence between them, as follows:”

$$
\begin{array}{ccc}
\underline{\mathbb{N}} & & \underline{\mathbb{W}} \\
1 & \rightarrow & 0 \\
2 & \rightarrow & 1 \\
3 & \rightarrow & 2 \\
. & & . \\
. & & . \\
. & & . \\
n & \rightarrow & n-1 \\
. & & . \\
. & & . \\
. & & .
\end{array}
$$

Correspondence #1

“Notice that one can devise other correspondences that are *not* 1-1 correspondences. For instance, look at this one:”

$$
\begin{array}{ccc}
\underline{\mathbb{N}} & & \underline{\mathbb{W}} \\
 & & 0 \\
1 & \rightarrow & 1 \\
2 & \rightarrow & 2 \\
. & & . \\
. & & . \\
. & & . \\
n & \rightarrow & n \\
. & & . \\
. & & . \\
. & & .
\end{array}
$$

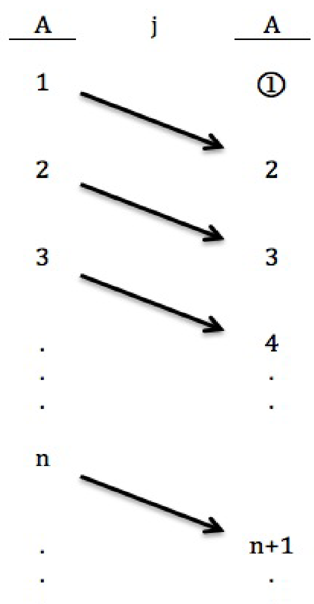
Correspondence #2

“In this example, the elements of $\mathbb{N}$ have been matched up with a *subset* of $\mathbb{W}$, namely, the set of all *nonzero* whole numbers; there is no element of $\mathbb{N}$ that is matched with 0 in this correspondence. This particular correspondence, therefore, is *not* a 1-1 correspondence. Sometimes, when we are trying to decide whether two sets A and B have the same size, we may think of a correspondence, like this one, that does not turn out to be a 1-1 correspondence. It is important to understand that even if we find a correspondence that is not a 1-1 correspondence, we cannot conclude from this that the two sets do *not* have the same size. As we have just seen, one of the correspondences we can think of from $\mathbb{N}$ to $\mathbb{W}$ is *not* a 1-1 correspondence (Correspondence #2), but another one *is* (Correspondence #1). If we can find even one correspondence that is indeed a 1-1 correspondence, then we may conclude that the two sets have the same size. So, $\mathbb{N}$ and $\mathbb{W}$ do indeed have the same size."

Dr. Wu paused for a moment, and this gave me a chance to absorb this new idea. It did seem a little strange to say that $\mathbb{W}$ and $\mathbb{N}$ *have the same size* since $\mathbb{W}$ obviously has one more element than $\mathbb{N}$ (namely, 0). But I could see that according to the definition Dr. Wu was using, this was not the issue; the point was that there was at least one way to match up the elements of $\mathbb{N}$ one for one with the elements of $\mathbb{W}$. This was enough to guarantee that the two sets had the same size. The "extra" 0 was shuffled into the correspondence in the same way the extra guy in the first Infinite Hotel problem was accommodated even though the Infinite Hotel was full. I seemed to be getting this point.

"By the way," he continued, "this particular kind of correspondence I showed you in this last example (Correspondence #2), where everything matches up except for the number 0 in $\mathbb{W}$, also turns out to be interesting for an altogether different reason. True, it is not a 1-1 correspondence, but it is 'almost' a 1-1 correspondence; we say that it is *1-1 but not onto.* 'Not onto' means that, in the set considered on the right side, some value has not been matched, like the number 0 in Correspondence #2. When all values in the right-hand set of a correspondence (expressed as a table, as we have been doing) *are* matched with values in the set on the left, the correspondence is *onto.* Next, we will consider correspondences from a set A *to itself* that are 1-1 but not onto. For example, consider the following correspondence j from the set $A = \{1, 2, 3, \ldots\}$ to itself."

He drew the following.



Correspondence #3

"The correspondence j takes 1 to 2, 2 to 3, and so on. As a result, there is no match for the element 1 in the right column. j in fact *is* a 1-1 correspondence between A and the subset $A - \{1\} = \{2, 3, 4, \ldots\}$, so we could say that j is *almost* a 1-1 correspondence from A to A. In this case, j is called a *Dedekind self-map*, and the special element 1 that was left out is called a *critical point for* j. In other words, any correspondence from a set to itself that is 1-1 but not onto is called a Dedekind self-map. We will discuss this type of correspondence in greater detail later on."

Dr. Wu paused for a moment. I had the feeling that he was sneaking in a very important concept here to prepare me for a more involved discussion later. At this point, he was mentioning this new concept of a Dedekind self-map in just a casual way, but I knew I should not treat the concept too casually. I decided to keep the picture of Correspondence #3 in my mind to help keep the idea straight. In Correspondence #3, j is the name of the correspondence from A to A; j is 1-1 because j is *almost* a 1-1 correspondence (and it *is* a 1-1 correspondence between A and $A - \{1\}$), but it is not onto because the number 1 in the right column has not been matched; and because it has not been matched, the number 1 is called a *critical point* of j. I think I had it.

Then Dr. Wu asked,

"Do you suppose that $\mathbb{N}$ and $\mathbf{E}$ have the same size?"

Now this was a different question—the set $\mathbf{E}$ was missing "half" the numbers that are in $\mathbb{N}$; namely, all the odd numbers. Surely $\mathbf{E}$ had to be a smaller set.

"Probably not," I said.

"Remember to use the definition," he began. "Although we can easily find a match-up between $\mathbf{E}$ and elements of $\mathbb{N}$ that is *not* a 1-1 correspondence, it is nonetheless possible to show that the two sets have the same size, by using the following correspondence from $\mathbb{N}$ to $\mathbf{E}$:"

$$\begin{array}{ccc}
\underline{\mathbb{N}} & & \underline{\mathbf{E}} \\
1 & \to & 2 \\
2 & \to & 4 \\
3 & \to & 6 \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
n & \to & 2n \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot
\end{array}$$

Oh, I got it. And you could show that $\mathbb{N}$ and $\mathbf{O}$ have the same size by doing something similar: Match 1 with 1, 2 with 3, 3 with 5, 4 with 7, and so on. Amazing.

