# ADDRESSING THE PROBLEM OF LARGE CARDINALS WITH VEDIC WISDOM

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Abstract. Shortly after Cantor's discovery of the existence in mathematics of an endless hierarchy of different sizes of infinite sets, a new challenge arose concerning the nature of infinity in mathematics. Enormous infinities, known as large cardinals, have turned out to be the key to solving many mainstream problems in mathematics, but because of their extraordinarily strong properties, large cardinals cannot be proven to exist at all. The Problem of Large Cardinals is to find a natural way to enrich the standard axioms of set theory so that large cardinals can be derived. To accomplish this goal, a much deeper intuition about the nature of the infinite than has been available so far is needed. We suggest that precisely such intuition can be extracted from the eternal Vedic wisdom. We formulate a new axiom of set theory, strongly motivated both by this ancient wisdom and by mathematical considerations, which provides a solution to the Problem of Large Cardinals.

One theme of research that has developed in the recent history of mathematics is the study of the infinite. In mathematics, the notion of the infinite is approached by studying sets having infinitely many members. The evolution of the mathematical investigation of the infinite has uncovered a fundamental question—known as the Problem of Large Cardinals—for which the usual tools and techniques of mathematics no longer seem to be adequate. A deeper insight into the structure of the mathematical universe itself seems to be necessary to provide a solution. This article discusses how Maharishi Vedic Science[1] has been used to provide the necessary insight, leading to a solution to the Problem of Large Cardinals—a solution that has appeared recently in the mathematics literature.[2]

## 1. The Classical Theory

The classical theory of the infinite began about 150 years ago. Progress in the classical theory is indicated by three significant milestones.

### 1.1. **Milestone #1: The Discovery That Infinity Exists.**

Prior to the second half of the nineteenth century, the subject of infinite sets was a forbidden topic. It was believed, for example, that, although we can imagine the natural numbers $1, 2, 3, \ldots$ continuing on indefinitely, any notion of a *single, completed set* containing

[1]Maharishi Vedic Science is Maharishi Mahesh Yogi's systematic presentation, both theoretical and practical, of the Veda and Vedic Literature. An introduction to Maharishi Vedic Science can be found in [7].

[2]The present work was presented at the Symposium *Maharishi Vedic Science: Illuminating the Cutting Edge of Modern Science*, April 27–29, 2012, Maharishi University of Management, Fairfield, Iowa, 52557. This article summarizes the work in [4] and updates [5].

all the natural numbers must be viewed as fanciful, lying outside the domain of rigorous mathematics.

There were many reasons for this taboo (see [8]). First, as Aristotle observed nearly two thousand years ago in his *Metaphysics* [1, Book 9, Chapter 6], we don't find completed infinities in nature (for instance, seasons return year after year, but at no point can it be said that "infinitely many seasons have passed," even though they could potentially continue forever), so one would not expect such a notion to make sense in mathematics either. A second reason had to do with theological beliefs: one objection [8, p. 13] asserted that a study of a completed infinity amounts to a study of God; but God cannot be bound by the mathematical conclusions of man. A third reason was that analysis of completed infinities seems to lead to paradoxes. For example, the infinite sum $1 - 1 + 1 - 1 + \cdots$ appears to have two values, depending on how parentheses are inserted:

$$(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0$$
$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 1.$$

The work of Georg Cantor (1845–1918) and other pioneers working during this time dispelled these doubts. Cantor not only addressed the mathematical issues, but also wrote extensive rebuttals to all philosophical and theological doubts that had been raised. In response to the long-held argument of Aristotle, Cantor observed that the long "potentially" infinite sequence $1, 2, 3, \ldots$ of natural numbers *presupposes* that all the natural numbers already exist as a completed collection, an "actual" infinite:

> ...in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on [22, p. 3].

Cantor and other researchers of that period resolved the apparent mathematical paradoxes that had been put forth. For instance, the problem about computing the sum of the terms $1-1+1-1+\cdots$ was resolved some years before Cantor's campaign by observing that, while some infinite summations like this do indeed have a value—a *sum*—others do not. We would not expect the summation $1 + 2 + 3 + 4 + \cdots$ to have a particular natural number value because *every natural number* is already a term in the series, so the "sum" would have to be bigger than any number. Such summations, like the series $1 - 1 + 1 - 1 + \cdots$, are said to *diverge*, to have no final sum. The precise notion of *convergence*, originated by A.L. Cauchy (1789–1857) in the first half of the nineteenth century, which put to rest this paradox, is now a core element of calculus and higher mathematics.

Cantor also addressed, in several ways, theological and philosophical concerns about studying completed infinities. One of his key arguments, which eventually transformed how mathematics was understood, was that mathematics is not inherently tied to any of the ways in which it is interpreted or applied. A large bulk of the problems that have been researched in mathematics have arisen from the sciences: the use of mathematical models to understand and predict nature's behavior has suggested hundreds of mathematical problems and has motivated significant advances in mathematics itself. But the mathematics in such cases, Cantor argued, is a *description* of natural phenomena, not the phenomena themselves. This separation of mathematics from its applications freed mathematics from irrelevant restrictions on the allowed topics of mathematical study, and, in particular, from non-mathematical views about non-mathematical notions of "the infinite."

Cantor's heroic efforts to argue the case for the mathematical infinite did not bear fruit, however, until a very practical need at the foundation of the mathematics of the day loomed large and was recognized as solvable only through the use of completed infinities. In Cantor's time, the basic tenets or "theorems" of calculus—the same calculus that is studied in mathematics curricula today—were well-understood but could not be rigorously proven using the tools available at the time. The difficulty, as Cantor and Dedekind observed, could be traced to an imprecise understanding of the idea of a "real number," a quantified mathematical point on a line. It was discovered that giving a precise definition of the real numbers *required* completed infinities (this point is familiar even in high school mathematics today: to represent a number like $\pi$ precisely as a decimal requires infinitely many decimal places).

With the recognition that completed infinities are necessary in mathematics, the first milestone in the classical era was achieved. With this discovery in place, Cantor went on to unveil another surprise about the infinite.

1.2. **Milestone #2: There is an endless hierarchy of infinite sizes.** Allowing the mathematical study of infinite *sets* gave the mathematician freedom to perform all the operations upon infinite sets that are ordinarily performed on finite sets. One such operation is the formation of the *power set* of a given set.

We can illustrate the power set operation with a simple example. Consider the set $S = \{1, 3, 4\}$. The set $S$ has three elements. The subsets of $S$ are $\{1\}, \{3\}, \{4\}$, $\{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}$, and the empty set, denoted $\emptyset$. These subsets can be arranged into a new set, denoted $\mathcal{P}(S)$, called the *power set* of $S$:

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}.$$

The set $\mathcal{P}(S)$ has *eight* elements, and so is larger than the set $S$ that we started with. In Cantor's time, it was well known that the power set of any finite set is always bigger than the original set: for any *finite* set $S, \mathcal{P}(S) > S$.

Cantor's surprising discovery was that the same could be said about *infinite* sets: for any set, finite or infinite, the power set must always be bigger.

The natural question, raised vigorously by Cantor's contemporaries (and perhaps equally vigorously by students even today) is: How can one infinite set be "bigger than" another?

Cantor was able to answer this question by developing a rigorous theory of infinite sets, which provided a precise definition of what it means for two sets to have the *same size.* Roughly speaking, two sets are said to have the same size if their elements can be matched up one for one. For instance, the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ of natural numbers is shown to have the same size as the set $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$ of whole numbers by matching the elements of $\mathbb{N}$ with those of $\mathbb{W}$ as shown in Figure 1.

Cantor then showed, with very clever reasoning, that, for any (infinite) set $S$, it is *impossible* for there to be a one-to-one correspondence between the elements of

| $\underline{\mathbb{N}}$ | | $\underline{\mathbb{W}}$ |
|---|---|---|
| 1 | $\rightarrow$ | 0 |
| 2 | $\rightarrow$ | 1 |
| 3 | $\rightarrow$ | 2 |
| . | | . |
| . | | . |
| . | | . |

FIGURE 1. A one-to-one correspondence between the natural numbers $\mathbb{N}$ and the whole numbers $\mathbb{W}$.

