Mathematics of Pure Consciousness

Paul Corazza
Department of Mathematics
Maharishi University of Management

Abstract: Adi Shankara, the foremost exponent of Advaita Vedanta, declared "Brahman alone is real, the world is mithya (not independently existent), and the individual self is nondifferent from Brahman." A fundamental question is, How does the diversity of existence appear when Brahman alone is? The *Yoga Vasistha* declares, "The world appearance arises only when the infinite consciousness sees itself as an object." Maharishi Mahesh Yogi has elaborated on this theme: Creation is nothing but the dynamics of pure consciousness, which are set in motion by the very fact that pure consciousness is conscious; being conscious, it assumes the role of knower, object of knowledge and process of knowing. To help clarify these issues, we offer a mathematical model of pure consciousness. We show that in a natural expansion of the universe of mathematics by ideal elements, there is a unique set Ω whose only element is itself, and which is equal to the set of all possible transformations from itself to itself. All "real" mathematical objects can be seen to arise from the internal dynamics of Ω. All differences among numbers, and among all mathematical objects, are seen to be ghostly mirages, hiding their true nature as permutations of one set, Ω.

1 Introduction

Adi Shankara, the foremost exponent of Advaita Vedanta, declared "Brahman alone is real, the world is mithya (not independently existent), and the individual self is nondifferent from Brahman."[1] How does the apparent diversity of existence arise when Brahman alone is? The *Yoga Vasistha* [16] declares,

> The world appearance arises only when the infinite consciousness sees itself as an object (p. 357).

Maharishi Mahesh Yogi [8, 9, 11] has elaborated on this theme by first observing that pure consciousness, the singularity, by virtue of being consciousness, is in fact *conscious* and therefore conscious of itself. Therefore, because it is *consciousness*, pure consciousness assumes the role of knower and object of knowledge. Moreover, the process of observing, perceiving, and knowing is itself the activity of consciousness, the activity of pure consciousness knowing itself. It is by virtue of the self-interacting dynamics of pure consciousness knowing itself that there is a sequential unfoldment, an unmanifest dynamism, that appears on the surface to be our manifest universe.

In this paper, we attempt to investigate these unmanifest dynamics using the tools of modern set theory. Providing a mathematical model of pure consciousness and its dynamics makes it possible to bring clarity to the mystery of creation, which, in the Vedantic view, is only an appearance.

We will see that the mathematical universe is like the material universe in that it is composed of a vast array of distinct individuals, interacting according to laws of the universe. We will identify an "ideal element" Ω of the universe that uniquely exhibits characteristics and dynamics that parallel those of pure consciousness. We will then be in a position to see the sense in which all mathematical objects are in reality nothing but Ω, but, because of a "mistake of the intellect," all sets are seen to be distinct and unconnected to their source.

We develop our thesis by first reviewing key elements of Advaita Vedanta. We take as our source for this knowledge expressions from the *Yoga Vasistha* [16], one of the most important scriptures of Vedantic philosophy [15, p. 37ff.], and the elaboration on these and other parts of the Vedic literature provided by Maharishi Mahesh Yogi, who has made the lofty heights of philosophy and wisdom of Advaita accessible to the common man through simple procedures of meditation and a scientific approach to study of Veda [8, 9, 11].

We then review the structure of the mathematical universe as it is understood today in modern foundational studies. Resources for this treatment include [6] and [1]. In studying the mathematical universe, we will discover a partial analogue to the field of pure consciousness and its dynamics. By observing how this analogue fails to fully capture the dynamics of consciousness, we are then led to the possibility of expanding the universe to include an ideal element, which could more fully embody these dynamics. Having located such an element, Ω, we then show how it provides the key to recognizing the deeper truth about every standard mathematical object as

[1] The transliterated Sanskrit is *Brahma satyam jagat mithya, jivo brahmaiva naparah.* This is a quotation from one of Shankara's famous works, *Vivekacudamani* or *Crest Jewel of Discrimination*. See [14, 67–68]. The translation given here comes from the Wikipedia article surveying the life and work of Shankara: `http://en.wikipedia.org/wiki/Adi_Shankara`.

being nothing other than Ω; at the same time, we will be able to give an account of the origin of the apparent separation of all mathematical objects from their source.

Most of the mathematical results mentioned in the paper are known. We have contributed a few new insights to the existing body of knowledge that support the coherence and cogency of our mathematical model. Seeing the realizations about the ultimate nature of reality modeled in this mathematical context will provide, we hope, a taste of this higher vision of life.

2 The Nature of the Singularity, Pure Consciousness

As we discussed in the Introduction, a key insight into the question, How does diversity arise from One? is that the "One," the singularity, is *pure consciousness*. By virtue of being consciousness, pure consciousness, the singularity, *automatically* assumes the roles of knower, known, and process of knowing. In Maharishi's treatment, the knower is referred to as Rishi, the known as Chhandas, and the process of knowing as Devata. Since the process of knowing has an impact on both the knower and the object of knowledge, this value of Devata is also to be appreciated as the principle of *transformation*. When these three are seen as one, they are referred to as Samhita (of Rishi, Devata, and Chhandas); Samhita in this context means *unity*.[2] The dynamics of pure consciousness can be seen here to be the dynamics by which pure consciousness knows itself and interacts with itself. This first step of diversification shows how unity can appear diversified without ever stepping out of itself, without ever really becoming anything other than One. In this section, we bring into focus aspects of these self-referral dynamics as they are expressed in the *Yoga Vasistha* and in Maharishi Vedic Science.

One theme in the dynamic unfoldment of pure consciousness within itself is the idea that, in knowing itself, in perceiving itself as an object, pure consciousness becomes as if focused on a point within itself, which the *Yoga Vasistha* [16] describes as a *seed of ideation* :

> My son, when, in the infinite consciousness, the consciousness becomes aware of itself as its own object, there is the seed of ideation. (p. 190)

In Maharishi's [11, pp. 171–174] treatment, the dynamics by which the infinitely expanded value of pure consciousness collapses to a point are displayed in the first syllable of Rig Veda: 'AK.' The letter 'A' is uttered with an open voice, making an unrestricted sound, whereas the sound 'K' represents a *stop* in the flow of sound; in this way the transformation of 'A' to 'K' expresses the collapse of the unbounded value of the singularity to a point. Maharishi [11, p. 171] explains that Rig Veda itself describes its own structuring dynamics; according to this description, the fundamental impulses and vibrational modes that arise in the process of pure consciousness

[2] Traditionally, each hymn in Rig Veda specifies the seer who saw (or heard) the hymn—Rishi; the meter of the hymn—Chhandas; and the *deva* or impulse of intelligence that is being expressed in the hymn. (See [4].) The connection to Maharishi's use of these terms should be clear in the case of Rishi; for Chhandas, the meter has to do with the objective *structure* of the hymn, rendering fine impulses of intelligence as concrete form; and Devata is what links the Rishi to the hymn (Chhandas). Samhita is usually translated as "collection"; Maharishi translates Samhita without introducing the notion of division or separation: What must be true of Rishi, Devata, and Chhandas, from the viewpoint of Vedanta, is that, in being collected together (in the form of hymns), they are in reality *one*—just dynamics of pure consciousness. In Maharishi's treatment, therefore, in his translation of the word "Samhita," the unified aspect of "collection" is emphasized.

knowing itself—the very structuring impulses of knowledge itself, of Veda itself—emerge in this collapse of 'A':

> *Richo akshare parame vyoman*
> The hymns of the Veda emerge in the collapse of 'A', the 'kshara' of 'A'.
> - Rig Veda 1.164.39

The dynamics indicated by the syllable 'AK,' representing the collapse of unboundedness to a point, are the dynamics inherent in pure consciousness, in *Atma*.[3] The unfoldment of the Veda and Vedic Literature from the first syllable AK is likewise, therefore, an elaboration of dynamics hidden within *Atma* [8, pp. 500–503].

Maharishi [10, p. 4] also points out that, as any kind of knowledge has *organizing power*—power to yield material consequences and effects—so likewise must the most concentrated knowledge, pure knowledge, have maximum, *infinite*, organizing power. Therefore, he concludes, from the Veda and its infinite organizing power arises all of manifest existence [9, p. 409]. From these observations, he concludes [9]:

> It is clear that Veda and *Vishwa*... [are] the inner content of *Atma*; the structure of knowledge and the structure of the universe is the inner nature of *Atma* (p. 409).

The dynamics of unfoldment from *Atma* to Veda to *Vishwa* have another important characteristic: Each impulse that arises, each expression that emerges, remains connected to its source. Nothing that arises in this process of unfoldment is separate from pure consciousness. The *Yoga Vasistha* [16] explains it in this way:

> Thus the pure consciousness brings into being this diversity with all its names and forms, without ever abandoning its indivisibility, just as you create a world in your dream (p. 638).