"One thing I would like you to think about for our next meeting is whether $\mathbb{N}$ and $\mathbb{Z}$ have the same size. For now, let's move on to our definition of 'infinite set'.

Subsets and Copies of Sets

"To begin, let's remind ourselves what it means for one set to be a *subset* of another set. Technically, if C and D are sets, then we say that C is a *subset* of D if every element of C is also an element of D, and in this case we write $C \subseteq D$. For instance, $\{2, 3\} \subseteq \{1, 2, 3, 4\}$."

This seemed like familiar territory.

"The notion of a subset provides us with a convenient way to test whether two sets are *equal*: We say that sets A and B are equal, and write $A = B$, if each is a subset of the other: $A \subseteq B$ and $B \subseteq A$.

"When $C \subseteq D$, we can express this fact in English by saying 'C is a subset of D,' 'C is contained in D (as a subset),' 'D includes C,' or 'D contains C (as a subset).'

"This word 'contains' can be tricky because sometimes it is used to express that something is an *element* of a set rather than being a *subset* of the set. For instance, we might say '2 is *contained in* $\{1, 2, 3\}$,' meaning that 2 is an element of (not a subset of) $\{1, 2, 3\}$.' To avoid this confusion, we will assume hereafter that 'contains' means 'contains as a subset,' but when we wish to use the other meaning of 'contains,' we will explicitly write 'contains as an element.' For instance, if $C \subseteq D$, we will say 'D contains C.' But, if x is an element of C, we can also use 'contains' in a qualified way by saying 'C contains x *as an element*.' By the way, the notation we will use to indicate that 'x is an element of C' is $x \in C$; the symbol '$\in$' is called the *membership relation* and is read 'is a member of' or 'is an element of.'

All this discussion of the fine meaning of words was getting tiresome. Later on though, as I reviewed some of Dr. Wu's definitions, I realized that spelling out the details now would save a lot of misinterpretation and confusion later on when these concepts would be used quite often.

"Let me mention here a concept that is naturally related to 'subsets'; a concept that must not be overlooked at this point: The concept of a set having no elements. A set with no elements is called the *empty set*, and is denoted with a special symbol."

He wrote:

The empty set is denoted $\emptyset$.

"The empty set has a unique status among sets. It is the only set with the property of being a subset of every set."

This was peculiar. I thought about a simple set, like $S = \{1, 2, 3\}$. Why is it true that S contains $\emptyset$ as a subset? I didn't see the logic of this, so I asked Dr. Wu to clarify. "Why must the empty set be a subset of this set S?" I asked.

"Yes, this claim I have made deserves a proof. Unfortunately, you will probably feel there is something strange about any proof I give you. But let's try. Let's use your set S to have something concrete to work with.

"To say that $\emptyset \subseteq S$ means, by our definition, that every element of $\emptyset$ is also an element of S. Is that true?"

"But there aren't any elements in $\emptyset$," I said.

"True," he continued. "That's why the statement is true! But this kind of reasoning may be difficult to accept right now, so let's try a different approach to proving this.

"Would you agree with this much, that it is either true or false that $\emptyset \subseteq S$?"

"Sure," I said.

"If you suspect that it is false, what must be done to prove it, to show $\emptyset \not\subseteq S$?"

I could see his point. We were going to need to prove that one of the two possibilities is correct—either that $\emptyset \subseteq S$ or $\emptyset \not\subseteq S$. I can't just scratch my head at the first possibility; if I want to claim that the empty set is *not* a subset of S, I need to prove that somehow. I thought for a moment how that could be done. If I had another set $T = \{2, 3, 4\}$, I could show that T is *not* a subset of S by observing that the number 4 belongs to T but not to S. In a similar way, if it is really true that $\emptyset \not\subseteq S$, it must be the case that some element of $\emptyset$ does not belong to S. I explained my reasoning to Dr. Wu.

"Excellent," he said. "So, if it is really true that $\emptyset$ is *not* a subset of S, you must be able to find an element of $\emptyset$ that does not belong to S. What is the conclusion, then?"

"Since $\emptyset$ has no elements, I cannot find such an element."

"And so?" he smiled.

The last step finally hit me. The only way it could be true that $\emptyset$ is *not* a subset of S is if I can find an element of $\emptyset$ that is not in S, but I can't. So it is *not* the case that $\emptyset \not\subseteq S$. The only other possibility is that $\emptyset \subseteq S$. And so that proves it.

"Yes, you've got it," he said. "We will have a chance to understand more deeply some of the logic that is involved here—and which makes the reasoning seem tricky—in a later lesson.[7] Now to prevent a possible misunderstanding in the future, let me ask you a question: Is it true to say that the empty set is an element of every set?"

I could see a gleam in Dr. Wu's eye. My first impulse was to say "yes," since it seemed we had just proven this to be the case. But "element of" and "subset of" are certainly not the same thing, so I held my tongue and thought a bit about my sample set $S = \{1, 2, 3\}$ again. Is the empty set an element of S? Certainly 2 is an element of S—I can see immediately that 2 lies between the curly braces of S. But I don't see $\emptyset$ between the curly braces. So it must not be an element. If I defined a new set $T = \{1, 2, 3, \emptyset\}$, then certainly T contains the empty set as an element. I responded to Dr. Wu : "No, Dr. Wu, not every set contains the empty set as an element."

"Very good," he smiled. "I could not trick you this time! This is a point that can be confusing when one first begins thinking about the two notions of membership and subset. While it is true that the empty set is a subset of every set, it is not an *element* of every set; in fact, most of the sets we will encounter will not contain the empty set as an element.