$S$ and the elements of its power set. Therefore, in particular, the power set $\mathcal{P}(\mathbb{N})$ of the infinite set $\mathbb{N}$ of natural numbers represents a *bigger* size of infinity than that of $\mathbb{N}$ itself. And his reasoning opens the door to yet bigger infinite sizes, since one can then apply the power set operation to $\mathcal{P}(\mathbb{N})$ to obtain a still larger infinite set. In fact, there is an infinite hierarchy of ever larger infinite sizes:

$$\mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \dots .$$

The discovery that for any set—even any *infinite* set—its power set is always bigger is known today as *Cantor's Theorem*. Cantor's Theorem marks the second big milestone in the classical theory of the infinite. From this second milestone we learn that not only does the mathematical infinite *exist*, but it also has a nature, a texture, an internal multiplicity, and even its own internal transformational dynamics.

Cantor's discoveries brought a long-sought sense of completion to the business of pure mathematics. Yet, soon after this sense of completion and balance had taken hold, a flaw was discovered—a flaw that would surprise even Cantor. To correct the problem, the foundations of mathematics were led to the third major milestone of this classical period.

1.3. **Milestone #3: To understand the very idea of a set, it is necessary to understand wholeness, the ultimate infinite.** As Cantor developed his theory of infinite sets, he apparently did not think to examine too closely his own definition of the concept of a *set*. Like most students of mathematics (and even many mathematicians today), he simply assumed that the meaning was obvious: a set is simply a collection of objects. At the turn of the century, Bertrand Russell [23] and others noticed, however, that this imprecise definition is flawed and leads to paradoxes that undermine the consistency of mathematics itself: using Cantor's naive notion of a set, Russell demonstrated that a certain paradoxical set $T$ must exist. This set $T$ is defined to be the "set" consisting of all sets that are not members of themselves. What makes $T$ paradoxical is that one can prove that it has the following property:

*$T$ is a member of itself if and only if $T$ is not a member of itself.*

If a set with an inconsistent property such as this were allowed into the mathematical universe, it would lead to an inconsistent mathematics, making it possible to prove absolutely anything.

To resolve the paradox, the approach was to return to Cantor's vision of the infinite. Cantor had not only shown that there is an endless hierarchy of infinite sizes—or infinite *cardinals* as they are called—but also declared [8, p. 42] that this multiplicity of infinite sizes in no way represents the *ultimate* infinite. For Cantor, the *Absolute Infinite*—the totality of all possible mathematics, beyond the possibility of increase or diminution, and beyond all mathematical determination—was the ultimate infinite, and provided the context in which mathematics should be done. Using Cantor's Absolute Infinite and a number of its properties as a guiding intuition, early set theorists developed an intuitive model for the universe of mathematics, the *universe of sets*, denoted $V$. The idea was that the legitimate sets, the sets that are to be used in mathematics, belong to $V$; but paradoxical sets would not appear in $V$.

With this intuitive model $V$, researchers formulated a collection of axioms that express the essential characteristics of $V$. The axioms describe which sets exist and how to obtain new sets from already existing sets. The axioms that were developed in this way, now known as the Zermelo-Fraenkel axioms with the Axiom of Choice, or ZFC, were sufficiently complete to support the formal construction of the stages of $V$, transforming the stages of $V$ from the realm of motivating intuition into formal, rigorously defined mathematical structures.

The construction of $V$ is simple and elegant; it is built in stages, $V_0, V_1, V_2, \ldots$. Then, $V$ itself is obtained by putting together all the stages, by forming the union of these stages. Stage $V_0$ is just the empty set $\emptyset$. Then, each successive stage is obtained from the previous stage by taking the power set of the previous stage; it follows that each successive stage *includes* all previous stages. This strategy leads to the following construction:

$$V_0 = \emptyset, \quad V_1 = \mathcal{P}(V_0) = \{\emptyset\}, \quad V_2 = \mathcal{P}(V_1) = \{\emptyset, \{\emptyset\}\} \ldots.$$

Basing their intuition on long years of experience with sets, coupled with the properties of sets that could be seen to hold in the universe $V$, the fathers of modern set theory formulated the axioms of ZFC. Below is a sampling of some of these axioms.

- *Axiom of Pairing.* If $A$ and $B$ are sets, there is another set $C$ whose only elements are $A$ and $B$; in other words, there is a set $C = \{A, B\}$.
- *Power Set Axiom.* If $A$ is a set, then the collection $\mathcal{P}(A)$ of all subsets of $A$ is also set.
- *Axiom of Infinity.* There is an infinite set.

The ZFC axioms, together with their natural model $V$, have provided a powerful unification of all areas of mathematics. This is because

1. Every mathematical notion can be represented as a set.[3]
2. Every set that is used in mathematics belongs to $V$.

[3] There are mathematical notions that entail collections that are too big to be sets; the category of all groups, for example, is such a notion. However, set theorists have devised ways of handling such collections as if they were sets.

3. Every mathematical theorem in any known field of mathematics can be formulated in the language of sets and derived directly from the ZFC axioms.

In hindsight, the fact that so much emerged from the simple question "What is a set?" is surprising. One might have expected that a more careful definition of the notion of a set would have solved the problem. Instead, the notion of "set" never was defined at all; in fact, in the solution we have outlined, a set is now to be understood as a *primitive*, an undefined notion, whose meaning emerges from the ZFC axioms. Another way to express this point is to say that a set is any member of the universe $V$. This means that the basic unit of all mathematics, the "point value" from which everything else is built—the notion of a set—can only be understood with reference to wholeness, with reference to the totality to which it belongs.[4]

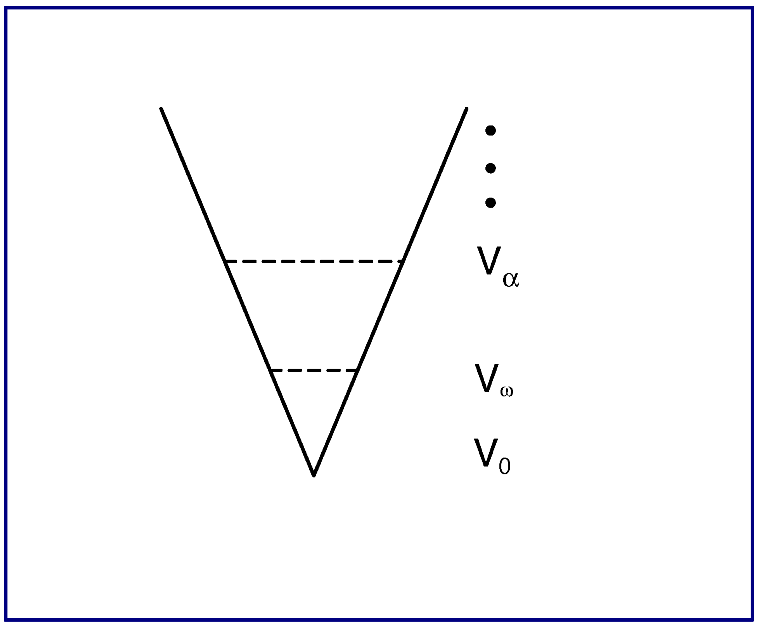


FIGURE 2. The universe of sets $V$.

