The Wholeness Axiom

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ABOUT THE AUTHOR

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ABSTRACT

The study of the Infinite in mathematics began with Cantor's demonstration that unless truly infinite sets were admitted in the realm of mainstream mathematical investigation, it would not be possible to formulate a rigorous foundation for calculus. Moreover, once one infinite size is allowed into mathematics, Cantor showed there must also exist a vast infinite hierarchy of ever bigger infinite sizes. No sooner had the mathematical world accepted Cantor's hierarchy of infinities than a new and deeper mystery about mathematical infinity appeared—the Problem of Large Cardinals. Large cardinals represent infinite sizes so vast that they cannot be proven to exist using the standard axioms (ZFC) of set theory. Yet, no one in 100 years has proven that they don't exist. A basic problem in foundations has been to give a natural, well-motivated set of axioms, to be added to the standard ZFC axioms, from which large cardinals can be derived. This paper suggests one such axiom, the Wholeness Axiom, which is formulated on the basis of an intuitive conception of the Infinite that comes from Maharishi Vedic Science. We view the mathematical universe V as an analogue to Vedic wholeness. We formulate intuitive principles regarding the nature of V based on principles and dynamics of wholeness as described in Maharishi Vedic Science. Based on these principles, we obtain the axiom schema that we call the Wholeness Axiom, and demonstrate that virtually all large cardinals can be derived from it. We pursue the analogy between wholeness and V, elaborating on the mathematical consequences that follow as we seek analogues to these Vedic principles in the realm of our new expanded set theory.

1. Introduction

For anyone who finds inspiration in the powerful conceptual unification provided by ZFC (or any of a number of other set theories), the need to give an account of large cardinal axioms[1] cannot be ignored. Efforts to prove that even the strongest large cardinal axioms are inconsistent have failed; large cardinals have turned up with increasing frequency as the central element in the solution of a variety of mathematical problems; and yet, even the weakest large cardinals cannot be proven to exist using ZFC alone (see [EM], [Je], [KM], [KV, ch. 15, 16], [MP, ch. 5]). It seems clear at this time in the history of research in the area that at least some large cardinal axioms ought to

1 See [We] and [Co3] for an introduction to large cardinals.

be admitted as legitimate additions to the axioms of ZFC, but which should be allowed?

Though large cardinals have been around at least since 1908 (see [Hf] in which the notion of "weakly inaccessible" first appears), they were not studied intensively until the 1960s. At that time, many who believed in the truth of certain large cardinal axioms attempted to justify them by drawing upon some form of intuition about the structure of the mathematical universe. Much of this intuition had its roots in Cantor's vision of the Absolute Infinite. Indeed, one of the main tools for justifying large cardinals in this period was the *Reflection Principle*, a principle that takes its inspiration directly from the "incomprehensibility" of Cantor's Absolute Infinite. It was quickly discovered, however, that only a handful of large cardinals could be justified using the Reflection Principle, and other intuitive principles emerged—principles which relied less and less on Cantor's original conceptual framework. By now, there are dozens of large cardinal axioms and equally many intuitive principles that have been used to justify their existence. In her article *Believing the Axioms I* [Ma1], Penelope Maddy, referring to this proliferation, remarks,

> . . . the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. ([Ma1] p.481)

Maddy's remarks make evident the need for a more unified conceptual framework in which the role of large cardinals in the universe could be more clearly grasped. Rather than approaching the problem of justifying large cardinals by inventing *ad hoc* principles as the need arises, we believe that what is needed is a comprehensive vision, like Cantor's vision of the Absolute Infinite, that would strongly suggest which large cardinals are "natural" and, for those deemed "unnatural," even suggest a direction for demonstrating their inconsistency. We, perhaps naively, believe that, had it turned out that Cantor's Absolute Infinite was rich enough to suggest the existence of even the largest large cardinals, these strong axioms of infinity would by now have found acceptance as true axiomatic assertions about sets, on an equal footing with the axioms of ZFC. The aim of this paper is to provide a natural enrichment of Cantor's notion of the Absolute Infinite in order to "fill in the details," as it were, in the hope that a vision of the mathematical universe will emerge in which all large cardinals are seen as an expected feature rather than an unexplainable mystery.

Our source of "new details" concerning the Absolute Infinite comes from the most ancient vision of the infinite on record—the detailed vision of the seers of the Vedic tradition of knowledge. In our study of this Vedic perspective on the infinite, we were surprised to discover that this tradition sets forth a number of basic principles concerning the nature of the Absolute Infinite that, on the one hand, seemed fully compatible with Cantor's own vision, and yet seemed at the same time to suggest the truth of the strongest large cardinal assertions, especially those assertions that are expressed in terms of elementary embeddings of the universe. Indeed, it has appeared to the author that these principles not only suggest the existence of the strongest possible type of elementary embedding of the universe (namely, a nontrivial elementary embedding from V to itself), but even hint at a way of "seeing through" a belief commonly held in the community of set theorists, that such an embedding is inconsistent with ZFC![2]

Our plan for this paper is to begin in Section 2 with some of Cantor's own assertions concerning the Absolute Infinite, especially those that have had a lasting impact on the collective intuition among set theorists concerning the structure of the mathematical universe. We then proceed to outline the highlights of the Vedic perspective, noting similarities with Cantor's perspective, but emphasizing new principles that will suggest a broader intuition concerning the structure of V and possible existence of large cardinals. In Section 3, on the basis of these Vedic principles, we introduce a new large cardinal axiom, which we call the *Wholeness Axiom* (WA), which asserts the existence of an *undefinable* elementary embedding from V to itself (and also specifies an additional technical requirement on V). In Section 4 we prove a number of mathematical consequences of the Wholeness Axiom. We prove that WA implies the existence of a cardinal that is super-n-huge for every natural number n and that WA is consistent relative to the large cardinal axiom

2 In a well-known and celebrated proof, K. Kunen [K2] did show that in the usual class theories, such as Bernays-Gödel Set Theory or Kelley-Morse Set Theory, such embeddings are inconsistent. The proof, however, makes essential use of the additional properties that are added to such an embedding when it is treated as a *class* in such a theory; therefore, his proof does not imply (as we discuss further in the present paper) inconsistency with ZFC. See [Co1] and [Co2] for detailed discussions of this point. [The unfounded belief that Kunen's proof demonstrates that such an embedding is inconsistent with ZFC has become less widespread since the time this article was first authored (1994). —Ed.]

I_3. We show how to formalize WA in an extension of the language of set theory obtained by adding a unary function symbol. And we prove a technical definability result that arises naturally in light of some of the structural consequences of WA. Finally, in Section 5, carrying the parallels with the Vedic paradigm even further, we are led to an analysis of the structure of Laver sequences, which seem to arise naturally when applying the Vedic paradigm.

We readily acknowledge that our use of the Vedic paradigm is somewhat unusual, and we do not claim to have the last word on the problem of justifying large cardinals. However, the parallels with this Vedic perspective have appeared quite striking and worthy of serious exploration. We believe our approach successfully provides a natural, unified framework for motivating virtually all large cardinals.

Let me conclude this introduction by expressing my gratitude to Maharishi Mahesh Yogi for his Vedic Science—the main inspiration for the present work.

2. The Cantorian Absolute and Vedic Wholeness

In this section, we review some of the main points concerning Cantor's view of the Absolute Infinite that have especially contributed to the modern-day intuition of the structure of the universe V. We suggest, relying on M. Hallett's perspective on Cantor's contribution to modern set theory, a simple picture of how certain features of Cantor's Absolute Infinite have naturally taken shape as intuitive principles and how these in turn have suggested the truth of certain assertions about the structure of V. We then give a brief overview of the Vedic perspective and extract a handful of central features of Vedic wholeness that seem to have set-theoretic relevance. Using our discussion of Cantor's work as a template, we formulate five intuitive principles based on these features, and then discuss the implications of these principles for the structure of V. These considerations will come together in the form of a rather pleasing axiomatic principle, the Wholeness Axiom, whose mathematical consequences will be discussed in the next section.

Cantor introduced his theory of the Absolute Infinite as part of a general conceptual framework to give intuitive meaning to the "actual infinites" that he had introduced into mathematics. Prior to his work, the only notion of the infinite that was deemed appropriate for mathematical formulation was the "potential infinite" since this sort of infinite is readily exemplified in nature (for example, seasons—indeed all cyclic processes—recur in endless succession and are thus potentially infinite). Hence, for instance, natural numbers were to be understood as continuing indefinitely but not to be treated as a completed totality.