"Now, moving forward on our path to a definition of 'the infinite,' I want to introduce you to a colloquialism that is often used by mathematicians in various contexts. Suppose $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. I want to say that 'B contains a copy of A'. What do you think I would mean by that?"

I remembered that 'contains' would mean 'contains as a subset.'

I said, "I can see that, even though B and A don't have any elements in common, still B does have subsets that have the same size as A. For example, $\{a, b, c\}$, $\{b, c, d\}$ are two subsets of B that have 3 elements. Since A has 3 elements, you could say, I suppose, that each of these 3-element subsets of B is a *copy* of A."

"Exactly right," he said. "B contains a copy of A because for example—as you say—B contains as a subset the set $\{a, b, c\}$, and we consider $\{a, b, c\}$ to be a *copy* of A because there is

[7] Lesson 9.

a 1-1 correspondence between A and $\{a, b, c\}$.

"Now, could you say that the set $C = \{0, -1, -2, -3, \ldots\}$ consisting of the negative integers, together with 0, contains a copy of $\mathbb{N}$?"

I had to think about this. Here the set C certainly didn't contain any of the natural numbers—but that wasn't the point. The question was: Is there a subset of C that has the same size as $\mathbb{N}$? Such a subset would be a "copy" of $\mathbb{N}$. Well, it looked to me as though the numbers $-1, -2, -3, \ldots$ matched up with the elements $1, 2, 3, \ldots$ of $\mathbb{N}$ pretty easily. So I wrote this on the board:

$$\begin{array}{ccc}
\underline{\mathbb{N}} & & \underline{C} \\
 & & 0 \\
1 & \rightarrow & -1 \\
2 & \rightarrow & -2 \\
3 & \rightarrow & -3 \\
. & & . \\
. & & . \\
. & & . \\
n & \rightarrow & -n \\
. & & . \\
. & & . \\
. & & .
\end{array}$$

"It seems to me that the subset of C consisting of the numbers $-1, -2, -3, \ldots$ is a copy of $\mathbb{N}$, and here is the 1-1 correspondence," I said, pointing to what I had written on the board.

The Definition of *Infinite* in Mathematics

"Excellent!" Dr. Wu seemed pleased. "Now we are ready for the definition of 'infinite set'."

He wrote:

Definition (Infinite Set)**.** A set A *infinite* if it contains a copy of $\mathbb{N}$.

He paused to let this definition sink in. I now had before me an awesome statement—the mathematical definition of Infinity. I couldn't get myself to feel any awe, though—I just wanted to make sure I was understanding the point.

"Now," he resumed, "let's test our definition. How can you tell that $\mathbb{W}$ is infinite?"

Hmmm. Well, this is just like the question about the set C we just did.

"That's not hard, Dr. Wu. $\mathbb{W}$ contains a copy of $\mathbb{N}$ pretty obviously, since the numbers $1, 2, 3, \ldots$ are all elements of $\mathbb{W}$."

"Good, good. What about $\mathbb{Z}$—why must it be infinite?"

"For the same reason," I laughed. "$\mathbb{Z}$ contains a copy of $\mathbb{N}$ since, again, the numbers $1, 2, 3, \ldots$ are all elements of $\mathbb{Z}$."

"Now," Dr. Wu said with a twinkle in his eye, "what about the set of real numbers between 0 and 1, inclusive? Why must they form an infinite set?"

This was harder. I had to find a copy of $\mathbb{N}$ that could be found between 0 and 1. But wait, I had already done something like that before. I could do the following matchup between $\mathbb{N}$ and some of the fractions between 0 and 1:

$$\begin{array}{ccc}
1 & \rightarrow & 1/2 \\
2 & \rightarrow & 1/3 \\
3 & \rightarrow & 1/4 \\
4 & \rightarrow & 1/5 \\
5 & \rightarrow & 1/6 \\
6 & \rightarrow & 1/7 \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot
\end{array}$$

This showed that the reals between 0 and 1 contain a copy of $\mathbb{N}$, so there must be infinitely many of them. I explained this to Dr. Wu.

"Very good, Paul. You are well on your way to understanding the Infinite!"

A Solution to the Infinite Superbowl Problem

I was getting the feeling that we were about done for the day when I remembered that I had figured out a solution to the Infinite Superbowl problem. I asked Dr. Wu if it would be all right for me to explain my solution. I was ready to put the following table on the board. Dr. Wu told me to go ahead with it.

Currently In	**Moves To**		**New Guest #**	**Will Stay In**
1	2		1	1
2	4		2	3
3	6		3	5
. . .			. . .	
n	$2n$		n	$2n-1$
. . .			. . .	

What I had realized when I saw the infinite sets Dr. Wu had written on the board earlier in the day was that **E** and **O** were both infinite sets, but without any elements in common. This gave me the idea to move the current guests in the Infinite Hotel to even-numbered rooms and move the new guests into the odd-numbered rooms. Dr. Wu was duly impressed.

More on the Infinite: The Axiom of Infinity

"Before you go," he continued, "I would like to mention a couple of other points.

"First of all, let me ask you a question. How did you feel when I told you the definition of 'infinite set'? After all, this is the starting point for understanding the entire field of mathematical infinity. Did it fill you with wonder?"

I recalled that actually, my reaction had been just the opposite, but I didn't want to tell Dr. Wu about it. I started to hem and haw. "Um, Dr. Wu, the definition was really interesting. Um, I'm not sure what to say. I—"

Dr. Wu laughed. "You are right," he said. "It's a remarkably unexciting definition of something as profound as 'the infinite'! There are actually a number of ways to define this idea of infinite set, and the one I have chosen will be useful because of its concreteness and its connection to the familiar world of natural numbers. One reason it lacks 'punch,' though, is that it assumes we already know about the set of natural numbers; what is needed is a definition of the natural numbers themselves. Where do they come from? What is really meant by 'the infinite'? Our definition of infinite set simply 'skipped over' the most important part!

