THE GENERALIZED BOREL CONJECTURE AND STRONGLY PROPER ORDERS

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Abstract. The Borel Conjecture is the statement that $C = [\mathbb{R}]^{<\omega_1}$, where C is the class of strong measure zero sets; it is known to be independent of ZFC. The Generalized Borel Conjecture is the statement that $C = [\mathbb{R}]^{<\mathbf{c}}$. We show that this statement is also independent. The construction involves forcing with an ω_2-stage iteration of strongly proper orders; this latter class of orders is shown to include several well-known orders, such as Sacks and Silver forcing, and to be properly contained in the class of ω-proper, ω^ω-bounding orders. The central lemma is the observation that A. W. Miller's proof that the statement $(*)$ "Every set of reals of power $\mathbf{c}$ can be mapped (uniformly) continuously onto $[0,1]$" holds in the iterated Sacks model actually holds in several other models as well. As a result, we show for example that $(*)$ is not restricted by the presence of large universal measure zero (U_0) sets (as it is by the presence of large C sets). We also investigate the σ-ideal $\mathcal{J} = \{X \subset \mathbb{R}: X$ cannot be mapped uniformly continuously onto $[0,1]\}$ and prove various consistency results concerning the relationships between $\mathcal{J}, \mathrm{U}_0$, and AFC (where AFC $= \{X \subset \mathbb{R}: X$ is always first category$\}$). These latter results partially answer two questions of J. Brown.

0. Introduction

A set $X \subset \mathbb{R}$ has strong measure zero, or property C, if for every sequence $\langle \varepsilon_n : n \in \omega\rangle$ of positive reals converging to zero, there is a sequence $\langle I_n : n \in \omega\rangle$ of intervals covering X such that for all $n \in \omega$, the length of I_n is less than ε_n. The Borel Conjecture is the statement

$$C = [\mathbb{R}]^{<\omega_1}.$$

It is well known that Martin's Axiom implies the failure of the Borel Conjecture (see [M3]); on the other hand, in [La], Laver builds a model in which the statement holds.

As a natural generalization, we say that the Generalized Borel Conjecture is the statement

$$C = [\mathbb{R}]^{<\mathbf{c}}.$$

The conjecture fails under MA; the main result of this paper is the construction of a model in which the conjecture holds. We have the following:

0.0 **Theorem** *The Generalized Borel Conjecture is independent of the axioms of set theory.*

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The result is somewhat surprising since a parallel conjecture for other well-known σ-ideals is false in ZFC; for example, if we replace C by the class U_0 of universal measure zero sets or by the class AFC of always first category sets (defined below), the conjecture is false by Theorem 0.7 and Lemma 0.9 below.

Our model is obtained by forcing with an ω_2-stage countable support mixed iteration of the Infinitely Often Equal Reals (IOER) order (see [M1, §7] and §1 below) and the Sacks order; in [M1], Miller observes that forcing with the IOER order makes the set of ground model reals have strong measure zero; thus, if one forces with an ω_2-stage iteration (from a model of, say, GCH), every set of reals of power $< \mathbf{c}$ has strong measure zero. In [M2], Miller shows that in the iterated Sacks model, every set of reals of power $\mathbf{c}$ can be mapped continuously onto the unit interval $[0, 1]$; as we show below, "continuous" can be replaced by "uniformly continuous"; one then shows—and this is the difficult part—that Miller's result holds even when the Sacks order is not used on all (but at least stationarily many) of the coordinates of the iteration. (Thus since uniformly continuous images of strong measure zero sets have strong measure zero (see [M3]), no such set has power $\mathbf{c}$ in our model.) In building our model, we noticed that the conditions under which a partial order P could be used in a mixed iteration with the Sacks order to produce a model of Miller's result were satisfied by several well-known orders; we have formulated these conditions as axioms for a class of orders we call strongly proper. Since the Sacks order is itself strongly proper, many of the more technical arguments are made more concise by this unified axiomatic approach.

Considering iterations of strongly proper orders also suggests an approach to another interesting problem which arises from a closer analysis of Miller's result. The question in its most general form is "Which sets of reals can be mapped uniformly continuously onto $[0, 1]$?" In ZFC, it follows from the Tietze Extension Theorem that every set of reals containing a perfect set has this property. (To see this for unbounded sets of reals, first use a uniformly continuous homeomorphism, such as $\tan^{-1}$, from $\mathbb{R}$ onto a bounded open interval.) This result suggests that the interesting answers to the question lie in the realm of totally imperfect sets, i.e., those sets which have no perfect subset; we call this class TI. Diagram 1 describes some of the relationships between many of the better known subclasses of TI.

We now define these classes (for a survey of results, see [Ku, §40; M3, or BrC]). $X \in L$ if X is Luzin, i.e., X is uncountable and $|X \cap F| \le \omega$ for every first category set F. X is concentrated on a set D if for every open $U \supset D$, $|X \backslash U| \le \omega$. $X \in P$ if X is concentrated on a countable subset of itself. $X \in \mathrm{con}$ if X is concentrated on some countable set of reals. $X \in C''$ if X has the Rothberger property, i.e., for every family $\mathcal{G}_n$ of open covers there is a diagonal sequence $\mathrm{U}_n \in \mathcal{G}_n$ such that $X \subseteq \bigcup_{n\in\omega} \mathrm{U}_n$. $X \in \mathrm{U}_0$ if X has universal measure zero, i.e., $\mu(X) = 0$ whenever μ is the completion of a finite Borel measure which takes singletons to zero. $X \in (s)_0$ if for each perfect $P \subset \mathbb{R}$ there is a perfect $Q \subset P$ such that $X \cap Q = \varnothing$. $X \in$ count if X is countable. $X \in [A]^{<\kappa}$, where κ is a cardinal, if $X \subset A$ and $|X| < \kappa$. $X \in S$ if X is Sierpinski, i.e., X is uncountable and $|X \cap N| \le \omega$ whenever N has Lebesgue measure zero. $X \in \lambda$ if for all $D \subset X$, if D is countable then D is a G_δ relative to X. $X \in \lambda'$ if for every countable set $E \in \mathbb{R}$, $X \cup E \in \lambda$. $X \in \mathrm{AFC}$ if X is always first category, i.e., $X \cap P$ is first category relative to P for each perfect set $P \subset \mathbb{R}$. $X \in \overline{\mathrm{AFC}}$ (see [G2]) if $f^{-1}(X) \in \mathrm{AFC}$ whenever f is 1-1 and continuous.

DIAGRAM 1

A class $\mathcal{K} \subset P(\mathbb{R})$ is *hereditary* if $P(Y) \subset \mathcal{K}$ for each $Y \in \mathcal{K}$; $\mathcal{K}$ is a *σ-ideal* if it is hereditary and closed under countable unions. In Diagram 1, all classes are hereditary except for L and S; all are closed under countable unions except C'', λ, and TI.

For sets $X \in \mathbb{R}$ (or occasionally $X \subset Y$, where Y is some compact metric space) we write $M(X)$ if X can be mapped uniformly continuously onto $[0,1]$. As we observed above, $M(X)$ holds whenever $X \notin \mathrm{TI}$.

In [Is], Isbell improved this result for $X \subset 2^\omega$ (where $2^\omega =$ the product space of ω copies of $2 = \{0,1\}$; recall that $2^\omega \cong$ the Cantor set) by showing that for all $X \notin (s)_0$, $M(X)$ holds. Let us show why his result holds for $X \subset \mathbb{R}$ as well: Using $\tan^{-1}$ as remarked above, we may assume $X \subset [0,1]$. Let $P \subset [0,1]$ witness that $X \notin (s)_0$; we may assume P is nowhere dense. Note that $X \cap P \notin (s)_0$ and that by compactness of P, any homeomorphism from P onto 2^ω is uniformly continuous. Thus since $M(X \cap P)$ holds, we can find $\varphi: P \to [0,1]$, whence a $\hat{\varphi}: [0,1] \to [0,1]$ extending φ, for which $\varphi''(X \cap P) = \hat{\varphi}''(X \cap P) = [0,1]$; $\hat{\varphi} \restriction X$ is the required function.

The class of sets satisfying M can be broadened further by observing that there is always an $X \in (s)_0$ for which $M(X)$ holds. We need the following proposition:

0.1 **Proposition** *If X and Y are compact metric spaces, $f: X \to Y$ is 1-1 and continuous, $S \subset X$, and $f''(S) \in (s)_0$ relative to Y, then $S \in (s)_0$ relative to X.*

Proof. Suppose $P \subset X$ is perfect; since $f''(P)$ is perfect, there is a perfect $Q \subset f''(P)$ missing $f''(S)$; now $f^{-1}(Q)$ misses S. $\square$

Now by [W, Theorem 2.2] (see also [M3, 5.10]), there is an $(s)_0$ set $S \subset 2^\omega$ of power $\mathbf{c}$, where $\mathbf{c}$ is the cardinality of the continuum. Let $f: S \to 2^\omega$ be a bijection and notice that by the proposition $f \subset 2^\omega \times 2^\omega$ is $(s)_0$. Thus if $\pi: 2^\omega \times 2^\omega \to 2^\omega$ is projection onto the second coordinate, $\pi \restriction f$ is a uniformly continuous map from an $(s)_0$ set onto 2^ω. Since the canonical homeomorphism from 2^ω onto $2^\omega \times 2^\omega$ and the usual continuous map from 2^ω onto $[0,1]$ are uniformly continuous, the result follows.

Thus even "fairly small" sets satisfy M. However in ZFC one cannot expect to map all $(s)_0$ sets of power $\mathbf{c}$ uniformly continuously onto $[0,1]$ because assuming

CH (or MA) there are Luzin and Sierpinski (**c**-Luzin and **c**-Serpinski) sets. (Say that X is *κ-Luzin* (or *κ-Sierpinski*) if $|X| \geq \kappa$ and $|X \cap F| < \kappa$ whenever F is first category (or Lebesgue measure zero).) Thus by the following corollary to an observation of Miller, and by the observation that κ-Luzin and κ-Sierpinski sets are $(s)_0$, it is consistent that $M(X)$ fails for some $X \in (s)_0$.

0.2 **Proposition [M2]** *No κ-Luzin or κ-Sierpinski set can be mapped continuously onto $[0,1]$.* $\square$

We now show that every set of reals of power **c** satisfies M in the iterated Sacks model; Miller [M2] actually proves this for $X \subset 2^\omega$. Suppose $X \in \mathbb{R}$ and $X \in (s)_0$; again assume X is bounded. Notice that $|X \cap P| = \mathbf{c}$ for some bounded nowhere dense perfect set P (otherwise X is **c**-Luzin and it follows that there is such a set in 2^ω, contradicting Proposition 0.2). Now map $X \cap P$ uniformly continuously onto $[0,1]$ using Miller's result (and a homeomorphism from P onto 2^ω) and extend this map to a $\varphi: I \to [0,1]$, where I is a closed interval containing $X \cup P$. Now $\varphi \restriction X$ is the required map.

Hence, Isbell's result on $X \notin (s)_0$ is extended to $X \notin [\mathbb{R}]^{<\mathbf{c}}$ in Miller's model, and we have the consistency of

$(*)$ Every set of reals of power **c** satisfies M.