Cantor (and independently, Dedekind) found it necessary to make use of actual infinites in order to give a rigorous definition of the real numbers. And, once one such an infinite is permitted in the arena of mathematics, Cantor showed that there is an endless hierarchy of them of ever increasing magnitude; his transfinite ordinals and cardinals map out the terrain of the actual infinite.

But if actual infinites do not exist[3] in nature, in what sense do they exist at all? For Cantor, the transfinite magnitudes represent thoughts in the Divine Intellect; certainly, even if *we* cannot conceive of the natural numbers as a completed totality, *God* can. Moreover, the Divine Intellect itself was to be understood as an Absolute Infinite, incapable of increase or diminution or any mathematical determination whatsoever (see [Ha, p. 13]). For Cantor, this Absolute Infinite represented the totality of all possible mathematical constructions, and, according to Hallett's account, provided much of the intuition for the modern-day universe of sets (see [Ha, pp. xii, 38, 43–44, 48, and Chapter 4]). Indeed, many of the intuitive principles at the heart of the construction of V can be traced directly to properties of the Absolute Infinite suggested by Cantor. We briefly discuss three such principles here: *Limitation of Size*, the *Principle of Maximum Possibility*, and the *Reflection Principle*. (See [Ha, pp. 20–23, 210–211, Chapter 4] for details concerning the first two of these, and [Ha, pp. 116–118] or [Re] for a discussion of the third.)

Limitation of Size. For Cantor, as far as mathematical practice is concerned, transfinite numbers resemble finite numbers in many important ways: both should be considered completed totalities on which we may perform operations, and both can be increased indefinitely and be given precise mathematical determination [Ha, Chapter 1.4]. By contrast, the Absolute Infinite is to be understood as an actual infinite of a very different kind; in Cantor's words, the Absolute is ". . . a 'true

3 Cantor argued that they must indeed exist in nature, but this is precisely because of their existence as thoughts of the Divine Intellect; see [Ha, pp. 23–24].

infinite' whose magnitude is capable of no increase or diminution, and is therefore to be looked upon quantitatively as an absolute maximum." (Quoted in [Ha, p. 44].) Indeed, Cantor makes the distinction between these two realms very clear when he says,

> . . . we must make a *fundamental* distinction here between
>
> IIa. Increasable actual-infinite or *transfinite*
>
> IIb. Unincreasable actual-infinite or *Absolute*.
>
> (Quoted in [Ha, p. 41].)

Clearly, Cantor has in mind what we would call nowadays a "type distinction": The objects that admit mathematical investigation are to be understood as different in kind, and different in size, from the ultimate infinite which is quite simply beyond mathematical determination, and an "absolute maximum." These notions contain the seeds of the Limitation of Size doctrine, according to which collections that are "too big" are not to be allowed as members of the mathematical universe; that is, they are not to be taken as *sets*. (See [Ha, 1.4 and Chapter 4].)

Principle of Maximum Possibility. For Cantor, "possibility implies existence;" if a notion is not blatantly inconsistent with the known theorems of mathematics, then this notion can be said to exist for the following reason: If the notion is not inconsistent then it is at least in principle possible for the object in question to exist; this means that it is creatable by the Divine Intellect. As such, the notion has existence as a thought in the Divine Intellect. Cantor used reasoning of this kind to conclude that his transfinite ordinals and cardinals exist. (See [Ha, pp. 21–22].) The principle here is that "as much as possible exists" and is clearly one of the precursors to the maximal iterative principle underlying the cumulative hierarchy, according to which "as many subsets as possible" are added to the universe with each application of the power-set operation (see [Ha], [W]).

Reflection Principle. According to Cantor, the Absolute Infinite cannot be grasped or comprehended by the rational mind and is certainly beyond any kind of mathematical description; in Cantor's words, "In a certain sense it [the Absolute Infinite] transcends the human power of comprehension, and in particular is beyond mathematical determination." (Quoted in [Ha, p. 13].) This basic intuition was largely responsible for the formulation and wide acceptance of the Reflection Principle, which asserts that any first-order property that is true of the universe V of sets(or the class ON of all ordinals) must already be true of some set (or ordinal). The rationale behind the Reflection Principle is simply that if it were false, then some first-order property would be sufficient to characterize V, in violation of the precept that V, like Cantor's Absolute, is too vast and complex to be determined by such a property.

The Reflection Principle has been used to motivate the inclusion of a number of large cardinal axioms among the basic axioms of set theory; see [Re]. For instance, because it is true that there are no cofinal sequences in ON and that the size of the power-set of an ordinal is always another ordinal—that is, because ON exhibits all the properties of an inaccessible ordinal—it follows from the Reflection Principle that some ordinal number should be inaccessible. (See [We] or [Co3] for more arguments of this kind.)

Having reviewed some of the more obvious ways in which Cantor's vision of the Absolute Infinite has impacted the structure of the universe of sets and motivated the acceptance of certain large cardinals, we turn now to the Vedic paradigm of the Absolute or *wholeness*. As was the case for Cantor's Absolute, Vedic wholeness is also unincreasable and of greater magnitude than any other existent thing—and indeed of an altogether different *type;*[4] beyond intellectual determination;[5] and the source of a manifest universe[6] of greatest possible richness and cre-

[4] The Upanishads (representing a part of the Vedic literature) declare wholeness to be "bigger than the biggest and smaller than the smallest":

aṇoraṇīyān mahatomahīyān

—Kaṭha Upanishad 1.2.20

A similar idea is expressed in the following, indicating that wholeness is an inexhaustibly rich field of existence:

pūrṇam adaḥ pūrṇam idam, pūrṇāt pūrṇam udachyate
pūrṇasya pūrṇam ādāya pūrṇam evāvaśiṣyate

That is full; this is full. The full comes out of the full.
Taking the full from the full, the full itself remains.
—Bṛhad-āraṇyaka Upanishad 5.2.1

(Translations both here and in the sequel are due to Maharishi Mahesh Yogi unless referenced otherwise.)

5 As the following passage indicates, the Veda holds the thinking mind, like everything else in the created universe, to be a derivative of wholeness and as such, an inadequate tool for fully apprehending its source:

yan manasā na manute yenāhur mano matam
tad eva brahma tvaṁ viddhi nedam yad idam upāsate

That which is not thought by the mind but by which, they
say, the mind is thought; that verily, know thou, is Brahman
[Wholeness], and not what people here adore.

—Kena Upanishad 1.6 [Ra]

agṛhyaḥ na hi gṛhyate

He is incomprehensible for he cannot be comprehended.
—Bṛhad-āraṇyaka Upanishad 4.5.15 [Ra]

[6] Maharishi explains that the manifest universe is the expression of the self-interacting dynamics of wholeness. He says, "It is the Creator, the process of creation, and the object of creation all together in itself" [M1, p. 213]. The Brahma Sutras (another aspect of the Vedic literature) declare wholeness or *Brahman* to be the source and ultimate reality of everything:

janmādyasya yataḥ

From which comes the birth, etc., of this (universe).
—Brahma Sutras, 1.1.2

ative potential.[7] However, in the Vedic view, wholeness is not static but, to the contrary, extremely dynamic by nature. Fundamentally, wholeness is pure wakefulness, and as such is awake to itself; consequently, it is perpetually in a state of self-knowing, self-transformation, and is always creating within itself, yet remaining fundamentally unaltered by these processes (much in the way that soft clay may be molded in various ways, as if bringing forth the potential of the clay, and yet the clay remains essentially unchanged by such transformations).

In the Vedic conception, the first step in the move of wholeness within itself involves a gathering of all its unbounded potential into a single focal point; this point of focus is said to be a point of "infinite dynamism," teeming with the tremendous power and possibilities of the Absolute. This point then begins to expand, step by step, into the fundamental frequencies from which the manifest universe is comprised. Collectively, these frequencies are known as the *Veda* and form a *blueprint* for the manifest universe. The term 'Veda' is translated as 'knowledge'; these fundamental frequencies represent the self-knowledge that emerges sequentially within the pure wakefulness of wholeness.

It must be emphasized that this entire process of self-unfoldment of wholeness is *unmanifest*, hidden from view, representing the hidden inner dynamics of the manifest universe. These dynamics are present deep within every grain of the universe, but are not obvious on the surface.

There are many passages in the Veda that elaborate upon these points at great length. We cite here one passage directly from Ṛk Veda, which expresses some of the main points just discussed, and then proceed to other quotations from Maharishi Mahesh Yogi's extensive commentary on the Veda; for further quotes and references, see [Co3] and [M1].

Ṛicho akshare parame vyoman
yasmindevā adhi vishve nisheduḥ

7 Consider the following passage from Maharishi's commentaries [M2, p. 30]: [Wholeness,] while remaining uninvolved with the creative process in nature, is an infinitely dynamic, inexhaustible source of energy and creativity. On that basis the whole creation goes on perpetually in its infinite variety, multiplying itself all the time. Change is perpetual, and change has come on the ground of non-change because of that non-changing value of infinite dynamism at the unmanifest basis of all creation.