"To understand what underlies this notion of 'infinity' in mathematics, and to see, to the extent that this is possible, what the 'origin' of the natural numbers might be, I want to take a brief look at the history of these topics in the world of mathematics, and to see what answers modern mathematics provides for these questions. The natural numbers have always been understood to be fundamental in mathematics. Before our modern era, they were considered by most to be the starting point of mathematics. And if we look back to certain ancient traditions of knowledge, they were even seen as the origin of everything in the universe. For that point of view to make sense, we would need to appreciate these numbers in a very different way; we will look more deeply into this perspective later in these lessons.

"However, something crucial has been recognized about the natural numbers in modern times that was not fully appreciated in earlier times: That all the natural numbers, even though endless and infinitely diverse, can be collected together into a single set; that they can be viewed as a totality and studied in the way mathematics studies other mathematical objects.

"As we discussed in the last lesson, it was because of the work of Georg Cantor that this recognition became part of the mathematical mainstream. Historically, Cantor's insight, that there is a *set* consisting of all the natural numbers, became a part of the fundamental axioms that form the basis for the development of all mathematics. These axioms, knowns as the *ZFC axioms*, provide the starting point for deriving all of today's mathematics. Among these, the axiom that asserts that there is an infinite set—one that contains all the natural numbers—is called the *Axiom of Infinity*.

"As it happens, the Axiom of Infinity is the *only* one of the axioms that talks about infinite sets explicitly. This fact suggests that, if we want to find out what modern mathematics has to say at a foundational level about the 'infinite,' a reasonable thing to do is to look closely at what this Axiom of Infinity actually says.

"The intuition behind the Axiom of Infinity is that, in our universe of sets, there should be at least one set that is *infinite*. This idea gives expression to Cantor's insight that infinite sets should exist. And, in the years when the axioms were actually being formulated, the most natural of all infinite sets was considered to be the set of natural numbers. So, the intention of the early founders of set theory was to formulate the Axiom of Infinity by stating, in the best way possible, that the collection $\{1, 2, 3, \ldots\}$ exists as a set.

"Let's take a look at how it was done. Here is what the Axiom of Infinity, as it is usually formulated, says."

He wrote:

Axiom of Infinity. There is an inductive set.

"I will give you a formal definition of *inductive set* in the exercises for today. But for now, the point I want to make is that to assert the existence of an inductive set accomplishes two things:

(1) We let ω ('omega') denote *smallest* inductive set. It can be shown that each element n of ω consists precisely of the elements of ω that precede it, and this leads to the following 'definition' of the whole numbers in terms of sets:

$$\begin{array}{rcl} 0 & = & \emptyset \\ 1 & = & \{0\} \\ 2 & = & \{0,1\} \\ . & & . \\ . & & . \\ . & & . \end{array}$$

(2) It guarantees the existence of the successor function s that adds 1 to any whole number ($s(n) = n + 1$).

"The axiom succeeds in getting the whole numbers into mathematics, collected together as a set. But, if we ask for more information about how the natural numbers arise, or what underlies our idea of 'infinite set,' we get only a tiny bit of information. One thing it tells us is that the natural numbers arise by starting with 0 or the empty set, and repeatedly adding 1: $s(0) = 0 + 1 = 1, s(1) = 1 + 1 = 2$, and, generally, $s(n) = n + 1$. But that's about it.

"Admittedly, questions like 'where do the natural numbers come from?' and 'what underlies the notion of infinite set?' are really more philosophical than mathematical. Nevertheless, this lack of substance in the Axiom of Infinity has in a way obscured potential solutions to deeper problems about the infinite that have arisen in mathematics in more modern times.

A New Perspective from Quantum Field Theory

"For this reason, I would like to suggest an alternative to the usual Axiom of Infinity. The alternative I have in mind is not due to me but actually has been known since the early days of set theory. This version, however, will tell us quite a bit more about the nature of the natural numbers and the infinite. In fact, the intuition implicit in this new axiom provides us, I believe, with significant clues about how to solve the famous Problem of Large Cardinals.

"This alternative Axiom of Infinity is naturally related to a rather different intuition concerning the notion 'infinite.' To prepare the way for this insight, I want to consider an analogy from modern physics. An old problem in physics has been to discover and classify the fundamental constituents of the universe; to find, if possible, the fundamental particles out of which the universe is built. The quest for a solution to this problem led to discoveries of finer and finer particles, until one day, a shift occurred. It was discovered that, in reality, particles are *not* the fundamental constituents of matter. What was discovered in quantum field theory is that discrete particles of all kinds are precipitations of something deeper, of an underlying *quantum field*. You have certainly heard of the gravitational field—an invisible influence that causes objects to have weight and that prevents planets from spinning out of orbit. This influence is *unbounded* in the sense that it has an impact throughout the universe. It was found that every particle has its own kind of field, called a *quantum field*. A quantum field is, like the gravitational field, unbounded in space and, in its unexcited state, invisible. Particles arise as precipitations or excitations of the underlying field. There is, for example, an *electron field* whose excitations are electrons. The main insight, then, is that what is fundamental in the physical world is not the observed discrete particles but unbounded, invisible, quantum fields, which give rise to these particles through their internal dynamics.[8]

"We propose to use this insight from physics as a model for a new way to look at the 'discrete particles' that make up the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers. Perhaps what is fundamentally true about $\mathbb{N}$ is not its 'particle nature'; perhaps these discrete quantities can more naturally be viewed as arising from something more fundamental—perhaps even from the dynamics of some kind of unbounded field.

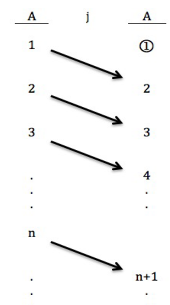
[8] A reference that summarizes this widely held perspective in the physics community is Hobson (2014).

"In fact, this idea was already worked out more than a century ago, though the philosophical paradigm I am using was probably not what the pioneers of these results had in mind.

Dedekind Self-Maps

"This new formulation of the 'underlying reality' of the natural numbers is based on the idea of Dedekind self-maps. Do you remember my mentioning this term earlier today?"

I thought for a moment. Dr. Wu had indicated that this new concept would be important so I had written a special note to myself about it. I remembered the picture (Correspondence #3) he had drawn before. I had a miniature version of it on hand.