Indeed, each step of unfoldment is nothing other than transformations within pure consciousness itself; in the language of the *Yoga Vasistha* [16],

> The ignorant regard this samsara as real. In reality it does not exist at all. What does exist is in fact the truth. But it has no name! (p. 528)

We also read,

> It is only in the state of ignorance that one sees a snake in the rope, not in an enlightened state. Even so, to the enlightened vision, only the infinite consciousness exists, naught else (p. 134).

This phenomenon—that we see undifferentiated pure consciousness as being a diversified manifest material universe—is referred to in Vedanta [14] as *vivarta*. Indeed, from this perspective, *vivarta* is responsible for each step of the apparent diversification of the singularity: from the

[3] *Atma* is understood to be the unbounded, unlimited nature inherent in individual awareness. "*Jiva*, then, is individualized cosmic existence; it is the individual spirit within the body. With its limitations removed, *jiva* is *Atma*, transcendent Being" [13, p.98].

analysis of one—*Atma*—into three (knower, known, process of knowing) and the appearance of the point value of pure consciousness within itself, to the emergence of impulses of self-knowing and the structuring of the Veda, to the appearance of the universe—each step in the process arises by virtue of this principle of *vivarta* [8]:

> Here, Unity (Samhita) appears to be diversity (Rishi, Devata, and Chhandas). This is the absolute eternal principle of *vivarta*, where something appears as something else. The very structure of knowledge (Samhita) has the principle of *vivarta* (Rishi, Devata, Chhandas) within it (p. 589).

Also, we read,

> The principle of *vivarta* makes the unmanifest quality of self-referral consciousness appear as the Veda and Vedic Literature, and makes the Veda and Vedic Literature appear as Vishwa (pp. 377, 589).

According to Maharishi, for the enlightened vision, for the knower of Brahman, the diversification that we see as the manifest universe is appreciated in terms of the one reality, wholeness, pure consciousness. Differences are seen but are as if transparent; what dominates is Unity.

> [In unity consciousness] the boundaries do not disappear . . . only they cease to dominate. Where before they were opaque . . . they are now fully transparent [7, p. 47].

The principle of *vivarta* is also responsible for the apparent reality that the world is different from, separate from, pure consciousness; that things really are separate and not connected to each other or to a fundamental source. This perspective Maharishi calls *praya-aparadh*—mistake of the intellect. Quantum field theorist John Hagelin elaborates on this point in Maharishi Vedic Science [5]:

> Hence the notion of diversity disconnected from unity is a fundamental misconception. This misconception is known as pragya-aparadh or "mistake of the intellect." Pragya-aparadh results when, in the mechanics of creation from the field of consciousness, the intellect loses sight of the essential unity which is the true nature of the self . . . The intellect gets caught up in its own creation, i.e., gets overshadowed by the perception of diversity to the exclusion of the unity which is the actual nature of the self being discriminated. According to Maharishi, this mistake of the intellect is so fundamental to the nature of human experience that it is responsible for all problems and suffering in life (p. 284).

3 Locating the Singularity in the Mathematical Universe

Our goal in this section is to examine to what extent the vision of Advaita can be modeled within the standard foundation of mathematics, ZFC set theory. It is reasonable to attempt to find such a model for several reasons. First, the universe of mathematics resembles, in several important ways, the material universe, in that it contains "everything" and consists of apparently distinct individuals that interact in endless ways. Secondly, as we describe in more detail below, there is a natural analogue to the "singularity" within the standard foundation, namely, the *empty set*—the set having no elements. As we will see, every mathematical object is built up from the empty set, and it is possible to locate within every mathematical object its "origin" in the empty set.

We will show that, while this model, using the empty set as an analogue to pure consciousness, does capture some of the relationships that have been identified as principles and dynamics of pure consciousness, it falls short in a number of important ways. For example, we will not find that this singularity is fundamentally self-interacting or "three-in-one" by nature. And we will find that the differences among mathematical objects are rigid; the unity that we are able to locate, though significant, is sufficiently hidden to prevent this unity from being a dominant characteristic of mathematical objects. Having identified these shortcomings, we will be in a position to significantly improve our model in the next section.

We begin with a brief introduction to modern mathematical foundations. At the beginning of the 20th century, modern mathematics became *one subject*; all the different fields of mathematics were at last seen to be limbs of a single tree of knowledge, the single field of *mathematics* [6]. This recognition can be described in three parts:

(1) *The recognition that every mathematical object can be represented as a set.* For instance, an ordered pair (a, b) can be represented as the set $\{\{a\}, \{a, b\}\}$. A function $f : A \to B$ can be represented as the set of ordered pairs $A_f = \{(x, y) \mid y = f(x)\}$. Whole numbers $0, 1, 2, \ldots$ are represented, respectively, by $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \ldots$, where $\emptyset$ denotes the empty set, and each successive set y in the list is obtained from the previous x by the rule: $y = x \cup \{x\}$.

(2) *The introduction of a standard set of axioms.* These axioms are called the *Zermelo-Fraenkel axioms with the Axiom of Choice*, or ZFC. Every theorem in mathematics can be re-stated in terms of the language of sets and derived directly from the axioms of ZFC. Here are three examples of ZFC axioms:

> *Axiom of Empty Set.* There is a set with no element.
> *Axiom of Pairing.* For any sets X, Y there is a set whose only elements are X, Y (denoted $\{X, Y\}$)
> *Power Set Axiom.* For any set X there is a set, denoted $\mathcal{P}(X)$, whose elements are precisely the subsets of X.

(3) *The universe V.* The universe V consists of all possible mathematical objects, represented as sets. The ZFC axioms "describe" how to build the universe V in stages $V_0, V_1, V_2, \ldots$. The zeroth stage V_0 is defined to be the empty set $\emptyset$. Each subsequent stage is obtained from the previous stage by an application of the *power set operator* $\mathcal{P}$. By definition, for any set A, $\mathcal{P}(A)$ is the set consisting precisely of the subsets of A. So, for example, $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Since the only subset of $\emptyset$ is $\emptyset$ itself,

$$V_1 = \mathcal{P}(V_0) = \mathcal{P}(\emptyset) = \{\emptyset\}.$$

Continuing in this way, the stages of V are defined in the following way:[4]

$$\begin{aligned}
V_0 &= \emptyset \\
V_1 &= \mathcal{P}(V_0) = \{\emptyset\} \\
V_2 &= \mathcal{P}(V_1) = \{\emptyset, \{\emptyset\}\} \\
V_3 &= \mathcal{P}(V_2) = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\} \\
\cdot &= \cdot \\
\cdot &= \cdot \\
\cdot &= \cdot
\end{aligned}$$

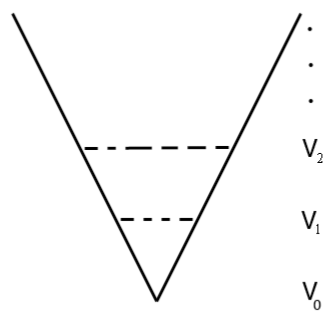


Figure 1. The Universe of Sets

If we actually display the elements of V in its first few stages (Figure 2), we observe an interesting pattern: Every set in the universe is, at its "core," just the empty set; every set is represented

[4]We have not given the full definition here for the sake of simplicity. A precise formulation requires the use of *infinite ordinal numbers*. Roughly speaking, the infinite ordinals extend the whole numbers, permitting enumerations of infinite sets of different sizes. Infinite ordinals are like whole numbers except that some of them do not have immediate predecessors. For example, if we let ω be the first infinite ordinal, the "number" that comes immediately after all the whole numbers, then ω has no immediate predecessor, whereas the number 5, for example, does have an immediate predecessor, namely 4. Ordinals with no immediate predecessor are called *limit ordinals*. The collection of all ordinals, including the usual whole numbers, is denoted ON. The formal definition of the stages of V is given by the following clauses:

$$\begin{aligned}
V_0 &= \emptyset \\
V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\
V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \qquad (\lambda \text{ a limit ordinal}) \\
V &= \bigcup_{\alpha\in \mathrm{ON}} V_\alpha.
\end{aligned}$$

syntactically by a sequence of curly braces and empty set symbols. One could say that every set is just another way of looking at the empty set.