The universe $V$, illustrated in Figure 2, represents the wholeness of all of mathematics, at least from the historical perspective that we have been discussing so far. It is in fact a concrete realization of Cantor's own vision of wholeness, of his Absolute Infinite: $V$ contains all mathematical constructions; it is bigger than any conceivable infinite size; it is not subject to increase or decrease in size (that is, one cannot perform an operation on $V$ to produce something bigger or smaller). Most of these characteristics follow from the simple fact that $V$ itself is *not a set!* It is

[4]The principle behind this phenomenon is expressed in [20, p. 538]: "Without reference to the transcendental basis of life all knowledge of life always remains incomplete—without reference to wholeness, parts will always remain undefined."

too big to be a set. One way to see this is to consider what would happen if $V$ were a set—if it were, then we could apply the power set operation to it to produce an even bigger set $\mathcal{P}(V)$—a set that would have to be bigger than the universe itself, which already contains everything! The impossibility of such a consequence leads to the conclusion that $V$ is, with respect to sets, *bigger than the biggest*, and therefore not a set.

In this way, the effort to provide a rigorous formulation of the notion of "set" resulted in the discovery of the biggest infinity of them all, the totality $V$, together with the laws that govern the unfoldment of sets within $V$—the ZFC axioms. This achievement marked the accomplishment of the third milestone in the classical theory of the infinite.

## 2. The Modern Era

Just as it seemed that the nature of the universe, the extent of mathematics, and the nature of the infinite were all beginning to be well understood, a new kind of infinity appeared. Certain combinations of the properties of the different infinite sizes—called *infinite cardinals*—were found to produce very strong notions of infinity, so strong that the ZFC axioms could not derive the existence of such infinities. Yet, these notions of enormous infinities started turning up as key elements in solutions to well-known research problems in analysis, topology, and algebra. These infinities have come to be known as *large cardinals*.

2.1. **Aleph Fixed Points.** To give a sense of the enormity of large cardinals, we spend a moment here considering one property that all large cardinals have in common. We begin with notation for infinite cardinals. Just as the familiar whole numbers $0, 1, 2, 3, \ldots$ are used to denote the sizes of finite sets (for instance, the size of the set $\{2, 9, 17\}$ is 3), likewise the sizes of infinite sets are specified using *infinite cardinal numbers*. Some of these infinite cardinals are

$$\omega_0, \omega_1, \omega_2, \omega_3, \ldots.$$

The smallest infinite size, $\omega_0$ (read "omega-zero", also denoted simply $\omega$), is the size of the set $\mathbb{N}$ of natural numbers (and is also the size of the sets of whole numbers, of integers, of rationals, and of algebraic numbers). The bigger cardinals, which come after $\omega_0$, represent sizes of much bigger sets. One well-known set of the bigger variety is the set $\mathbb{R}$ of real numbers. Though it is impossible to determine exactly which of the cardinals $\omega_1, \omega_2, \omega_3, \ldots$ is the exact size of $\mathbb{R}$, it can be shown that the size of $\mathbb{R}$ must lie among these; the size of $\mathbb{R}$ is at least bigger than $\omega_0$.

An easily observed feature of the list of infinite cardinals displayed above is that, at least at the beginning of the list, we find that the *index* of a cardinal is always smaller than the cardinal itself. For instance the index of $\omega_0$ is 0, and certainly 0 is smaller than $\omega_0$. Likewise, the index of $\omega_1$ is 1, and 1 is smaller than $\omega_1$. This obvious pattern continues far into the endless list of infinite cardinals. However, eventually, something new appears. Eventually, one arrives at a cardinal whose index is equal to the cardinal itself.[5] In other words, there must exist a cardinal $\omega_\kappa$ with the property that

$$\kappa = \omega_\kappa.$$

Such a cardinal is called an *aleph fixed point.*

Certainly this is an extraordinary property of infinite cardinals, and it is one that belongs to every large cardinal. However, the first aleph fixed point that one encounters on the list is not big enough to be a large cardinal. Nor is the second or third or even the $\omega_0$th. In fact, no conceivable mathematical procedure[6] could ever result in a precise specification of a large cardinal. And this limitation is not simply the result of lack of persistence or skill on the part of mathematicians. Rather, this limitation is a provable theoretical result: *It is impossible to prove from ZFC that any large cardinal exists at all.*[7]

Why, then, may we not conclude that such "large cardinals" simply don't exist? Strangely enough, large cardinals do appear as key elements in the solutions of quite a number of significant mathematical problems that have arisen in the past century, so they cannot be dismissed so easily.

We list below some of the most widely used large cardinals (see [9]), in increasing order of strength, and then give two examples of well-known mathematical problems whose solutions do depend on large cardinals.

2.2. **Some Common Large Cardinals.**

- Inaccessible
- Mahlo
- Weakly Compact
- Ramsey
- Measurable
- Strong
- Woodin
- Supercompact
- Extendible
- Huge
- Superhuge

[5]The least such cardinal $\kappa$ can be obtained as the supremum of the countable sequence $\omega, \omega_\omega, \omega_{\omega_\omega}, \dots$ . Note that the form of this supremum is $\omega_\kappa$, where

$$\kappa = \omega_{\omega_{\omega_{\ddots}}}.$$

A moment's reflection reveals that $\kappa = \omega_\kappa$.

[6]By "conceivable" we mean "formalizable in ZFC."

[7]This is a direct consequence of Kurt Gödel's celebrated *Second Incompleteness Theorem*, which states that no reasonable formulation of set theory (such as ZFC) can prove its own consistency, unless the set theory is itself inconsistent. Stated another way, no construction of a universe of sets can be carried out in its entirety using only the axioms themselves as the basis for the construction (unless the set of axioms is inconsistent); here, by "universe of sets" we mean a collection, something like $V$, in which all the axioms hold true. Now assume for the moment that ZFC is consistent and that we could derive from ZFC the existence of a large cardinal $\kappa$. It can be shown that the $\kappa$th stage $V_\kappa$ of the universe, viewed as a mini-universe, satisfies all the axioms of set theory. We would therefore have proven from ZFC that ZFC is consistent, in violation of the Second Incompleteness Theorem. For more on this topic, see [9].

- ∘ Super-$n$-huge for every natural number $n$

2.3. **Two Examples of Theorems That Depend on Large Cardinals.** The many examples of mathematical theorems that are tied to large cardinals have the unfortunate characteristic of being difficult to understand for the non-expert. For the sake of the mathematically experienced reader, we take a short detour to examine two such examples. The reader who does not wish to follow this detour may safely skip to the next section.

1. *The Normal Moore Space Conjecture.* A *metric space* is a set that admits a metric or distance function. A familiar example is the real line $\mathbb{R}$ whose distance function $d$ is defined using absolute value: the distance between reals $x$ and $y$ is the absolute value of their difference: $d(x,y) = |x - y|$. The Pythagorean Theorem is used to obtain the usual distance function for the plane $\mathbb{R} \times \mathbb{R}$: the distance between points $(x, y)$ and $(u, v)$ in the plane is the square root of $(x - u)^2 + (y - v)^2$. And there are many other more advanced examples.

Metric spaces have a number of very nice properties that make them easier to work with than more general topological spaces. One such property is *normality*: in any metric space, disjoint closed sets can be separated by disjoint open sets.

A generalization of metric spaces, called *Moore spaces*, named after their inventor R.L. Moore (1882–1974), replaced the use of the metric in a topological space with a countable sequence of covers of the space (called a *development*) having special separation properties. A Moore space is defined to be a developable, regular[8] Hausdorff space.

Many of the nice properties of metric spaces also hold in Moore spaces, but Moore spaces are more general: many examples of nonmetrizable Moore spaces are known (a topological space $X$ is *metrizable* if a metric can be defined on $X$ that is compatible with the topology on $X$). Moreover, as of the mid-twentieth century, all the known examples of nonmetrizable Moore spaces were also non-normal. In this context, the question arose: Is every normal Moore space metrizable? The conjecture that this is indeed the case is called the *Normal Moore Space Conjecture (NMSC)*.