The verses of the Veda exist in the collapse of fullness in the transcendental field, in which reside the fundamental frequencies responsible for the whole manifest universe.

—Ṛk Veda 1.164.39

The expression 'verses' in the above passage is also sometimes translated 'frequencies'. The quotation indicates how, within wholeness, the 'transcendental field,' the basic frequencies of the universe emerge in the collapse of the unbounded fullness of the Absolute. The fact that this collapse is a gathering of potential to a point is important and is brought out by a basic feature in the structure of Vedic Sanskrit: the structure of the Vedic literature and the Vedic language is such that the first word in a verse, hymn, or longer section in the Veda contains in seed form the total content of this verse, hymn or section. Moreover, the sound of this first word—indeed, even the first syllable—is said to contain the fundamental impulses or frequencies that structure the content of the entire verse, hymn, or section.[8]

An example of this phenomenon of central importance here is the first syllable of the Veda itself: 'AK.' The first letter 'A' is a sound that is made with the throat fully open and embodies "fullness." The second letter 'K' is a sound made with the throat closed and is called a "stop"; it stands for the focal point within wholeness, the "point of all possibilities," in which is located the "infinite creative potential of nature that in one stroke can give expression to the infinite diversity of creation" [M3, p. 278].

Commenting on the significance of the syllable 'AK,' in relation to the passage from Ṛk Veda quoted above, Maharishi remarks,

> The first syllable of Ṛk Veda, AK, expresses the dynamics of *akshara*—the '*kshara* of A' or collapse of infinity to its point value, which is the source of all the mechanics of self-interaction [M4, p. 1].

We now extract from this brief description of the Vedic dynamics of wholeness certain principles that we will use in the next section to motivate the Wholeness Axiom.

8 This observation about the structure of the Vedic Literature is due to Maharishi, and is known as Maharishi's *Apaurusheya Bhāshya*, the "uncreated commentary" of the Veda. See Section 5, and also [Ch], for a fuller discussion.

The Principle of Self-Transformation. This principle asserts that wholeness by nature moves within itself, knows itself, and creates within itself, and yet remains unchanged by these transformations.

The Focal Point Principle. When wholeness moves, the move becomes focused at a central point within wholeness in which all the knowledge and power of the Absolute becomes concentrated, in preparation for infinite expansion into the universe.

The Blueprint Principle. When the focal point of the move of wholeness begins to expand, it first takes shape as a blueprint of the domain that is about to be created. (In the Vedic paradigm, the Veda itself is this blueprint, and it is a blueprint for the manifest universe; but the principle we are giving here simply asserts that whenever wholeness moves, the focal point of the move unfolds into a basic blueprint of the thing being created.)

The Principle of Global Undefinability. This principle asserts that the self-interacting dynamics of wholeness take place not on the surface of existence but, rather, are totally unmanifest, logically prior to the manifest universe.

The Principle of Local Existence. The dynamics of wholeness are present at every point in its manifestation. (The point here is that, although the dynamics of wholeness are unmanifest and transcendental to the manifest universe, they are not divorced from it; rather, these dynamics are present in every grain of the manifest universe.)

This list of five Vedic principles provides us with the main "new details" that we wish to add to Cantor's paradigm of the Absolute Infinite and apply to the structure of the mathematical universe. The new feature that the Vedic paradigm offers is a sequence of dynamic stages of unfoldment within the Absolute that we find nowhere in Cantor's treatment. We find these new principles rather suggestive of a view in which an elementary embedding of the universe (to itself) would be an expected feature and in which the critical point of the embedding would play a crucial role. In the next section, we show how these new principles can be used to motivate the Wholeness Axiom.

3. The Wholeness Axiom

Let us recall that our aim in introducing the Vedic version of the Absolute has been to provide a more detailed conceptual framework on which to base intuitions concerning the structure of the universe. Our hypothesis is that, given such a sufficiently rich, natural extension of Cantor's Absolute Infinite as an intuitive model of the universe, it will be "obvious" that even the strongest large cardinal axioms should hold true.

The reader familiar with large cardinals will recall that the strongest large cardinal axioms assert the existence of nontrivial elementary embeddings of the universe into transitive proper class models of ZFC (in this paper, we reserve the letter M to denote such models); the first ordinal moved by such an embedding, called the *critical point* of the embedding, is the large cardinal defined by the embedding. Generally speaking, the stronger the large cardinal axiom becomes, the more completely the image model M is required to resemble V. From this perspective, the strongest large cardinal axiom of all would be the assertion of the existence of a nontrivial elementary embedding from V to itself. Kunen [K2], however, showed (in the context of class theories) that such an embedding is too strong to be consistent, and so this type of embedding has not been studied extensively.

Because an embedding of this kind represents a natural upper limit to the strongest large cardinal axioms, our strategy has been to determine whether such an embedding is suggested to us by our Vedic principles; that is, does the picture of the universe V that is painted using our Vedic principles suggest to us that an elementary embedding from V to itself ought to exist? It was surprising to find that it does and that it even points to certain assumptions that are buried in Kunen's proof, which are essential for the proof to go through, but which are not guaranteed to hold true in the context of ZFC (but which do hold true in the usual class theories).

To begin our analysis, let us recall from the last section that Self-Transformation suggests that the universe, the mathematical analogue of wholeness, "moves within itself, is transformed within itself, yet remains unchanged by the transformation." This feature suggests that there is some sort of mapping naturally associated with the universe, with domain and codomain being the universe itself, and with the property that the "universe remains unchanged" by the mapping—a phrase we could interpret as meaning "all relationships are preserved." These requirements on a mapping seem most naturally satisfied by an *elementary embedding*. Note that since the requirement is that there be some kind of "move" effected by the map, we can eliminate the identity mapping from consideration, and we seem to be left with a well-motivated nontrivial elementary embedding from V to itself.[9]

If we turn to the Focal Point Principle, we encounter additional requirements on the embedding: The embedding should give rise to a focal point within V in which "all knowledge and power" of V are found in seed form. A natural candidate for this "point" in the universe is the critical point κ of the embedding. Certainly, of all sets in the universe, the critical point κ is endowed by the embedding with the most powerful properties. As we will show, κ necessarily has virtually all large cardinal properties; it also acts as a "seed" from which all sets in the universe can be located, as we will discuss more fully in the context of the Blueprint Principle, below.

These observations make a case for the "power" part of the focal point principle. As for the "knowledge" part, we draw upon one more mathematical result that we will demonstrate below: namely, that once we have such an embedding with critical point κ, it follows that the stage V_κ is an elementary submodel of V; in symbols, $V_\kappa \prec V$. (This means that , for any possible relationship among sets $a_1, \ldots, a_n$ belonging to V_κ, this relationship holds in V_κ if and only if it holds in V; more precisely, the inclusion map incl: $V_\kappa \to V : x \mapsto x$ is an elementary embedding. In particular, V_κ and V satisfy the same first-order sentences.) This relation provides a believable realization of the requirement that "all knowledge of V can be found in κ in seed form": every first order statement true in V, and no false statements, must also hold at the κth stage of the universe.

9 Roughly speaking, an *elementary embedding* j from V to a transitive model M of ZFC is a function that "preserves all relationships" among sets; that is, for any first-order formula $\phi(x_1, \ldots, x_n)$ and any sets $a_1, \ldots, a_n$, $\phi(a_1, \ldots, a_n)$ holds true in V if and only if $\phi(j(a_1), \ldots, j(a_n)$ holds true in M. Also, j is *nontrivial* if for some ordinal $\alpha, j(\alpha) \neq \alpha$. See [We], [Co3], or [Je].

The Blueprint Principle suggests to us that the point value that we locate should in some way give rise to a blueprint that encodes the essential information about the structure of the "manifest universe." We take "sets" to correspond to the manifest universe here. To fill the role of a "blueprint," we have found it natural to consider a Laver sequence. A Laver sequence [La] (see also [Co3]) at κ is a function $f: \kappa \to V_\kappa$ such that for every set A there is a supercompact embedding $i: V \to M$ such that $A = i(f)(\kappa)$, where M is a transitive class model of ZFC. In other words, a Laver sequence "codes up" the information about the location of each set in the universe: Each set occurs as the κth term of the image of the Laver sequence under a suitable supercompact embedding. As we will show below, given $j: V \to V$, the necessary supercompact embeddings arise naturally from j.