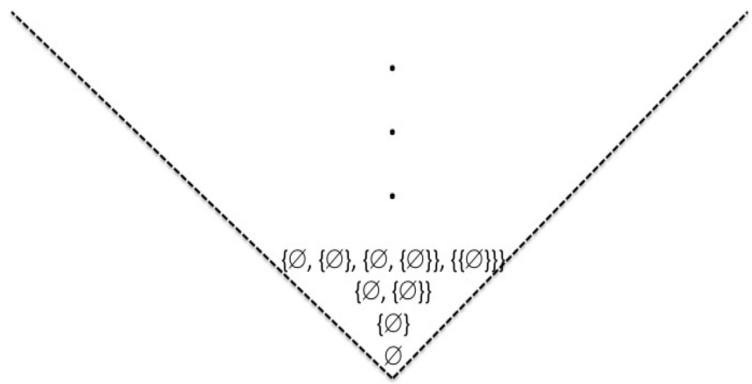


Figure 2. V As Permutations of the Empty Set

If we consider the empty set as a representation of the singularity, pure consciousness, then Figure 2 provides, to some extent at least, a model of the principle that pure consciousness "pervades everything inside and out" [16, p. 513]. Moreover, it is always possible to locate this "transcendental value" of any set in a finite sequence of steps, as we now describe. We first make several observations about sets in the universe [6]:

(1) By virtue of the construction of V, every nonempty set belonging to V is composed of elements that are themselves sets.

(2) Every set X has a *rank*, which signifies the least stage in the construction in which X occurs as a subset. For instance, $\{\emptyset\}$ is a subset of V_1 but not of V_0, so $\text{rank}(\{\emptyset\}) = 1$. Likewise, $\{2\}$ can be shown to be a subset of V_3 but not of V_2, and so $\text{rank}(\{2\}) = 3$.

(3) For every X in the universe and for every $x \in X$, $\text{rank}(x) < \text{rank}(X)$.

(4) *Infinite $\in$-chains* do not exist in V; that is, there do not exist sets $x_0, x_1, x_2, \ldots$ in the universe for which the following holds:

$$\cdots \in x_2 \in x_1 \in x_0.$$

To see this, suppose such an infinite $\in$-chain $\cdots x_2 \in x_1 \in x_0$ does exist; we call x_0 the *starting point* of the chain. Now pick such a chain whose starting point has the least possible rank. We denote this chain $\cdots y_2 \in y_1 \in y_0$. But now $\cdots y_3 \in y_2 \in y_1$ (removing y_0 from the list) is also an infinite $\in$-chain whose starting point y_1 has rank less than $\text{rank}(y_0)$ (since $y_0 \in y_1$), and this contradicts the leastness of y_0.

The property (4) is expressed by saying that every set in V is *well-founded*: No set can be the starting point of an infinite $\in$-chain. It is equivalent to one of the axioms of ZFC, the Axiom of Foundation.

We observe next that for any nonempty set X in the universe, at least one element x of X is an *$\in$-minimal element*; this means that, for any $y \in x$, no element of y belongs to X. So, for example (recalling that whole numbers $0, 1, 2, \ldots$ are represented as the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \ldots$), the set $X = \{1, \{2\}, 3\}$ has two $\in$-minimal elements, 1 and $\{2\}$: The only element of 1 is $\emptyset$, and it does not belong to X; and the only element of $\{2\}$, namely, 2, also does not belong to X. On the other hand, one of the elements of 3, namely, 1, *does* belong to X, and so 3 is *not* $\in$-minimal in X. In general, one may find an $\in$-minimal element of a given nonempty set X by noticing that any element x of X having least rank among all elements of X—that is, $x \in \{y \in X \mid \text{for all } z \in X, \text{rank}(y) \leq \text{rank}(z)\}$—must be $\in$-minimal in X.

Finally, given any nonempty set X in the universe, we show how to arrive at its "core" in finitely many steps: Let $x_0 = X$ and let $x_1 \in x_0$ be an $\in$-minimal element of x_0. If x_1 is empty, we stop the construction; otherwise, obtain an $\in$-minimal element x_2 of x_1. Again, if x_2 is empty, stop; otherwise obtain an $\in$-minimal element x_3 of x_2. This process leads to a possibly finite sequence of sets $\cdots \in x_2 \in x_1 \in x_0$. If the sequence is finite, by construction, the leftmost set must be the empty set, since the construction will terminate only if the next $\in$-minimal element selected is nonempty. But now we observe that the sequence *must* be finite in every case because, by (4), there are no infinite $\in$-chains in the universe. Therefore, we have extracted from X, in finitely many steps, the "core" of X, namely, $\emptyset$.

We have seen that the universe V, being the all-inclusive container of all diversity in mathematics, is a mathematical parallel for the manifest universe. The fact that all sets are built from a singularity—a set devoid of all content; the fact that all sets can in fact be seen as different ways of viewing this singularity $\emptyset$; and the fact that this source of all sets can be located deep within any set in finitely many steps—these facts show that, in certain respects, V, together with the empty set, model the Vedantic view that the manifest universe is permeated by pure consciousness. However, there are aspects of the Vedantic view that have no counterpart in our mathematical model.

One aspect of the Vedantic view of pure consciousness that is missing from our model is the *dynamics* of consciousness itself. For one thing, our mathematical representative of the singularity has no internal characteristics that would lead to the emergence of analogues to "knower," "known," and "process of knowing."

On the other hand, we do find the principle "unboundedness collapses to a point" modeled partially: The dynamics by which $\emptyset$ is transformed into $\{\emptyset\}$ is a transformation from abstract emptiness to the objectification of that emptiness as the set $\{\emptyset\}$ which contains it; in a sense, then, the boundary-less value of emptiness, embodied in $\emptyset$, collapses to emptiness-with-boundary, embodied in $\{\emptyset\}$, as the universe of sets emerges in its stage-by-stage unfoldment. What is missing here is the fact that the point that appears within pure consciousness is in reality no different from pure consciousness itself. Recall from the *Yoga Vasistha* [16],

> When this understanding arises in one, though there is self-awareness, even that ceases for there is no division between the observer and the observed (p. 513).

This reality is obscured in the first transition step from V_0 to V_1 in the construction of V. It is not the case that $\emptyset = \{\emptyset\}$. What we do see modeled here is the emergence of what Maharishi has called *pragya-aparadh*, mistake of the intellect, the notion of diversity disconnected from unity (see Section 2).

This separateness of parts and disconnection of parts from their source then is propagated as stages of V continue to emerge. The fact that differences dominate in the construction of V is obvious when one compares any distinct sets in V. For instance, consider $\{1, 3\}$ and $\{\{4\}\}$. There is no mathematical sense in which these sets can be seen as "fundamentally the same"; and the fact that they have a common source in $\emptyset$, while true, is not obvious, but is rather hidden from view. The result is that the construction of the universe departs from the self-referral unfoldment that occurs within pure consciousness by which everything emerges; what is created instead is a world of *concepts* rather than a progressive unfoldment of *reality*. Again, from the *Yoga Vasistha* [16], we read,

> Hence, O Rama, abandon all forms of division—division in terms of time or of parts of substance —and rest in pure existence. These divisions are conducive to the arising of concepts (p. 319).

Maharishi makes the same point explicitly with regard to the way in which sets and mathematical structures unfold in the mathematical universe:

> [The dynamics of pure consciousness are]...unknown to the mathematics of the conceptual world, unkwown to the mathematics of the world of diversity, which has its basis in the notion of reality—not the reality but the notion about it, the concept of it. Born of a notion means born not of reality but of the idea of reality, the shadow of reality...

To improve our model, one could try to replace the empty set, as a representative of pure consciousness, with a set x having the property that $x = \{x\}$.[5] However, it is easy to see that, because every set in V is well-founded, V contains no such set x: If it did, it would give rise to the following infinite $\in$-chain:

$$\cdots \in x \in x \in x.$$

Despite this theoretical restriction, it is possible to create an expansion of the standard universe of sets in which an element of this kind does indeed exist. Such a universe can be created in a way that is analogous to the way in which the number $i = \sqrt{-1}$ can be added to the usual set of real numbers to produce an expanded field, the field of complex numbers. The number i arises as a solution to a certain equation over the set $\mathbb{R}$ of reals, namely $x^2 + 1 = 0$. In like manner, an expansion of V can be created by introducing a solution to the equation $x = \{x\}$, together with a large number of other such equations that have this sort of self-referential flavor. In this expansion, there is a unique set, denoted Ω, that satisfies this equation: $\Omega = \{\Omega\}$. We will see in the next section that Ω provides a more suitable representative of the singularity in the realm of sets.

Our study of Ω in the next section will begin by introducing a new way of representing sets: via *directed graphs*. To give a flavor of what is to come, consider the set $X = \{0, 1\}$. We can consider the elements of X to be vertices of a directed graph in which, for any vertices x, y, we have an edge $x \to y$ if and only if $y \in x$. With this scheme, one way to depict X as a graph is as follows:

[5] A similar point is made in [8, p. 628].