The conjecture was settled through the use of advanced techniques in set theory. There are many aspects of the solution, but the result we wish to mention here is that the truth of NMSC is intimately tied to the existence (or at least the consistency) of large cardinals. More precisely:

**Theorem 2.1** (Nyikos, Fleissner)**.**

1. *Assuming there is a strongly compact cardinal, there is a model of set theory (a universe of mathematics) in which NMSC holds.*
2. *If NMSC is true, then there is a model of set theory in which there is a measurable cardinal.*

The conclusion is that if NMSC is true, then large cardinals must be lurking in at least some of the universes of set theory, and, conversely, if there exists a sufficiently strong large cardinal, then NMSC must hold in at least some of the universes of set

[8]A topological space is *regular* if, for each closed subset $C$ and each point $p$ not in $C$, there are disjoint open sets separating $C$ and $p$.

theory. For a detailed discussion on NMSC with references, see [12].

2. *Determinacy of Analytic Games.* An *infinite game* based on a subset $A$ of the unit interval $[0,1]$ involves two players, Player I and Player II, who take turns picking elements of the two-element set {0,1}. Their successive plays result in an infinite sequence of 0s and 1s, which we denote $s = s_1 s_2 s_3 \cdots$. Player I wins the game if the sum

$$\frac{s_1}{2} + \frac{s_2}{4} + \frac{s_3}{8} + \cdots$$

lies in $A$; otherwise Player II wins. The set $A$ is said to be *determined* if one of the two players has a winning strategy.

The natural question that arises in this context is whether *every* subset of $[0,1]$ is determined. The Axiom of Choice shows that the answer is "no," but there are other related questions for which a positive answer does not conflict with AC.

To arrive at these related questions, we consider "nicely defined" subsets of $[0,1]$. These nice sets are called the *Borel* sets; they are obtained by beginning with the closed subsets of $[0,1]$, then considering all possible countable unions of closed sets, then all countable intersections of these, all countable unions of these, and so forth. After the process is repeated $\omega_1$ times, no more new sets can be obtained by this procedure; the sets that are obtained in this way are the Borel sets. One can obtain a somewhat larger class than the Borel sets—a class that also has many nice properties—by considering *continuous images* of Borel sets. The class of all continuous images of Borel sets is called the class of *analytic sets*.

Two questions that relate the notion of determinacy with these classes of sets of reals are:

1. Is every Borel set determined?
2. Is every analytic set determined?

The work of set theorists D.A. Martin and L.A. Harrington established the following:

**Theorem 2.2** (Martin, Harrington)**.**

1. *Every Borel set is determined.*
2. *If there is a measurable cardinal, every analytic set is determined.*
3. *If every analytic set is determined, there is a model of set theory in which there is a proper class of weakly compact cardinals.*

This theorem demonstrates, perhaps even more dramatically than the results on NMSC, how large cardinals can be inextricably tied to the solution of a research problem in mathematics. For a fuller discussion of determinacy, with references, see Jech [9].

## 3. Where Do Large Cardinals Come From?

In the 1960s, many new kinds of large cardinals began to emerge from various quarters. At this time, a more pressing need was felt in the set-theoretic community to come to terms with this phenomenon. Reactions to the challenge varied among researchers. Among those who participated in moving toward a solution, some hoped to "debunk" large cardinals, while others sought to provide a foundational basis for them. Some in the former category—including some of the brightest set theorists of the time—dedicated many years in an attempt to prove that some or all large cardinals are inconsistent with ZFC. And, although many deep results in set theory emerged in these research projects, none of them resulted in a proof that any large cardinal, big or small, is inconsistent.

The view of the other group of researchers was that some or all large cardinals should indeed be thought of as an authentic part of the mathematical universe. To travel this course required answers to the following questions.

1. Which large cardinals are *legitimate*? It is possible that some may have arisen in such an arbitrary and ill-motivated way that there is no justification for them.
2. How can those large cardinals that are considered legitimate be derived from the foundational axioms? Certainly ZFC is not strong enough to derive any large cardinal, but can we find an axiom (or possibly several) that expresses some intuitively clear truth about the universe and, at the same time, is strong enough to provide a proof of the existence of these large cardinals?

These are the central questions of the *Problem of Large Cardinals*. To address these questions, many researchers in the field turned to Cantor's original vision of the universe $V$ as a guide to intuition—what is it about the structure of $V$ that would suggest that large cardinals should really exist?

One approach was to recognize that $V$ itself represents the "ultimate infinite," and an intuition that emerged was that large cardinals are "reflections" of that totality into the realm of ordinary sets. Large cardinal properties that seemed to hold true of $V$ itself were thereby legitimized, and the result was that some of the smaller large cardinals, such as inaccessible and Mahlo cardinals, found a high degree of acceptance.

Another approach was the observation that many large cardinal properties happen to belong to the smallest infinite cardinal, $\omega_0$, the size of the set of natural numbers. Being the least among the infinite cardinals $\omega_0, \omega_1, \omega_2, \omega_3, \ldots$, the cardinal $\omega_0$ cannot actually be a large cardinal (for example, it is not an aleph fixed point). Nevertheless, the cardinal $\omega_0$ does possess many large cardinal characteristics, which, were they to belong to any cardinal $\lambda$ bigger than $\omega_0$, would cause $\lambda$ to be a *bona fide* large cardinal. It is accurate to say that $\omega_0$ is to the world of the finite what large cardinals are to the world of the infinite; indeed, to the world of the finite, $\omega_0$ appears to be a "large cardinal."[9] It is for this reason, one can argue,

[9]The "world of the finite" is captured by the axiom system ZFC − Infinity (that is, standard set theory ZFC with the Axiom of Infinity removed). A universe built from this set of axioms may or may not contain an infinite set. For instance, the stage $V_\omega$, obtained as the union of all the finite stages of the universe ($V_\omega = V_0 \cup V_1 \cup V_2 \cup \cdots$), satisfies all the axioms of ZFC − Infinity, and it contains no infinite set. On the other hand, the usual ZFC universe $V$, which also satisfies the axioms of ZFC − Infinity, does contain infinite sets. From the perspective of the "ZFC world," it is obvious that infinite sets exist, but this perspective is not available in the ZFC − Infinity world because the axiom system is not rich enough; in that world, it "might be" the case that infinite sets exist, but it is impossible to determine the truth of the matter. The question of the existence of large cardinals is, we suggest, similar: from a certain vantage point (a vantage point we attempt to articulate in this article, in which the ZFC axioms are supplemented with additional axioms), it is "obvious" that large cardinals exist, but from ZFC alone, this perspective is not available because the axiom system is not rich enough.

that $\omega_0$ exhibits so many large cardinal characteristics. In any case, the fact that these properties belong to one infinite cardinal was used as a justification for the belief that *other* infinite cardinals should have the same properties.

The logic for this justification again comes from Cantor; in Cantor's vision, the landscape of infinite cardinals exhibits a certain *uniformity*: properties found to hold for one infinite cardinal should be found in other cardinals throughout the universe. Justifying large cardinals on the basis of properties found to hold for $\omega_0$ is known as *generalization*. Generalization was used to legitimize several large cardinals, such as weakly compact and measurable cardinals.[10]

Efforts to justify large cardinals using such heuristics have met with limited success. The really big and often complex large cardinals, such as supercompact and superhuge cardinals, could not be justified using these approaches. Cantor's vision has been able to carry us only so far in our understanding of the structure of the universe $V$—a fact that should not be surprising since Cantor himself was entirely unaware of the phenomenon of large cardinals.

To make further progress toward a solution to the Problem of Large Cardinals, the following questions present themselves:

1. Beyond Cantor's vision of the wholeness of the universe $V$, what source of intuition can we draw upon to decide which large cardinals really do belong in the universe?
2. Can we draw upon this new source of intuition to help in the formulation of a new axiom for set theory, which would provide an axiomatic basis for these large cardinals?

## 4. Insights from Maharishi Vedic Science[SM]

A natural approach to consider in addressing these questions is Maharishi Vedic Science. A Vedic mathematician's hunch, using this approach, might be something like this:[11]

*Everything to do with the infinite arises from*
*the self-interacting dynamics of wholeness.*

In attempting to use this apparently non-mathematical principle as an intuitive guideline that could provide insight into the structure of the universe and even possibly a new axiom of set theory, we need to identify the mathematical analogues to the notions of "wholeness" and "self-interacting dynamics."

