Introduction

"Dr. Wu, what is Infinity?"

The question popped up in a math class I was taking in my freshman year at college. Dr. Wu, a top-notch professor at the university and teacher of the course, was obviously delighted with the question.

"What an excellent question!" he began.

Dr. Wu had been giving us "math questions" to exercise our skills. I think we were all ready for a break, and the question provided an immediate distraction. Clearly, this was a question that interested the professor, so we could expect a substantial break from the course material till he had said his piece about it. Since Dr. Wu was blessed with a talent for making ideas of all kinds entertaining and even inspiring, I was looking forward to his response to the question.

"There are many types of answers that could be given," Dr. Wu continued. "I think we all have some inkling of what is meant by 'infinity.' Does anyone want to volunteer some ideas about it?"

After a few moments, a couple of hands went up, somewhat reluctantly. Dr. Wu called on one of the students.

"Is it like eternity? Something like 'forever'?"

"Yes, certainly, this is a way of expressing the idea of infinity. Anyone else?"

Another student said, "What about limitlessness, or being unlimited? That seems like another expression for 'infinity.'"

"Yes, very good. Other ideas?"

"What if we just say 'infinity is that which is not finite'?"

Dr. Wu enjoyed that one. "No way to deny it. Infinite must be 'not finite.' In ancient times, in certain philosophies and religions, it was believed that the Infinite could not be described, except in terms of negations such as this. In that view, we can't say what the infinite *is*, only what it *is not*."

Dr. Wu called on one final student whose idea about infinity had a different feel to it.

"Infinity is the totality of everything," the student said confidently.

"Yes, very good. This is a different sense of infinity. It suggests that even if 'every thing' is, on its own, *finite*, still, the *totality* is not finite. This point of view is very interesting.

"Philosophers, theologians, mystics, and mathematicians have all studied the Infinite in their own ways. Mathematics in particular has studied the Infinite intensively for about 100 years. The subject began with the work of Georg Cantor who demonstrated the need for a rigorous study of infinite sets in mathematics—sets like the set of natural numbers $1, 2, 3, 4, \ldots$. Prior to Cantor's work, it was believed that the natural numbers, being infinitely vast, could not be assembled

into a single collection. The work of Cantor and other researchers of the day demonstrated that the doubts about studying the natural numbers as a completed collection were little more than superstitious beliefs, as amazing as that may sound.

"As Cantor's view began to win support, he unveiled other surprises. He showed that in fact there are different *sizes* of infinity. For instance, he showed that there are 'more' points on a line than there are natural numbers. Even more amazingly, he showed that there is an endless hierarchy of ever larger infinite sets. In fact, he showed that there are more different sizes of infinite sets than there are points on a line!

"As mathematicians became accustomed to Cantor's landscape of infinities, other researchers dug deeper still and began to unearth some staggeringly big infinite sets. In Cantor's terminology, sizes of infinite sets are called *infinite cardinals*; researchers began to discover some truly enormous infinite cardinals, which came to be known as *large cardinals*.

"In those early days, no one knew what to make of these large cardinals. They had such strong properties, it was difficult to prove that they existed at all. By the 1930s, it had actually been proved that it was *impossible to prove that large cardinals exist or that they are even consistent* with the rest of mathematics. Yet, over the decades, large cardinals have proven to be key ingredients in the solutions of many significant problems in many areas of mathematics.

"The problem of giving some sort of mathematical account of large cardinals is sometimes called the *Problem of Large Cardinals*. Cantor's profound intuition about the mathematical landscape led to the acceptance of *infinite* sets; the mathematical community has, in a sense, been seeking some equally profound intuition that could point the way toward a resolution of the Problem of Large Cardinals.

"In any case, I think these few points should give you some sense of the modern history of the mathematical infinite. It's an exciting area of research!"

I couldn't help being intrigued. More points on a line than natural numbers? Different sizes of infinity? And what in the world could large cardinals be? If an infinite cardinal is supposed to be the size of some set, some collection, then what sort of set would have as its size a *large cardinal*? How bizarre it all seemed. And if they can't prove these large cardinals exist, then how could they be turning up in mathematical theorems?

Dr. Wu paused and appeared to be getting ready to get back to the main topic for class. Instead, he asked a question that caught me by surprise. "Do you suppose that the Infinity that is studied by mathematics is the same Infinity that philosophers and theologians and mystics talk about?"

Some brave student offered to answer Dr. Wu's question.