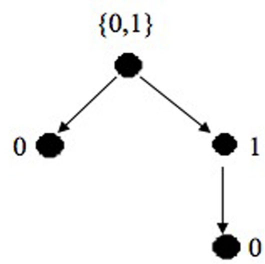


Figure 3. The Set $\{0, 1\}$ As a Directed Graph

With graphs, we have a visual way to examine the structure of sets. Now consider the following two very elementary graphs:

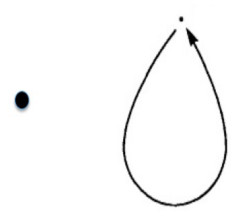


Figure 4. Representing $\emptyset$ and Ω As Directed Graphs

The first graph represents the empty set; the second graph represents Ω. What is missing in the first graph, but present in the second, is a *self-referral relationship* of the set with itself. Whatever self-referral dynamics may be present *within* the empty set, speaking metaphysically, these dynamics are not accessible *mathematically*; we do not find self-referral dynamics within $\emptyset$. However, we do find such dynamics explicitly represented in the graph for Ω: Ω is related to itself—it itself is its only element. As we will show, structuring the universe on the basis of Ω rather than $\emptyset$ leads to a vision of sets that reveals the fundamental unity of all diversity in a very explicit way. The fact that a self-referral loop is at the basis of this new model accords well with an insight expressed by Maharishi [11] regarding the structuring dynamics of pure consciousness:

> The evolution of consciousness into its object-referral expessions, ever maintaining the memory of its self-referral source—ever-evolving structure of consciousness maintaining the memory of its source—progresses in self-referral loops—every step of progress is in terms of a self-referral loop (p. 64).

4 Unfoldment of the Universe Within the Unmanifest

In the previous section, we argued that the universe of sets, together with the empty set, models certain aspects of the unfoldment of pure consciousness, but fails to capture other aspects. One lack in this respect is that, although, in the construction of V, we find a "collapse" from the boundary-less and content-free set $\emptyset$ to a concrete single-element set $\{\emptyset\}$, what is lost is the connection of this expressed value to its source in $\emptyset$; we argued that this transition from $\emptyset$ to $\{\emptyset\}$ is the sprouting of *pragya-aparadh*, the beginning of separation between individuals and their source. We also mentioned that if, in place of $\emptyset$, we could obtain a set x that satisfies the equation $x = \{x\}$, such a set would give expression both to the dynamics of "collapse from unboundedness to a point" and also to the theme of *self-referral* dynamics, by which expressions remain connected to their source in their unfoldment.

We also observed that within pure consciousness emerge three values: knower (Rishi), the known (Chhandas), and process of knowing and transformation (Devata). This emergence of three from unity has no parallel in our analogy in the realm of sets, where sets emerge from the empty set. On the other hand, if there could exist a set x that not only satisfies the equation $x = \{x\}$, but also has the property that x is a transformation from itself to itself—that is, $x : x \to x$—then we would have given mathematical expression to two of these three: To the idea that x is an *object of knowledge* of itself, assuming the role of Chhandas, because of the relation $x = \{x\}$, and also to the notion that x is a *transformation* within itself, assuming the role of Devata, because of the fact that x is the map $x : x \to x$.

The dynamics of pure consciousness could be modeled even more profoundly if, in addition, x satisfies $x = x^x$, where x^x signifies the set of *all possible* transformations from x to itself. Notice that in that case, since $x \in x = x^x$, it follows that $x \in x^x$ and so x is a transformation $x : x \to x$. Therefore, from these two equations, we also give expression to the Chhandas and Devata values. But, because $x = x^x$, we can say more: Now, x includes within itself all possible transformations of itself; in this sense, it is the *witness* of all its internal transformations, and in that role, it is the knower, the Rishi. The connection between "witness" and "Rishi" is discussed in [5] in which Maharishi's approach to this topic is elaborated:

> In the structure of knowledge, Rishi is the knower—the lively, discriminative but unmanifest basis of knowledge, which stands as a witness to the known and process of knowing (p. 255).

The discussion above suggests that, in order to capture more of the dynamics of pure consciousness than we have been able to achieve using the empty set as a model, it would be desirable to find a solution to the following equations:

$$(+) \qquad x^x = x = \{x\}.$$

A solution Ω to (+) would have the following characteristics:

(1) Ω consists precisely of all possible transformations of itself to itself.

(2) One of the transformations of Ω to itself is Ω itself: $\Omega : \Omega \to \Omega$ (since $\Omega \in \Omega^\Omega$).

(3) Ω and its collapsed value $\{\Omega\}$ are one and the same

In this section we show there is a natural (and unique) mathematical solution Ω to the equations (+), using a slight expansion of the usual ZFC universe. We show also how it is possibe to derive all sets in the universe from this one point Ω, and at the same time, how all sets naturally collapse back to Ω. Moreover, the entire mathematical landscape will be seen, in this view, to be nothing but patterns and permutations of this one "reality" Ω; differences between individuals will no longer dominate.

One other consequence of (+) that we mention here is that Ω embodies, in an abstract sense, the very dynamics of nature's functioning, of the functioning of the laws of nature. This can be seen by the fact that Ω is itself equal to the *evaluation map* $eval : \Omega^\Omega \times \Omega \to \Omega$ defined by $eval(f, p) = f(p)$. The evaluation map expresses the way in which natural law is applied to each point in existence to carry it forward to the next stage of its unfoldment. Each impulse of evolution can be represented by a function, a kind of transformation, and "points in the universe" correspond to sets. The evaluation map is, in this sense, the master plan for all evolutionary dynamics, governing the application of each evolutionary impulse f to each point p of the manifest universe, producing a new, "more evolved" value, $f(p)$.

We summarize the parallels we have identified so far between dynamics of consciousness and dynamics of Ω:

Ω As Samhita of Rishi, Devata, and Chhandas, and Administrator of the Cosmos

(1) *Rishi:* $\Omega = \Omega^\Omega$

(2) *Devata:* $\Omega : \Omega \to \Omega$

(3) *Chhandas:* $\Omega = \{\Omega\}$

(4) *Structuring Dynamics of the Universe*:

$$\Omega = eval : \Omega^\Omega \times \Omega \to \Omega : (\Omega, \Omega) \mapsto \Omega.$$

We demonstrate the fact that

$$\Omega = eval$$

in the following Proposition. We take as our background theory the ZFC axioms without the Axiom of Foundation (denoted ZFC^-), together with the assumption that Ω is indeed a solution to the equation $x = \{x\}$. The proof will show that, once we know Ω is a solution to $x = \{x\}$, we can conclude without additional assumptions that it is a solution to $x = x^x$ as well. To draw further conclusions, we will assume somewhat more later on.

Proposition 1. (ZFC^-) *Suppose Ω is a solution to the equation $x = \{x\}$. Then the following statements hold true:*

(A) $\Omega = (\Omega, \Omega)$

(B) $\Omega = \Omega \times \Omega = \Omega^\Omega = \Omega^\Omega \times \Omega.$

(C) $\Omega \in \Omega^\Omega$, *so we may write* $\Omega : \Omega \to \Omega.$

(D) For all $x \in \Omega$, $\Omega(x) = \Omega$ *(using the representation of* Ω *in (C)).*

(E) $\Omega = eval$.

Proof of (A). We have the following derivation (using the fact that, by definition, for any sets x, y, $(x, y) = \{\{x\}, \{x, y\}\}$).

$$\begin{aligned}(\Omega,\Omega) &= \{\{\Omega\},\{\Omega,\Omega\}\}\\ &= \{\{\Omega\},\{\Omega\}\}\\ &= \{\{\Omega\}\}\\ &= \{\Omega\}\\ &= \Omega.\end{aligned}$$

Proof of (B). For the first equality, we have:

$$\begin{aligned}\Omega\times\Omega &= \{(x,y)\mid x\in\Omega \text{ and } y\in\Omega\}\\ &= \{(\Omega,\Omega)\}\\ &= \{\Omega\} \quad \text{by (A)}\\ &= \Omega\end{aligned}$$

To show $\Omega = \Omega^\Omega$, we compute as follows: Notice first that there is only one function $f : \{\Omega\} \to \{\Omega\}$, namely, the function f defined by $f(\Omega) = \Omega$. We denote this function f_Ω. Note that as a set of ordered pairs, $f_\Omega = \{(\Omega, \Omega)\}$. Therefore:

$$\begin{aligned}\Omega^\Omega &= \{f\mid f:\Omega\to\Omega\}\\ &= \{f\mid f:\{\Omega\}\to\{\Omega\}\}\\ &= \{f_\Omega\}\\ &= \{\{(\Omega,\Omega)\}\}\\ &= \{\{\Omega\}\} \quad \text{by (A)}\\ &= \{\Omega\}\\ &= \Omega.\end{aligned}$$

Finally, the fact that $\Omega \times \Omega = \Omega^\Omega \times \Omega$ follows from $\Omega = \Omega^\Omega$.