We have already seen that, from the mathematical point of view, $V$ already naturally represents a kind of wholeness for mathematics. Examining its properties further, we can see even more clearly that it is a natural analogue to the notion

[10]A discussion can be found in [10].

[11]For example, consider the following quote from *Maharishi's Absolute Theory of Defence* [20, p. 626]: "Vedic Mathematics starts from the total reality of the Absolute Number, which is defined as that which functions from within itself and thereby accounts for the world of infinities—the world of the Absolute—because there are many infinities, and these cannot be expressed by finite numbers." Also, "This is how everything in the objective world is the expression of wholeness. This presents to us the need for an Absolute Number in the field of Mathematics, a number that can help us to account for the infinite number of wholenesses within the universe—a number that will help us to account for the theme of creation and evolution in terms of wholeness" [20, p. 611].

of wholeness in Maharishi Vedic Science.[12] The universe $V$ is the source and container of all sets, and yet is not itself a set; since it is not a set, and therefore cannot be directly referred to in the formal theory, it exhibits the property of being *unmanifest*. Also, being bigger than any possible set, it exhibits the properties of being *unbounded* and *bigger than the biggest*; and, being the container of all possible mathematical structures from any area of mathematics, it exhibits the quality of *omnipresence*. In addition, $V$ can be considered to be the *total potential of natural law* in the sense that the laws that govern the unfoldment of sets—the ZFC axioms—occur in $V$ coded as sets[13]; moreover, $V$ can regenerate its own stages using its own internally coded ZFC axioms, thereby expressing its *self-sufficient* quality.[14]

Next, to represent transformational dynamics in a mathematical way, it is natural to consider the mathematical concept of a *function*. A function from one collection $A$ to another collection $B$ is a rule that uniquely associates to each element of $A$ an element of $B$. As a simple example, if one were to take a straight piece of wire and bend it so that it forms a circle, one could represent this transformation with a function that assigns to the position of each point on the wire in its starting position the corresponding position of that same point after the wire has been bent; such a function gives a meaningful and precise description of the physical change applied to the wire. In a similar way, all types of transformation in the sciences are represented by functions.

With these analogies in mind, we can now ask whether the universe $V$, as it is presently understood in set theory, comes equipped with some kind of function that transforms $V$ to $V$ and mirrors essential features of the dynamics of wholeness described in Maharishi Vedic Science. If we can locate such a function, we can examine it closely and see whether it provides hints about the origin of large cardinals.

To narrow the search somewhat and to aim for the fullest use of Maharishi Vedic Science, we will attempt to find a function $j$ transforming $V$ to $V$ that has some additional characteristics. The dynamics represented by $j$ should

1. transform wholeness and yet leave wholeness unchanged by the transformation;
2. be unmanifest;
3. be present at each point in the universe.

To meet the first requirement (1), the function $j$ must, as much as possible, preserve the integrity of the structure of $V$.[15] Structure-preserving functions are a

[12]Over the years, on the basis of both experience and an analysis of the Vedic Literature, Maharishi formulated a large number of *qualities of pure consciousness*—distinctive properties that characterize this field of existence. A list of these can be found in [20, pp. 602–605]. Among these are the qualities of *bigger than the biggest* [21, pp. 16], *unmanifest, unboundedness, total potential of natural law, omnipresence,* and *self-sufficiency*, which we mention in this article.

[13]This encoding is described in [5, pp. 141–142].

[14]More precisely, the axioms of ZFC are rich enough to build each of the stages $V_0, V_1, V_2, \ldots$ of $V$, but not rich enough to establish the existence of $V$ itself. This limitation is a consequence of Gödel's Second Incompleteness Theorem, and is discussed later in this article.

[15]This requirement is expressed in Maharishi Vedic Science in the following way: "The essential and ultimate constituent of creation is the absolute state of Being or the state of pure consciousness.

key notion in nearly every field of mathematics: Continuous functions preserve limits of sequences. Homomorphisms preserve the operations of an algebraic structure. Order-morphisms preserve the relation of an ordered structure. Likewise, whatever relationships exist within the structure of $V$ should be preserved by $j$. At the same time, $j$ must do *something*—one could mistakenly let $j$ be simply the identity function that has no transforming effect at all. The identity function $id$ is the function that assigns to each set $x$ the value $x$ itself: $id(x) = x$. Certainly the identity function preserves all relationships in $V$, but no transformation occurs either. So, we require $j$ to be a *non-trivial*, structure-preserving function: there must be some $x$ for which $j(x) \neq x$.

For the second point (2), the idea that $j$ should be *unmanifest* also has a reasonable mathematical analogue. To make this point, we begin with the observation that a function from $V$ to $V$ has such enormous scope, it cannot be considered a function in the ordinary sense. The usual functions in mathematics are actually members of $V$ (represented in a standard way as sets of ordered pairs). But a function defined on all of $V$ cannot be represented as a set (since $V$ itself is not a set). A usual maneuver to get a handle on such enormous transformations in set theory is to consider whether such a transformation is *definable*. Definability of such a function allows one to say things about the function almost as if it were an actual set. Requiring our function to be *unmanifest* can then be done by insisting that it *not* be definable. The Vedic Science perspective suggests that the transformational dynamics represented by $j$ are hidden from ordinary view[16] and therefore, mathematically speaking, undefinable.

Finally, to address (3), we wish to ensure that the behavior of our function $j$, being undefinable, is not divorced from the reality of sets in $V$; $j$ needs to be somehow "present" everywhere within $V$.[17] This requirement is realized by declaring that the *restriction*[18] of $j$ to any set in $V$ also belongs to $V$. The function $j$ itself does not belong to $V$; it is not even definable in $V$. But we require that every restriction of $j$ to a set belongs to $V$.

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This absolute state of pure consciousness is of unmanifested nature which is ever maintained as that by virtue of the never-changing cosmic law. Pure consciousness, pure Being, is maintained as pure consciousness and pure Being all the time, and yet it is transformed into all the different forms and phenomena. Here is the cosmic law, one law which never changes and which never allows absolute Being to change. Absolute Being remains absolute Being throughout, although it is found in changed qualities here and there in all the different strata" [14, p. 12].

[16]For instance, Maharishi remarks, "The self-referral state of consciousness is that one element in nature on the ground of which the infinite variety of creation is continuously emerging, growing, and dissolving. The whole field of change emerges from this field of non-change, from this self-referral, immortal state of consciousness" [16, p. 25]. He goes on to say, "This state of consciousness is completely self-sufficient. How it emerges from within its own self-referral performance, which is going on eternally at the unmanifest basis of all creation, is Vedic Science" [16, p. 26].

[17]The parallel with Maharishi Vedic Science is described by Maharishi as follows: "The deepest level of every grain of creation is the self-referral field, the transcendental level of pure intelligence, the self-referral state of Unity—pure wakefulness, pure intelligence—*Chiti Shaktiriti*—as expressed by the last *Yog-sūtra*—that self-referral intelligence which is the common basis of all expressions of Natural Law" [20, p. 425].

[18]If $h : A \to C$ is a function and $B \subseteq A$, the *restriction of $h$ to $B$*, denoted $h \restriction B$, is the function having domain $B$ that acts on elements of $B$ in the same way as $h$; that is, for all $b \in B$, $(h \restriction B)(b) = h(b)$.

Summarizing these requirements, we can say that we are looking for some evidence of a naturally occurring function $j: V \to V$ with these characteristics:

1. $j$ preserves the internal structure of $V$;
2. $j$ is undefinable in $V$;
3. the restriction $j \restriction X$ of $j$ to any set $X$ in $V$ must itself belong to $V$.