In this model, the σ-ideals more restrictive than $(s)_0$ (i.e., U_0, AFC and their subideals; see Diagram 1) are properly contained in $[\mathbb{R}]^{<\mathbf{c}}$. (Laver [La] showed $\mathrm{U}_0 \subset [\mathbb{R}]^{<\mathbf{c}}$; Miller [M2] showed AFC $\subset [\mathbb{R}]^{<\mathbf{c}}$; and it follows from 0.7 and 0.9 below that the inclusions are proper.) It is natural to ask whether $(*)$ can still hold in a model in which these smaller ideals have members of power **c**. As Theorem 0.0 shows, it is consistent for every $X \notin C$ to satisfy M. Moreover, in this model there are many U_0 sets of power **c**:

0.3 **Theorem** *If ZFC is consistent, so is ZFC + "Every set of reals of power **c** satisfies M and there are $2^{\mathbf{c}}$ sets in U_0 of power **c**".*

0.4 **Remark** The situation in the category direction is less clear; the only known "lower bound" for $(*)$ in this direction is S (by Proposition 0.2). In particular, it is unknown whether $(*)$ can hold in a model in which there are AFC sets of power **c**. A natural strategy to build such a model is to find a strongly proper order P which forces the ground model reals to be meager. Then by 0.7(b) and 0.9(b) below, there are $2^{\mathbf{c}}$ sets in AFC in a model obtained by forcing with an ω_2-stage countable support mixed iteration of the Sacks order with P.

It is apparent from Isbell's result that the sets for which M fails are "small"; in fact, as we show in §3, if $\mathcal{J} = \{X \subset \mathbb{R}: \neg M(X)\}$, then $\mathcal{J}$ is a σ-ideal. We have seen that $C \subseteq \mathcal{J} \subsetneq (s)_0$. In light of Diagram 1, it is natural to ask what the relationship is between $\mathcal{J}$ and U_0 and between $\mathcal{J}$ and AFC. (The relationship between $\mathcal{J}$ and $[\mathbb{R}]^{<\mathbf{c}}$ is easily described: clearly $[\mathbb{R}]^{<\mathbf{c}} \subseteq \mathcal{J}$; CH implies $[\mathbb{R}]^{<\mathbf{c}} \neq \mathcal{J}$; in Miller's model described above, $[\mathbb{R}]^{<\mathbf{c}} = \mathcal{J}$.) Apart from the question of whether "$\mathcal{J} \subseteq$ AFC" is consistent (which is still open) we give a complete description of these relationships as follows.

0.5 **Theorem** (a) ZFC $\vdash \mathrm{U}_0 \neq \mathcal{J}$ and AFC $\neq \mathcal{J}$;

(b) ZFC + CH $\vdash \mathrm{U}_0 \cap \mathrm{AFC} \subset \mathcal{J}$ and $\mathcal{J} \subset \mathrm{U}_0 \cup \mathrm{AFC}$;

(c) Con(ZFC) $\to$ Con(ZFC $+ \mathrm{U}_0 \subseteq \mathcal{J} +$ AFC $\subseteq \mathcal{J}$);

(d) Con(ZFC) $\to$ Con(ZFC $+ \mathcal{J} \subseteq \mathrm{U}_0$).

Part (d) is established using the model in Theorem 0.0.

As a final application of our techniques, we discuss two problems raised by J. Brown [Brl] (see also [BrC] and [Br2]). The first question is whether there is a ZFC example of a set in $\mathrm{U}_0 \backslash$AFC or in AFC$\backslash \mathrm{U}_0$. As a partial answer, we prove the following.

0.6 **Theorem** (a) In the random real model (or if $\mathbf{c}$ is real-valued measurable) we have $\mathrm{U}_0 \subsetneq \overline{\mathrm{AFC}} \subseteq \mathrm{AFC}$.

(b) In the Cohen real model, $\overline{\mathrm{AFC}} \subsetneq \mathrm{U}_0$ (in fact, $\overline{\mathrm{AFC}} \subsetneq C''$).

(c) Con(ZFC) $\to$ Con(ZFC + either φ or ψ) where $\varphi \equiv$ "AFC $\subsetneq \mathrm{U}_0$" and $\psi \equiv$ "every set of reals of power $\mathbf{c}$ satisfies M, and there are $2^{\mathbf{c}}$ many AFC sets of power **c**."

Part (b) of the theorem is related to the second question which concerns us. Sierpiǹski [Si] showed that $\lambda' \cap \mathrm{con} = \mathrm{count}$. Referring to Diagram 1, one naturally asks whether

$$(**) \qquad \lambda' \cap C'' = \mathrm{count}$$

is true. Since "$C'' = \mathrm{count}$" is possible (in Laver's model, see [La]; but note also that "$\lambda' \supsetneq \mathrm{count}$" is always true, see [M3]), it is natural to look for a model in which C'' has uncountable members and $(**)$ fails. As a case of special interest, Brown asks if CH $\to \neg(**)$. Todorcevic and Miller brought to the author's attention the fact that a minor modification of Todorcevic's proof of Theorem 4 in [GM] yields a positive answer to Brown's question. (In fact, in unpublished work, Todorcevic has constructed an uncountable set in $\lambda' \cap C''$ which is also a σ set (i.e. a set whose relative F_σ subsets are relative G_δ's) under the weaker assumption of MA.) Here we wish to observe simply that $(**)$ can fail badly since Theorem 0.6(b) provides a model of $\lambda' \subset C''$.

Let $\mathcal{I} = \{X \subset \mathbb{R}: X$ cannot be mapped continuously onto $[0,1]\}$. In this paper, we emphasize the study of $\mathcal{J}$ rather than $\mathcal{I}$ because the former seems to be the more natural of the two classes. For example, while $\mathcal{J}$ is a σ-ideal, whether $\mathcal{I}$ is also a σ-ideal is independent: in Miller's model, $\mathcal{I} = \mathcal{J} = [\mathbb{R}]^{<\mathbf{c}}$; on the other hand, assuming CH, there is a scale in ω^ω (recall $\omega^\omega \cong$\{irrationals\}) which can be mapped continuously onto $[0,1]$; however, if S is any scale and $Q =$\{rationals\}, then $S \cup Q \in \mathcal{I}$ (see [M3]). Thus $\mathcal{I}$ is not hereditary under CH. ($\mathcal{I}$ *is* closed under countable unions; the argument is the same as that used for $\mathcal{J}$ in §3.) Another undecidable property of $\mathcal{I}$ is whether $C \subset \mathcal{I}$: On the one hand, a scale is in $C \backslash \mathcal{I}$; on the other hand in Laver's model [La], $C = \mathrm{count} \subset \mathcal{I}$.

The paper is organized as follows. In §1, we introduce strongly proper orders and discuss their relationship to other well-known classes of orders. In §2 we develop the machinery for iterating these orders, and in §3 we apply this machinery to prove the results stated in this introduction.

We will make liberal use of Cichon's Diagram [F], which is presented as Diagram

2 below. (We use the notation of [P].)

$$
\begin{array}{ccccccc}
\operatorname{cov}(\mathcal{L}) & \rightarrow & \operatorname{non}(\mathcal{K}) & \rightarrow & \operatorname{cf}(\mathcal{K}) & \rightarrow & \operatorname{cf}(\mathcal{L}) \\
 & & \uparrow & & \uparrow & & \\
\uparrow & & \mathbf{b} & & \mathbf{d} & & \uparrow \\
 & & \uparrow & & \uparrow & & \\
\operatorname{add}(\mathcal{L}) & \rightarrow & \operatorname{add}(\mathcal{K}) & \rightarrow & \operatorname{cov}(\mathcal{K}) & \rightarrow & \operatorname{non}(\mathcal{L})
\end{array}
$$

Diagram 2

If $T \subset P(\mathbb{R})$, $\operatorname{non}(T) = \min\{|X| : X \notin T\}$, $\operatorname{cov}(T) = \min\{|\mathcal{U}| : \mathcal{U} \subset T$ and $\bigcup \mathcal{U} = \mathbb{R}\}$, $\operatorname{add}(T) = \min\{|\mathcal{U}| : \mathcal{U} \subset T$ and $\bigcup \mathcal{U} \notin T\}$, and $\operatorname{cf}(T) = \min\{|\mathcal{U}| : \mathcal{U} \subset T$ and every $F \in T$ is contained in a member of $\mathcal{U}\}$. Let $\mathcal{K}$ and $\mathcal{L}$ denote the σ-ideals of first category and Lebesgue measure zero sets, respectively. If $f, g \in \omega^\omega$, we write $f <^* g$ iff there is an $n \in \omega$ so that for all $m \geq n$, $f(m) < g(m)$. Then $\mathbf{b} = \min\{|\mathcal{V}| : \mathcal{V} \subset \omega^\omega$ and $\mathcal{V}$ is $<^*$-unbounded$\}$ and $\mathbf{d} = \min\{|\mathcal{V}| : \mathcal{V} \subset \omega^\omega$ and $\mathcal{V}$ is $<^*$-cofinal$\}$. In the diagram, an arrow $\rightarrow$ indicates that $\leq$ is provable in ZFC. We state two results which illustrate the connection between these cardinals and the classes we have been considering.

0.7 **Theorem** [G1, G2, G3] (a) There is a set $X \in \mathrm{U}_0$ of power $\operatorname{non}(\mathcal{L})$.

(b) There is a set $X \in \overline{\mathrm{AFC}}$ of power $\operatorname{non}(\mathcal{K})$.

0.8 **Theorem** [FM]. $\operatorname{cov}(\mathcal{K}) \leq \operatorname{non}(C'')$.

In §3 we prove the following:

0.9 **Lemma** (a) $\operatorname{non}(\mathcal{L}) = \operatorname{non}(\mathrm{U}_0)$.

(b) $\operatorname{non}(\mathcal{K}) = \operatorname{non}(\overline{\mathrm{AFC}}) = \operatorname{non}(\mathrm{AFC})$.

Actually part (a) follows immediately from the fact that $X \in \mathrm{U}_0$ if and only if $h(X) \in \mathcal{L}$ for every homeomorphism h (see [M3]).

In proving results about relationships between various classes, we often work in $[0,1]$, 2^ω, or ω^ω instead of in $\mathbb{R}$; the fact that there are measure-preserving and category-preserving maps between these spaces (see [M3]) is generally enough to justify this laxity. Whenever these maps do not suffice for the argument at hand (as in some of the arguments above) we supply the necessary additional details in the proof. In particular, in discussing strong measure zero sets, we work only in $\mathbb{R}$, $[0,1]$ or 2^ω (see [Ba, §9]).

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1. Strongly proper orders

Let P denote the Sacks order (see [S or BL] for definitions and basic results), and for all $p \in P$ and $s \in p$ let $p_s = \{t \in p : t \supset s$ or $s \subset t\}$. For $m, n \in \omega$ and $q, p \in P$, write $(q, m) < (p, n)$ if $m > n$, $q \leq p$, and for each $s \in p \cap 2^n$ there are $t \neq t'$ in $q \cap 2^m$ which extend s. It is well known that if $n \in \omega$ and $p \Vdash \dot{a} \in V$, then there are $q \in P$, $m \in \omega$ and $x_s \in V$ for each $s \in q \cap 2^n$ such that $(q, m) < (p, n)$

and $q_s \Vdash \dot{a} = x_s$; moreover, since the set $\{q_s: s \in q \cap 2^n\}$ is a maximal antichain below q, it follows that $q \Vdash \dot{a} \in \{x_s: s \in q \cap 2^n\}$.

The properties described above are to a large extent preserved by countable support iterations of the Sacks order, and Miller's consistency result [M2] relies heavily on this fact. Since several well-known partial orders have these properties, and since much of Miller's machinery goes through for any such partial order, we have formulated the properties as axioms for a class of orders which we will call strongly proper. Ultimately, we will use a particular strongly proper order along with the Sacks order in a countable support iteration to obtain most of the results discussed in §0.