Proof of (C). This follows from the fact that $\Omega = \Omega^\Omega$, shown in (B), and the fact that $\Omega \in \Omega$, which follows from the fact that $\Omega = \{\Omega\}$.

Proof of (D). Since $\Omega = \{\Omega\}$, the only element of Ω is Ω. Therefore, it suffices to show that $\Omega(\Omega) = \Omega$. But this is equivalent to the assertion that $(\Omega, \Omega) \in \Omega$, and this follows from (A) and the fact that $\Omega \in \Omega$.

Proof of (E). The set-theoretic definition of *eval* is:

$$eval = \{(x, y, z) \in \Omega^\Omega \times \Omega \times \Omega \mid x(y) = z\}.$$

For any $(x,y,z) \in \mathit{eval}$, the fact that $(x,y) \in \Omega^\Omega \times \Omega$ implies that $(x,y) = (\Omega,\Omega)$; and the fact that $z \in \Omega$ implies $z = \Omega$. Therefore, the only element of $\Omega^\Omega \times \Omega \times \Omega$ is (Ω,Ω,Ω), and this element (x,y,z) does satisfy $x(y) = z$ (as shown in (D)). Therefore,

$$\mathit{eval} = \{(\Omega,\Omega,\Omega)\} = \{(\Omega,\Omega)\} = \{\Omega\} = \Omega,$$

as required. ∎

Proposition 1 has been established in the theory ZFC$^-$ together with the assumption that there is a solution to the equation $x = \{x\}$—more formally, from the theory ZFC$^- + \exists x\, x = \{x\}$—but we have not yet demonstrated that this theory is consistent. We handle this issue by showing that existence of a solution to $x = \{x\}$ is provable from the theory ZFC$^-$ + AFA, where AFA stands for the *Anti-Foundation Axiom*, a well-known alternative to the Axiom of Foundation, due to Forti and Honsell [3] and popularized by P. Aczel [1]. It is known that if ZFC is consistent, so is the theory ZFC$^-$ + AFA. Later in this article, we will discuss some important points about the proof of this fact.

We give a quick introduction to this axiom AFA so that we can use it to gain additional insights into the model of pure consciousness that we have proposed. We begin with several definitions. A (directed) graph G is a pair (M,E) consisting of a set M of vertices and a set E of edges. Edges are represented by pairs of vertices. So, if u, v are vertices in a graph G and G has an edge from u to v, this edge is denoted (u,v). We also write $u \to v$ to indicate that $(u,v) \in E$.

A *pointed* graph is a graph with a designated vertex, called its *point*. When we draw pointed graphs, the point of the graph is the top vertex (whenever that makes sense) and descendants of the point evolve downward. (See examples below.)

A pointed graph $(G,p) = (M,E,p)$ is *accessible* if for all $v \in M$ there is a path from p to v in G. Accessible pointed graphs are referred to by the acronym *apg*. If, for all v, there is exactly one path from p to v, then G is called a *tree*. A graph is *well-founded* if it has no infinite path.

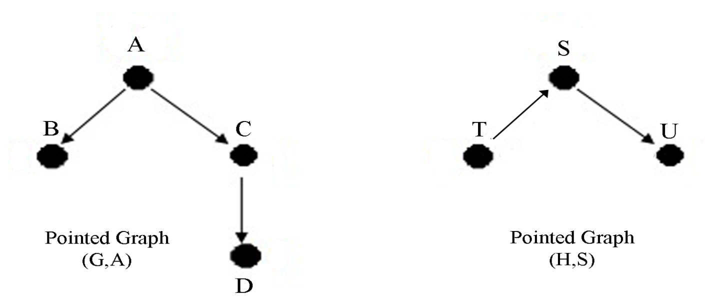


Figure 5. Examples of Pointed Graphs

In Figure 5, the left graph (G,A) has point A; it is well-founded and accessible. The right graph (H,S) has point S, but since there is no path from the point S to the vertex T, (H,S) is not accessible. Notice that if we change the point of H to be T, (H,T) is now an accessible pointed graph.

A *decoration* of a graph is an assignment of a set to each vertex of the graph in such a way that the elements of a set assigned to a vertex are always assigned to the children of that vertex. In symbols, a decoration of G is a map $d : G \to V$ such that, for all vertices v, w of G,

$$v \to w \text{ if and only if } d(w) \in d(v).$$

A *picture* of a set X is an apg that has a decoration in which X is assigned to the point.

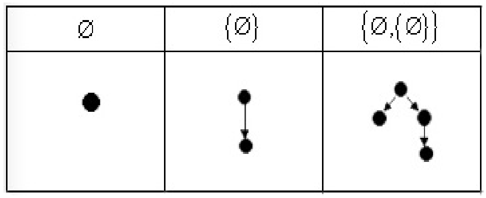


Figure 6. Pictures of Well-founded Sets

Figure 6 exhibits examples of well-founded apgs; each apg shown is a picture of the set that labels it. The first apg was introduced at the end of the last section. This graph has just one vertex and no edges; this means that the set it represents can have no elements. Accordingly, the unique set that is represented by this apg is the empty set $\emptyset$. The designated point for the second apg has just one child, which, in turn, has no children; accordingly, the set it represents is $\{\emptyset\}$, the set whose only element is $\emptyset$. Similarly, the third apg shown represents the set having as children the empty set and the set whose only element is the empty set, namely, $\{\emptyset, \{\emptyset\}\}$.

We state some important facts about representing the sets in a ZFC universe with graphs. The following theorem does not require AFA; it follows from ZFC:

Theorem 2.

(A) Every well-founded graph has a unique decoration.

(B) Every well-founded apg is a picture of a unique set.

(C) Every well-founded set has a picture.

The examples of Figure 6 illustrate Theorem 2(A); in these simple cases, it is easy to see that there is only one way to decorate the vertices of the given apgs with sets. A reasonable generalization of Theorem 2(A) to all possible apgs is the Anti-Foundation Axiom:

The Anti-Foundation Axiom (AFA). *Every graph has a unique decoration.*

An immediate consequence is the following:

Proposition 3 (Uniqueness Theorem). *Every apg is a picture of a unique set.*

A consequence of the Uniqueness Theorem is that the following apg uniquely determines a set:

Figure 7. A Single-Loop Graph

The unique way to decorate this graph is with a set whose only element is itself; as we proved earlier, such sets cannot exist in any universe built from the standard ZFC axioms because existence of such sets contradicts the Axiom of Foundation. Since AFA contradicts the Axiom of Foundation, in order to work with AFA in a consistent way, we must remove the Axiom of Foundation from our basic set of axioms. In the theory $\mathrm{ZFC}^- + \mathrm{AFA}$, the Uniqueness Theorem does indeed hold true.

The unique set pictured by the single-loop graph of Figure 7 is usually denoted Ω.

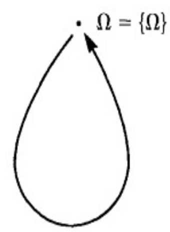


Figure 8. Single-Loop Graph with Unique Decoration Ω

We observe that, with regard to Figure 7, AFA tells us two things: First, that the single-loop graph is a picture for *some* set; and second, that there is *only one* set for which this graph is a picture. This latter point is as important as the first. Without AFA, even if we assume that the single-loop graph of Figure 7 can be decorated with a set $X = \{X\}$, there is no guarantee that it is unique (there could be X and Y such that $X = \{X\}$ and $Y = \{Y\}$ and $X \neq Y$).

In our work in this section so far, we have used the symbol Ω in two different ways: as a solution to $x = \{x\}$ and also as the unique decoration for the single-loop graph. We show now that this ambiguous usage is justified.

Theorem 4. (ZFC^-+AFA). *Let Ω be the unique decoration of the loop graph shown in Figure 7. Then Ω is the unique solution to the equations* (+)*; that is, Ω is the unique set for which $\Omega = \{\Omega\}$ and $\Omega = \Omega^\Omega$.*

Proof. The fact that $\Omega = \{\Omega\}$ follows from the structure of the apg for Ω: Certainly the designated point of the single-loop graph is a child of itself, so it follows $\Omega \in \Omega$. But it is also clear that the designated point is its *only* child. Therefore, Ω is the *only* element of Ω, and so $\Omega = \{\Omega\}$. By Proposition 1, $\Omega = \Omega^\Omega$ as well.