## 5. Locating the Seed for a Solution to the Problem of Large Cardinals

In the 1960s, William Lawvere [13] observed that the usual Axiom of Infinity is actually *equivalent* to the existence of a certain very interesting function $j: V \to V$. This $j$ is obtained as the composition of two functors $F$ and $G$ which have a highly coherent relationship with each other (they are *adjoint functors*):

$$j = G \circ F.$$

A *functor* is a certain kind of function that exhibits special characteristics when it is applied to other *functions*; it is perfectly legitimate here to think of functors as just another kind of function. Because of the adjoint relationship between $G$ and $F$, both exhibit strong preservation properties: relationships in the domains of each of the functors are preserved by these functors. In the language of category theory, $F$ preserves all *limits* and $G$ preserves all *colimits*. This gives a hint that even the existence of an infinite set implies that certain truth-preserving dynamics are at work within the wholeness of $V$.

We shall refer to $j$ as the *Lawvere functor.* The Lawvere functor suggests an alternative form of the Axiom of Infinity:

> *Lawvere Axiom of Infinity.* There is a functor $j: V \to V$ that factors as $j = G \circ F$, where $G$ is the forgetful functor from the category of self-maps[19] on sets to the category of sets, and $F$ is left adjoint to $G$.

It is important to keep in mind that this $j$ has a special status among functions that one could define from $V$ to $V$. The function $j$ is special because its existence is *equivalent* to the Axiom of Infinity. The Axiom of Infinity simply states that there is an infinite set—asserting nothing more than the existence of $\mathbb{N}$, the set of natural numbers. This is a very localized phenomenon: a single set is declared to exist someplace in the universe. On the other hand, this particular $j$ provides transformational dynamics of the entire universe, exhibiting important structure-preserving characteristics. The fact that the existence of $j$ is equivalent to the Axiom of Infinity tells us that the presence of a companion transformation $j: V \to V$, with its structure-preserving characteristics, is an essential characteristic of the universe $V$. The seed of the vision of wholeness from the cognition of the ancient seers seems therefore to be already present in the design of the universe $V$.

When we look at the properties exhibited by the Lawvere functor $j$ through the eyes of a Vedic mathematician, however, we notice that something about $j$ seems

[19] A *self-map on a set* $A$ is simply a function $f : A \to A$; in other words, the domain and codomain of a self-map are equal. Such self-maps can be collected together to form a category. The forgetful functor $G$ on this category acts on a self-map $f : A \to A$ by stripping away its structure: $G(f) = A$.

amiss. To be a full expression of the transformational dynamics that belong to wholeness, as understood in Maharishi Vedic Science, we expect $j$ to have more fully developed properties. In particular, it would be reasonable to expect that $j$ itself, rather than just its factors $F$ and $G$, should exhibit strong preservation properties. This gap between what we expect to find based on our guiding intuition and what we actually find suggests a direction for improvement.

We are expecting that, by implementing principles of Vedic Science in a mathematical context, motivation for large cardinals will naturally appear. What we have now discovered is that our candidate for giving mathematical expression to the dynamics of wholeness is missing some desirable characteristics. A reasonable hope is that if we attempt to strengthen[20] the axiomatic properties of $j$, we will strengthen the ZFC Axiom of Infinity in such a way that deeper properties of wholeness can be brought to light and illuminate the issue of large cardinals.

## 6. Enhancing the Preservation Properties of $j$

In the 1970s, Blass and Trnková [2] took the step we have just been discussing. They asked what happens if a functor $j: V \to V$ is required to have essentially the same preservation properties as those of the *factors* $F$ and $G$ of the Lawvere functor $j$. Such a function, in precise mathematical terms, is called an *exact functor*; an exact functor preserves all *finite* limits and colimits. Blass and Trnková were able to prove the following interesting theorem.

**Theorem 6.1** (Blass, Trnková). *The following are equivalent:*

1. *There is a nontrivial exact functor* $j: V \to V$.
2. *There is a measurable cardinal.*

The theorem shows that our program of axiomatically enhancing the Lawvere functor so that it exhibits more preservation properties is on the right track. The Blass-Trnková functor is already a much fuller expression of the functional dynamics we are seeking. If we replace the Lawvere Axiom of Infinity that we stated before with this new Blass-Trnková version,

"There is a nontrivial exact functor $j: V \to V$,"

then we have as an immediate and perfectly natural consequence that the universe must contain a measurable cardinal.

[20]To clarify a possibly confusing point, we explain a bit more what we mean by "strengthening the properties of $j$." We have found an equivalent axiomatic formulation of the Axiom of Infinity that asserts the existence of a function $j : V \to V$ having certain properties, namely, Lawvere's Axiom of Infinity. We wish to construct new, stronger axioms that also assert the existence of a $j: V \to V$. These new axioms will be stronger because of the stronger properties that the function $j$ will be declared to have. Therefore, when we say that we wish to strengthen the axiomatic properties of $j$, what we mean is that we wish to find another axiom, involving the notion of $j: V \to V$, so that the declared properties of $j$ render the new axiom stronger than previous versions of the axiom. Note that the kinds of new properties we will ask the $j$ of the axiom to have will be guided by the principles of Maharishi Vedic Science that we described earlier.

## 7. The Wholeness Axiom

Taking the next step, we can ask: Can the preservation properties of the Blass-Trnková functor be enhanced even further so that $j$ preserves *all* properties of $V$? Can we even require that $j$ have the added characteristics, in accordance with properties (2) and (3) mentioned earlier, that it is undefinable but its restrictions to sets belong to $V$?

In the language of set theory, the way to require a function to preserve *all* properties is to make it an *elementary embedding*.[21] Therefore, the requirements on a function from $V$ to $V$ mentioned above will be met if we require, axiomatically, that $j$ be an undefinable elementary embedding whose restrictions lie in $V$. Below, we give the statement of the *Wholeness Axiom*, which asserts the existence of such an elementary embedding; the Wholeness Axiom represents a kind of "ultimate" enhancement of the Axiom of Infinity. Following the statement of the axiom, we examine the new features of wholeness that are brought to light by adding the axiom to the standard set theory axioms, ZFC.

> **Wholeness Axiom (WA).** There is a nontrivial elementary embedding $j\colon V \to V$ with the property that for every set $X$, the restriction $j \restriction X$ is also a set.

Notice that the requirement that $j$ should be undefinable has not been mentioned in the definition of the Wholeness Axiom. The reason is that undefinability of $j$ actually can be *proven*: it follows from a theorem by K. Kunen [11] that if such an embedding exists at all, it cannot be definable.

Also notice that we have required $j$ to be *nontrivial*. This means that for some set $x$, $j(x) \neq x$; we say that $j$ *moves* $x$. In fact, it can be shown that some *infinite cardinal* is moved by $j$. The *least* cardinal moved by $j$ is called the *critical point* of $j$ and is usually denoted by the Greek letter $\kappa$ (read "kappa"). It can be shown that $j$ moves $\kappa$ to another infinite cardinal $j(\kappa)$ that is bigger than $\kappa$; that is, $j(\kappa) > \kappa$.

We can now state the main result of this article, which shows that our efforts to provide a solution to the Problem of Large Cardinals have been successful.

**Theorem 7.1** ([3]). *Assume WA and let $j\colon V \to V$ denote the WA-embedding. Let $\kappa$ be the first cardinal moved by $j$. Then $\kappa$ is the $\kappa$th cardinal that is super-$n$-huge for every $n$ in $\mathbb{N}$. In particular, the critical point $\kappa$ has virtually all large cardinal properties.*

[21] Formally, to say that $j : V \to V$ is an elementary embedding means that for every formula $\phi(x_1, x_2, \ldots, x_n)$ (where the free variables of $\phi$ are precisely $x_1, x_2, \ldots, x_n$) in the language of set theory, and for all sets $a_1, a_2, \ldots, a_n$, the formula $\phi[a_1, a_2, \ldots, a_n]$ (obtained by substituting each $a_i$ for $x_i$) holds true in $V$ if and only if the formula $\phi[j(a_1), j(a_2), \ldots, j(a_n)]$ holds true in $V$. Speaking more intuitively, to say that $j$ is an elementary embedding means that it preserves all possible finitary relationships among sets (more precisely, those relationships that are first-order expressible using the membership relation). Note that if $j$ is an elementary embedding, it must be a functor. To illustrate one step of the verification of this fact, suppose $1_A : A \to A$ is the identity map on a set $A$. We check that $j(1_A)$ is the identity map $j(A) \to j(A)$. First note that $j(1_A)\colon j(A) \to j(A)$ is a function, since, applying $j$ to the formula $\forall x \in A\, \exists! y\, 1_A(x) = y$ yields $\forall x \in j(A)\, \exists! y\, j(1_A)(x) = y$. Next, observe that applying $j$ to the formula $\forall x \in A\, 1_A(x) = x$ yields $\forall x \in j(A)\, j(1_A)(x) = x$, from which it follows that $j(1_A)$ is the identity map on $j(A)$.