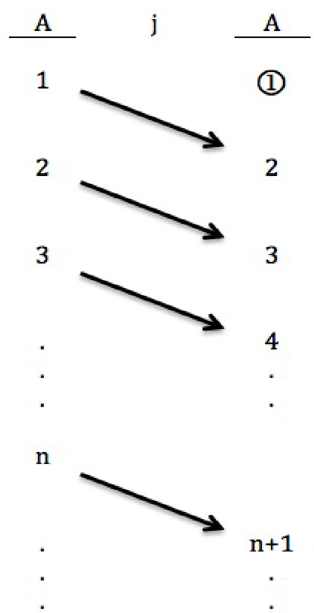
In the diagram, we have a correspondence from a set A to itself, which has been given the name j. Here, j is almost a 1-1 correspondence, but not quite; the value 1 on the right side is not matched with any value on the left. Dr. Wu had said that j is a correspondence that is 1-1 but not onto. Also, he had said that when such a correspondence is from a set A to itself, it is called a *Dedekind self-map*. Any element on the right side of the diagram, like the number 1 in this example, which is not matched with any element on the left side is called a *critical point* of the self-map.

So, putting all this together, I said,

"Dr. Wu, as I understand it, a Dedekind self-map is a 1-1 but not onto correspondence from a set A to itself; any element of A that is not matched with something in A by the correspondence is called a critical point of the self-map."

"Exactly right," he said, obviously pleased that I had remembered all of these points. "Now, let's look a bit more closely at one of these Dedekind self-maps. Let's return to the example I gave before."

Dr. Wu then reproduced on the board the diagram that I had just been reviewing.



"Let's recall that the diagram depicts a Dedekind self-map j from A to itself that has critical point 1. As we noticed before, 1 is the one and only element of A (on the right side) that has been left out of the correspondence; we say that 1 is *not in the range of* j. What I mean by the 'range' of j is the set of all values on the right side that have been matched with elements on the left side. Notice that all the numbers $2, 3, 4, \ldots$ are matched with values on the left. We say, then, that the range of j is the set $\{2, 3, 4, \ldots\}$. Let's call this set B; that is, $B = \{2, 3, 4, \ldots\}$.

"Paul, I want you to try looking at this diagram in a slightly different way. Our perspective so far has been that j is a correspondence that is *almost* a 1-1 correspondence from A to A. But another way to look at it is that it *is* a 1-1 correspondence from A to B. Can you see that this is true?"

I drew the diagram in another way that made the point clearer. In the new diagram, we don't list the number 1 on the right side, and just restrict attention to the set B. From this perspective, j is now a 1-1 correspondence, not from A to A, but from A to B.

$$\begin{array}{ccc}
\underline{A} & j & \underline{B} \\
1 & \to & 2 \\
2 & \to & 3 \\
3 & \to & 4 \\
. & & . \\
. & & . \\
. & & . \\
n & \to & n+1 \\
. & & . \\
. & & . \\
. & & .
\end{array}$$

I explained my reasoning to Dr. Wu and showed him my diagram.

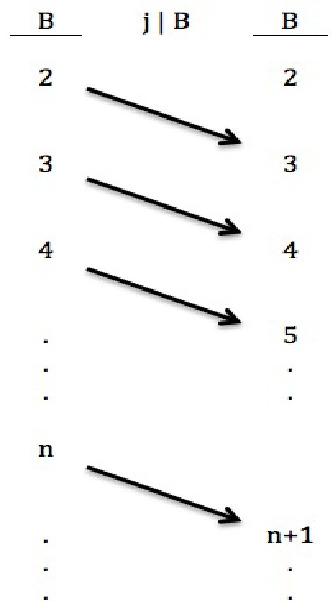
"Yes, very good, this is right," he said. "As your diagram shows, j now represents a 1-1 correspondence between A and a *proper subset* of A, namely B. We say a subset of A is *proper* if it is not actually equal to A itself. This tells us that any Dedekind self-map from a set S to itself can also be seen as a 1-1 correspondence between S and a proper subset of itself.

"Do you think it is possible for a *finite* set to have this unusual property that it can be put in 1-1 correspondence with a proper subset of itself?"

I tried thinking of an example. The set $\{a, b, c\}$ is finite, having just 3 elements. The proper subsets of this set are sets like $\{a, b\}, \{b, c\}, \{c\}$. Each has size less than 3. There's certainly no way to match the elements of $\{a, b, c\}$ one for one with the elements of any of these subsets since they don't have enough elements. It seemed that this same problem would arise if we were to start with any finite set; it would have too many elements for us to put them in 1-1 correspondence with a proper subset.

"That's right, Paul," he continued, after hearing my point. "A set that can be put in 1-1 correspondence with a proper subset of itself is necessarily infinite, although to prove this carefully, we need to return to our original definition of 'infinite,' and we will do this in a moment. In mathematics, a set that can be put in 1-1 correspondence with a proper subset of itself is called *Dedekind infinite*. What we have just seen, then, is that whenever j is a Dedekind self-map from a set A to itself, A must in fact be a Dedekind infinite set. And, conversely, if we have a 1-1 correspondence j from a set A to a proper subset B of A, j may also be viewed as a self-map from A to A that is 1-1 but not onto; therefore, j can be seen as a Dedekind self-map from A to A.

"Now what do you think will happen if we look at how j behaves relative to B only? We can picture it like this:



"Since we are considering j only as it acts on elements of B, we name this correspondence $j \upharpoonright B$—j restricted to B. This restricted version of j matches elements of B with other elements of B, and, like the original, it has a critical point—in this case, it is the number 2: The number 2 on the right side is not matched with any element on the left side.

"So, what can we say about the correspondence $j \upharpoonright B$ and about the set B itself?"

I could see that $j \upharpoonright B$ was just like j—the diagram made it clear that $j \upharpoonright B$ is a Dedekind self-map from B to B. And this means that B is in 1-1 correspondence with a proper subset itself; in this case, that subset is $\{3, 4, 5, \ldots\}$. Therefore B is a Dedekind infinite set.

