Equivalent Forms of the Axiom of Infinity

Axiom of Infinity 1. *There is a set that contains each finite ordinal as an element.*

"The Axiom of Infinity is the axiom of Set Theory that explicitly asserts that 'an actual infinity does exist'. In the formulation of this axiom that I have given here — version 1 — I am using the fact that the other axioms of Set Theory provide us with enough axiomatic strength to build the usual finite ordinals:

$$0 = \emptyset$$

$$n + 1 = \mathrm{ss}(n)$$

"But, without the Axiom of Infinity, there is no guarantee that the finite ordinals can be collected together to form a *set*; for the theory ZFC−Infinity, the finite ordinals must be viewed as a *class*.

"Once we add this Axiom of Infinity to the other axioms, we can *define* the usual set ω of finite ordinals using the Axiom of Separation. Let X be a set containing all the finite ordinals. Then

$$\omega := \{x \in X \mid x \text{ is a finite ordinal}\}.$$

"Here, the statement 'x is a finite ordinal' is an English version of the formula that defines the class of finite ordinals.

"The Axiom of Infinity therefore makes the familiar set ω available. An important next step is then to restate our definition of 'infinite set': A set is to be considered *infinite* if it includes, as a subset, a *copy* of ω. Or, stated more precisely in the language of functions:

He wrote:

Definition. A set Y is *infinite* if there is a 1-1 function $f : \omega \to Y$.

"Under this definition, f tells you precisely what the copy of ω is: Let

$$Z = \{y \in Y \mid \text{ for some } n \in \omega, y = f(n)\} = \mathrm{ran}(f)$$

Then Z is a copy of ω that is a subset of Y.

"You may notice that I have not expressed this axiom using logical symbols yet. This is because our way of expressing the axiom does not permit a very simple formulation in the language of set theory, because we are making use of an auxiliary concept 'finite ordinal'.

"For this reason, it is common to state this axiom in a slightly less intuitive, but equivalent, way. Let us say that a set Y is *inductive* if it satisfies the following two properties:

(1) $\emptyset \in Y$
(2) for any set x, if $x \in Y$, then $x \cup \{x\}$ is also in Y

"If we apply (2) with $x = \emptyset = 0$, we find that $1 = 0 \cup \{0\} \in Y$. Apply (2) again using $x = 1$, and we find that $2 = 1 \cup \{1\} \in Y$. In this way, we see how the statement that Y contains all the finite ordinals is conveniently embedded in the definition of 'inductive set'.

"This leads to an alternative formulation of the Axiom of Infinity:

Axiom of Infinity 2 There is an inductive set.

$$\exists w\,(\emptyset \in w \,\wedge\, \forall x(x \in w \longrightarrow x \cup \{x\} \in w)).$$

"Let's verify that, working in the theory ZFC − Infinity, these two versions of the axiom are equivalent."

He wrote:

Proposition. From the theory ZFC − Infinity, it is provable that the Axiom of Infinity 1 and the Axiom of Infinity 2 are equivalent.

"To prove the theorem, suppose Axiom of Infinity 1 is true, so we assume there is a set X that contains every finite ordinal. Again, recall that ZFC − Infinity is enough to give us the usual definition of the finite ordinals. As described above, we may define ω to be the set $\{x \in X \mid x \text{ is a finite ordinal}\}$. We now verify that ω itself is an inductive set.

"ω satisfies (1) in the definition of inductive set, since $\emptyset \in \omega$. For (2), suppose $x \in \omega$. Then x is a finite ordinal. Clearly the successor $\mathrm{ss}(x) = x \cup \{x\}$ of x is another finite ordinal. Thus, condition (2) of the definition of inductive set is also satisfied. Therefore, we have prove Axiom of Infinity 2.