We have made use of our first three Vedic principles to motivate the existence of a $j: V \to V$ and have found that the mathematical implications of such an embedding correspond nicely with the Vedic requirements. At this point, we can no longer postpone our obligation to address the first major obstacle to our program so far:

Obstacle #1: Kunen's Theorem. In [K2], K. Kunen showed, in the context of class theories, that there is no nontrivial elementary embedding from the universe to itself. A viewpoint held by many set theorists, especially when the discovery was first made, was that Kunen's theorem proved that such embeddings are inconsistent with ZFC (which is *not* a class theory). However, since the existence of such an embedding cannot be formalized in the language of ZFC, care is needed in determining the impact of such an embedding on ZFC. As it turns out, what Kunen's result forbids, in the context of ZFC, is elementary embeddings $j: V \to V$ that are *weakly definable*[10] in V. We can rephrase Kunen's theorem as follows:

3.1 Theorem. (Kunen) Every nontrivial elementary embedding of V to itself is not weakly definable in V.

Kunen's proof cannot be used to prove the inconsistency of a

[6] Suppose $\langle M, \epsilon\rangle$ is a model of ZF and $X \subseteq M$. Then X is *weakly definable* in M if the expanded model $\langle M, \epsilon, X\rangle$ satisfies all instances of Replacement for formulas of the expanded language. It is straightforward to show that whenever X is definable in M, it must also be weakly definable in M.

$j\colon V \to V$ with critical point κ if j is not weakly definable since, in the proof, we must be able to take the supremum of the sequence κ, $j(\kappa)$, $j^2(\kappa)$, . . . , and this step requires an application of Replacement for formulas in the expanded language (formulas that have an occurrence of j). Since j is not weakly definable, we are not allowed to freely apply Replacement to j-formulas, and so the supremum may not exist. Thus, the proof is not applicable to such an embedding. We propose, then, to introduce an axiom that asserts the existence of such a j that is *not* weakly definable in V.

Mathematically speaking, then, Obstacle #1 has been overcome. Still, we must ask whether or not it follows from our Vedic vision that an elementary embedding from V to itself—if such a thing exists at all—really ought to be undefinable (or more precisely, *not weakly definable*). The applicable principle here is the Principle of Global Undefinability. This principle says that the self-transformation of wholeness, which in our context is embodied in a $j\colon V \to V$, is fundamentally unmanifest and ungraspable from the perspective of the manifest universe. This view suggests that if there is going to be a $j\colon V \to V$ at all, we should expect that it would be highly undefinable in V.

Based on this analysis, we would like to assert as a basic axiom that there is a nontrivial elementary embedding $j\colon V \to V$ that is not weakly definable in V. However, one other obstacle seems to darken our hopes:

Obstacle #2: Embeddings $j\colon V \to V$ that are not weakly definable can be weak. The fact is that elementary embeddings from a model of ZFC to itself that fail to be weakly definable are quite common: For instance, under certain mild large cardinal assumptions (for instance, the existence of a measurable cardinal), there is such an embedding from L to L, where L is the constructible universe (see [Je]). We didn't have such an embedding in mind in declaring our axiom, but clearly a $j\colon L \to L$ is a possible interpretation. By observing what makes this latter type of embedding so weak, we can see what else needs to be added to elementarity of the embedding to capture our intention. What becomes apparent is that the action of a $j\colon L \to L$ is *discoordinated* from the structure of the model. For instance, if we try to form the canonical ultrafilter[11] $U = \{X \subseteq \kappa : \kappa \in j(X)\}$ using such a j, we find that relative

11 When a collection U is defined in this way from an elementary embedding $j\colon V \to M$ with critical point κ, U is certainly a set and has the properties of a *normal measure* on κ, which makes

to L, U is not a set. Not only is j globally unrecognizable to L (being highly undefinable), but it is also locally unrecognizable (since sets cannot always be defined using formulas—even bounded formulas—that refer to j).

Now from our Vedic perspective, we would not expect our embedding to be locally out of synchrony with the universe. Our final Vedic principle, Local Existence, implies that the dynamics of wholeness, represented by j, while being highly undefinable, should nevertheless be well-coordinated with the universe; indeed the dynamics are supposed to be "present at every point." One natural way to implement Local Existence is to require not only that j not be weakly definable, but, for each set A, that $j \restriction A$ also be a set.[12] This eliminates situations that are like embeddings from L to L, and seems very much in accord with our fifth Vedic principle.

We are now ready to state our Wholeness Axiom:

Wholeness Axiom

> There is a nontrivial elementary embedding $j\colon V \to V$ that is not weakly definable in V, such that for all sets A, $j \restriction A$ is a set.

Here, by "nontrivial," we mean that j is not the identity map, that it moves at least one set. It can be shown, as we observe in Proposition 3.2, that such an embedding must actually move some *ordinal*. The least such ordinal is called the *critical point* of the embedding.

We will call an embedding j given by the Wholeness Axiom a *WA-embedding*. Given a nontrivial elementary embedding $j\colon V \to V$ that is not weakly definable, if V has the property that for every set A, $j \restriction A$ is a set, we will say that V is *j-closed*.

We should observe here that WA is not formally expressible in the language of ZFC set theory. In Theorem 4.10, we show how to formalize WA in a language in which there is, in addition to the usual binary relation symbol $\in$, one unary function symbol $\mathbf{j}$.

κ a measurable cardinal. Embeddings of the form $j\colon L \to L$ are therefore too weak to give rise to a normal measure on κ; in fact it can be shown that L does not contain a measurable cardinal.

[12] In the literature on this subject (developed subsequent to the present paper), the Wholeness Axiom is defined so that Local Existence is translated into an even stronger mathematical requirement: that Separation holds for all **j**-formulas. See[Co2].

Before systematically examining some mathematical consequences of the Wholeness Axiom, which we do in the next section, we pause here to prove some claims that we made earlier to motivate WA. In particular, we need to show that

(1) The critical point κ of the embedding j has virtually all large cardinal properties.
(2) $V_\kappa \prec V$.
(3) j gives rise to a Laver sequence in a fairly natural way.

We prove (2) and (3) here; we will also prove (1) up through supercompactness and reserve the proof of the stronger assertion (that κ is super-n-huge for every n) for the next section.

To begin, we dispense with an easy but essential observation, which is proved using the standard argument (see [Je, Lemma 28.5]):

3.2 Proposition. If $j: V \to V$ is a WA-embedding, there is an ordinal κ such that $j(\kappa) \neq \kappa$. □

Given a WA embedding j, as we mentioned before, the sequence κ, $j(\kappa), j^2(\kappa), \ldots$ is not weakly definable, for, if it were, by Replacement, its range would have a supremum λ, and we could reproduce Kunen's inconsistency argument (see [KM, p. 202]) for embeddings $i: V_{\lambda+2} \to V_{\lambda+2}$. Thus we have the following:

3.3 Proposition. If $j: V \to V$ is a WA-embedding, then the sequence κ, $j(\kappa), j^2(\kappa), \ldots$ is not weakly definable in V and is unbounded in ON. □

Using this ω-cofinal sequence, we can reason[13] as though we had an I_3-embedding $V_\alpha \to V_\alpha$ to show that

$$V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)} \prec \ldots$$

[13] The remarks here mask certain subtleties that need to be handled carefully in the formal proof of Proposition 3.4. Though the reasoning, at a high level, is like that used in the context of I_3 embeddings, there are technical issues that arise in the context of WA (for instance, it is possible that $j^n(\kappa)$ fails to exist for certain n), which make the proof considerably more difficult. See [Co1, Proposition 8.13ff.] for a detailed treatment of these issues. A similar point applies to other applications of the j-closed property in subsequent proofs.

and hence that for each n, $V_{j^n(\kappa)} \prec V$, since in this case V is the union of an elementary chain (see [KM, p. 203]). Thus, the following proposition takes care of (2):

3.4 Proposition. If j: $V \to V$ is a WA-embedding, then for each n, $V_{j^n(\kappa)} \prec V$. □

Our last proposition in this section establishes the existence of Laver sequences in the presence of WA (taking care of (3)); it also tells us that the critical point of a WA-embedding is supercompact. We begin with a review of the definition of a Laver sequence at κ:

3.5 Definition. Suppose κ is an infinite cardinal. A Laver sequence at κ is a function f: $\kappa \to V_\kappa$ such that for every cardinal $\lambda \geq \kappa$ and every set A, if $|\mathrm{TC}(A)| \leq \lambda$, then there is a canonical supercompact embedding i: $V \to M$ (where M is a transitive class model of ZFC) defined from a supercompact ultrafilter over $P_\kappa\lambda$ such that $A = i(f)(\kappa)$.