To see that Ω is the *unique* solution to the equations (+), note that any solution to these equations—even to the single equation $x = \{x\}$—is a decoration of the single-loop apg pictured above. By AFA, there is only one such decoration. Therefore, there is only one solution to (+). ■

Theorem 4, together with the fact that ZFC$^-$+AFA is consistent whenever ZFC is consistent, shows that ZFC$^-$ is consistent with the statement $\exists x\, x = \{x\}$, and so our earlier work in this section is fully legitimized.

Before embarking on a discussion about how all real sets can be seen to arise from, and return to, the single non-well-founded set Ω, we spend some time exploring how a ZFC$^-$ + AFA universe is built.

Building a ZFC$^-$ + AFA Universe

Let V denote the usual universe of sets—a model of ZFC—as discussed earlier. One can build, within V, a model $\hat{V}$ of ZFC$^-$ + AFA. Moreover, any such model will have a well-founded part WF $=$ WF$_{\hat{V}}$ (consisting of all the well-founded sets in $\hat{V}$) that is isomorphic to the original ZFC model V: WF $\cong V$.

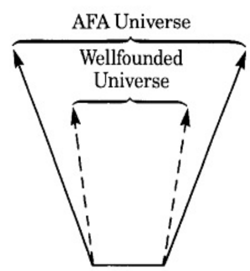


Figure 9. An AFA Universe Is an Expansion of the Well-founded Universe

In this way, we obtain the intuitive picture that a ZFC$^-$ + AFA universe is an expansion of the standard cumulative hierarchy of well-founded sets—an expansion consisting of well-founded sets together with *ideal elements*, which in the present context are the non-well-founded sets, like Ω.

We take a moment to describe, at a high level, how a universe for ZFC$^-$ + AFA can be constructed. We begin with the usual well-founded universe V of ZFC. Roughly speaking, we wish to think of the sets of our new universe as being precisely the apgs that live in V. This isn't quite right though because different (nonisomorphic) apgs can picture the same set, even in the well-founded case. For example, each of the apgs in Figure 10 (below) is a picture of the (well-founded) set $\{0, 1\}$, but the underlying graphs are nonisomorphic (since, for example, their vertex sets have different sizes).

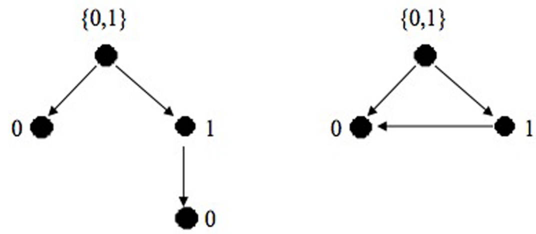
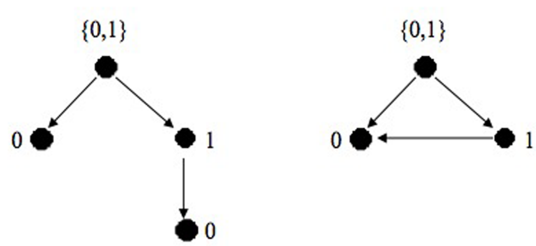


Figure 10. Nonisomorphic Pictures of the Same Set

This situation exactly parallels the situation one faces in attempting a construction of the reals from the rationals—a first try is to declare that a real is a Cauchy sequence of rationals. Just as many apgs may represent the same set, so likewise in the context of constructing the real line, we face the fact that many Cauchy sequences converge to the same real. The solution in the latter case is to form a quotient by an appropriate equivalence relation. In constructing the real line, one would like to declare two Cauchy sequences to be equivalent if they converge to the same real, but, since the reals have not yet been constructed, this approach cannot be used (though it serves to guide the intuition about it). Likewise, we would like to declare that two apgs are equivalent if they picture the same set, but since we have not yet constructed all the sets of our new universe, this statement of the equivalence relation is not formally correct. To capture this idea without assuming existence of the non-well-founded sets we are trying to build, the necessary equivalence relation, called *bisimilarity*, is formulated in another way. Ultimately, the universe that we build will consist of all the equivalence classes of apgs under the bisimilarity relation.

Since the definition of the bisimilarity equivalence relation is somewhat technical, we save a discussion of those details for the Appendix. For our discussion here, it will be enough to rely on the guiding intuition that two apgs are equivalent if they picture the same set, and we consider a couple of examples.

When apgs happen to be well-founded, we already know which sets they picture, because we are starting from the universe V of well-founded sets. As we observed before (see Figure 10), the following two apgs picture the same set, namely, $\{0, 1\}$, and so they are bisimilar:



For a second example, consider the following non-wellfounded apgs, each of which pictures the same non-wellfounded set, Ω.

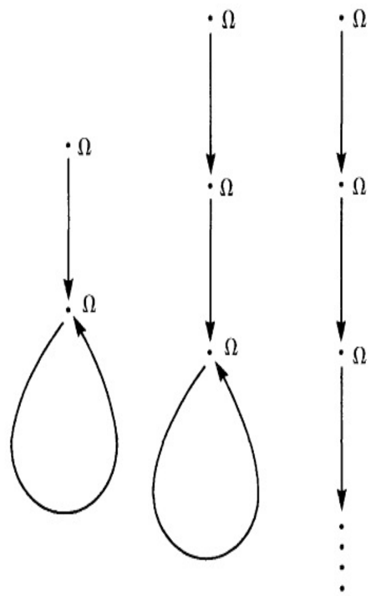


Figure 11. Pictures of Ω

In Figure 11, notice that each of the apgs shown *can be* decorated with Ω. But then by the uniqueness part of AFA, Ω is the *only* set that can decorate these apgs. Here again these apgs are bisimilar.

Starting from the standard ZFC universe V, then, we build a subclass $\hat{V}$ consisting of these equivalence classes of apgs. For $\hat{V}$ to be a valid "universe of sets," it needs to have its own version of the membership relation. We describe this in an intuitive way here and give more details in the Appendix. The following example will illustrate the idea:

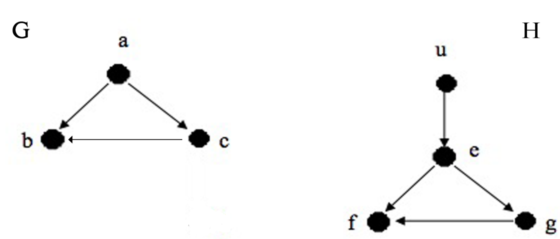


Figure 12. Membership in $\hat{V}$.

In Figure 12, the (equivalence class of the) apg (G, a) is to be thought of as a "member" of the (equivalence class of the) apg (H, u) because, if we look at the sub-apg of H having point e—and we denote this subgraph He—then the apgs (He, e) and (G, a) are essentially the same (in this case, they are actually isomorphic); in addition, e itself is a child of the point u of H.

This example is typical: In general, (the equivalence class of) an apg (G, p) is a "member" of the (equivalence class of the) graph (H, q) if there is a child r of q—that is, we have $q \to r$—such that the sub-apg (Hr, r) is essentially the same as the apg (G, p) (here, "essentially the same as" means "bisimilar to").

The Ideal Elements of $\hat{V}$ As Solutions to Equations

Earlier in this article, we mentioned that a ZFC$^-$ + AFA universe could be viewed as an expansion of the usual class V of well-founded sets by adding "ideal" elements, like Ω, and that the procedure for forming such an expansion is analogous to adjoining the ideal, pure imaginary element i to the field $\mathbb{R}$ to obtain the complex field $\mathbf{C} = \mathbb{R}(i)$. In this subsection, we give an overview of how this can be done.

We recall that the pure imaginary number i arises as a solution to the equation $x^2+1 = 0$ over $\mathbb{R}$. We show that one may also view the expansion from V to $\hat{V}$ as arising from the introduction of "ideal" solutions to—in this case—*classes* of equations. One such equation, as we have seen, is $x = \{x\}$. Another is $x = (0, x)$. An example of a small system of such equations is:

$$\begin{aligned} x &= \{0, x, y\} \\ y &= \{\{x\}\} \end{aligned}$$

We can give a precise formulation of the relevant equations as follows: By analogy with the expansion from $\mathbb{R}$ to $\mathbf{C}$, we need to introduce indeterminate elements. To take the step from $\mathbb{R}$ to $\mathbf{C}$, we first need to obtain the domain $\mathbb{R}[x]$ of polynomials in the indeterminate x; then $x^2 + 1 \in \mathbb{R}[x]$ is an expression for which we seek a root; moreover, any root will be an expression that does not implicitly contain the indeterminate x. Likewise, we will expand V with a *class* X of indeterminates, one for each set in V: $X = \{x_a : a \in V\}$.[6] And the "polynomials" we obtain—which we will call *complex sets*—are sets built up from other sets together with elements of X. As a simple example, consider the complex set $A(x_a, x_b) = \{0, \{1, x_a\}, (x_b, 2)\}$, where $x_a, x_b \in X$. One of the equations that we wish to be able to solve is

$$x_a = A(x_a, x_b).$$

A solution to such an equation will be a set whose build-up does not contain any of the elements of X; such a set is called a *pure set*.