"To prove the converse, assume the Axiom of Infinity 2 and let w be an inductive set. We need to check that w contains all the finite ordinals. Recall that the theory ZFC − Infinity allows us to perform induction arguments pretty much in the usual way, over the finite ordinals. We let $P(u, v)$ be the property that u is a finite ordinal and $u \in v$. We prove $P(n, w)$ for all finite ordinals, by induciton. For the Base Case, we know by definition of inductive set that $\emptyset \in w$, so $P(0, w)$ holds. For the Induction Step, we assume $P(n, w)$; that is, $n \in w$. By (2) of the definition, $n + 1 = \mathrm{ss}(n) = n \cup \{n\} \in w$, so $P(n + 1, w)$ is true. By induction, therefore, $P(n, w)$ holds for all n, and so w is a set that contains all finite ordinals. This proves Axiom of Infinity 1.

Other Formulations of the Axiom of Infinity

"So we have seen that, from the point of view of setting up basic axioms for Set Theory, a 'cleaner' formulation of the Axiom of Infinity is the simple assertion that there exists an *inductive set*; this notion does not require us to work out the definition of 'finite ordinal' in order to state the axiom. As we showed last time, these two forms of the Axiom of Infinity are provably equivalent

— provable from the axioms of set theory *without* the use of the Axiom of Infinity; that is, the equivalence is provable from ZFC − Infinity.

"Next, I would like to consider a few other equivalent formulations of the Axiom of Infinity. Each formulation captures a particular intuition that many mathematicians have had about the essential characteristics that make a set *infinite*. It is a testament to the clarity of the intuitions behind these formulations that they all turned out to be equivalent, relative to ZFC − Infinity.

"To prove the equivalence of these statements, we will rely just on the theory ZFC − Infinity. It will be helpful at times to consider that we are working in a universe of mathematics that provides a model of this theory. We will denote such a universe with the letter V even though, up till now, we have reserved this letter for the standard model of ZFC. We adopt this convention just because of the familiarity of the notation. The fact is, though, nothing that we do today requires the use of such a model; all the equivalences are provable directly from the theory ZFC − Infinity.

"Now let's think for a moment about some other ways to assert 'there is an infinite set'. The way to come up with ideas for this is to think of properties that an infinite set must have. Some of these properties may turn out to be significant enough to capture the idea that an 'infinite set exists'. For instance, the intuition we began with in this course is that, if we are going to accept some infinite set into mathematics, then surely we should accept the set of natural numbers. So, though we didn't make a formal statement about it, we began our study with the idea that 'there exists an infinite set' can be captured by declaring 'the set of natural numbers exists' (or, as we developed the ideas further, 'the set ω exists').

"How else might we have begun? You may remember that even back then, I mentioned another way to get started."

Knowing that today's topic was going to be equivalent forms of the Axiom of Infinity, I had gone through my earlier notes to look for ideas. Way back in the beginning, Dr. Wu had mentioned the simple idea that 'infinite' should mean 'not finite'. And 'finite' means 'either empty or in 1-1 correspondence with $\{0, 1, 2, \ldots, n\}$ for some n'. So, this led to another way of saying it.

"Dr. Wu," I said. "What about this statement."

There is a set that cannot be put in 1-1 correspondence with any finite ordinal.

"Excellent," Dr. Wu smiled. "As we discussed in the last lesson, we may assume we know about 'finite ordinals' because ZFC − Infinity is strong enough to permit us to define them, and the idea of '1-1 correspondence' is also definable from this theory. So yes, we would expect this formulation to be an equivalent form of the Axiom of Infinity. Any other ideas?"

Xinyuan had an idea. "One thing that is noticeably different about infinite sets is the way they can be shuffled around to establish 1-1 correspondences. At first, it is surprising that $\mathbb{N}$ and $\mathbf{W}$ have the same size. It's a bigger surprise that $\mathbb{N}$ and $\mathbf{Z}$ have the same size. And an even bigger surprise that $\mathbb{N}$ and $\mathbb{Q}$ have the same size. To establish the correspondence in each case, we had to shuffle around the elements in one of the sets. This kind of trick could never by done with finite sets, so I think it must be an essential feature of the infinite, but I'm not sure how to formulate it precisely."

"Yes, good," Dr. Wu said. "You are on the right track. The examples you mentioned have one thing in common that may help in the formulation."