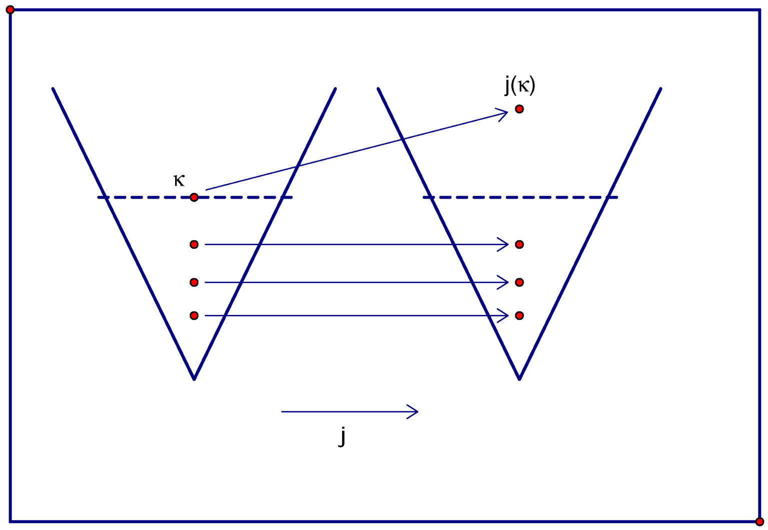


FIGURE 3. The non-trivial elementary embedding $j$.

The theorem tells us where large cardinals come from: in the transformational dynamics of the wholeness $V$, large cardinal properties arise as the characteristics of the first point that is moved under the transformation. At the precise moment when complete silence, represented by the behavior of $j$ as simply the identity function below $\kappa$, changes to dynamism—in that first impulse of activity, represented by and concentrated at $\kappa$—we find that $\kappa$ is filled with a powerful dynamism, evidenced by the fact that it has essentially all known large cardinal properties.

In fact, it is accurate to say that $\kappa$ acquires the status of a point in the universe which stands as a *representative* of the totality of $V$. This is seen by the fact that the stage $V_\kappa$ of the universe indicated and coded[22] by $\kappa$ is in fact an *elementary submodel* of $V$.[23] This means that all the relationships that hold inside $V$ also hold inside $V_\kappa$.[24] It also means that $V_\kappa$ knows all there is to know about the wholeness $V$. Truly, $\kappa$ can declare "I am wholeness."

These dynamics closely parallel the way in which the unbounded nature of wholeness, as described in Maharishi Vedic Science, collapses to its own point value in the emergence of manifest existence. Maharishi explains that manifest existence arises from the Veda, which can be seen as an unmanifest blueprint for the manifest world, giving rise to everything by way of its own self-interacting dynamics.[25] Moreover, the Veda describes, in one of its own verses (Ṛk Veda I.164.39), how the Veda itself arises. The verse states, "The verses of the Veda exist in the collapse of fullness

[22]In the sense that there is a bijection between $\kappa$ and $V_\kappa$.

[23]See, for example, [3].

[24]Formally, this implies that for any sentence $\sigma$ of set theory, $\sigma$ holds true in $V$ if and only if $\sigma$ holds true in $V_\kappa$.

[25]Citing Maharishi, R.K. Wallace [24, p. 218] writes, "Maharishi describes the four Vedas as 'a beautiful, sequentially available script of nature in its own unmanifest state, eternally functioning within itself, and, on that basis of self-interaction, creating the whole universe and governing it.'"

(the *kshara* of अ (A)) in the transcendental field, in which reside all the *Devas*, the impulses of Creative Intelligence, the Laws of Nature responsible for the whole manifest universe" [21, pp. 52–53]. Maharishi explains that this collapse of fullness is represented by the very first syllable of Ṛk Veda, *AK*. In the syllable AK, the letter 'A' represents fullness (pronounced without restriction in intonation), while 'K' represents a *stop*, uttered with a closed throat.[26] Being the focal point of the collapse of the unbounded totality, 'K' represents a point of infinite dynamism, all possibilities, that can burst forth into the diversity of creation.[27] Therefore, the syllable AK, he says, embodies in seed form the entire transformational dynamics of the unfoldment of the Veda, which in turn gives rise to manifest life.[28]

Analogously, we have seen that the set-theoretic universe $V$ is a natural analogue for wholeness, and its transformational dynamics $j : V \to V$, characterized by the Wholeness Axiom, bring into view a special point, the *critical point* of $j$, denoted $\kappa$. As we have seen, $\kappa$ has essentially all large cardinal properties and is so "infused" with the properties of the wholeness $V$ that the stage $V_\kappa$ can be said to know all there is to know about $V$. We see a parallel between the wholeness indicated by 'A' and the mathematical wholeness $V$, and also between the point 'K' of infinite dynamism and the critical point $\kappa$, which encodes all first-order properties of $V$.

The analogy goes considerably further. Just as from the syllable AK, the Veda emerges, which in turn gives rise to manifest creation, so likewise we find that the "collapse" of $V$ to $\kappa$ gives rise to a kind of blueprint for the sets of $V$. This blueprint is known to mathematicians as a *Laver sequence* and arises in the following way. From $j : V \to V$, there arises a class $\mathcal{E}$ of *extendible embeddings*, which are elementary embeddings of the form $i\colon V_\alpha \to V_\beta$, all with critical point $\kappa$; the class $\mathcal{E}$ can be viewed as a class of approximations to $j$. These extendible embeddings

[26]"The pronunciation of अ (A) requires full opening of the mouth, indicating that अ (A) is the expression of the total value of speech. अ (A) presents unbounded totality, अ (A) is the total potential of speech. Pronunciation of क् (K) requires complete closing of the channels of speech (the throat). अ (A) fully opens the channels of speech; क् (K) closes the channels of speech. Full opening followed by full closing displays the phenomenon of collapse of the unbounded field (of speech) to the point value (of speech). The whole range of speech is contained in this collapse; all sounds are in this collapse, and all the mechanics of transformation of one sound into the other are also contained in this collapse" [18, pp. 171, 354].

[27]Maharishi explains, "It is interesting to observe that अ (A) in its continuum stands for the continuum of silence, and the collapse of this infinite value onto its point value generates dynamism within the nature of silence, and this silent dynamism is the lively home of all the Laws of Nature from where specific Laws of Nature emerge within the total potential of Natural Law in अ (A)—eternal silence" [20, p. 620]. This "point value," as he explains elsewhere, is represented by 'K': "The total potential of अ (A) is available between the infinity of अ (A) and its point क् (K). The liveliness of the inner structure of अ (A), the liveliness of the Constitution of the Universe, is represented by अक् (Ak). क् (K), the point of the Constitution of the Universe, is the total Constitution of the Universe concentrated at the point of WHOLENESS, अ (A)" [19, p. 454].