1.0 **Definition** (strongly proper orders) A partial order $(P, \leq)$ is strongly proper if there are orderings $\{<_{m,n}: n < m \in \omega\}$ and a constructible sequence $T = \langle T_n: n \in \omega\rangle$ of finite sets satisfying:

(1) $T_0 = \{\varnothing\}$.

(2) If $q <_{m,n} p$ (which we will write $(q, m) < (p, n)$), then $m > n, q \leq p$, and whenever $m' \geq m$ and $n' \leq n$ then $(q, m') < (p, n')$.

(3) (Fusion) If $\{(p_n, m_n): n \in \omega\}$ is a sequence such that for all $n \in \omega$, $(p_{n+1}, m_{n+1}) < (p_n, m_n)$, there is a $p \in P$ such that for all $n \in \omega$, $(p, m_{n+1}) < (p_n, m_n)$. Such a sequence will be called a *fusion sequence* and the condition p will be called a *fusion* of this sequence.

(4) For each $p \in P$, $n \in \omega$, there is a nonempty $A_{p,n} \subset T_n$ and a map $\varphi_{p,n}: A_{p,n} \to P$ (whose images $\varphi_{p,n}(s)$ we denote by p_s whenever n is fixed) such that

(a) $\varphi_{p,0}(s) = p$;
(b) the set $\{p_s: s \in A_{p,n}\}$ is a maximal antichain below p;
(c) $\forall p, q \in P((q, m) < (p, n) \to A_{q,n} = A_{p,n})$;
(d) $\forall p, q, r \in P[((r, m) < (q, n) \wedge q \leq p \wedge A_{q,n} = A_{p,n}) \to (r, m) < (p, n)]$;
(e) If $p \Vdash$ "$\dot{a} \in V$" and $n \in \omega$, then there are $m > n$ and $q \in P$ such that $(q, m) < (p, n)$ and for each $s \in A_{q,n}$ there is an $x_s \in V$ so that $q_s \Vdash \dot{a} = x_s$.

1.1 *Notation.* For $p, q \in P$ and $n \in \omega$ we write $(q, n) \leq (p, n)$ if $q \leq p$ and $A_{q,n} = A_{p,n}$.

Miller points out that much of the strength of these axioms is embodied in the notion of ω^ω-bounding orders, discussed by Shelah in [Sh]. (A partial order Q is ω^ω-bounding if for each $\dot{f} \in V^Q$ for which $\Vdash_Q \dot{f}: \omega \to \omega$, there is $g \in V$, $g: \omega \to \omega$, such that $\Vdash_Q \forall n\, \dot{f}(n) \leq g(n)$; note that strongly proper orders have this property by Axioms 3, 4(e).) Shelah [Sh, p. 169] shows that "ω-proper + ω^ω-bounding" is preserved by countable support iterations. One might hope to replace our somewhat lengthy list of axioms with this more concise list of two. In §2, however, we show that our central lemma (Theorem 2.23) fails if "strongly proper" is replaced by "ω-proper + ω^ω-bounding" (see Remark 2.24).

We now briefly examine the relationship between strongly proper orders and other well-known classes of orders. We show that the strongly proper orders are properly included in the class of ω-proper, ω^ω-bounding orders and that ω_1-closed orders are strongly proper, as are several other familiar orders. We first show the following:

1.2 **Theorem** *Strongly proper orders do not add Cohen or random reals.*

Proof. That Cohen reals are not added follows from the ω^ω-bounding property: If $\dot{x} \in V^P$, $\Vdash_p \dot{x} \in \omega^\omega$, and $f\colon \omega \to \omega$ is in V with $\Vdash_p$ "$\forall n(\dot{x}(n) \leq f(n))$" then the set $\{g \in \omega^\omega | \forall n(g(n) \leq f(n))\}$ is nowhere dense, coded in V, and contains $\dot{x}$.

That random reals are not added follows from Remark 2.23; we give a direct proof which does not involve the machinery of iterations in an essential way.

Lemma *If P is strongly proper, $p \in P$, $p \Vdash$ "$\dot{a} \notin V$ and $\dot{a} \in 2^\omega$", and $n \in \omega$, then there is a finite $X \in V$ and a $q \in P$ such that $(q, n) \leq (p, n)$ and*

$$q \Vdash \exists x \in X[x \restriction 2(|X| + n) = \dot{a} \restriction 2(|X| + n)].$$

Proof. We postpone the proof to §2 where we prove a much more general statement (see Lemma 2.15).

Lemma *If P is strongly proper, $p \in P$, and $p \Vdash$ "$\dot{a} \notin V$ and $\dot{a} \in 2^\omega$", then there is a sequence $\langle X_n \colon n \in \omega \rangle \in V$ of finite sets and a condition $q \leq p$ so that for all $n \in \omega$,*

$$q \Vdash \exists x \in X_n[x \restriction 2(|X_n| + n) = \dot{a} \restriction 2(|X_n| + n)].$$

Proof. We use fusion and Axiom 4(e). By induction, build a fusion sequence $\{(p_n, m_n)\colon n \in \omega\}$ and a sequence $\langle X_n \colon n \in \omega \rangle$ of finite sets so that

(1) $p_0 = p$ and m_0 is arbitrary;
(2) $p_{n+1} \Vdash \exists x \in X_n[x \restriction 2(|X_n| + n) = \dot{a} \restriction 2(|X_n| + n)]$.

If (p_n, m_n) have been defined, let $(r, m_n) \leq (p_n, m_n)$ and X_n be as in the first lemma, and let $(p_{n+1}, m_{n+1}) < (r, m_n)$ be as in Axiom 4(e). Now by Axiom 4(d), $(p_{n+1}, m_{n+1}) < (p_n, m_n)$, and the induction is complete. Clearly, if q is a fusion of the (p_n, m_n), q satisfies the conclusion of the lemma.

We now finish the proof of Theorem 1.2. Suppose $p \Vdash$ "$\dot{a} \notin V$ and $\dot{a} \in 2^\omega$". Let $q \leq p$ and $\langle X_n \colon n \in \omega \rangle$ be as in the second lemma. Then if $q \in G$, G P-generic over V, then in $V[G]$ the set

$$\bigcap_n \bigcup_{x \in X_n} \{f \in 2^\omega \colon f \restriction 2(|X_n| + n) = x \restriction 2(|X_n| + n)\}$$

is coded in V, contains $\dot{a}$, and has measure zero. (To see the last part, notice that before taking the intersection, the nth union has measure $\leq 1/2^{|X_n|+n}$.) $\square$

1.3 **Corollary** *The class of strongly proper orders is properly included in the class of ω-proper, ω^ω-bounding orders.*

Proof. Because of our earlier remarks, to prove inclusion we have only to prove that strongly proper orders are ω-proper; the proof of the latter is a straightforward modification of the argument that Axiom A orders are proper (using the model-theoretic definition of proper) (see [Sh, p. 169] for definitions).

To see that inclusion is proper, we show that the random real order is ω-proper and ω^ω-bounding (we have already seen it is not strongly proper): Being ccc, it is ω-proper; that it is also ω^ω-bounding is folklore (see [J2, p. 14] for a proof). □

1.4 **Remark** The similarity between strongly proper orders and Axiom A orders is evident; 1.2 shows the classes are different and suggests the following questions.

1.5 *Questions.* (i) Does every strongly proper order satisfy Axiom A?
(ii) Is there a nonatomic ccc order which is strongly proper?

We proceed to several examples.

1.6 **Proposition** *The Sacks order is strongly proper.*

Proof. Define T_n, $A_{p,n}$, $\varphi_{p,n}$, and $<_{m,n}$ as follows: $T_n = 2^n$; $A_{p,n} = p \cap 2^n$; $\varphi_{p,n}(s) = p_s$; and $(q,m) < (p,n)$ is defined as in the first paragraph of this section. □

1.7 **Remark** One reason we opted for the $<_{m,n}$ orderings rather than the simpler Axiom A orderings $\leq_n$ is that the analogue to Axiom 4(e) fails for the Sacks order if the usual $\leq_n$ orderings are used in place of the $<_{m,n}$ (see [Ba, §7] for definitions and results).

1.8 **Proposition** *Each ω_1-closed order is strongly proper.*

Proof. Let $T_n = A_{p,n} = \{\varnothing\}$, $\varphi_{p,n}(\varnothing) = p$, and let $<_{m,n}$ be $\leq$. □

1.9 **Definition** Let $P = \{p: A \to 2^{<\omega} | A \subset \omega$ is coinfinite and for all $n \in A$, $p(n) \in 2^n\}$ and write $q \leq p$ if $q \supset p$. P is called the Infinitely Often Equal Reals (IOER) order (see [M1, §7]).

1.10 **Proposition** *The IOER order is strongly proper.*

Proof. We define $T_n, A_{p,n}, \varphi_{p,n}$, and $<_{m,n}$ as follows:

$$\begin{aligned} T_n = \{s: s &\text{ is a partial function from } n \text{ into } 2^{<n} \\ &\text{such that for all } i \in \operatorname{dom} s,\ s(i) \in 2^i\}; \\ A_{p,n} &= \{s \in T_n: \operatorname{dom} s = n \backslash \operatorname{dom} p\}; \\ \varphi_{p,n}(s) &= p \cup s; \\ (q,m) < (p,n) &\text{ if } m > n, q \leq p, A_{p,n} = A_{q,n}, \text{ and} \\ &\text{there is } i,\ n < i \leq m, \text{ such that } i \notin \operatorname{dom} q. \end{aligned}$$

We verify that Axioms 4(b) and 4(e) hold; the others are immediate.

It is clear that the set $\{p \cup s: s \in A_{p,n}\}$ is an antichain; if $q \leq p$, then q must agree with some $s \in A_{p,n}$ on their common domain; thus the set is in fact maximal below p.

For 4(e), assume $p \Vdash$ "$\dot{a} \in V$", $n \in \omega$. Write $A_{p,n} = \{s_1, \ldots, s_k\}$. Build $(q,n) \geq (q_2,n) \geq \cdots \geq (q_k,n)$ and $\{x_1,\ldots,x_k\} \in V$ so that for $1 \leq i \leq k$, $q_i \cup s_i \Vdash \dot{a} = x_i$. Given q_i, obtain $q' \leq q_i \cup s_{i+1}$ and $x_{i+1} \in V$ so that $q' \Vdash \dot{a} = x_{i+1}$. Let $q_{i+1} = q' \backslash s_{i+1}$. Now let $q = q_k$ and let m be large enough so that $(q,m) < (q,n)$. Then (q,m) is the required pair. □

1.11 **Remark** (i) The verification of 4(e) above is a minor modification of a similar proof by Miller [M1, §7].

(ii) Let $P = \{p: A \to 2 | A \subset \omega \text{ is coinfinite}\}$ and say $q \le p$ if $q \supset p$. P is Silver forcing and an argument similar to the one given above shows that P is strongly proper.

2. Iterations

In this section we develop the machinery for iterating strongly proper orders. Our notation and terminology for general iterated forcing follow [Ba]; our arguments are patterned after [BL] and [M2]. Recall that if P_α is an α-stage iteration and $p \in P_\alpha$, then for all $\beta < \alpha$, $p \restriction \beta \in P_\beta$ and $\Vdash_\beta p(\beta) \in Q_\beta$. However, if we need to verify that a particular p (having the right kind of support) is in P_α, it suffices to check that for all $\beta < \alpha$, $p \restriction \beta \in P_\beta$ and $p \restriction \beta \Vdash_\beta p(\beta) \in Q_\beta$ (see [Ba]); we use this fact without special mention.