"Excellent," Dr. Wu said, on hearing my reasoning. "And what is the critical point of this new map $j \upharpoonright B$?"

"The number 2 was left out; so 2 is the critical point," I said.

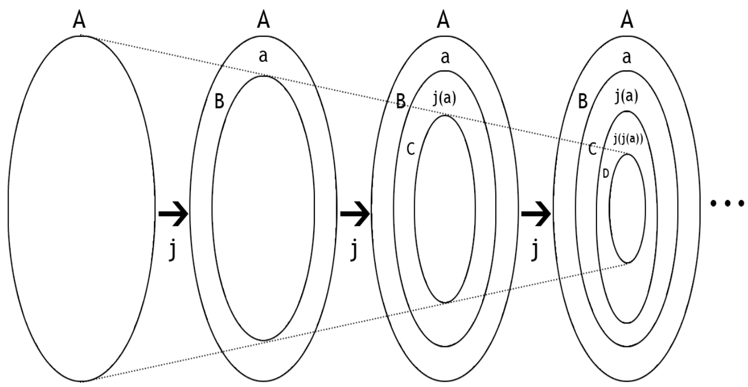
"Right," he continued. "Also recall that $2 = j(1)$. All these points tell us something very important about a Dedekind self-map $j : A \to A$: The effect of j is to transform A into another set B, which is a proper subset of A. This new set B continues to be Dedekind infinite. Moreover, B gives rise to another Dedekind self-map, from B to itself, by considering the restriction $j \upharpoonright B$ of j to B. Moreover, j takes its critical point 1 to the critical point 2 of the new Dedekind self-map $j \upharpoonright B$.

"What I am getting at with all these observations is that, although j transforms A—in fact, in this case, *every* element of A is taken by j to a value different from its original value (i.e. $j(x) \neq x$ for all $x \in A$)—j also *preserves* the essential characteristics of A, namely that of being a Dedekind infinite set, and being the domain of a Dedekind self-map.

"What's more, if we look at the range of $j \upharpoonright B$, we uncover another Dedekind infinite set C, and the correspondence $j \upharpoonright C$ is another Dedekind self-map, now having critical point $j(j(1))$. This process continues as we repeatedly apply j and restrict it to the new range each time.

"Let me show you a diagram of the situation. I will start with any Dedekind self-map $j : A \to A$ with a critical point a. In our previous example, a was the number 1."

He drew the following:



Transformational dynamics of a Dedekind self-map $j : A \to A$

"If we collect together the sequence of critical points of all these new Dedekind self-maps, we obtain

$$(*) \qquad a, j(a), j(j(a)), \ldots$$

"This sequence looks a lot like the sequence of whole numbers $0, 1, 2, \ldots$. And, as a matter of fact, the sequence $(*)$ of critical points can be seen as a *blueprint* for all the whole numbers. In particular, using the terminology I introduced earlier today, this sequence of critical points, collected into a set, is a *copy* of the set $\mathbb{N}$ of natural numbers. Therefore, any Dedekind infinite set is also infinite. I will leave it as an exercise for you to check that, conversely, every infinite set is also Dedekind infinite.[9] These facts tell us that the existence of an infinite set is equivalent, mathematically speaking, to the existence of a Dedekind infinite set, and to the existence of a Dedekind self-map.

A New Axiom of Infinity and a Perspective from the Ancients

"These observations lead us to a new version of the Axiom of Infinity."

He wrote,

New Axiom of Infinity. There exists a Dedekind self-map.

[9] A fact that is usually mentioned in textbooks concerning Dedekind infinite sets is that the equivalence between the usual definition of "infinite set" and that of Dedekind infinite sets requires the Axiom of Choice. This is not the case here, however, since the definition of "infinite set" that Dr. Wu is using is not the usual one. What is true is that any proof of the equivalence between Dr. Wu's definition of infinite set and the usual the textbook definition (a set is infinite if it *cannot* be put in 1-1 correspondence with any of the sets $\{1, 2, \ldots, n\}$ for any $n \in \mathbb{W}$) does require the Axiom of Choice.

"Now notice that this New Axiom of Infinity does not directly assert anything about the existence of the set of natural numbers, or any other infinite set. Instead, it asserts the existence of a certain type of map, from an 'unbounded' set to itself—unbounded in the sense that, because of the characteristics of this self-map, such a set could never be *finite*. Moreover, this map, as we have just discussed, actually gives rise to the set of natural numbers. Like our physics model, this view about the infinite takes the dynamics of an underlying field as primary, and views the discrete quantities that are derived from it as secondary. Now, even though the mathematical content of the New Axiom of Infinity is equivalent to that of the original, it provides quite a bit more intuition about the nature of infinite sets and about 'where the natural numbers come from.'

"We will see in later lessons that this added intuition will be significant enough to suggest a direction for addressing the Problem of Large Cardinals. If we want to know whether certain types of exotic infinities really exist in the universe, we now have the insight that, if they do exist, we may naturally expect them to arise from the dynamics of some kind of Dedekind self-map, possibly a Dedekind self-map from the universe to itself; and perhaps they may even arise from the critical point of such a self-map, just as a primitive form of the natural numbers did.

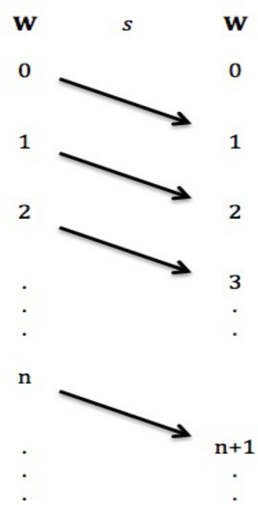
"This view, that the natural numbers arise from the dynamics of an underlying field, not only accords with the model suggested by quantum field theory, as we have discussed, but also is in agreement with some of the ancient philosophies, both from East and West, that attempt to give an account of the origin of the natural numbers. For instance, the early Greek philosopher Pythagoras considered the natural numbers to be rooted in a profound *source*, which he called the *Number of numbers*.[10] In his view, each whole number had its own value, distinct from all others, but each also was rooted in its source and in this way was to be appreciated as a unique expression of the Ultimate. A more recent perspective, due to the modern-day sage Maharishi Mahesh Yogi, and based on the ancient Vedic wisdom, also regards each whole number as embodying 'wholeness,' each giving expression to the *Absolute Number*.[11] Maharishi's Absolute Number is understood to be an 'infinitely dynamic field,' which gives rise to each whole number, with its own unique properties, on the ground of 'infinite silence.'