3.6 Proposition. If j is a WA-embedding with critical point κ, then there is a Laver sequence at κ.

Proof. By [La, p. 386], it suffices to show that κ is λ-supercompact for every $\lambda \geq \kappa$. Given $\lambda \geq \kappa$, let n_λ be the least integer n such that $j^n(\kappa) > \lambda$ (using Proposition 3.3).

Let

$$U = \{A \subseteq P_\kappa \lambda : (j^{n_\lambda})"\lambda \in j^{n_\lambda}(A)\}$$

Set $n = n_\lambda$. We use the fact that V is j-closed to verify that U is a set. First, notice that j^n may be replaced by $j^n \restriction PP_\kappa\lambda$. We also have

$$\left(j^n\right)"\lambda \subseteq j^{2n}(\kappa).$$

Therefore,

$$(j^n)"\lambda = \{\beta \in i(\kappa) : (\exists \alpha < \lambda)\ i(\alpha) = \beta\}$$

where $i = j^{2n}$.

Now since n_λ was chosen large enough, $P_\kappa\lambda \in U$, and the usual arguments [Je, Chapter 33] show that U is a λ-supercompact ultrafilter over $P_\kappa\lambda$. □

4. The Theory ZFC + WA

In this section we prove several mathematical consequences of assuming the Wholeness Axiom. In the context of the general theme of our paper, the most important of these is that the critical point of a WA-embedding has virtually all large cardinal properties. We also show that the large cardinal axiom I_3 is strictly stronger than WA and discuss a formalization of ZFC + WA.

We begin with a review of the definition of some of the strongest large cardinal axioms. For a thorough treatment of other large cardinal axioms and related notions, see [Je] or [KM].

4.1 Definition. Suppose κ is an infinite cardinal and $n \in \omega$. Then κ is *n-huge* if there is a transitive model M of ZFC containing all the ordinals and an elementary embedding $i: V \to M$ with critical point κ such that every function $g: i^n(\kappa) \to M$ belongs to M. Such a map i is called an *n-huge embedding*.

4.2 Definition. [Ba] Suppose κ is an infinite cardinal, $n \in \omega$, and α is an ordinal. Then κ is said to be *n-huge α times* if there exist a sequence $\langle M_\beta : \beta < \alpha\rangle$ of transitive models of ZFC containing all the ordinals and a sequence of elementary embeddings $\langle i_\beta : V \to M \mid \beta < \alpha\rangle$, each with critical point κ, such that each i_β is an n-huge embedding, and the sequence of targets $\langle i_\beta(\kappa) : \beta < \alpha\rangle$ is strictly increasing. The cardinal κ is called *super-n-huge* if κ is n-huge α times for every α.

4.3 Definition. The axiom I_3 is the statement that there is a limit ordinal α and a nontrivial elementary embedding $i: V_\alpha \to V_\alpha$. The axiom I_1 is the statement that there is a limit ordinal α and a nontrivial elementary embedding $i: V_{\alpha+1} \to V_{\alpha+1}$.

4.4 Proposition. [Ba] If κ is super-n-huge for every n, then κ is supercompact, extendible, and n-huge for every n.

The property of being super-n-huge for every n is the strongest among the well-known large cardinal properties that are weaker than I_3. (The fact that I_3 is consistency-wise stronger than super-n-huge for every n follows from the proofs of Theorems 4.5 and 4.6 below.)

4.5 Theorem. Suppose $j: V \to V$ is a WA-embedding with critical point κ. Then κ is super-n-huge for every natural number n.

Proof. We first verify that κ is n-huge for every n. To see that κ is huge, let $U = \{A \subseteq P(j(\kappa)) : j''(j(\kappa)) \in j(A)\}$. Because V is j-closed, U is a set. The usual arguments (see [Je] or [KM]) show that $[j(\kappa)]^\kappa = \{B \subseteq j(\kappa)$: ordertype$(B) = \kappa\} \in U$, that U is a normal, fine ultrafilter over $P(j(\kappa))$, and hence that κ is huge. To see that κ is n-huge for every n, reason as above, replacing $j(\kappa)$ with $j^n(\kappa)$.

Now, to prove super-n-hugeness for every n, it suffices to show that for all $m, n \in \omega$, κ is n-huge κ_m times, where $\kappa_m = j^m(\kappa)$. We will first show that there is a stationary subset S_1 of $j(\kappa)$ each of whose elements is the target of an n-huge embedding with critical point κ; then we apply a suitable elementary embedding repeatedly to S_1 to show that similar stationary sets exist below each κ_m.

Let

$$U = \{X \subseteq j(\kappa) : j(\kappa) \in j(j \restriction V_{j(\kappa)})(X)\}.$$

U is a set because V is j-closed. The usual arguments show that U is a normal, nonprincipal ultrafilter over $j(\kappa)$. Note that the critical point of $j(j \restriction V_{j(\kappa)})$ is $j(\kappa)$.

Now, setting

$$S_1 = \{\alpha < j(\kappa) : \alpha \text{ is a target of some } n\text{-huge embedding having critical point } \kappa\},$$

it is easy to see that $S_1 \in U$ since $j(\kappa)$ is a target of an n-huge embedding having critical point κ, as we showed in the first paragraph. Hence, S_1 is stationary.

For each $m > 0$, let $i_m = j(j \restriction V_{\kappa_{m+1}})$. Now for each m, inductively define

$$S_{m+1} = i_m(S_m).$$

By elementarity, S_m is a stationary subset of κ_m each of whose elements is a target of an n-huge embedding with critical point κ. □

In order for the last result to be significant, we need to know that WA is consistent. On the one hand, our hope is that our intuitive model is compelling enough to make it "obvious" that a universe with such an embedding exists—at least, perhaps, as compelling as Cantor's Absolute Infinite as an intuitive model to justify the existence of a model V of ZFC. On the other hand, from a strictly mathematical point of view, we can at least show that the relative consistency of ZFC + WA is implied by a known large cardinal axiom, namely I_3; for, suppose α is a limit and $j\colon V_\alpha \to V_\alpha$ is an elementary embedding with critical point κ. Then as in [KM, p. 203], V_α is a model of ZFC. But it is also clear that V_α is j-closed, and, by Kunen's Theorem, that j is not weakly definable in V_α. We have therefore:

4.6 Theorem. Con(ZFC + I_3) implies Con(ZFC + WA).

What about the converse? Notice that if α is the least limit for which there is a nontrivial elementary embedding $V_\alpha \to V_\alpha$, then we have the following:

$$V_\alpha \models \mathrm{WA} + \quad I_3.$$

Thus, I_3 bears the same relationship to the ZFC + WA universe as an inaccessible bears to the ZFC universe, and it is not even possible to prove from ZFC + WA the consistency of ZFC + I_3.

4.7 Theorem. Con(ZFC + WA) does not imply Con(ZFC + I_3).

From our Vedic perspective, how is the axiom I_3 to be viewed, since it isn't derivable from ZFC + WA? Since our Vedic model for the universe is compatible with Cantor's, we may still legitimately apply the Reflection Principle; doing so tells us that if there is a WA-embedding $j\colon V \to V$, the same ought to be true relative to some stage V_α of the universe.

Nevertheless, even with Reflection, we do not have a way, using our Vedic model, to justify the existence of the large cardinals given by any of the axioms I_2 to I_0. The strongest of these, I_0, arose in Woodin's work to prove the consistency of AD relative to large cardinals (see [Ma2] for a discussion). The fact that no inconsistency arose in the rather involved uses to which Woodin put this axiom has offered some hope that even this strong assertion may be consistent. In the present context, however, it is not obvious how to motivate I_0 using the Vedic model described here. We leave this problem as an open question.

4.8 Open Question #1. Can the Vedic perspective given here be deepened (in a natural way) so that the truth of any of the axioms $I_2 - I_0$ becomes "obvious"?

Our applications of WA so far have necessarily been metatheoretic since WA cannot be expressed in the language of ZFC. We now offer a more formal way of presenting the Wholeness Axiom and indicate why the formal version matches our intended intuition for the axiom; see [Co2] for a detailed account. To formulate our axiom, we will work in the usual language of set theory, with one additional (unary) function symbol **j**, intended to stand for an elementary embedding of the universe. The introduction of this new function symbol expands the range of formulas that are now used in the formal theory to represent mathematical assertions. Formulas in which the new symbol **j** occurs will be called **j**-formulas (or **j**-sentences if there is no occurrence of a free variable). Formulas in which **j** does not occur are called ϵ-formulas. The Wholeness Axiom (WA) is now formally defined as a schema of axioms consisting of the following:

Elementarity Schema. Each of the following **j**-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ denotes an ϵ-formula:

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Leftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right).$$

Nontriviality. For some set x, $\mathbf{j}(x) \neq x$.