For the rest of this section, P_α will denote a countable support α-stage iteration and for all $\beta < \alpha$, $T^\beta = \{T^\beta_n : n \in \omega\}$ is a constructible set of finite sets, and $\{\dot{A}^\beta_{\tau,n} : n \in \omega \text{ and } \Vdash_\beta \tau \in \dot{Q}_\beta\}$, $\{\dot{\varphi}^\beta_{\tau,n} : n \in \omega \text{ and } \Vdash_\beta \tau \in \dot{Q}_\beta\}$ are sets of terms such that

$$\Vdash_\beta \text{“}T^\beta, \{\dot{A}^\beta_{\tau,n}\}_{\tau,n}, \text{ and } \{\dot{\varphi}^\beta_{\tau,n}\}_{\tau,n} \text{ witness that } \dot{Q}_\beta \text{ is strongly proper”}.$$

In practice, the fact that we use only canonical terms for the T^β is not a restriction at all since in any application of such an iteration, we would have a particular (constructible) definition of T^β in mind for each factor $\dot{Q}_\beta$ (and canonical names would be perfectly general). (Of course for a general theory of iterated strongly proper orders, arbitrary terms for the T^β would have to be allowed.)

2.0 **Definition** If $F \in [\alpha]^{<\omega}$ and $n < m \in \omega$, then $(q, m) <_F (p, n)$ if $q \le p$, $m > n$, and $q \restriction \beta \Vdash$ “$(q(\beta), m) < (p(\beta), n)$” for all $\beta \in F$.

2.1 **Lemma** (Fusion)**.** *Suppose $\{(p_n, F_n, m_n) : n \in \omega\}$ is a sequence such that for all n, $p_n \in P_\alpha$, $F_n \in [\alpha]^{<\omega}$, $(p_{n+1}, m_{n+1}) <_{F_n} (p_n, m_n)$, $F_n \subset F_{n+1}$ and $\bigcup_n F_n = \bigcup_n \operatorname{suppt}(p_n)$. Then there is $p \in P_\alpha$ such that for all n, $(p, m_{n+1}) <_{F_n} (p_n, m_n)$.*

Proof. Define $p \restriction \beta$ by induction on $\beta \le \alpha$ so that for all n, $(p \restriction \beta, m_{n+1}) <_{F_n} (p_n \restriction \beta, m_n)$ and $\operatorname{suppt}(p \restriction \beta) = (\bigcup_n F_n) \cap \beta$. If β is a limit, take the union of the restricitons defined below β. To obtain $p \restriction \beta + 1$ from $p \restriction \beta$ where $\beta < \alpha$, let $p(\beta) = \dot{1}$ if $\beta \notin \bigcup_n F_n$; if $\beta \in \bigcup_n F_n$, use the fact $p \restriction \beta$ forces Axiom (4) to hold for $\dot{Q}_\beta$ to obtain $p(\beta)$ as follows: Let n be least such that $\beta \in F_n$. For $k \in \omega$, let $\dot{q}_k = p_{n+k}(\beta)$ and let $j_k = m_{n+k}$. Now

$$p \restriction \beta \Vdash \text{“}\langle (\dot{q}_k, j_k) : k \in \omega \rangle \text{ is a fusion sequence”}.$$

Let $p(\beta)$ be a term forced by $p \restriction \beta$ to be the fusion of the $(\dot{q}_k, j_k)$. To complete the proof it suffices to verify that $(p \restriction \beta + 1, m_{i+1}) <_{F_i} (p_i \restriction \beta + 1, m_i)$ for all $i \in \omega$. For $i \ge n$, this follows from the definition of $p(\beta)$; as a consequence we have that

$$\forall i \in \omega (p \restriction \beta + 1 \le p_i \restriction \beta + 1).$$

From this and the induction hypothesis we have the result for $i < n$ as well. □

2.2 **Remark** The sequence $\{(p_n, F_n, m_n): n \in \omega\}$ will be called a *fusion sequence* and the condition p constructed above will be called a *fusion* of $\{(p_n, F_n, m_n): n \in \omega\}$.

2.3 **Definition** For $F \in [\alpha]^{<\omega}$, $n \in \omega$, a function σ on F is called an (F,n)-*function* if for all $\beta \in F$, $\sigma(\beta) \in T^\beta_n$. If σ is an (F,n)-function define $p|\sigma$ by

$$p|\sigma(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F, \\ \dot\varphi^\beta_{p(\beta),n}(\sigma(\nu)) & \text{if } \beta \in F. \end{cases}$$

(In other words, if $\beta \in F$, $p|\sigma(\beta) = p(\beta)_{\sigma(\beta)}$.)

Notice that, in general, $p|\sigma$ need not be in P_α.

2.4 **Definition** A function σ on F is (F,n)-*consistent with* p if σ is an (F,n)-function and, for all $\beta \in F$,

$$(p|\sigma) \restriction \beta \Vdash \sigma(\beta) \in \dot A^\beta_{p(\beta),n}.$$

(In other words, modulo our convention, $p|\sigma \in P_\alpha$.)

2.5 **Definition** The condition $p \in P_\alpha$ is (F,n)-*determined* if for each (F,n)-function σ, either σ is (F,n)-consistent with p or

$$\exists \beta \in F(\sigma \restriction F \cap \beta \text{ is } (F \cap \beta, n)\text{-consistent with } p \restriction \beta \\ \text{and } (p|\sigma) \restriction \beta \Vdash \text{“}\sigma(\beta) \notin \dot A^\beta_{p(\beta),n}\text{”}).$$

2.6 *Notation.* We write $\Sigma(p, F, n)$ for the set $\{\sigma: \sigma \text{ is } (F,n)\text{-consistent with } p\}$.

2.7 **Definition** If $p, q \in P_\alpha$, $n \in \omega$, $F \in [\alpha]^{<\omega}$, then $(q,n) \le_F (p,n)$ if $q \le p$ and $q \restriction \beta \Vdash (q(\beta), n) \le (p(\beta), n)$ for all $\beta \in F$.

2.8 **Proposition** (i) If $(q,m) <_F (p,n)$, $m' \ge m$, $n' \le n$, then $(q,m') <_F (p,n')$.

(ii) If $(q,m) <_F (p,n)$, then $(q,n) \le_F (p,n)$.

(iii) If $(r,m) <_F (q,n)$ and $(q,n) \le_F (p,n)$, then $(r,m) <_F (p,n)$.

(iv) If $\sigma \ne \tau$ are in $\Sigma(p,F,n)$, then $p|\sigma \perp p|\tau$.

Proof. Proceed by induction on $\beta \in F$; for (i) use Axiom (2); for (ii) use Axiom 4(c); for (iii) use Axiom 4(d); and for (iv) use Axiom 4(b). □

2.9 **Lemma** For all $p \in P_\alpha$, if p is (F,n)-determined, $\{p|\sigma: \sigma \in \Sigma(p,F,n)\}$ is a maximal antichain below p.

Proof. By 2.8, it suffices to prove maximality. Suppose $q \le p$. By induction on $\beta \le \alpha$ we find $q \restriction \beta$, $\sigma \restriction F \cap \beta$, and $r \restriction \beta$ so that $r \restriction \beta \le q \restriction \beta$, $(p|\sigma \restriction F \cap \beta) \restriction \beta$. The cases in which β is a limit and $\beta = \gamma + 1$ with $\gamma \notin F$ are easy. We assume $\beta = \gamma + 1$ and $\gamma \in F$. Assuming $r \restriction \gamma$, $\sigma \restriction F \cap \gamma$, and $q \restriction \gamma$ have been defined, it follows from 4(b) that there is a term $\dot s$ such that

$$(*) \qquad r \restriction \gamma \Vdash \text{“}\dot s \in \dot A^\gamma_{p(\gamma),n} \text{ and } p(\gamma)_{\dot s} \text{ is compatible with } q(\gamma)\text{”}.$$

Let $r' \leq r \restriction \gamma$ and $s \in T_n^\gamma$ be such that $r' \Vdash s = \dot{s}$, and let $\sigma(\gamma) = s$.

Claim. $\sigma \restriction F \cap \beta$ is $(F \cap \beta, n)$-consistent with p.

Proof of Claim. If $p|\sigma \restriction \gamma \nVdash \sigma(\gamma) \in \dot{A}^\gamma_{p(\gamma),n}$ then by (F,n)-determinedness of p, "$\sigma(\gamma) \notin \dot{A}^\gamma_{p(\gamma),n}$" is forced by $p|\sigma \restriction \gamma$, and hence by r'. But this contradicts $(*)$, and the claim is proven.

Again by $(*)$ we can find a term $r(\gamma)$ such that $r \restriction \gamma \Vdash r(\gamma) \leq q(\gamma), p(\gamma)_{\sigma(\gamma)}$ (note that $r \restriction \gamma \Vdash \sigma(\gamma) \in \dot{A}^\gamma_{p(\gamma),n}$). This completes the induction step and the proof of Lemma 2.9. □

2.10 **Lemma** *If p is (F,n)-determined and $(q,n) \leq_F (p,n)$, then q is (F,n)-determined.*

Proof. Suppose σ is (F,n)-consistent with q. First observe that σ is (F,n)-consistent with p as well: if not, let $\beta < \alpha$ be least for which this is false, i.e.,

$$p|\sigma \restriction \beta \nVdash \sigma(\beta) \in \dot{A}^\beta_{p(\beta),n}.$$

By (F,n)-determinedness of p and the fact that $q \leq p$ we have

$$q|\sigma \restriction \beta \Vdash \sigma(\beta) \notin \dot{A}^\beta_{p(\beta),n}.$$

But since

$$(**) \qquad q|\sigma \restriction \beta \Vdash \dot{A}^\beta_{p(\beta),n} = \dot{A}^\beta_{q(\beta),n},$$

we get the contradiction

$$q|\sigma \restriction \beta \Vdash \sigma(\beta) \notin \dot{A}^\beta_{q(\beta),n}.$$

Now to prove the lemma, let σ be an (F,n)-function and let β be least in F such that

$$q|\sigma \restriction \beta \nVdash \sigma(\beta) \in \dot{A}^\beta_{q(\beta),n}.$$

Use $(**)$ again, the fact that $q \leq p$, and the observation above to get

$$p|\sigma \restriction \beta \nVdash \sigma(\beta) \in \dot{A}^\beta_{p(\beta),n}.$$

Now $(**)$ and (F,n)-determinedness of p give us

$$q|\sigma \restriction \beta \leq p|\sigma \restriction \beta \Vdash \sigma(\beta) \notin \dot{A}^\beta_{p(\beta),n} = \dot{A}^\beta_{q(\beta),n}$$

as required. □

2.11 **Lemma** *If p is (F,n)-determined and $(q,n) \leq_F (p,n)$, then $\Sigma(q,F,n) = \Sigma(p,F,n)$.*

Proof. The proof of Lemma 2.10 shows that $\Sigma(q,F,n) \subseteq \Sigma(p,F,n)$. To get inclusion in the other direction, use (F,n)-determinedness of q. □

The next theorem is the analogue to Axiom 4(e); most of our machinery has been set up to make this theorem true. We include parts (ii) and (iii) in order to make the induction work; they are (essentially) true for any proper order.

2.12 **Theorem** (i) If $p \in P_\alpha$, $F \in [\alpha]^{<\omega}$, $n \in \omega$, and $p \Vdash \dot{a} \in V$, then there are q, m and, for each $\sigma \in \Sigma(q, F, n)$, a set $x_\sigma \in V$ such that $(q, m) <_F (p, n)$, q is (F, n)-determined, and for each $\sigma \in \Sigma(q, F, n)$,

$$q|\sigma \Vdash \dot{a} = x_\sigma.$$

Moreover, if $x = \{x_\sigma : \sigma \in \Sigma(q, F, n)\}$, then

$$q \Vdash \dot{a} \in x.$$

(ii) If $p \in P_\alpha$, $F \in [\alpha]^{<\omega}$, $n \in \omega$, and $p \Vdash$ "$\dot{A} \subset V \wedge \dot{A}$ is countable", then there are q, m, and a countable set $B \in V$ such that $(q, m) <_F (p, n)$ and $q \Vdash \dot{A} \subset B$.