The *Solution Lemma*, which is equivalent in ZFC$^-$ to AFA, gives a precise statement of classes of equations that we wish to consider, and asserts that any such system of equations always has a unique solution.

Theorem 5 (Solution Lemma). *Suppose A_x is a complex set, for each $x \in X$. Then the system of equations*

$$(**) \qquad x = A_x$$

[6] More formally, we are re-building the universe starting with atoms or *urelements* at the 0th stage. Urelements are sets, different from the empty set, that have no elements. See [2] for a full treatment.

has a unique solution; that is, there is a unique family $\{b_x \mid x \in X\}$ *of pure sets such that for each* $x \in X$*, and each indeterminate* x_a *occurring in* A_x*, if we replace in* A_x *each such occurrence of* x_a *with* b_{x_a}*, and if we denote the resulting set* B_x*, then, for each* $x \in X$*,*

$$b_x = B_x.$$

Therefore, a $\mathrm{ZFC}^- + \mathrm{AFA}$ universe is obtained as a universe that includes the well-founded sets and provides unique solutions to all equations of the form ($**$).

We now return to our discussion Ω as a model of pure consciousness. We consider next how all sets can be seen to arise from and return to Ω, in such a way that unity dominates.

Ω As the Only Reality

So far in this article we have seen how Ω in a $\mathrm{ZFC}^- + \mathrm{AFA}$ universe, as a model of pure consciousness, captures the dynamics of pure consciousness in ways that the empty set, in a ZFC universe, cannot. We have seen that dynamics of Ω originate with its "collapse" to a point—indicated by the fact that it satisfies the equation $x = \{x\}$—paralleling the Vedantic perspective, elaborated by Maharishi, that the dynamics of unfoldment of consciousness within itself, into the Veda and the universe, begin with the collapse of 'A' to 'K,' of unboundedness of *Atma* to a point within *Atma*. Although the empty set also exhibits the dynamics of collapse from $\emptyset$ to $\{\emptyset\}$, in this case, this "collapse" results in a separation of the expressed value from its source—we have $\emptyset \neq \{\emptyset\}$ in contrast to $\Omega = \{\Omega\}$—and in that sense represents the sprouting instead of *pragya-aparadh*. We have also seen how the appearance of three from one, displayed in the dynamics of pure consciousness as the emergence of Rishi, Devata, and Chhandas, has a parallel in the dynamics of Ω, as evidenced by the facts that, respectively, $\Omega = \Omega^\Omega$, $\Omega = \Omega : \Omega \to \Omega$, and $\Omega = \{\Omega\}$.

The Vedantic insight we wish to explore in this final section is the view that everything is "nothing but" pure consciounsness; that everything is nothing but the dynamics of pure consciousness. The reality is one; differences arise as a point of view, a way of looking at or conceiving, this one reality. The reality of the all-pervasiveness of pure consciousness is expressed this way in the *Yoga Vasistha* [16]:

> What appears as the world to the conditioned mind is seen by the unconditioned mind as Brahman (p. 506).

Also:

> When pure consciousness alone exists, pervading everything inside and out, how does the notion of division arise, and where? (p. 513).

And the answer to this rhetorical question given in Maharishi's [8] Vedic commentaries is that division arises by virtue of the principle *vivarta*; that is, division is only an appearance:

> Here, unity [in the Samhita of Rishi, Devata, and Chhandas] appears to be diversity (Rishi, Devata, and Chhandas). This is the absolute eternal principle of *vivarta*, where something appears as something else (p. 589).

Once again, using the empty set within a ZFC universe as a model of pure consciousness falls short as we seek to model the "nothing but pure consciousness" principle. Even though the empty set is at the core of every set, differences among sets in the universe dominate. The entire enterprise of modern mathematics relies on the fact that sets that do not have precisely the same elements are *different sets*. In no mathematical sense can it be said, for example, that $\{1, 2\}$ and $\{0, 4, 9\}$ are "the same." The fact that these sets have a common source in the empty set is a *hidden reality*, not a "living" reality.

The reason that this fundamental unity is not seen "on the surface" of mathematics is, we suggest, because of the fact that even the dynamics of $\emptyset$ are hidden from view, in contrast with Ω whose internal dynamics are seen explicitly in its representation as a graph. In fact, as we observed at the end of the last section, the difference between the empty set and Ω is captured nicely in contrasting their respective apgs:

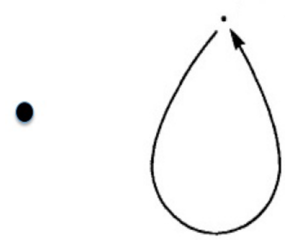


Figure 13. Contrasting the Graphs of $\emptyset$ and Ω.

In the single-loop graph, we see a picture of a self-relationship, an inherent dynamism between the vertex and itself. One way to view the single-loop graph that pictures Ω is as a kind of *refinement* of the single-vertex graph that pictures $\emptyset$, in the sense that the self-referral dynamics that one may imagine are "hidden" deep within the empty set have been brought into plain view. The edge from the single vertex to itself that is added to the single-vertex graph can be seen as symbolic of "re-connecting" the point to itself.

This viewpoint provides a new way of viewing all sets in the standard ZFC universe V. In V, all sets are seen as distinct and unrelated, even though the "core" of every set is always simply the empty set. We can use our insights about the relationship between $\emptyset$ and Ω to explicitly *re-connect* every set to its "source" in the following way.

First, let us observe that every well-founded set can be pictured by an apg that has exactly one childless vertex, and in every case, this childless vertex is decorated with the empty set. This is true because, for every set A, by the Axiom of Foundation (as we have observed), every maximal $\in$-chain starting at A is finite and terminates in $\emptyset$; in particular, there is a natural number n and sets $x_0, x_1, \ldots, x_n$ such that:

$$\emptyset = x_0 \in x_1 \in \ldots \in x_{n-1} \in x_n = A.$$

Therefore, as in the different apg pictures of the set $\{0, 1\}$ discussed earlier (shown below in Figure 14), all edges pointing to $\emptyset$ can be directed to a single vertex decorated with $\emptyset$, and this vertex is necessarily childless (since $\emptyset$ has no elements). We shall call any such picture of a well-founded set a *canonical picture* of the set.

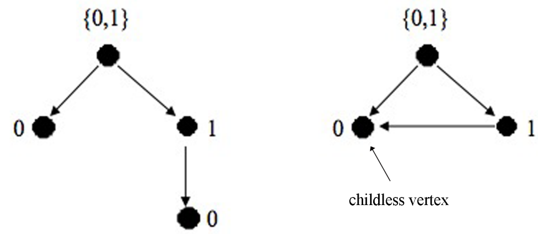


Figure 14. Canonical and Non-canonical Pictures of the Same Set

Then, to reconnect any well-founded set A to its source, we can simply add one edge to a canonical picture from the vertex decorated with $\emptyset$ to the designated point, decorated with A; that is (if for the moment we name vertices by their decorations), we add to the graph the edge $\emptyset \to A$. We will call the edge that is added in this way the *reconnecting edge*; notice that there is, for any canonical picture of a well-founded set, *just one* reconnecting edge. Here is an example:

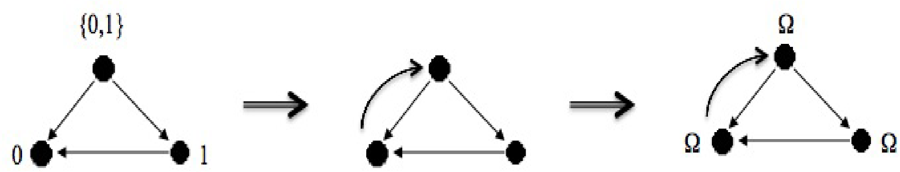


Figure 15. Adding a Reconnecting Edge to a Canonical APG

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In Figure 15, we begin with a canonical picture of the well-founded set $\{0, 1\}$; recall that $0 = \emptyset$ and $1 = \{\emptyset\}$, so the graph on the left is also a canonical picture of $\{\emptyset, \{\emptyset\}\}$. Its unique childless vertex is located in the lower left of the picture, labelled by $0 = \emptyset$. The middle graph shows what happens when we reconnect this vertex to its source by adding an edge from 0 to $\{0, 1\}$. The decoration will necessarily change because of the addition of the reconnecting edge (so no decoration is shown in the middle graph). Finally, in the third apg, we attempt to decorate the graph in the middle with some set. Certainly Ω can be used, as one may easily verify. But now because every apg has a *unique* decoration, the *only* way to decorate this middle graph is by placing Ω at every vertex.