Amenability. For every set x, $\mathbf{j} \restriction x$ is a set.

The expanded set theory we are proposing is ZFC + WA. We note that if $i\colon V_\alpha \to V_\alpha$ is an I_3 embedding, then $\langle V_\alpha, \in, i\rangle$ is a model of ZFC + WA, so the formal theory is consistent relative to the axiom I_3, exactly as we described above. Also, we note that in any model of ZFC + WA, the interpretation j of $\mathbf{j}$ in the model can never be weakly definable, because of Kunen's Theorem, so the formally defined $\mathbf{j}$ must fail to be weakly undefinable in any model in which it is realized.

5. Laver Sequences

Our application of the Vedic paradigm to set theory has placed the concept of a Laver sequence in a prominent role. In this section we study certain properties of Laver sequences more closely, as they relate to our Vedic model. This analysis will lead to the observation that a cardinal κ admits a generalized Laver sequence if and only if κ is strong, and will highlight several simple properties of Laver sequences that have clear parallels to features of the structure of the Veda.

We begin by recalling the significance of Laver sequences in our application of the Vedic model. In applying the Blueprint Principle, our goal was to locate some compact collection that would code up the essential information about every set in the universe, and that would emerge in some natural way from the interaction of a WA-embedding j and its critical point κ. What made the notion of a Laver sequence suitable in this context was that, first, we could obtain such a sequence f from j and κ, and second, that every set occurs as the κth term of some image of f by a suitable elementary embedding.[14] Actually, the sort of Laver sequence that was defined by Laver imposed a more stringent requirement: namely, the only elementary embeddings that one is allowed to apply to f to satisfy the condition are canonical embeddings that are defined from λ-supercompact ultrafilters for various $\lambda \geq \kappa$. While this requirement is essential in certain applications (for example, in the proof of the consistency of the Proper Forcing Axiom), it isn't necessary to meet the more general philosophical requirement

14 We make a few observations here about elementary embeddings and Laver sequences. Nontrivial elementary embeddings of the form $i\colon V \to M$ (where M may or may not be equal to V itself) with critical point κ always have the property that $i(\kappa) > \kappa$. By elementarity of such an i, one can show that for any function $f\colon A \to B$, $i(f)$ is also a function having domain $i(A)$ and codomain $i(B)$. In particular, if $f\colon \kappa \to V_\kappa$ is a Laver function, whenever $i\colon V \to M$ is elementary with critical point κ, $i(f)$ is a function with domain $i(\kappa)$. Since $\kappa \in i(\kappa)$, we can apply $i(f)$ to κ. The key property of a Laver sequence is that, for any set A, we can find i so that $A = i(f)(\kappa)$.

we are seeking. These considerations lead us to define generalized Laver sequences:

5.1 Definition. Suppose κ is an infinite cardinal. Then a *generalized Laver sequence at* κ is a function $f: \kappa \to V_\kappa$ such that for every set A, there is an elementary embedding $i: V \to M$, where M is a transitive model of ZFC containing all the ordinals and κ is the critical point of i, so that $A = i(f)(\kappa)$. In this case, we say that κ *admits a generalized Laver sequence.*

The definition suggests the following question: What is the consistency strength of the statement, "κ admits a generalized Laver sequence"? If we replace "generalized Laver sequence" with "Laver sequence," it is obvious that the supercompactness of κ is both necessary and sufficient because of the dependence of the concept of Laver sequence on supercompact embeddings (and because of Laver's theorem [La, p. 386]). For the generalized case, we first observe that if κ admits a generalized Laver sequence, then κ must at least be a strong cardinal. (Recall that κ is strong if for each $\lambda \geq \kappa$ there is an elementary embedding $i: V \to M$ such that M is a transitive model of ZFC containing all the ordinals, κ is the critical point of i, and $V_\lambda \subseteq M$. See [MS].) This holds because every V_λ must be an $i(f)(\kappa)$ for some $i: V \to M$ with critical point κ, and hence $V_\lambda \in M$. The proof of the converse requires more work; details may be found in [Co2].

5.2 Theorem. An infinite cardinal κ admits a generalized Laver sequence if and only if κ is strong.

If Laver sequences—especially those defined from a WA-embedding—are supposed to be an analogue to the Veda, how far does the analogy go? To what extent does the structure of a Laver sequence mirror that of the Veda? We make some observations here as a starting point for deeper research to appear in later work.

For our discussion, the commutative diagram pictured below will be useful. In the diagram, $\lambda \geq \kappa$, U_λ is a fine, normal measure on $P_\kappa\lambda$, i_λ is the canonical embedding, $U = \{X : \kappa \in i_\lambda(X)\}$ is the measurable ultrafilter defined from i_λ, i is the canonical embedding defined from U, and k is defined by $k([g]) = i_\lambda(g)(\kappa)$. (See [Je, Chapter 28].)

$$\begin{tikzcd} V \arrow[r, "i_\lambda"] \arrow[d, "i"'] & V^{P_\kappa\lambda}/U_\lambda \\ V^\kappa/U \arrow[ur, "k"'] & \end{tikzcd}$$

A. *The Veda "contains everything."* Maharishi has explained that every item in the manifest universe occurs in seed form in the Veda. It is interesting to note that in order for a Laver sequence f at κ to "capture" every set, it must be the case that each member of V_κ occurs stationarily often in f. Moreover, any set A must be represented by a function $g\colon P_\kappa\lambda \to V$ and stationarily often, g "agrees with" f. We state this more precisely in the following proposition:

5.3 Proposition. Suppose $f\colon \kappa \to V_\kappa$ is a Laver sequence at κ, A is a set, and $\lambda \geq \max(\kappa, |\mathrm{TC}(A)|)$. Then there are U_λ, i_λ, U, i as in the diagram and a function $g\colon P_\kappa\lambda \to V$ representing A in the ultrapower by U_λ such that

$$\{P \in P_\kappa\lambda : g(P) = f(P \cap \kappa)\} \in U_\lambda$$

and

$$\{\alpha < \kappa : \exists P\, (\alpha = P \cap \kappa \wedge g(P) = f(\alpha))\} \in U.$$

Proof. Recall (see [Je, p. 409]) that for all $\alpha \leq \lambda$, α is represented in M_λ by the function t_α defined by

$$t_\alpha(P) = \mathrm{ordertype}(P \cap \alpha),$$

and $i_\lambda''(\alpha)$ is represented by r_α where

$$r_\alpha(P) = P \cap \alpha.$$

Note also that because $B \in U$ iff $i_\lambda''(\lambda) \in i_\lambda(B)$, we have

$$\{P \in P_\kappa\lambda : \text{ordertype}(P \cap \kappa) = P \cap \kappa\} \in U_\lambda.$$

Now, given A and $\lambda \geq \max(\kappa, |\text{TC}(A)|)$, let i_λ be such that

$$(*) \qquad i_\lambda(f)(\kappa) = A.$$

Clearly $A \in M_\lambda$. Let g: $P_\kappa\lambda \to V$ represent A in M_λ. Since the constant function c_f represents f, and t_κ represents κ, we can rewrite (∗) as

$$[c_f]([t_\kappa]) = [g].$$

By Łoś' Theorem and the fact that ordertype$(P \cap \kappa) = P \cap \kappa$ on a set in U_λ,

$$\{P \in P_\kappa\lambda : f(P \cap \kappa) = g(P)\} \in U_\lambda.$$

Next, observe that

1. $i_\lambda(f)(\kappa) = [g] = i_\lambda(g)(i_\lambda''\lambda)$;
2. $\kappa = i_\lambda''\lambda \cap i_\lambda(\kappa)$; and
3. $i''_\lambda \lambda \in P_{i_\lambda(\kappa)} i_\lambda\lambda = i_\lambda(P_\kappa\lambda)$.

Hence,

$$\kappa \in \{\alpha < i_\lambda(\kappa) : \exists P\,(\alpha = P \cap i_\lambda(\kappa) \wedge g(P) = f(\alpha))\}.$$

It follows that $\{\alpha < \kappa : \exists P\,(\alpha = P \cap \kappa \wedge g(P) = f(\alpha))\} \in U$. □

B. *Infinite dynamism/infinite silence*. The structure of the Veda is said to embrace a wide range of opposite values; in particular, its structure is said to be at once infinitely dynamic and infinitely silent. The coexistence of these opposite values is said to make possible the full range of activity in the manifest universe. If we wanted to formulate the notion of dynamism as a property of a mathematical structure, we might think of fast-growing functions; and the most nonactive or "silent" behavior a function could exhibit would probably be that of being the identity map. We might conjecture that for any Laver sequence f, the function

$\alpha \mapsto |f(\alpha)|$ dominates, on a stationary set, any function in ${}^{\kappa}\kappa$ that is definable in V_κ and that f is the identity on another stationary set.[15] Note that the first half of the conjecture would imply that Laver sequences at κ are not definable in V_κ. This accords with our Vedic view, since the Veda, like the dynamics of wholeness corresponding to j, is to be understood as entirely unmanifest in nature.