(iii) If $p \in P_\alpha$, $F \in [\alpha]^{<\omega}$, $n \in \omega$, $\gamma > \alpha$, and $p \Vdash \dot{f} \in \dot{P}_{\alpha\gamma}$, then there are q, m, and $g \in P_{\alpha\gamma}$ such that $(q, m) <_F (p, n)$ and $q \Vdash \dot{f} = g$.

Proof. We prove (i), (ii) and (iii) simultaneously by induction on α. Assume the result holds for all $\beta < \alpha$.

(i) *Case* 1. $\alpha = \beta + 1$ and $\beta \in F$.

Since $p \restriction \beta \Vdash p(\beta) \Vdash \dot{a} \in V$, applying Axiom 4(e) in $\dot{Q}_\beta$, we obtain terms $\dot{m}, \dot{r}$, and for each $s \in T^\beta_n$, $\dot{x}_s$ such that

$$\begin{aligned} p \restriction \beta \Vdash \text{“}\forall s \in T^\beta_n [\dot{x}_s \in V \wedge (s \in \dot{A}^\beta_{p(\beta),n} \rightarrow \dot{r} \Vdash \dot{a} = \dot{x}_s) \\ \wedge (\dot{r}, \dot{m}) < (p(\beta), n)]\text{”}. \end{aligned} \tag{1}$$

Since

$$p \restriction \beta \Vdash (\dot{m}, \dot{A}^\beta_{\dot{r},n}, \langle \dot{x}_s : s \in T^\beta_n \rangle) \in V,$$

by the induction hypothesis there are q', m_0, and for each $\sigma \in \Sigma(q', F \cap \beta, n)$, sets $(m_\sigma, \beta_\sigma, \langle x_{\sigma^\wedge s} : s \in T^\beta_n \rangle) \in V$ such that $(q', m_0) <_{F\cap\beta} (p \restriction \beta, n)$, q' is $(F \cap \beta, n)$-determined for each $\sigma \in \Sigma(q', F \cap \beta, n)$, and, for all $s \in T^\beta_n$,

$$q'|\sigma \Vdash \text{“}\dot{m} = m_\sigma, \dot{A}^\beta_{\dot{r},n} = B_\sigma, \quad \text{and } \dot{x}_s = x_\sigma\text{”};$$

moreover, if $m_1 = \max\{m_\sigma : \sigma \in \Sigma(q', F \cap \beta, n)\}$, then

$$q' \Vdash \dot{m} \leq m_1. \tag{2}$$

Now let $m = \max\{m_0, m_1\}$ and let $q = q'^\wedge \dot{r}$.

Claim 1. $(q, m) <_F (p, n)$.

Proof. $(q \restriction \beta, m) <_{F\cap\beta} (p \restriction \beta, n)$ by Proposition 2.8(i); that $q \restriction \beta \Vdash (q(\beta), m) < (p(\beta), n)$ follows from (1) and (2).

Claim 2. q is (F, n)-determined.

Proof. By the induction hypothesis, it suffices to consider the case in which τ is an (F,n)-function, $\sigma = \tau \restriction F \cap \beta$, $s = \tau(\beta)$, and $q|\sigma \restriction \beta \nVdash s \in \dot{A}^{\beta}_{q(\beta),n}$. We have the following implications:

$$\begin{aligned} q|\sigma \restriction \beta \Vdash \dot{A}^{\beta}_{q(\beta),n} = B_\sigma &\Rightarrow q|\sigma \restriction \beta \nVdash s \in B_\sigma \\ &\Rightarrow s \notin B_\sigma \\ &\Rightarrow q|\sigma \restriction \beta \Vdash s \notin \dot{A}^{\beta}_{q(\beta),n}. \end{aligned}$$

Claim 3. For each $\tau \in \Sigma(q,F,n)$, there is $x_\tau \in V$ such that $q \Vdash \dot{a} = x_\tau$; moreover, if $x = \{x_\tau \colon \tau \in \Sigma(q,F,n)\}$, then $q \Vdash \dot{a} \in x$.

Proof. The second clause follows from the first because of Lemma 2.9. Let $\sigma = \tau \restriction \beta$ and $s = \tau(\beta)$. Then

$$q \restriction \beta \Vdash \text{“}(q(\beta) \Vdash \dot{a} = x_s) \wedge (1 \Vdash \dot{x}_s = x_{\sigma^\wedge s})\text{”}.$$

Case 2. α is a limit or $\alpha = \beta + 1$ where $\beta \notin F$.

In case α is a limit, choose β so that $\max F < \beta < \alpha$. Then

$$p \restriction \beta \Vdash p^\beta \Vdash \dot{a} \in V.$$

Let $\dot{f}, \dot{b} \in V^{P_\beta}$ be such that

$$p \restriction \beta \Vdash \text{“}\dot{f} \le p^\beta, \dot{b} \in V, \text{ and } \dot{f} \Vdash \dot{a} = \dot{b}\text{”}.$$

Use the induction hypothesis to obtain q_1 and m_1 such that $(q_1, m_1) <_{F\cap\beta} (p \restriction \beta, n)$, q_1 is $(F \cap \beta, n)$-determined, and for each $\sigma \in \Sigma(q_1, F \cap \beta, n)$ there is $x_\sigma \in V$ such that

$$q_1|\sigma \Vdash \dot{b} = x_\sigma.$$

By 2.8(ii), $(q,n) \le_{F\cap\beta} (p \restriction \beta, n)$; use part (iii) of the induction hypothesis to obtain q_2, m_2, and $g \in P_{\beta\alpha}$ so that $(q_2, m_2) <_{F\cap\beta} (q_1, n)$ and

$$q_2 \Vdash \dot{f} = g.$$

Note that by 2.8(ii) and 2.10, q_2 is $(F \cap \beta, n)$-determined.

Now, using the fact that $F \subset \beta$, one easily shows that $(q_2 \cup g, m)$ is the required pair.

(ii) Let p, F, n, and $\dot{A}$ be as in the hypothesis and let $\dot{f} \in V^{P_\alpha}$ be such that $p \Vdash$ “$\dot{f} \colon \omega \to \dot{A}$ is a bijection”. Obtain a fusion sequence $\{(q_n, F_n, m_n) \colon n \in \omega\}$ and a sequence $\{x_n \colon n \in \omega\}$ of finite sets so that $q_0 = p$ and $q_{n+1} \Vdash \dot{f}(n) \in x_n$. Now let $B = \bigcup_n x_n$ and let q be the fusion of the (q_n, F_n, m_n).

(iii) Given $\gamma > \alpha, p, F, n$, and $\dot{f}$ with $p \Vdash_\alpha \dot{f} \in \dot{P}_{\alpha\gamma}$, let q, m, B be such that $B \in V$ is countable, $(q,m) <_F (p,n)$ and $q \Vdash \operatorname{suppt} \dot{f} \subset B$. For each $\mu \in B$, notice that p forces $\dot{f}(\mu)$ to be a term in the language of forcing over V. Thus we let $g(\mu)$ be such a term denoting the same object in $V[G_\mu]$ (where $q \in G_\mu \restriction \alpha$) as $\dot{f}(\mu)$

denotes in $V[G_\alpha][G_{\alpha\mu}]$ (where $q \in G_\alpha$). Now $g \in P_{\alpha\gamma}$ and $q \Vdash \dot{f} = g$. (See [Ba, §7] for a similar argument.) $\square$

2.13 **Corollary** P_α *does not collapse* ω_1. $\square$

As a further application which we will use later in the proof of Theorem 2.22, we show the following:

2.14 **Lemma** *Suppose* $\mathrm{cf}(\alpha) = \omega_1$ *and* G_α *is* P_α*-generic over* V*. Then for every* $f \in V[G_\alpha] \cap 2^\omega$*, there is* $\beta < \alpha$ *such that* $f \in V[G_\beta]$*.*

Proof. Let $p \Vdash \dot{f}\colon \omega \to 2$ and define a fusion sequence $\{(q_n, F_n, m_n)\colon n \in \omega\}$ so that $q_0 = p$, q_{n+1} is (F_n, m_n)-determined for all $n \in \omega$, and for all $\sigma \in \Sigma(q_{n+1}, F_n, m_n)$,

$$q_{n+1}|\sigma \Vdash \dot{f}(n) = x_\sigma.$$

Let q be a fusion of the q_n and let β be such that $\sup(\mathrm{suppt}\, q) < \beta < \alpha$. Now if $q \restriction \beta \in G_\beta$, define $g \in {}^\omega 2 \cap V[G_\beta]$ by

$$g(n) = x_\sigma \quad \text{iff} \quad \sigma \text{ is the unique member of } \Sigma(q \restriction \beta, F_n, m_n) \text{ for which } (q|\sigma) \restriction \beta \in G_\beta.$$

Notice g is well defined because

$$\Sigma(q \restriction \beta, F_n, m_n) = \Sigma(q_n, F_n, m_n) = \Sigma(q_{n+1}, F_n, m_n).$$

Thus $q \Vdash_\alpha$ "$\dot{g} \in V[G_\beta] \wedge \dot{f} = \dot{g}$", as required. $\square$

As a final technical lemma, we show that arbitrarily large initial segments of a new real are determined by a finite set of old reals.

2.15 **Lemma** *If* $p \Vdash$ "$\dot{a} \notin V$ *and* $\dot{a} \in 2^\omega$", $n \in \omega$, $F \in [\alpha]^{<\omega}$, $Y \in [2^\omega]^{<\omega} \cap V$ *and* p *is* (F, n)*-determined, there are* $X \in [2^\omega]^{<\omega} \cap V$ *and for each* $k < \omega$ *a condition* q^k *such that*

(a) $(q^k, n) \leq_F (p, n)$**;**
(b) q^k **is** (F, n)**-determined;**
(c) $q^k \Vdash \exists x \in X (x \restriction k = \dot{a} \restriction k)$; **and**
(d) $X \cap Y = \varnothing$**.**

Proof. List $\Sigma(p, F, n)$ as $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\}$. We build a sequence of terms $\langle \dot{s}_i\colon i \in \omega\rangle$ and conditions $\langle q_i\colon i \in \omega\rangle$ and an $\omega \times N$ matrix $[t_{ij}]$ so that for all $i \in \omega$, $j \in N$,

(1) $q_i|\sigma_j \Vdash$ "$t_{ij} = \dot{s}_i$, $\dot{s}_i$ is an initial segment of $\dot{a}$, and $\forall y \in Y (y \supset \dot{s}_i)$";
(2) $(q_{i+1}, n) \leq_F (q_i, n)$;
(3) q_i is (F, n)-determined.

To get the 0th row, note there is a term $\dot{s}_0$ such that

$$p \Vdash \text{"}\dot{s}_0 \text{ is an initial segment of } \dot{a} \text{ and } \forall y \in Y (y \supset \dot{s}_0)\text{"}.$$

Use Theorem 2.12 to obtain q_0, m_0, and t_{0j}, $j < N$, so that $(q, m_0) <_F (p, n)$ (whence $(q_0, n) \leq_F (p, n)$) and for all $j < N$, $q_0|\sigma_j \Vdash \dot{s}_0 = t_{0j}$. Note that

$\Sigma(p,F,n) = \Sigma(q_0,F,n)$ and q_0 is (F,n)-determined (by (F,n)-determinedness of p).