[28]On this point, Maharishi remarks, "*Ṛk Veda* says, 'All the verses of Veda are within अक् (Ak), the first syllable of *Ṛk Veda*, and all the *Devas* (the administering intelligence of the universe) are lively within अक् (Ak)—the universe is lively within अक् (Ak)—the entire dynamism of the Veda and Vedic Literature, and the corresponding expression of the Veda and Vedic Literature in terms of the physical expression within the unmanifest structure of self-referral consciousness, presents the self-referral ocean of consciousness as the invincible Cosmic Catalyst (*Purusha*) of the entire ever-expanding universe'" [20, p. 545].

code up a Laver sequence $S$, which can be used to locate every set in the universe.[29] Indeed, if

$$S = \langle s_0, s_1, s_2, \ldots, s_\alpha, \ldots \rangle_{\alpha<\kappa}$$

is the Laver sequence that is obtained from $j$ and its derived extendible embeddings, then, for any set $X$ in $V$, there is an extendible embedding $i$ with the property that the $\kappa$th term of the expanded sequence $i(S)$ must be $X$ itself. These dynamics parallel the sequential unfoldment of the Veda from AK, giving rise, in turn, to all of creation. We can summarize these results as follows:

**Theorem 7.2** ([3]). *Assume WA. Let $j$ be the WA-embedding and let $\kappa$ denote the critical point of $j$. Then there is a $\kappa$-sequence $S = \langle s_0, s_1, s_2, \ldots, s_a, \ldots \rangle_{\alpha<\kappa}$ of elements of $V_\kappa$ with the following property: for every set $X$, there is an extendible embedding $i$ with critical point $\kappa$ such that if $i(S)$ denotes the sequence obtained by elementarily[30] expanding $S$ by $i$, then $X$ occurs as the $\kappa$th term of $i(S)$; that is,*

$$X = i(S)_\kappa$$

As a final point of interest, once the existence of a WA-embedding is known, the structure of the universe $V$ is seen in a new way. Whereas before, even the existence of a single large cardinal—even an inaccessible cardinal—was cause for doubt, now in the presence of a WA-embedding, *almost all cardinals in the universe are large cardinals!* This phenomenon is formulated precisely in the following theorem:

**Theorem 7.3** ([3]). *Assume WA and let $j$ denote the WA-embedding. The sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is unbounded in $V$ and each term is a WA-cardinal.[31] Moreover, each of these cardinals $\lambda$ admits a normal measure[32] with the property that the set of cardinals less than $\lambda$ that are super-$n$-huge for every $n$ has normal measure 1. More succinctly, almost all cardinals in the universe are super-$n$-huge for every $n$.*

## 8. Conclusion

In this article, we have reviewed the evolution of the mathematical analysis of the infinite. The classical period in this history achieved important milestones, including the initial recognition that infinitely many objects could be collected together into a single set; that there are many different sizes of infinite sets; and that all of

[29]We outline the definition of $S$: define a function $f : \kappa \to V_\kappa$ recursively as follows:

$$f(\alpha) = \begin{cases} \emptyset \text{ if } f \restriction \alpha \text{ is Laver at } \alpha, \text{ or } \alpha \text{ is not a cardinal} \\ x \text{ otherwise, where } x \text{ is a counterexample for Laver-ness of } f \restriction \alpha \end{cases}$$

Then letting $s_\alpha = f(\alpha)$ for $0 \le \alpha < \kappa$, it can be shown (assuming the Wholeness Axiom) that $S = \langle s_\alpha \mid \alpha < \kappa \rangle$ is a Laver sequence. See [4] for details.

[30]When $i$ acts on $S$, it produces a longer sequence, of length $i(\kappa)$, whose first $\kappa$ terms are those of the original sequence $S$.

[31]A *WA-embedding* is a map $j : V \to V$ for which $(V, \in, j)$ is a model of ZFC + WA. A *WA-cardinal* is the critical point of a WA-embedding. The statement that each of the elements of $\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$ is a WA-cardinal can be formalized in the first-order language of WA [4].

[32]A normal measure on a cardinal $\lambda$ partitions the subsets of $\lambda$ into two collections $X$ and $Y$, where $X$ is the collection of "big" subsets of $\lambda$, with the property that every set in $X$ has *measure 1*, and $Y$ is the collection of "small" subsets of $\lambda$ with the property that every set in $Y$ has *measure 0.* In mathematical parlance, one says a normal measure 1 subset of $\lambda$ contains *almost all* elements of $\lambda$.

mathematics can be viewed as taking place within—indeed, *originating* within—a vast wholeness $V$, beyond the limits of any particular set or infinite size. The modern era of this analysis began with the discovery of large cardinals and their underivability from the axioms of set theory. A persistent theme in this period has been the quest to provide an axiomatic account for the presence of large cardinals in the universe.

We found that Cantor's vision of the universe of mathematics as an embodiment of the Absolute Infinite was able to guide the mathematical formulation of the axioms of ZFC and even provided techniques for justifying many of the smaller large cardinals. However, the need for a deeper insight into the structure of the wholeness $V$ led us to seek a deepening of the intuition offered by Cantor.

Our proposed approach to address this need has been to make use of the principles of Maharishi Vedic Science. Maharishi Vedic Science identifies qualities and dynamics of wholeness itself—the wholeness of life and of consciousness. We have applied these to formulate a strategy for locating in the mathematical wholeness $V$ heretofore unrecognized characteristics that could provide natural justification for large cardinals.

In this effort, we discovered, in Lawvere's equivalent formulation of the Axiom of Infinity, the beginnings of a natural parallel to the self-interacting dynamics of wholeness. Refining Lawvere's results to their logical conclusion, aiming toward the fullest possible representation of Maharishi Vedic Science principles within this context, we were led to the formulation of the Wholeness Axiom. The Wholeness Axiom asserts, in a precise mathematical way, that the wholeness $V$ has at its unmanifest basis transformational dynamics (represented by $j$) which preserve the internal structure of $V$ and which are present at every point within the universe.

From the Wholeness Axiom, we were able to derive a solution to the Problem of Large Cardinals. The solution shows that large cardinal properties arise as special properties that appear in the first impulse of change arising in the transformational dynamics embodied in $j$; in particular, that all large cardinal properties arise as properties of the first cardinal $\kappa$ moved by $j$. Further examination of the interactions that occur between $j$ and $\kappa$ led to the observation that a certain sequence $S$ of sets—known as a Laver sequence—naturally arises within the $\kappa$th stage of the universe. This sequence has the special property that it encodes all sets in the universe. In particular, all sets in the universe can be seen to emerge through the interaction of $j, \kappa$, and $S$. These dynamics provide a strong analogy to the dynamics of wholeness described in Maharishi Vedic Science according to which wholeness, represented by the first letter 'A' of Ṛk Veda, collapses to its point, 'K' (the second letter), in the sequential unfoldment of the entire Veda, which in turn, through its own self-referral dynamics, gives rise to all manifest existence.

The evolution of mathematical insight about the infinite suggests another parallel, a parallel between the quest for the Infinite in mathematics and in the life of the individual. When the quest begins, the "infinite" seems to be an unrealistic fairy tale. In the mathematical world, actual infinities were barred from the mainstream for centuries; and later, in the modern era, large cardinals were viewed with great skepticism for many decades after their initial discovery. So likewise in the life of the individual there is often an initial skepticism at the prospect that something as grandiose as the "Infinite" could really exist, really be experienced.

Then, after a taste of the Infinite, a change occurs. In mathematics, once the infinite was recognized as a reality, the *nature* of the infinite was found to be vast and textured, and its unfolding dynamics were found to be contained in a wholeness vaster than even the biggest notion of "infinity." And then in the modern era, as more attention was paid to the phenomenon of large cardinals and certainty of their validity grew, they became a central tool in contemporary foundational research. In a similar way, once an individual has tasted the Infinite and its influence in life, the doorway to a clear perception of the nature and hidden dynamics of the Infinite gradually starts to open.

Finally, there is a deeper realization. In the world of mathematics, a notion of infinity that seemed hardly possible or imaginable is finally seen to be nearly omnipresent: under the Wholeness Axiom, nearly all cardinals are discovered to have essentially all large cardinal properties. And in the life of the individual, the tall tale of the "Infinite," once ignored and pushed aside, at last is seen to be the truest of all realities, awake and present in every aspect of experience.

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