"There are several points to verify regarding the consequences of a Dedekind self-map; in particular, the formal derivation of the sequence of natural numbers from such a self-map takes a bit of work. The main point is really just to show how the successor function s (recall, the successor function from $\mathbb{W}$ to $\mathbb{W}$ takes each whole number to the next larger whole number: $s(n) = n + 1$) emerges from a sequence of critical points $a, j(a), j(j(a)), \ldots$, as described earlier. Notice that the successor function is itself a Dedekind self-map, since its range leaves out the number 0."

He drew the following:

[10] See D'Olivet, F. and Redfield, N.L. (1992, p. 137).

[11] This insight is part of the Vedic philosophy of Maharishi Mahesh Yogi and will be elaborated further in subsequent lessons. A good reference is Maharishi Mahesh Yogi (1996).



"A very interesting consequence of this fact," he continued, "and of this derivation of s from the sequence of critical points, is the fact that the successor function is the 'smallest' possible Dedekind self-map; more precisely, the successor function s is *uniquely embedded in* every Dedekind self-map. A fancy way to say this is that *s is initial in the category of Dedekind self-maps*. This result is an elaboration of the fact that every infinite set contains a copy of $\mathbb{N}$; that $\mathbb{N}$ is 'embedded in' every infinite set.

Initiality of the Successor Function

"For fun, I will show you a diagram, called a *commutative diagram*, that expresses this embedding theorem in a very compact way. Later in the course it will be useful to look at commutative diagrams more deeply, but for now, I am showing it to you as a form of entertainment only! There is no need to master any of the details, though I will let you try a few exercises to get familiar with the concept of an 'initial Dedekind self-map.' The definition of 'initial' in this context will be part of the exercises.

"The embedding theorem can be formulated in the following way."

He wrote:

Theorem (*Successor Map Embedding Theorem*)**.** *Given any Dedekind self-map $j : A \to A$ with critical point a, there is a unique transformation $\tau : \mathbb{W} \to A$ such that $\tau(0) = a$ and the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{W} \arrow[r, "s"] \arrow[d, "\tau"] & \mathbb{W} \arrow[d, "\tau"] \\ A \arrow[r, "j"] & A \end{tikzcd}$$

To say that the diagram is commutative means

$$\text{for all } n \in \mathbb{W},\ \tau(s(n)) = j(\tau(n)).$$

In other words, if you start with an element n of $\mathbb{W}$ in the upper left corner, and first follow its path going to the right (using s) and then down (using τ), resulting in an output z_1, and then you follow the path n follows by initially going down from the upper left (using τ) and then going to the right (using j), resulting in output z_2, the end result is that $z_1 = z_2$.

$$\begin{tikzcd} n \arrow[r, "s"] \arrow[d, "\tau"] & s(n) \arrow[d, "\tau"] \\ \tau(n) \arrow[r, "j"] & z_2 = z_1 \end{tikzcd}$$

"The diagrams show how the downward pointing map τ embeds $\mathbb{W}$ in A and s in j.

Toward the Experience of the Infinite

"To sum up, we found our definition of 'infinite set,' which states that X is infinite if and only if it contains a copy of $\mathbb{N}$, to be somewhat unsatisfying. When we looked to the modern-day solution offered by axiomatic set theory, we found that the axiom that guarantees the existence of an infinite set in the universe of sets is also rather unsatisfying—that axiom, called the Axiom of Infinity, asserts little more than the existence of the set of whole numbers, formalized as particular sets. We are left wondering about the possible origin of the whole numbers and the real meaning of 'infinity' in mathematics. We then applied an ancient perspective, that the enormous collection of discrete natural numbers is, fundamentally, an expression of a unified source. We noted the strong parallel with the quantum field theory view that discrete particles are simply excited states of an unbounded invisible field. We then made use of the fact, well-known to mathematicians, that an equivalent form of the Axiom of Infinity is the statement that a Dedekind-infinite set exists, or, equivalently again, that a Dedekind self-map exists. We arrived at a New Axiom of Infinity which asserts the existence of a Dedekind self-map. Shifting the emphasis from the existence of a vast collection of distinct objects to the dynamics of an unbounded field, the New Axiom of Infinity gives us, intuitively speaking anyway, more information about what the mathematical infinite is all about; natural numbers are seen to arise as a sequence of critical points emerging from the dynamics of a Dedekind self-map.

"This New Axiom of Infinity provides the sort of information we might want to consider when trying to answer other questions about mathematical infinities. I mentioned to you in class some time ago that mathematicians have found it difficult to explain the presence of 'large cardinals'—enormous infinities that are so big, they can't be proven to exist—in the mathematical landscape. As I have mentioned, we will attempt to address this problem in our work together. Today, we have landed upon a big clue: To account for any kind of mathematical infinity, it is reasonable to ask whether it could arise from some kind of Dedekind self-map. As we suggested before, it is reasonable to conjecture that large cardinals arise from a Dedekind self-map from the universe to itself in a way similar to emergence of the natural numbers from a Dedekind self-map from some set to itself. We will explore this conjecture as we move forward in our class.