"Dr. Wu," he began, "the infinite studied by mathematics just seems to be a concept. Philosophers and theologians are more interested in examining the 'Infinite' as something that really exists. I would guess that these are different worlds."

"This is an excellent point," Dr. Wu replied. "And perhaps you are right. But here is something to consider. Consider an idea like 'justice' or 'virtue.' Much has been written about these topics and efforts have even been made to make a rigorous study of them. Yet, in this case, the rigorous theory is about something that most of us have experienced. Most of us would agree that we have had our own encounters both with 'justice being done' and with injustice, and similarly for 'virtue.' These experiences are quite different from a philosophical exercise in which we examine hypothetical situations and try to determine what is just or virtuous. The immediate,

often inspiring experience we have when we hear about a virtuous act or how dedicated effort is rewarded in just the right way is not the result of any sort of intellectual activity, though it can of course be reflected upon later. Now, hypothetically, imagine a society that evolves in such a way that the experience of 'virtue' and 'justice' erodes away completely. Such a society would be left with intricate theories about these notions without any point of reference in human experience. A dweller in such a world, when asked what the words 'virtue' and 'justice' are actually referring to, would say, 'These notions are just concepts that are part of a formal theory, which we use for scientific purposes. They don't point to any underlying reality. I mean, how could there be such a thing as 'virtue' in the real world?'

"I'm sure you can see where I am going with this. Perhaps the 'reality of the Infinite' that is discussed by certain philosophers, mystics, and theologians throughout history is really part of our heritage as human beings, though mostly forgotten. What we have today is an elaborate mathematical theory of the infinite. What if the theory is actually talking about something real? What if, as the mystics say, the Infinite can be experienced? Would the Infinite experienced in this way look like the Infinite that is studied in mathematics? If you brought scientific tools to an experience of the Infinite, would you find yourself reconstructing the modern mathematics of the infinite?

"And if the Infinite can be experienced, and if the ancient wisdom concerning the Infinite, found in scriptures and ancient teaching around the world, happens to be true, do you think it might be possible to apply this wisdom to address a modern problem in mathematics? Maybe even to get a hint about a solution to the Problem of Large Cardinals?

"This probably just seems too far-fetched to be true. And yet a solution to the Problem of Large cardinals has indeed been proposed using exactly this approach. Using principles distilled from ancient texts on the nature of the 'Infinite,' the research provides a plausible and rigorous mathematical account of large cardinals. I find research of this kind to be very intriguing, to say the least.

"Anyway, I know you are all only too happy to distract me from diving too deeply into the material for today's class. I don't think it's really too far off the topic though—it's good to have a little appreciation of the history of the mathematical infinite, since it is directly related to the topics we will be studying in our course."

With these transitional remarks, Dr. Wu returned to his lecture, but I was just not able to pay much attention. This idea that ancient wisdom concerning the Infinite could have a bearing on modern mathematical research just seemed so appealing. It seemed so crazy, and yet, someone had actually taken the idea seriously and gotten somewhere with it. It was amazing to me.

In the following days, I found myself thinking and wondering about the things that Dr. Wu had said about Infinity. I kept hoping he would say something more about it, but subsequent classes stayed closely with the topics of the syllabus.

I started poking around our university library in search of books on these subjects. To my amazement, I did find several texts that made explicit mention of large cardinals. But these were graduate texts in mathematics, for students of set theory—it was all incomprehensible to me. I wasn't even able to find a definition of *large cardinal* in those books.

So I started looking for books written for a more general audience. This time I was more successful—I found quite a number of books about infinity, and all of them looked very interesting. But because I had been "bitten" by Dr. Wu's remarks that day, I felt driven to find his perspective

reproduced in at least one of these books—unfortunately, without success. I did find some books that discussed Cantor and different sizes of infinity, though even their more accessible treatments were hard for me to understand. But only one of these books actually mentioned large cardinals — a book by Rudy Rucker called *Infinity and the Mind*.[1] This looked like a very interesting book, and one that I would come back to later. Rucker did address large cardinals and treated them with the same awe and respect that seemed to inspire Dr. Wu. And he also discussed ways of broadening the foundation of mathematics to explain where some of them come from. I also noticed that perspectives from mystical traditions were woven throughout his treatment, though the kinds of mystical experiences he talked about didn't seem to be accessible to ordinary people.

I checked his book out and took it home. As I sat down with it, I was fascinated with all the concepts and new ideas that seemed to fill every page. One thing I was still very curious about, which I didn't find in this book, was this "solution" to the Problem of Large Cardinals, which Dr. Wu had mentioned. And I had this odd feeling that Dr. Wu's perspective on these topics was in some way different, and I was still very curious to know what more he would have to say about them. Not sure where to go from here, I called it a night and hoped some insight about my next step would come in the morning.