Xinyuan thought for a moment. "Well, one common feature is that, in each case, the set that looks bigger actually contains the 'smaller' set as a proper subset. I mean, we have $\mathbb{N} \subseteq \mathbf{W}$, $\mathbb{N} \subseteq \mathbf{Z}$, and $\mathbb{N} \subseteq \mathbb{Q}$, and in each case, the subsets are *proper* subsets. But, if A and B are *finite sets* and A is a proper subset of B, it couldn't possibly be true that A and B have the same size. So perhaps the point is that an infinite set is capable of having the same size as one of its proper subsets."

"Exactly right," Dr. Wu smiled. "You have hit upon another equivalence to the Axiom of Infinity: A set is infinite if and only if it can be put in 1-1 correspondence with one of its proper subsets.

"Let's organize these ideas in the form of a theorem:"

He wrote:

Equivalents Of Infinite Theorem. The theory ZFC − Infinity proves that the following statements are equivalent.

(1) There is a set that contains every finite ordinal.
(2) There is an inductive set.
(3) There is a set that cannot be placed in 1-1 correspondence with any finite ordinal.
(4) There is a set that has the same size as one of its proper subsets

"In the last lesson, we verified that (1) and (2) are equivalent. Next, I will show why the existence of an inductive set implies (3). Let's first recall from our lesson on the Axioms of Set Theory that an inductive set X must contain every finite ordinal, and so we can define ω, using the Axiom of Separation, by $\omega := \{x \in X \mid x \text{ is a finite ordinal}\}$. Let us define the *natural numbers* $\mathbb{N}$ to be the set $\omega - \{0\}$; that is, the familiar notation $\mathbb{N}$ will now be used to denote all the nonzero elements of ω. Since we are thinking now that mathematics takes place inside the universe and since in that context, ω is the 'real' set of whole numbers, this use of the notation $\mathbb{N}$ is reasonable. We need the following lemma:

Lemma 1. Suppose ω, defined as the set of all finite ordinals, exists. Suppose there is a 1-1 correspondence between X and ω and suppose $x \in X$. Let $Y = X - \{x\}$. Then there is a 1-1 correspondence between Y and ω.

"Of course, way back when we were first starting our study of the infinite, we proved results like this, but at that stage of the course, our proofs were more informal. Now we take the same ideas and proceed more rigorously.

"You will recall that when we were proving in an earlier lesson that ω is a structural duplicate of $\mathbf{W}$, we showed that ss is a successor function for ω relative to 0; translating this fact into the language of functions, that proof (which is essentially the same in the present context) tells us that ss is a 1-1 function with range $\mathbb{N}$; another way to express this fact is to say that $\text{ss} : \omega \to \mathbb{N}$ is 1-1 and onto. Well, to prove the Lemma, we use essentially the same reasoning to show that, because there is a 1-1 correspondence between X and ω, there is also a 1-1 correspondence h between Y and $\mathbb{N}$. Then, by cleverly combining ss and h (actually, ss and h^{-1}), we obtain a 1-1 correspondence between ω and Y. I will outline more detailed steps in the argument for you to work out in the exercises.

"Now, we may freely use Lemma 1 to establish (3), assuming that ω exists (since we are assuming (2) is true). We let A be the set of all elements of ω for which it is true that *there is no 1-1 correspondence between n and ω*. That is,

$$A := \{n \in \omega \,|\, \text{there is no 1-1 correspondence between } n \text{ and } \omega\}$$

"We will show now that A is an inductive set. As we showed earlier, an inductive set must contain all finite ordinals, so establishing that A is inductive will guarantee that $A = \omega$; in other words, once we show A is inductive, we will know that for every finite ordinal n, there is *no* 1-1 correspondence between n and ω, thereby establishing (3).

"To show A is inductive, notice first that $0 = \emptyset \in A$ since there is no 1-1 correspondence between the empty set, which has 0 elements, and a set having more than zero elements. Now assume $n \in A$. We will show $\text{ss}(n) \in A$ with an indirect argument. Thus, assume that there is a 1-1 correspondance between $\text{ss}(n)$ and ω. By Lemma 1, however, there must also be a 1-1 corresondence between n and ω; this contradicts our induction hypothesis, and so we have a contradiction. We have therefore shown that (2) $\Rightarrow$ (3).