In spirit, our conjecture does indeed turn out to be true. Although it is possible for a Laver sequence to be definable in V_κ, this can only happen (assuming WA) if a significant restriction is placed on the structure of the universe itself: Assuming WA, there is such a definable Laver sequence if and only if there is a definable well-ordering of the universe. Intuitively, we may consider V, as a model of wholeness, to be "too ungraspable" to admit such a well-ordering; from this perspective, Laver sequences are always undefinable in V_κ. Also, though we cannot show that Laver sequences always dominate definable functions in ${}^{\kappa}\kappa$, we are able to provide (Proposition 5.10) a construction that produces Laver sequences that do have this property.

The next proposition verifies that Laver sequences exhibit "infinite silence" in the sense that they behave like the identity function on a large set.

5.4 Proposition. If $f\colon \kappa \to V_\kappa$ is a Laver sequence at κ, there is a stationary set $S \subseteq \kappa$ such that for all $\alpha \in S$, $f(\alpha) = \alpha$.

15 In set theory, stationary sets represent "rather big" subsets of a regular cardinal κ, much more substantial than an arbitrary subset of size κ. Intuitively, we wish to demonstrate that "infinite silence" and "infinite dynamism" are both displayed in the behavior of a Laver function simultaneously on different big subsets of its domain, and we can achieve this if by "big" we mean "stationary." A helpful analogy is to consider different kinds of subsets of the unit interval [0,1] of size $\mathbf{c}$ (the cardinality of the continuum). The "thickest" among such sets would be a (Lebesgue) measure 1 set. Next would be sets that have *outer* measure 1 (it is possible for there to be two such sets that are disjoint, but this is not possible for measure 1 sets; note that if X has outer measure 1 and is disjoint from another outer measure 1 set, then X must *meet* every measure 1 set). Finally we have the "thinnest" such sets—sets of *measure zero*. The analogy to subsets of a regular cardinal κ is this: The thickest subsets of κ are those that contain a *closed unbounded* set ("closed" means that it contains the supremum of each of its subsets); these are so thick that intersecting fewer than κ of these yields another such set. Next are the stationary sets, which are those sets that meet every closed unbounded set; one can always build disjoint stationary sets. Finally, the "thinnest" subsets of κ are those that lie in the complement of a closed unbounded set.

Proof. Choose λ so that $i_\lambda(f)(\kappa) = \kappa$. It follows immediately from Łoś' Theorem that

$$\{\alpha < \kappa : f(\alpha) = \alpha\} \in U. \square$$

We turn to a canonical construction for Laver sequences; the construction, in the presence of WA, has enough flexibility to allow us to create very fast growing Laver sequences, as described earlier. The parameter t in the construction represents an arbitrary sequence of sets $t: \kappa \to V_\kappa$. The function f defined in the construction will be shown to be Laver, under the assumption of WA.

5.5 Canonical Construction CC(t). Define $f: \kappa \to V_\kappa$ by recursion as follows:

$$f(\alpha) = \begin{cases} \text{some } t_\alpha \in V_\kappa, \text{ if } f \restriction \alpha \text{ is Laver at } \alpha \\ \text{some } x \in V_\kappa, \text{ where } x \text{ witnesses } \phi(f \restriction \alpha, \lambda_{f\restriction\alpha}) \end{cases}$$

where $\phi(g,\delta)$ denotes the following formula:

> "There exists a cardinal α such that g is a function $\alpha \to V_\alpha$ and δ is the least cardinal such that for some set y we have $|\mathrm{TC}(y)| \leq \delta$ and $i(g)(\alpha) \neq y$ for all i canonically generated from a supercompact ultrafilter over $P_\alpha\delta$."

5.6 Proposition. Assuming WA, the function f given by the construction CC(t) is well-defined and is in fact a Laver sequence at κ. Moreover, the sequence t of sets may be defined so that the function $\alpha \mapsto |f(\alpha)|$ dominates, on stationary set, every function $\kappa \to \kappa$ that is definable in V_κ.

Proof. To see f is well-defined, first recall that by WA we have $V_\kappa \prec V$. Note that if there is a pair (x, λ) witnessing $\phi(f \restriction \alpha, \lambda_{f\restriction\alpha})$ at all, then, since $f \restriction \alpha \in V_\kappa$, there is, by elementarity of V_κ in V, such a pair in V_κ itself.

Now, suppose $j: V \to V$ is a WA-embedding with critical point κ. Let $D = \{A \subseteq \kappa : \kappa \in j(A)\}$ and let $i: V \to V^\kappa / D \cong M$ be the canonical embedding. We observe first that if $\{\alpha : f \restriction \alpha \text{ is Laver at } \alpha\} \in D$ then, as $\kappa \in j(\{\alpha : f \restriction \alpha \text{ is Laver at } \alpha\})$, f is Laver at κ. So, to complete the proof, it suffices to show that this set is indeed in D.

Working toward a contradiction, assume that

$$\{\alpha : f\restriction\alpha \text{ is Laver at } \alpha\} \notin D.$$

Then we have $\{\alpha : f(\alpha)$ witnesses $\phi(f\restriction\alpha, \lambda_{f\restriction\alpha})\} \notin D$, whence $j(f)(\kappa)$ witnesses $\phi(f, \lambda_f)$. To obtain a contradiction, we find an embedding i canonically derived from a supercompact ultrafilter U for which

$$i(f)(\kappa) = j(f)(\kappa).$$

Obtain i by setting $U = \{X \subseteq P_\kappa\lambda : j''\lambda_f \in j(X)\}$ and letting i be the canonical embedding $i : V \to V^{P_\kappa\lambda_f}/U \cong N$. Let k: $N \to V$ be defined by $k([g]) = j(g)(j''\lambda_f)$. By the usual argument, k is elementary, $j = k \circ i$, and it is straightforward to verify that $k(x) = x$ whenever $|\mathrm{TC}(x)| \leq \lambda_f$. Hence, setting $x = j(f)(\kappa)$ and recalling that x is a witness to $\phi(f, \lambda_f)$, we conclude from the previous observation that $k(x) = x$. Thus,

$$\begin{aligned} k(x) &= x \\ &= j(f)\kappa \\ &= k(j(f))\kappa \\ &= k(i(f))k(\kappa) \\ &= k(i(f)\kappa), \end{aligned}$$

whence $x = i(f)\kappa$, a contradiction.

Now, to ensure that $\alpha \mapsto |f(\alpha)|$ dominates, on a stationary set, every function $\kappa \to \kappa$ that is definable in V_κ, we define a sequence t of sets, to be used in $\mathrm{CC}(t)$, as follows: Let $\langle h_\xi : \xi < \kappa\rangle$ enumerate the members of ${}^\kappa\kappa$ that are definable in V_κ. Then, whenever $f\restriction\alpha$ is a Laver sequence at α, we let $t_\alpha = \sup\{h_\xi(\alpha) : \xi < \alpha\} + 1$. Since the set of α for which $f\restriction\alpha$ is Laver at α has D-measure 1, f dominates each h_ξ on a D-measure 1 set, and hence on a stationary set. □

Using ideas from the proposition, we may now show that, assuming WA, unless there happens to be a definable well-ordering of the universe, no Laver sequence at κ is definable in V_κ.

5.7 Corollary. Assume WA. Then the following are equivalent:

1. There is a Laver sequence at κ that is definable in V_κ.
2. There is a definable well-ordering of the universe.

Proof. ($1 \Rightarrow 2$) Suppose $f: \kappa \to V_\kappa$ is a Laver sequence definable in V_κ. Define $g: V_\kappa \to \kappa$ by

$$g(x) = \text{least } \alpha \text{ such that } f(\alpha) = x.$$

Let $h: \kappa \to g''V_\kappa$ be the increasing enumeration of $g''V_\kappa$. Define $k: \kappa \to V_\kappa$ by $k = g^{-1} \circ h$. Now k is a well-ordering of V_κ, and since f is definable in V_κ, so is k. But now because $V_\kappa \prec V$, the defining formula for k defines a well-ordering of V as well.