Having satisfied (1) and (2) at stage i, note that there is a term $\dot{s}_{i+1}$ such that

$$q_i \Vdash \text{“}\dot{s}_{i+1} \supsetneq \dot{s}_i \text{ and } \dot{s}_{i+1} \text{ is an initial segment of } \dot{a}\text{”}.$$

Find $q_{i+1}, m_{i+1}, t_{i+1,j}$ $(j < N)$ so that $(q_{i+1}, m_{i+1}) <_F (q_i, n)$ and for each j, $q_{i+1}|\sigma_j \Vdash \dot{s}_{i+1} = t_{i+1,j}$. Note that $\Sigma(q_{i+1},F,n) = \Sigma(q_i,F,n) = \Sigma(p,F,n)$ and q_{i+1} is (F,n)-determined.

Having completed the induction, we let X be the set of unions of the columns of $[t_{ij}]$, i.e.,

$$X = \left\{ x \colon \exists_j \left(x = \bigcup_{i\in\omega} t_{ij} \right) \right\} = \{x_0, x_1, \ldots, x_{N-1}\}.$$

$X \in [2^\omega]^{<\omega}$ because, as one shows by induction, the $t_{i,j}$ are strictly increasing for fixed j.

Now we modify the q_i slightly to satisfy condition (c): To get q^k, let n_k be so large that

$$\forall j < N(\operatorname{dom} t_{n_k,j}) \supset k \text{ and } \forall k' < k(n_k > n_{k'})$$

and let $q^k = q_{n_k}$. Then for all $j < N$,

$$q^k|\sigma_j = q_{n_k}|\sigma_j \Vdash \dot{a} \restriction k = t_{ij} \restriction k$$

and so

$$q^k|\sigma_j \Vdash \exists x \in X(x \restriction k = \dot{a} \restriction k).$$

Now by Propositon 2.9, condition (c) is satisfied and we are done. □

From now on we will be a little more specific about the particulars of the iteration P_α. We shall assume that $V \vDash$ “$2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$”. Let P be some strongly proper order defined in V by $\theta(x)$ and witnessed by a constructible sequence of finite sets $T = \{T_n \colon n \in \omega\}$. Let P' denote the Sacks order. Let $Z = \{\alpha < \omega_2 \colon \operatorname{cf}\alpha = \omega\}$. Let P_{ω_2} denote an ω_2-stage countable support iteration such that for all $\beta \in Z$, $T^\beta = T$, and $\{\dot{A}^\beta_{\tau,n} \colon n \in \omega \wedge \Vdash_\beta \tau \in \dot{Q}_\beta\}$, $\{\dot{\varphi}^\beta_{\tau,n} \colon n \in \omega \wedge \Vdash_\beta \tau \in \dot{Q}_\beta\}$ are terms so that $\Vdash_\beta$ “$|\dot{Q}_\beta| \leq \mathbf{c}$ and $\theta(\dot{Q}_\beta)$ and T^β, $\{\dot{A}^\beta_{\tau,n}\}_{\tau,n}, \{\dot{\varphi}^\beta_{\tau,n}\}_{\tau,n}$ witness that $\dot{Q}_\beta$ is strongly proper”. For $\beta \in Z' = \omega_2 \backslash Z$, assume $\Vdash_\beta \dot{Q}_\beta$ is the Sacks order, and we will have corresponding $T^\beta, \dot{A}^\beta_{\tau,n}, \dot{\varphi}^\beta_{\tau,n}$ for these coordinates as well. Using familiar techniques (which can be found in [Ba] and [J2]), we have the following theorem.

2.16 **Theorem** (i) If $\beta \in Z'$ and $\dot{P}_{\omega_2}$ is a term for the iteration P_{ω_2} defined in V^{P_β}, then $\Vdash_\beta \dot{P}_{\beta\omega_2} \cong \dot{P}_{\omega_2}$;

(ii) $\forall \beta < \omega_2 \Vdash_\beta$ CH;

(iii) P_{ω_2} has the $\aleph_2$-cc and all cardinals and cofinalities are preserved;

(iv) $\Vdash_{\omega_2} \mathbf{c} = \omega_2$.

Proof (Outline). (i) See [Ba, §5; J2, 7.13]; for each β, the βth stage forces $\dot{P}_{\beta,\omega_2}$ to be an iteration of some kind in V^{P_β}; then since (a) for $\beta \in Z'$ the factors “line up”, and (b) the type of iteration being considered (i.e. countable support) is absolute for V, $V[G_\beta]$ (by 2.12(ii)), the iteration in V^{P_β} is isomorphic to $\dot{P}_{\beta,\omega_2}$.

(ii) One shows by induction on $\beta < \omega_2$ that (a) $\Vdash_\beta$ CH and (b) there is a dense set $D_\beta \subset P_\beta$ of power $\aleph_1$. To construct the dense set, each factor must be forced to have power $\leq \aleph_1$ and CH is used; since $|D_\beta| \leq \aleph_1$, an unpublished observation of Baumgartner (which says that whenever CH holds, posets of power $\leq \aleph_1$ which do not collapse $\aleph_1$ preserve CH) guarantees that $\Vdash_\beta$ CH.

(iii) A Δ-system argument gives the $\aleph_2$-cc. The rest follows from 2.13.

(iv) We have added $\aleph_2$ Sacks reals; because of CH and the fact that $2^{\omega_1} = \omega_2$, the continuum is at most ω_2. $\square$

2.17 **Remark** In our applications, a mixed iteration P_{ω_2} of just two orders of cardinality $\mathbf{c}$, as described above, will suffice. As we shall see, the only restrictions to increasing the number of factors are (a) the set Y of coordinates $\beta < \omega_2$ for which $\Vdash_\beta$ "$\dot{Q}_\beta$ is the Sacks order" should include the set $\{0\} \cup \{\gamma\colon \mathrm{cf}\,\gamma = \omega_1\}$, and (b) for all $\beta \in Y$, $\Vdash_\beta \dot{P}_{\beta\omega_2} \cong \dot{P}_{\omega_2}$. Thus, for example, if we let $Y = \{0\} \cup \{\gamma\colon \mathrm{cf}\,\gamma = \omega_1\} \cup \{\mu \cdot \nu\colon \mathrm{cf}\,\mu = \omega_1 \text{ and } \mathrm{cf}\,\nu = \omega\}$, then since for every pair $\mu < \nu$ of successive members of Y, the interval (μ, ν) is a copy of $(0, \omega_1)$, we can obtain an iteration of ω_1 many strongly proper orders—so that (a) and (b) are satisfied—by (essentially) repeating the sequence of orders defined over $(0, \omega_1)$ over each interval (μ, ν), and putting the Sacks order elsewhere.

In the remainder of this section, we show how iterations of two factors described above yield forcing extensions which model Miller's result, i.e., that for every $X \subset 2^\omega$ of power $\mathbf{c}$, there is a continuous $f\colon 2^\omega \to 2^\omega$ such that $f''X = 2^\omega$. The main idea is that each new real can be mapped continuously onto the first Sacks real by a map coded in V. Then, given $X \in V[G_{\omega_2}]$, $X \subset 2^\omega$, which cannot be mapped continuously onto 2^ω, one shows that $X \subset V[G_\alpha]$ for some $\alpha < \omega_2$ by using the map above to force each new real not in $V[G_\alpha]$ to lie outside of X. The proofs follow [M2] closely. As a notational convenience, we will henceforth identify $s \in 2^{<\omega}$ with the map $\hat{s}\colon \{0\} \to 2^{<\omega}$ defined by $\hat{s}(0) = s$.

2.18 **Lemma** *If* $p \Vdash$ "$\dot{a} \notin V$ and $\dot{a} \in 2^\omega$", $n \in \omega$, $F \in [\omega_2]^{<\omega}$, and p is (F, n)-determined, then there are $q \in P_{\omega_2}$ and a collection $\{C_s\colon s \in q(0) \cap 2^n\}$ of disjoint clopen subsets of 2^ω such that

(i) $(q, n) \leq_F (p, n)$, and
(ii) $\forall s \in q(0) \cap 2^n (q|s \Vdash \dot{a} \in C_s)$.

Proof. Let $\{s_0, \ldots, s_{N-1}\}$ enumerate $p(0) \cap 2^n$. Apply Lemma 2.15 using $p|s_0$ and $Y = \varnothing$ to obtain a finite set $X_0 \subset 2^\omega \cap V$ and conditions $q_0^k \leq p|s_0$, $k \in \omega$, such that
$$q_0^k \Vdash \exists x \in X_0 (\dot{a} \restriction k = x \restriction k).$$

Having defined X_i, q_i^k for $i < N-1$, apply Lemma 2.15 again, using $p|s_{i+1}$ and $Y = \bigcup_{j \leq i} X_j$, to obtain $q_{i+1}^k \leq p|s_{i+1}$ and a finite set $X_{i+1} \subset 2^\omega \cap V$ disjoint from Y such that
$$q_{i+1}^k \Vdash \exists x \in X_{i+1} (\dot{a} \restriction k = x \restriction k).$$

Let k be large enough so that if $i \neq j$, $x_i \in X_i$, $x_j \in X_j$ then $x_i \restriction k \neq x_j \restriction k$. Let $C_{s_i} = \bigcup_{x \in X_i} U_{x \restriction k}$ (where $U_t = \{f \in 2^\omega\colon f \supset t\}$). Now glue together the q_i^k

as follows: Let $q(0) = \bigcup_{i<N} q_i^k(0)$. Notice $(q(0), n) \leq (p(0), n)$. For $0 < \beta < \omega_2$, proceed by induction to define $q \restriction \beta$ using as the induction hypothesis the following:

(a) $(q \restriction \beta, n) \leq_{F\cap\beta} (p \restriction \beta, n)$, and
(b) $(q|s_i) \restriction \beta = q_i^k \restriction \beta$.

For limit β, let $q \restriction \beta$ be the union of the $q \restriction \beta'$, $\beta' < \beta$. If $q \restriction \beta$ is defined, let $q(\beta)$ be a term defined by cases:

$$(\forall i < N)(q|s_i) \restriction \beta \Vdash q(\beta) = q_i^k(\beta).$$

By (b) of the induction hypothesis, $q|s_i \restriction \beta \Vdash$ "$(q(\beta), n) \leq (p(\beta), n)$" if $\beta \in F$, and so (a) is satisfied. (Note that $q \restriction \beta$ is $(\{0\}, n)$-determined.)

It is clear that the resulting q satisfies the conclusion of the lemma. □

The fact that any new real appearing at any stage can be mapped continuously onto the first Sacks real by a V-coded map derives from this lemma, and in particular, from the fact that having the Sacks order on the first coordinate allows us to paste together conditions below p of our choosing to obtain a condition which is "fat" with respect to p (and hence preparing for fusion). This property appears to be unique to the Sacks order (at least among strongly proper orders which add reals).

2.19 **Theorem** *If* $p \Vdash \dot{a} \notin V$ *and* $\dot{a} \in 2^\omega$, *there is* $q \leq p$ *and for each* $n \in \omega$ *there is* $m \geq n$ *and a family* $\{C_s : s \in q(0) \cap 2^m\}$ *of disjoint clopen subsets of* 2^ω *such that*

$$\forall s \in p(0) \cap 2^m (q|s \Vdash \dot{a} \in C_s).$$

Proof. Build a fusion sequence $\{(q_n, F_n, m_n) : n \in \omega\}$ and disjoint clopen sets $\{C_s : s \in q_n(0) \cap 2^{m_n}\}$ for $n > 0$ so that

(a) q_{n+1} is (F, n)-determined;
(b) $(q_{n+1}, m_{n+1}) <_{F_n} (q_n, m_n)$;
(c) $\forall s \in q_n(0) \cap 2^{m_n} (q_{n+1}|s \Vdash \dot{a} \in C_s)$.