"Next, let's prove (3) $\Rightarrow$ (1). This will establish that all three statements (1)–(3) are equivalent. So, assuming (3) holds, let X be a set for which there is no 1-1 correspondence between X and any finite ordinal. Now recall that one of the consequences of the Axiom of Choice that we discussed in the last lesson is that every set has a cardinality — that is, for every set A, there is a 1-1 correspondence between A and some cardinal number [1]. And since every cardinal is an ordinal, there must be a 1-1 correspondence between our set X and some ordinal number α. But, by assumption, α is not a finite ordinal. Recall that the membership relation $\in$ on α is a linear ordering (in fact, a well-ordering), so that, for every finite ordinal n, we have either that $n \in \alpha$ or $\alpha \in n$. But if $\alpha \in n$ for a finite ordinal, then α itself would have to be a finite ordinal, and this contradicts our assumption about X. Thus, α contains every finite ordinal as an element, and this establishes (1).

[1] This fact does not depend on the Axiom of Infinity.

"We move now to statement (4). Notice that Lemma 1 immediately implies that (4) is true because it tells us that ω is in 1-1 correspondence with its proper subset $\mathbb{N}$. Lemma 1 depends on (1) and (2), so, we get immediately that (1) and (2) together imply (4).

"For the other direction of the equivalence, we show that (4) $\Rightarrow$ (1). Let's begin by noticing that no finite ordinal can be put in 1-1 correspondence with a proper subset of itself. This is proved by induction on the finite ordinals. Again, we resort to the version of induction that can be used in ZFC $-$ Infinity, without assuming that ω exists. For our property that depends on the finite ordinals n, we define

$$P(n): \ n \text{ cannot be placed in 1-1 correspondence with a proper subset of itself}$$

"Certainly $P(0)$ holds since $\emptyset$ has no proper subsets. Assuming $P(n)$ holds, we will show that $\mathrm{ss}(n)$ cannot be placed in 1-1 correspondence with a proper subset of itself. Suppose, by way of contradiction, that we have a 1-1 onto function $f : \mathrm{ss}(n) \to A$ where A is a proper subset of $\mathrm{ss}(n) = n \cup \{n\}$.

"Notice that $A \neq \emptyset$ since there can be no 1-1 correspondence between a set with no elements and a set with > 0 elements. But then $|A| > 0$. Let $x \in A$ be the image of n; that is $f(n) = x$. Define a proper subset of n as follows: If $n \notin A - \{x\}$, just let $B = A - \{x\}$. If $n \in A - \{x\}$, let $B = A - \{n\}$. We will show there is a 1-1 correspondence between n and B, and this will contradict our induction hypothesis since B is a proper subset of n. To do this, we just need to show there is a 1-1 correspondence between n and $A - \{x\}$; we will get the required 1-1 correspondence with B in the second case mentioned by noticing that there is also a 1-1 correspondence between $A - \{x\}$ and $A - \{n\}$.

"So, why is there a 1-1 correspondence between n and $A - \{x\}$? Think of f as a set of ordered pairs. One of its ordered pairs is (n, x). Let g be the function you get by removing this pair from f. Clearly, this is a 1-1 correspondence between n and $A - \{x\}$. Therefore, we have found the correspondence between n and one of its proper subsets, leading to a contradiction. This completes the induction that establishes our intermediary result: No finite ordinal can be put in 1-1 correspondence with a proper subset of itself.

"Now we complete the proof of (4) $\Rightarrow$ (1). Let X be a set that can be put in 1-1 correspondence with one of its subsets. Using ZFC $-$ Infinity, we can show, as usual, that there is a 1-1 correspondence between X and some ordinal α. So, α must also have this special property that it can be put into 1-1 correspondence with a proper subset of itself. But by what we just proved, we must conclude that α is not a *finite ordinal*. Arguing as we did earlier, we may conclude that every finite ordinal must be an element of α. Therefore, (1) holds.