($2 \Rightarrow 1$) Assume $\lhd$ is a definable well-ordering of V; since $V_\kappa \prec V$, the induced wellordering $\lhd \restriction V_\kappa \times V_\kappa$ is definable in V_κ with the same formula. Use the construction $CC(t)$ to define a Laver sequence $f: \kappa \to V_\kappa$, with $t_\alpha = \emptyset$ whenever $f \restriction \alpha$ is Laver at α, and with the additional refinement that, in case $f \restriction \alpha$ is not Laver at α, $f(\alpha)$ is chosen to be the $\lhd$-least set satisfying the second condition of the construction. Clearly, f is the required Laver sequence. □

C. *Apaurusheya Bhāshya—uncreated commentary.* As was mentioned in the Introduction, according to Maharishi, the Veda is structured in such a way that it provides its own commentary on the knowledge it contains; this self-commentary is called *Maharishi's Apaurusheya Bhāshya*.[16] The way that this self-commentary takes shape is as follows: The totality of knowledge contained in the Veda is considered present in seed form even in the first letter of the Veda, 'A'; it is present in progressively more "elaborated" forms in the first syllable 'AK,' the first word 'AGNIM,' the first verse, the first hymn, and the first *Maṇdala*. Successively larger "packets" of this kind express the same totality of knowledge as earlier stages, but in more elaborated form.

Assuming WA, we can observe a similar flavor in the structure of Laver sequences: stationarily many initial segments of f are themselves Laver sequences. This means that the global property of being a Laver sequence can be located at progressively smaller "micro" scales within

[16] This feature of the structure of the Veda, discovered by Maharishi Mahesh Yogi, is discussed in [Ch], [Wa], and [Co3].

the structure of f itself, very much like finding the totality of the Veda in progressively smaller "packets" within its own structure. We state the result in the following proposition:

5.8 Proposition. If j is a WA-embedding with critical point κ and $f: \kappa \to V_\kappa$ is a Laver sequence at κ, then the set

$$\{\alpha < \kappa : f \restriction \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha\}$$

is stationary.

Proof. Let $D = \{A \subseteq \kappa : \kappa \in j(A)\}$. By the usual argument, D is a measurable ultrafilter, and, by j-closedness, D is a set in V. Since $j(f) \restriction \kappa = f$, we have

$$\{\alpha < \kappa : f \restriction \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha\} \in D,$$

as required. □

Our analogy here between Maharishi's Apaurusheya Bhāshya of the Veda and the property of Laver sequences demonstrated in Proposition 5.8 has its limitations. Perhaps the most evident of these is the fact that the obvious analogue to the first letter of the Veda is the first term of a Laver sequence f; however, the latter has no special significance. Indeed, the first term could be replaced by any other set in V_κ, and the sequence f' so obtained is indistinguishable from f in that for all canonical supercompact i, $i(f) = i(f')$.[17] We could modify the canonical construction defined above by artificially requiring the first term of the Laver sequence to be the first V_α for which $V_\alpha \prec V$, and this would capture a certain sense of wholeness at the outset; but a Laver sequence of this kind does not emerge in any natural or canonical way from WA and so we consider the question to remain open.

5.9 Open Question #2. Is there a canonical way to define a Laver sequence from WA that even more closely reflects the Apaurusheya Bhāshya characteristic of the Veda?

[17] In fact, this is also true for any $f': \kappa \to V_\kappa$ that disagrees with f only on a nonstationary set.

6. Conclusion

Intuitive models have always been used to guide mathematical research; a good model often suggests the truth of important conjectures and methods for proving them. And the usefulness of such models has clearly played a role in the history of research in large cardinals. Many set theorists have taken Cantor's Absolute Infinite quite seriously as a tool to motivate large cardinals; the unified perspective offered by this intuitive model adds cogency to arguments in favor of accepting the large cardinals suggested by the model. On the other hand, *ad hoc* intuitive principles for justifying other large cardinals lose considerable cogency precisely because they do not emerge from a compelling conceptual framework. Based on these considerations alone, it has seemed reasonable to attempt to broaden Cantor's intuitive model in a philosophically natural and mathematically useful way.

Though largely unfamiliar to the mathematical community, the Vedic model of wholeness has seemed a particularly natural philosophical extension of Cantor's Absolute Infinite because, on the one hand, it agrees with Cantor's view as far as the latter goes, and, on the other hand, it arose as a fundamental detailed insight—by an entire culture—into the nature of the infinite, without the ulterior motive of justifying a pet mathematical theory. The fact that such extensive non-mathematical research offers structural principles that seem, even in detail, to be closely related to the modern notion of elementary embeddings of the universe argues well for the suitability of the Vedic model as a "natural" extension of Cantor's model with genuine mathematical utility.

It is often argued that the ultimate criterion for deciding among candidates for large cardinal axioms is the reasonableness of their mathematical consequences, and not some philosophical viewpoint imposed from the outside and divorced from mathematical practice. The fact is, however, that the enormous range of appealing mathematical consequences of the dozens of known large cardinal axioms has not been sufficient to establish the latter among the basic axioms of set theory. This diversity of results has served more to legitimize large cardinals as a worthy set of supplemental notions to be used as sparingly as possible. Our view is that a compelling conceptual framework, like Cantor's Absolute Infinite, contributes a key ingredient in the determination of the primitive notions of mathematics. Moreover, when such a conceptual framework is sufficiently natural, it serves to unify diverse notions and to stimulate deeper insight into the mathematical entities under investigation—an effect that can hardly be classified as originating from an artificial imposition of philosophical dogma.

In actual mathematical practice, set theorists freely use large cardinals as the need arises; moreover, a perusal of applications of large cardinals in recent years reveals that most authors no longer even feel the need to apologize for their use of these axioms. It seems reasonable to the author that there be an overarching mathematical context in which all mathematical research takes place instead of a long sequence of viewpoints that decrease in their plausibility as more large cardinal axioms are included. The Wholeness Axiom, based on the Vedic model, is, we feel, one worthy attempt at providing such a context.

In some circles it is argued that the view that there ought to be such an "absolute" underlying mathematical framework is old-fashioned; we should rather seek to build different mathematical universes to suit the requirements of the particular needs of different areas of mathematics, rather than insist on "stuffing" all mathematics into a single universe (see for example [Be, pp. 235–239]). Our view, again motivated by the Vedic model, is that within the context of "wholeness" there is room for all particularized mathematical models and universes. Each "world" of mathematics has its own validity and its own range of applications. But the diversity of such models does not conflict with the possibility of an underlying unity in which these diverse world views coexist. Certainly, for example, the world of Choice and the world of Determinacy are quite different; yet in a large enough universe (if there exists a supercompact cardinal for instance) these two worlds coexist (AC holds in V while AD holds in $L(\mathbf{R})$).

Our view that all foundational models ought to have a place within an absolute foundation also explains our lack of concern to resolve undecidable propositions like CH or statements having large cardinal strength like the existence of saturated ideals. It seems to the author quite reasonable, and in accord with our Vedic model, that different "portions" of the universe ought to reflect different possibilities concerning the size of the continuum, the existence of saturated ideals, and other such questions. On the other hand, if one adopts the view that certain large cardinals should be banned from mathematics (not for mathematical, but rather, philosophical reasons), this situation would seem to be a matter of genuine concern, for then some significant portion of mathematics is supposed to be understood as taking place, rather artificially, outside the realm of "legitimate mathematics." Thus, our efforts in supplementing ZFC with an axiom schema such as WA have been concerned with providing a context for all large cardinals, that is, all possible consistency strengths, and not at all with resolving any of the well-known propositions that are undecidable by ZFC alone.

On a practical note, because of its richness, the Vedic model suggests many principles and properties for which one might expect to find natural mathematical analogues in the structure of the universe. We have pursued some of these here and others in [Co3]. Pursuing these analogies results in interesting mathematical conjectures and a stream of research that might otherwise never have emerged. Moreover, the results so obtained contribute to the general program of erecting a unified foundational framework that encompasses all of mathematics. It is the author's belief that there is a great need to restore a sense of purpose and direction to foundational studies; we hope our work here will provide a step toward meeting this need.

We have found the research here compelling because of the striking relevance of the Vedic paradigm to deep structural questions of the modern set-theoretic universe. Why should insights arising from ancient methods of contemplating the ultimate structure of reality exhibit strong parallels with modern-day large cardinal research? An intuition about it, again suggested by our Vedic model, is that the "infinity" that has been studied in mathematics for more than a century may well be a conceptualization of the "Infinity" that is pointed to in the ancient Vedic wisdom. From this point of view, the relevance of the ancient wisdom no longer seems surprising but instead, just what one would expect. Our hope is that, as the field of mathematics continues to evolve, Maharishi Vedic Science and the wisdom of the ancients will find their place among the resources and tools upon which mathematical researchers rely as they continue to advance their discipline.

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