WLOG, assume $0 \in \mathrm{suppt}(p)$ and begin building the sequence by letting $0 \in F_0$, $q_0 = p$, and letting m_0 be arbitrary. Now if (q_n, F_n, m_n), $n > 0$, has been defined, use 2.12(i) to get q' so that

$$(*) \qquad (q', m_n) <_{F_n} (q_n, m_n) \quad \text{and} \quad q' \text{ is } (F, n)\text{-determined}.$$

Let q'', $\{C_s : s \in q_n \cap 2^{m_n}\}$ be as in Lemma 2.17 and finally let $(q_{n+1}, m_{n+1}) <_{F_n} (q'', m_n)$ be as in 2.12(i) again. Choose $F_{n+1} \supseteq F_n$ using an appropriate recipe to ensure $\bigcup_n F_n = \bigcup_n \mathrm{suppt}\, q_n$. Now any fusion of $\{(q_n, F_n, m_n) : n \in \omega\}$ is the desired condition. □

2.20 **Corollary** *If* $p \Vdash a \notin V$ *and* $\dot{a} \in 2^\omega$, *there is* $q \leq p$ *and for each* $n \in \omega$ *a family* $\mathcal{C}_n = \{C_s : s \in q(0) \cap 2^n\}$ *of disjoint clopen subsets of* 2^ω *such that*

(a) $\mathcal{C}_n$ *is a partition of* 2^ω;
(b) *if* $s \subset t$ *then* $C_t \subset C_s$;
(c) $q|s \Vdash \dot{a} \in C_s$.

Proof. This is straightforward (see [M2, §4]). $\square$

2.21 **Theorem** (Key Lemma)**.** *Suppose* $p \Vdash \dot{a} \notin V$ and $\dot{a} \in 2^\omega$. Then there are $q \le p$ and f coded in V such that

$$q \Vdash_{\omega_2} \text{“}\dot{f}\colon 2^\omega \to [q(0)],\ \dot{f} \textit{ is continuous},\ \textit{and } \dot{f}(\dot{a}) = x_0\text{”},$$

where x_0 denotes the first Sacks real.

2.22 **Remark** g has a code in V if the sequence $\langle (g^{-1}(U_s)\colon s \in 2^{<\omega}\rangle$ is coded in V. Note this sequence is coded as a subset of ω. As above, we denote the evaluation in V of such a code with an undotted letter (say f) and its evaluation in an extension by a dotted letter (say $\dot{f}$).

Proof of 2.21. Let $q \le p$ and $\{\mathcal{C}_n\colon n \in \omega\}$ be as in the corollary. Define $f\colon 2^\omega \to [q(0)]$ in V by putting

$$s \subset f(x) \quad \text{iff} \quad x \in C_s.$$

Now, referring to 2.19, f is a function because of (a) and (b); f is continuous by clopenness of the C_s; and $q \Vdash \dot{f}(\dot{a}) = x_0$ because of (c). $\square$

2.23 **Theorem** $\Vdash_{\omega_2}$ *“Every set of reals of power* $\mathbf{c}$ *is mapped uniformly continuously onto* $[0,1]$*”.*

Proof. By our remarks in the Introduction, it suffices to reproduce Miller's proof (for iterated Sacks forcing) that for each $X \subset 2^\omega$, $|X| = \mathbf{c}$, there is a continuous $f\colon 2^\omega \to 2^\omega$ for which $f''X = 2^\omega$. Assume there is an X for which the statement fails; for each $f\colon 2^\omega \to 2^\omega$, let $z_f \in 2^\omega \backslash f''X$. Let $C^\beta(2^\omega) = \{f \in 2^\omega \cap V[G_{\omega_2}]\colon f$ is coded in $V[G_\beta]\}$, where G_{ω_2} is P_{ω_2}-generic over V.

Claim. There is $\beta \in Z'$ such that for all $f \in C^\beta(2^\omega)$, $z_f \in V[G_\beta] \cap 2^\omega$.

Proof of Claim. Use the $\aleph_2$-cc to construct a Löwenheim-Skolem type argument as in [BL, §4.5], which yields a club of β with the property that for all $\gamma < \beta$ if $f \in C^\gamma(2^\omega)$ then $z_f \in V[G_{\gamma'}]$ for some $\gamma' < \beta$. Thus there is such a β of cofinality ω_1 (hence $\beta \in Z'$); now if $f \in C^\beta(2^\omega)$ with code $c \in 2^\omega$, use 2.14 to obtain $\gamma < \beta$ such that $c \in V[G_\gamma]$. Then by the choice of β, $z_f \in V[G_\beta]$.

Now by 2.16(i) we may assume that for each $f \in C^0(2^\omega)$, $z_f \in 2^\omega \cap V$. We show $\Vdash_{\omega_2} X \subset V$ by showing that if $p \Vdash$ “$\dot{a} \in 2^\omega$ and $\dot{a} \notin V$” then there is $\hat{q} \le p$ for which $\hat{q} \Vdash \dot{a} \notin X$.

Let $q \le p$ and $f\colon 2^\omega \to [q(0)]$ be as in 2.20. Define $g = g_1 \circ g_2\colon [q(0)] \to 2^\omega$ as follows: $g_2\colon [q(0)] \to 2^\omega \times 2^\omega$ is a homeomorphism and $g_1\colon 2^\omega \times 2^\omega \to 2^\omega$ is projection onto the first coordinate. Note that for each $z \in 2^\omega$, $g^{-1}(z) = [r_z]$ is a perfect subset of $[q(0)]$. Define $q_z(0) = r_z$ and for $\beta > 0$, $q_z(\beta) = q(\beta)$; then $q_z \Vdash \dot{g}(x_0) = z$. Thus let $\hat{q} = q_z$ where $z = z_{g \circ f}$. Then $\hat{q} \Vdash \dot{g}(\dot{f}(\dot{a})) = z$ and $\hat{q} \Vdash z \notin \dot{g}(\dot{f}(X))$. It follows that $\hat{q} \Vdash \dot{a} \notin X$. $\square$

2.24 **Remark** Note that Theorem 2.23 cannot be proven if we allow arbitrary ω-proper, ω^ω-bounding orders in place of strongly proper orders in the iteration; for example, if the random real order is used as a factor on the Z-coordinates, $\Vdash_{\omega_2}$ “There is a $\mathbf{c}$-Sierpinski set” (recall Proposition 0.2).

3. Proofs and questions

In this section we complete the proofs of the results introduced in §0 and state several open problems along the way. Henceforth, let us denote by V' the model obtained by forcing with a mixed iteration as in §2 using the IOER order on the Z-coordinates. We observe that V' satisfies the Generalized Borel Conjecture by Theorem 2.23, the fact that forcing with the IOER order makes the ground model reals have strong measure zero, and that the mixed iteration is ω^ω-bounding. (This proves Theorem 0.0.) By the same fact,

$$V' \models \mathrm{non}(U_0) = \mathbf{c}.$$

As was observed in §0 (see remark following 0.9) $\mathrm{non}(U_0) = \mathrm{non}(\mathcal{L})$; so by Grzegorek's result (0.7(a)), we have in V' a U_0 set (hence $2^{\mathbf{c}}$ U_0 sets) of power $\mathbf{c}$. This proves Theorem 0.3.

As noted before, Theorem 0.0 is a somewhat unusual result for hereditary classes over $\mathbb{R}$. For a class $\mathcal{A} \subset (s)_0$, we write $\mathrm{GBC}(\mathcal{A})$ if it is consistent that $\mathcal{A} = [\mathbb{R}]^{<\mathbf{c}}$. We observed in the introduction that $\neg\mathrm{GBC}(\mathrm{U}_0)$ and $\neg\mathrm{GBC}(\mathrm{AFC})$; similarly, $\neg\mathrm{GBC}(\overline{\mathrm{AFC}})$. We now briefly consider the other classes in Diagram 1. Since L and S are not hereditary, we augment them with count: Let $L' = L \cup \mathrm{count}$ and $S' = S \cup \mathrm{count}$. Still, $\neg\mathrm{GBC}(L')$ and $\neg\mathrm{GBC}(S')$ hold because of the easily proved fact that there is in ZFC a set of reals of power $\aleph_1$ which is neither Luzin nor Sierpinski. Thus if CH holds we have sets $X, Y \subset \mathbb{R}$ with $X \in L\backslash[\mathbb{R}]^{<\mathbf{c}}$ and $Y \in S\backslash[\mathbb{R}]^{<\mathbf{c}}$ and if $\neg$CH holds we have a set in $[\mathbb{R}]^{<\mathbf{c}}\backslash(S' \cup L')$. Turning to λ sets, we use the fact (proven in an article by van Douwen [vD]) that $\mathbf{b}$ is the least cardinal κ for which there is a λ set of power κ that is not a λ' set to show that $\neg\mathrm{GBC}(\lambda)$: The case in which $\mathbf{b} = \mathbf{c}$ is immediate; if $\mathbf{b} < \mathbf{c}$, let $A \subset \mathbb{R}$, A countable, and $X \in \lambda$ be such that $|X| = \mathbf{b}$ and $X \cup A \notin \lambda$ (i.e., A witnesses that $X \notin \lambda'$). Then $X \cup A \in [\mathbb{R}]^{<\mathbf{c}}\backslash\lambda$. Miller points out that $\neg\mathrm{GBC}(C'')$ (see [FM]) and $\neg\mathrm{GBC}(\gamma)$ (γ sets are defined in [GM]).

On the other hand, GBC does hold for a few classes. Letting $C' = \{X \subset \mathbb{R}\colon f''X \in C$ for all continuous $f\}$ (C' was introduced by Rothberger in [R]), it is clear that $\mathrm{GBC}(C) \to \mathrm{GBC}(C')$. It is well known that under MA,

$$\{X\colon X \text{ is a } Q \text{ set}\} = [\mathbb{R}]^{<\mathbf{c}}.$$

(A Q set is a set all of whose subsets are relative G_δ's; see [M3].) Finally, an unpublished result of Miller states that $\mathrm{GBC}(\lambda')$ holds in Laver's model [La]. We are left with the following questions:

3.0 *Question.* Does GBC hold for con or P?

Recall that $(*)$ is the statement that every set of reals of power $\mathbf{c}$ satisfies M. As was mentioned in the introduction, a model of "$(*)+$ there is an AFC set of power $\mathbf{c}$" could be obtained if there were a strongly proper order forcing the ground model reals to be meager. Although all our examples of strongly proper orders force the ground reals to be "badly" nonmeager (what is actually forced is the statement "Every meager set is contained in a meager set coded in the ground model"), the axioms in 1.0 do not seem to imply this. The reason appears to be that the Sacks, IOER, and Silver orders all satisfy the following additional axiom which does not follow readily from the other axioms:

(5) For each $s \in A_{p,n}$ and each $q \leq p|s$ there is $(r,n) \leq (p,n)$ so that $r_s = p$.

Adding (5) to the list in 1.0, we can show that iterations satisfy a similar property and hence that each meager set in $V[G_{\omega_2}]$ is covered by a meager set coded in V (see [M1, §7.2] for a prototypical proof). A natural question is:

3.1 *Question.* Does Axiom (5) follow from Axioms 1–4?