"Although I think it is fair to say that our New Axiom of Infinity is somewhat more satisfying than the original, we can still ask if even this new notion gives us all that we might hope for. I would like to suggest that *any* mathematical concept that we come up with may still fail to satisfy completely. A fundamental difficulty is that, for any definition I could possibly give, what it actually *defines* is never anything more than a *concept*. Just like the point about 'tasting the fruit' that I was discussing with you last time: Without tasting the fruit, you may not be convinced it exists; and even if you are, your knowledge of the fruit will not satisfy the thirst for knowledge as long as you have just the concept of it. Concepts can be stimulating, but it is often helpful to view them as an invitation to direct experience. I would say that a very profound satisfaction awaits those who embark on the path of *experiencing* the infinite, just as we discussed in our last meeting. As you will see, there is an extraordinary connection between the concepts that are explored in the mathematical study of the infinite and the threads of insight that come from a direct experience of this field.

"In fact, the insights we have mentioned in our attempt to formulate a more suitable Axiom of Infinity suggest where to look for a direct experience of the Infinite. The ancient Greek and Vedic perspectives on the matter consider not just numbers, but also all of what we could call 'manifest existence' as a discrete and sequential expression of an unmanifest continuous field. This perspective accords with the quantum field theory viewpoint, as we discussed earlier, according to which all discrete expressions are excitations of invisible, continuous quantum fields. In each case, the 'Infinite' has been located by going beyond the obvious, discrete expressions and stepping into something far vaster, coherent, and unbroken. By analogy, if we want to discover the Infinite experientially, within our own subjective life, it seems natural to seek that which underlies immediate obvious content of experience. Just as the particles of objective existence are excitations of hidden quantum fields; just as natural numbers may be seen as discrete expressions of a more fundamental unbounded field; so likewise the discrete impulses of thought that constitute our subjective life may also be seen as excitations of another kind of field—a field of consciousness. So I offer this as another hint about experiencing the Infinite: the Infinite can be found as the field that lies beyond thought, beyond the thinking process.

"The traditional way to gain this direct experience is through some form of *meditation*. Laozi urged *xīn shàn yuān*—become skilled in diving deep into the heart.[12] However, I need to

[12]This is Dr. Wu's translation of a line in verse 8 of the *Tao Te Ching*.

point out that the term 'meditation' has come to mean many things over the centuries. Not all forms of meditation lead to an experience of the Infinite, and this happens to be true for two reasons. First, some forms of meditation do not even aim for this experience; there are plenty of good things that can be achieved with certain techniques that have nothing to do with Infinity! Some meditation techniques focus exclusively on breathing, on exploring the meaning of certain expressions, or on a variety of other things. Secondly, there are some forms of meditation that view experience of the Infinite as a kind of objective, but an effective procedure for reaching this goal has been lost. Certain techniques that require a great deal of focus or mental effort are in this category. As you will discover, the internal path to the Infinite is necessarily an easy-going, effortless process; strenuous efforts will not reach the mark.

"What is needed is that the process of thinking should be given a chance to settle down. If the mind can be given a way to turn inward and just take it easy, it will naturally unfold into a more expanded state and eventually just drop into the Infinite. Just like that. So so simple. Straining for this result will just lead to frustration because the very act of straining and effort is opposed to 'natural unfoldment.' The mind can't be forced to relax into its unbounded state—but it will be only too happy to ease into it on its own!"

"Well, that sounds great, Dr. Wu," I said, feeling that maybe at last he was going to reveal the secret. "But how do you do that kind of meditation?"

"Here's what I'd like you to do. We happen to live in an area where meditation techniques of all kinds are popular. Why don't you look around a bit and see if you can find a technique that suits our purpose for this course. You will want to find a technique that is easy to do and promotes a state of relaxation. And the technique should very effortlessly allow the mind to relax into the Infinite, to the field beyond the field of thinking. Use these points as criteria. You may find that there are other techniques that don't meet these criteria, but which also interest you. For now though, let's focus on finding one solid technique that does meet the criteria, for the purpose of our work together. When you find something, let me know and if it sounds good, we'll go with it. This will be the tool that will help you appreciate the insights into Infinity that are spoken of by the ancient sages."

This was great. I was hoping Dr. Wu would show me how to do this kind of meditation right then and there, but at least he seemed to be confident that such a technique would be available in this part of the world—no trips to the Far East required! I was ready to begin my search.

"As a final remark for today," Dr. Wu resumed, "I would like to put our discussion about 'the infinite' in mathematics in perspective. Even though we have found an interesting new Axiom of Infinity, it will be useful for us in the upcoming lessons to rely on our original definition of 'infinite set'; namely, that a set is infinite if it contains a copy of $\mathbb{N}$. This definition will prove to be very convenient, and would arise no matter which version of the Axiom of Infinity we started with. I will give you some exercises about both the original and the new Axiom of Infinity, but for subsequent lessons, we will use the original version and our first definition of 'infinite set.' "

As I headed home, I found myself absorbed in thoughts about mathematical infinity. The idea of stating as an axiom that the *source* of the set of natural numbers exists, rather than saying that the set itself exists was really interesting to me. To begin with, I had never considered the possibility that the natural numbers *had* a source. And I remembered this expression he had said—that the set of natural numbers arise from the *dynamics of an unbounded field.* This seemed profound to me, and the analogy with physics seemed reasonable: Natural numbers appear as special values in the dynamics of a Dedekind self-map, just as electrons appear as excited states of a quantum field. Although I didn't yet fully grasp the significance of the "Problem of Large Cardinals," I did know it required some insight into the nature of the mathematical infinite. So, at least with Dr. Wu's New Axiom of Infinity, the axiom suggests a direction for investigating more deeply: Perhaps, like the natural numbers, these "large cardinals" arise from the dynamics of a Dedekind self-map. Maybe the fact that such a self-map $j : A \to A$ has a strong kind of preservation property (that when you look at the restriction of j to its range, you get another Dedekind self-map) or the fact that its critical point generates the natural numbers—maybe these characteristics have something to do with generating stronger kinds of infinities. This is what he seemed to be suggesting.

As I came closer to the downtown area where my favorite pizza place could be found, my busy mind started to settle down a bit. It was definitely time for a pizza. As I settled into the restaurant, I remembered that I would be looking for some kind of "infinity practice" in the coming days—some form of meditation that would offer this experience of the Infinite in an effortless way. It almost seemed too much to believe that such practices existed. I would find out soon enough.