When I woke up the next morning, the obvious answer popped into my head: If I wanted Dr. Wu's perspective, why don't I talk to Dr. Wu himself? I felt a little foolish for not thinking of this right away, but now at least it was clear that this was my next step.

After class that same day, I approached Dr. Wu.

"Dr. Wu, I wanted to ask you something about what you said several weeks ago about large cardinals and ancient philosophies."

"Oh, yes, yes, what is your question?" he replied with an interested tone.

"Well, I guess I have many questions that have been racing through my mind ever since your lecture about all of this, but it's hard to think of one at the moment. You know, what I would really like to know is how I could learn about this subject. Do math majors learn about this? Or have you written anything about it?"

"You would like to learn more?" he smiled. "Yes, it's a very wonderful subject. Unfortunately none of the popularized treatments of mathematics in print today address all the points I mentioned to you in class, though I suspect someone will write such a book soon. There are technical mathematics texts and papers that can tell you all about large cardinals, but reading these requires a great deal of technical training, and still the role of 'ancient wisdom' will not be apparent even in these works. The fact is, this particular line of investigation is fairly new. Though many of the most fascinating parts of the subject can be grasped easily by non-experts, the material has not found its way into any of the standard mathematics courses. So, where does that leave you?"

"Yes, how can I learn more?"

After a bit more discussion, Dr. Wu suggested that I might like to try a directed reading with him on these topics. I was worried whether I would have the necessary skills, but was very excited about the possibility.

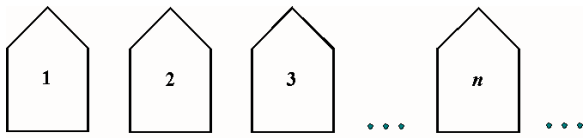
"As a pre-requisite to this Directed Reading," he said, "I am going to ask you to solve a little problem for me, a problem about infinity. You won't have any idea how to solve it at first. But

[1] See Rucker, R. (2004).

it's certainly something you have the ability to solve. I want you to show me that you are willing to do what it takes. If you can find a correct answer, then we can begin a course of study."

Dr. Wu then proceeded to tell me the problem. It had to do with a hotel with infinitely many rooms. I wrote down the problem as he described it and felt some excitement at the prospect of starting down this unusual path.

Dr. Wu's problem for me was this: Imagine that there is a hotel somewhere with infinitely many rooms. This means that every one of the natural numbers $1, 2, 3, \ldots$, appears as a room number for some room in the hotel. (It does *not* mean that some room has an infinity symbol ∞ on it!)



The Infinite Hotel

Imagine that every room in the hotel is occupied. (This means the same thing as it does for any Holdiay Inn—if you go into a hotel that has "No Vacancy" and try walking into any of the rooms, you'll find that someone is staying there. Don't worry about the fact that there aren't enough people in the world to really fill all the rooms in the hotel—this isn't intended to be realistic.) One day a fellow named Bob walks into the hotel and, not having seen the "No Vacancy" sign, comes up to the hotel manager and asks for a room. Without hesitating, the hotel manager says, "Yes sir, we can find a room for you tonight." The hotel manager announces on the public announcement system of the hotel (which pipes into every room in the hotel) that there have been some changes in room assignments, and that everyone in the hotel should come in to check to see if his/her room number has changed. What does the hotel manager have in mind? How does he plan to find a room for Bob? A correct answer requires that you specify which room Bob will stay in, and also which room each of the current guests of the hotel will also stay in. For instance, you must say where the guest currently staying in Room 1 will stay tonight; where the guest currently in Room 2 will stay; and so forth.

I had to spend a few days working on Dr. Wu's problem. An answer finally came to me—an answer so simple I couldn't understand why I hadn't seen it much sooner. But I was hooked. I liked thinking about infinity. I didn't know what else Dr. Wu had in store for me, but I wanted to take the ride.

Exercises

Exercise Intro1. What is your view about this idea of "infinity"? What is it? Is it real? Does it exist? If so, where? In what sense? Is it a religious idea, a mathematical idea, a philosophical idea? Or not an idea at all?

Exercise Intro2. Look up "Infinity" on Wikipedia, the online encyclopedia, to see how people think about this notion in various traditions of knowledge. The url is: `http://en.wikipedia.org/wiki/Infinity`