Models obtained by forcing as in §2 must satisfy "$\mathrm{cov}(\mathcal{L}) < \mathbf{c}$ and $\mathrm{cov}(\mathcal{K}) < \mathbf{c}$" since $(*)$ itself implies these relations (for example, if $\mathrm{cov}(\mathcal{K}) = \mathbf{c}$ there is a $\mathbf{c}$-Luzin set). However, though our models must satisfy "$\mathbf{d} < \mathbf{c}$" (since the iterations are ω^ω-bounding), it is unclear whether $(*)$ implies this. More generally, we ask:

3.2 *Question.* Is "$(*) + \mathbf{b} = \mathbf{c}$" consistent?

Note that a positive answer would give a model of "$(*)$ + there are $2^{\mathbf{c}}$ λ sets of power $\mathbf{c}$".

We now turn to a discussion of the class $\mathcal{J} = \{X \subset \mathbb{R} \colon \neg M(X)\}$ introduced earlier. We first note that unlike C, $\mathcal{J}$ must contain uncountable members, for if CH holds, there is a Luzin set; if CH fails, there is a set $X \subset \mathbb{R}$ with $\omega < |X| < \mathbf{c}$ (and such a set is in $\mathcal{J}$). We have

3.3 **Proposition** *$\mathcal{J}$ has uncountable members.* □

We now show that $\mathcal{J}$ is a σ-ideal; we use an argument of Miller's which he used to show that $\mathcal{I}$ (mentioned in the Introduction) is closed under countable unions.

3.4 **Proposition** *$\mathcal{J}$ is a σ-ideal.*

Proof. $\mathcal{J}$ is hereditary since uniformly continuous maps extend to the whole space. To see $\mathcal{J}$ is closed under countable unions, first note that $\mathcal{J}$ is closed under uniformly continuous maps and recall that $\mathcal{J}$ is a subclass of the σ-ideal $(s)_0$. Thus, if $X_n \in \mathcal{J}$ and $X = \bigcup_n X_n$ for each $n \in \omega$, and $f \colon X \to [0,1]$ is uniformly continuous, then

$$f''X = \bigcup_n f''X_n \in (s)_0;$$

thus $f''X \neq [0,1]$. Now since f was arbitrary, it follows that $X \in \mathcal{J}$. □

Next, we restate and prove Theorem 0.5:

Theorem (a) $\mathrm{ZFC} \vdash \mathrm{U}_0 \neq \mathcal{J}$ and $\mathrm{AFC} \neq \mathcal{J}$;

(b) $\mathrm{ZFC} + \mathrm{CH} \vdash \mathrm{U}_0 \cap \mathrm{AFC} \subset \mathcal{J}$ and $\mathcal{J} \subset \mathrm{U}_0 \cup \mathrm{AFC}$;

(c) $\mathrm{Con(ZFC)} \to \mathrm{Con(ZFC)} + \mathrm{U}_0 \subsetneq \mathcal{J} + \mathrm{AFC} \subsetneq \mathcal{J})$;

(d) $\mathrm{Con(ZFC)} \to \mathrm{Con}(\mathrm{ZFC} + \mathcal{J} \subsetneq \mathrm{U}_0)$.

Proof. (a) We use 0.7. If $\mathrm{non}(\mathcal{L}) < \mathbf{c}$, there is $X \in [\mathbb{R}]^{<\mathbf{c}} \setminus U_0$ and clearly such X is in $\mathcal{J}$. If $\mathrm{non}(\mathcal{L}) = \mathbf{c}$, argue as in the proof that in ZFC there is always a member of $(s)_0$ satisfying M: Note that U_0 is closed under 1-1 continuous preimages (see [M3, 9.3.1]) and so if f is a bijection from a U_0 set in 2^ω onto 2^ω, then $f \subset 2^\omega \times 2^\omega$ is U_0. Now projection onto the second coordinate gives rise to a uniformly continuous map from a U_0 set onto $[0,1]$.

For AFC, if $\mathrm{non}(\mathcal{K}) < \mathbf{c}$ we have an $X \in [\mathbb{R}]^{<\mathbf{c}}\backslash\mathrm{AFC}$ as before. If $\mathrm{non}(\mathcal{K}) = \mathbf{c}$, argue as above using $\overline{\mathrm{AFC}}$ (which is closed under 1-1 continuous preimages; see [G2]) in place of U_0. We get a uniformly continuous map from an $\overline{\mathrm{AFC}}$ (hence AFC) set onto $[0,1]$.

(b) To see $\mathrm{U}_0 \cap \mathrm{AFC} \neq \mathcal{J}$, we take any uncountable set $X \in \mathrm{U}_0 \cap \overline{\mathrm{AFC}}$ (a Hausdorff gap for example; see [La, M3]). By CH, $|X| = \mathbf{c}$. Now argue as in part (a) to get a uniformly continuous map from X onto $[0,1]$. To see that $\mathcal{J} \subset \mathrm{U}_0 \cup \mathrm{AFC}$, consider the union of a Luzin set and a Sierpinski set.

(c) The required model is obtained by iterated perfect set forcing [M2]; it has already been observed that $\mathrm{U}_0 \cup \mathrm{AFC} \subset [\mathbb{R}]^{<\mathbf{c}}$ in this model.

(d) The model is that of Theorem 0.0 in which $\mathcal{J} = [\mathbb{R}]^{<\mathbf{c}}$ and $\mathrm{non}(\mathcal{L}) = \mathbf{c}$. $\square$

Next we prove Lemma 0.9; we begin with the following lemma.

3.5 **Lemma** *The following are equivalent for any perfect Polish space Z and $X \subset Z$:*

(i) $X \in \mathrm{AFC}$.

(ii) X is meager and for all nowhere dense perfect sets P, $X \cap P$ is meager relative to P.

(iii) For all $P \subset Z$, if P is perfect nowhere dense or if P is the closure of a basic open set, then $X \cap P$ is meager relative to P.

Proof. (i) → (ii) and (ii) → (iii) are immediate. For (iii) → (i) let $Q' = P\backslash \mathrm{int}\, P$ and write $Q' = Q \cup C$, where Q is perfect nowhere dense and C is countable. Note that for each basic open set $B \subset Z$, $X \cap B$ is meager relative to B. Thus $X \cap \mathrm{int}\, P$ is meager relative to $\mathrm{int}\, P$, hence to P; also, $X \cap Q$, hence $X \cap Q'$, is meager relative to P. The result follows. $\square$

We now restate and prove Lemma 0.9:

Lemma (a) $\mathrm{non}(\mathcal{L}) = \mathrm{non}(U_0)$; and (b) $\mathrm{non}(\mathcal{K}) = \mathrm{non}(\overline{\mathrm{AFC}}) = \mathrm{non}(\mathrm{AFC})$.

Proof. (a) was proven in the remarks following 0.9. For (b), first note that $\mathrm{non}(\overline{\mathrm{AFC}}) \leq \mathrm{non}(\mathrm{AFC}) \leq \mathrm{non}(\mathcal{K})$. We prove $\mathrm{non}(\mathcal{K}) \leq \mathrm{non}(\overline{\mathrm{AFC}})$: Suppose $Y \notin \overline{\mathrm{AFC}}$. Let $f\colon X \to Y$ be 1-1 continuous so that $X \notin \mathrm{AFC}$. By 3.5(iii), there is a perfect nowhere dense set P such that $X \cap P$ is nonmeager relative to P. Let $h\colon P \to 2^\omega$ be a homeomorphism and $g\colon 2^\omega \to [0,1]$ the canonical onto map. Then $g(h(X \cap P)) \notin \mathcal{K}$ and $|g(h(X \cap P))| \leq |Y|$. $\square$

Finally we prove Theorem 0.6:

Theorem (a) In the random real model (or if $\mathbf{c}$ is real-valued measurable) we have $\mathrm{U}_0 \subsetneq \overline{\mathrm{AFC}}$.

(b) In the Cohen model, $\overline{\mathrm{AFC}} \subsetneq \mathrm{U}_0$ (in fact, $\overline{\mathrm{AFC}} \subsetneq C''$).

(c) Con(ZFC) → (Con(ZFC) + *either* φ *or* ψ) where $\varphi \equiv$ "$\mathrm{AFC} \subsetneq \mathrm{U}_0$" and $\psi \equiv$ "every set of reals of power $\mathbf{c}$ satisfies M and there are $2^{\mathbf{c}}$ many AFC sets of power $\mathbf{c}$".

Proof. (a) In [M2], Miller shows that in the model obtained by adding ω_2 random reals to a model of GCH, every U_0 set has power $< \mathbf{c}$. Now, using Lemma 0.9(b) and the fact the model also satisfies "$\mathrm{non}(\mathcal{K}) = \mathbf{c}$", we get that "$\mathrm{U}_0 \subset \overline{\mathrm{AFC}}$" holds as well. Using 0.7(b), we conclude that inclusion is proper.

To obtain the result from the theory "ZFC + $\mathbf{c}$ is real-valued measurable", begin with an atomless, nonzero, $\mathbf{c}$-additive probability measure μ defined on $\mathcal{P}(\mathbb{R})$. By

[Ma, 3.1(i)], $X \in \mathrm{U}_0$ iff each diffused measure on $\mathcal{B}(X)$ (i.e. the Borel sets relative to X) vanishes identically. (Say μ is diffused if it takes singletons to zero.) Thus, since μ is diffused (being atomless) and its properties are preserved under bijections, there is no U_0 set of cardinality $\mathbf{c}$. Thus, as above, it suffices to prove $\mathrm{non}(\mathcal{K}) = \mathbf{c}$. Referring to Diagram 2, it is enough to prove $\mathrm{cov}(\mathcal{L}) = \mathbf{c}$. Suppose $\{N_\alpha : \alpha < \kappa\}$ is a cover of $[0,1]$ by Lebesgue measure zero sets where $\kappa \leq \mathbf{c}$; we may assume each N_α is a Borel set. If $\mathcal{B} = \{\text{Borel sets on } [0,1]\}$, then $\mu \restriction \mathcal{B}$ is a finite diffused Borel measure and there is a continuous function $F : [0,1] \to [0,1]$ such that $\lambda \restriction \mathcal{B} = (\mu \circ F^{-1}) \restriction \mathcal{B}$ (where λ is Lebesgue measure). Hence $\{F^{-1}(N_\alpha) : \alpha < \kappa\}$ is a cover of $[0,1]$ by μ-measure zero sets; by $\mathbf{c}$-additivity of M, $\kappa = \mathbf{c}$, and the result follows.

(b) In [M2], Miller shows that in the model obtained by adding ω_2 Cohen reals to a model of GCH, every set of reals of power $\mathbf{c}$ contains a 1-1 continuous image of a Luzin set. Since such sets are not $\overline{\mathrm{AFC}}$ and since the latter is closed under 1-1 continuous preimages, it follows that no $\overline{\mathrm{AFC}}$ set has cardinality $\mathbf{c}$. Since the Cohen model satisfies "$\mathrm{cov}(\mathcal{K}) = \mathbf{c}$" (see [K]), and since $\mathrm{cov}(\mathcal{K}) \leq \mathrm{non}(C'')$ (Theorem 0.8 above), it follows that $\overline{\mathrm{AFC}} \subset C''$. The inclusion is proper since there is a Luzin set of power $\mathbf{c}$.

(c) Consider the model V' of Theorem 0.0. If there is an AFC set of power $\mathbf{c}$ in this model, then there are $2^{\mathbf{c}}$ of them, and ψ holds. If every AFC set has cardinality $< \mathbf{c}$, then since $\mathrm{non}(\mathcal{L}) = \mathbf{c}$, we can use 0.7(a) and 0.9(a) again to conclude that $\mathrm{AFC} \subsetneq \mathrm{U}_0$.

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