What the example shows is that, by adding the reconnecting edge to a canonical apg of a well-founded set, everything about the set, including its elements and internal relationships, reveal themselves to be nothing but Ω.

We can look at this example in a somewhat different way. As we observed above, the graph on the left in Figure 15 pictures the set $\{\emptyset, \{\emptyset\}\}$. When we add the reconnecting edge to the apg, the effect is the same as substituting Ω for each occurrence of $\emptyset$, and so the apg on the right in

Figure 15 is in fact $\{\Omega, \{\Omega\}\}$, which can easily be seen to equal Ω. The point here is that the set obtained by adding the reconnecting edge is built up in exactly the same way from Ω as the original set was built from $\emptyset$. But in the Ω case, though the dynamics are the same, all that is actually ever built in the process is Ω.

This example provides a good analogy for the Vedantic insight that all the transformational dynamics of pure consciousness are self-referral dynamics in which pure consciousness remains pure consciousness:

> Thus the pure consciousness brings into being this diversity with all its names and forms, without ever abandoning its indivisibility.... [16, p. 638].

In our example, it was clear that adding the reconnecting edge produces an apg that pictures Ω. However, whenever we start with a canonical apg, when we join the only childless vertex to the distinguished point, the result is an apg in which every vertex has a child. By the following Theorem, it follows that adding the reconnecting edge always produces an apg that pictures Ω.

Theorem 6 (The Ω-Theorem). *An apg is a picture of Ω if and only if every vertex of the apg has a child.*

By the Ω-Theorem, the "fundamental reality" underlying each well-founded set can be discovered by adding a single reconnecting edge to its canonical picture, from the vertex labelled with 0 (or $\emptyset$) to the designated point of the apg.

This insight about the structure of sets captures in a mathematical way the dawning of the vision of Vedanta, in which every object is recognized to be nothing but pure consciousness.

Even certain experiential aspects of this awakening are modeled here: In Maharishi's [7] treatment, the full awakening to Brahman occurs first in the blossoming of experience and then is completed with one final stroke of knowledge. That final stroke of knowledge comes from the imparting of a *mahavakya* when the student's experience is "ripe." Well-known examples of mahavakyas from the Vedic literature include *tat tvam asi* (That thou art)[7] and *sarvam khalvidam brahma* (all this is Brahman).[8]

> Brahman becomes an all-time reality through the mahavakyas. See, through the experience everything is recognized in terms of the Self, but that experience in terms of the Self becomes significant through the teaching, because through the teaching it comes onto the level of understanding. Experience is one thing, understanding is another, and only when it comes onto the level of understanding does it become established everywhere. Then its all-pervadingness becomes a living reality . . . when the experience is *ripe* and the teacher says "tat tvam asi—really you are That," it's a *revelation*. He may have known "tat tvam asi" before, but that "tat tvam asi" did not pinpoint that experience (pp. 316, 318).

In our model based on Ω, we see that the "awakening" to the reality that every set is nothing but Ω arises from a "final stroke," represented by a single reconnecting edge. This one small change

[7] *Chhandogya Upanishad,* 6.11
[8] *Chhandogya Upanishad,* 3.14.1.

in the viewpoint regarding any given set reveals that the set's true nature, including the very way it is built up from its origin, is nothing but Ω.

We make one further observation about our model: One can apply the procedure of introducing reconnecting edges to every stage V_α of the universe. In Figure 16, this is illustrated with a canonical apg for V_3 and the result of adding the reconnecting edge. The example shows that the "reality" of V_3, accessed when the reconnecting edge is introduced is, as described above, simply Ω. Likewise, each V_α is transformed into Ω by introducing a reconnecting edge. Forming the union of all these refined stages, as one does to build the usual universe V, results in a union of many copies of Ω; in the end we just end up with Ω.

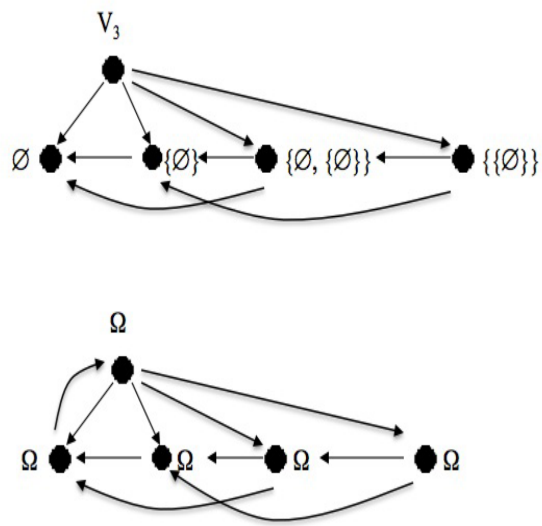

Figure 16. V_3 Before and After Adding the Reconnecting Edge

The same thing happens if we picture the universe V itself with one enormous canonical apg: When we introduce just one reconnecting edge, the entire universe is seen to be nothing other than Ω, and yet the dynamics of set formation can still be seen in the apg after that edge is inserted. Here, with one "final stroke of knowledge," the detailed inner dynamics of Ω are fully displayed, and yet all transformations are seen to be nothing other than dynamics of Ω. This example provides a parallel for the reality embodied in the mahavakya *sarvam khalvidam brahma* (all this is Brahman).

We have used our examples in Figures 15 and 16 to illustrate how Ω can be appreciated as the pervasive reality of any well-founded set. These examples also show how all well-founded sets *originate from* Ω. This becomes apparent as we observe in these examples that by *removing* the reconnecting edge, the original apg, and hence the original set, comes back into view. For example, removing the reconnecting edge in Figure 15, Ω is transformed back to the set $\{0, 1\}$.

Generalizing a bit, let us define U to be the set of all apgs (G, p) that picture Ω and in which there is an edge $e : u \to p$, removing which produces a well-founded apg with point p. U consists of a vast class of apgs, all of which are pictures of Ω. At the same time, one can say that every well-founded set "arises from" the act of removing a single edge from some apg in U, and thereby breaking the apg's (and the set's) connection to its source.

As we discussed in the first section, from the Vedantic perspective, the viewpoint that takes an expressed value to be disconnected from its source is the nature of *pragya-aparadh*, and arises because of the principle of *vivarta*. We see the mechanics of the emergence of *pragya-aparadh* in this mathematical model. First, when the perfectly balanced state in which the large apg for V, plus reconnecting edge, which pictures Ω and all its internal dynamics, undergoes the *transition* to the apg for V *without* its reconnecting edge, we see the actuality of diversified values of sets. This is precisely the nature of *vivarta* according to Maharishi [8]: "The actuality of *vivarta* is realized in the transition state, where ... unity appears to be diversity" (p. 589). At the same time, the diversity that emerges as the universe V results in a loss of unity; the connection of each set in V to its source (accessible by introducing a reconnecting edge) is obscured.

5 Conclusion

With the aim of clarifying the vision of Vedanta, we have sought in this article a foundations-based mathematical model that could give adequate expression to the internal dynamics of pure consciousness and the relationship of those dynamics to the manifest field of existence, the universe itself. We showed that the standard ZFC universe provides a reasonable analogy for manifest existence and the empty set naturally plays the role of pure consciousness. Because the empty set is, like pure consciousness itself, devoid of individual content, and because it is (as we showed) at the core of every individual in the universe, we suggested that, at least in these respects, the mathematical universe together with the empty set provides a reasonable model for manifest existence and its relationship to pure consciousness.

On closer examination, though, we found that the empty set fails to exhibit characteristics and dynamics of pure consciousness that are key elements in the Vedantic vision. These elements can be summarized as follows:

(1) *Rishi, Devata, Chhandas.* Pure consciousness, being conscious, automatically assumes the roles of knower, known, and process of knowing, unfolding a three-in-one structure within itself

(2) *Akshara: Collapse of infinity to a point.* In the process of knowing itself, it locates a point within itself; the dynamics that follow, indicated by a verse within the Rig Veda itself, involve a "collapse" of unboundedness to the point. From this collapse arises a sequential unfoldment of the structuring impulses of the Veda and *Vishwa*. In these dynamics, unboundedness and point are nothing other than pure consciousness in different modes.

(3) *Unity consciousness.* The reality of the manifest universe is that it is nothing other than the internal dynamics of pure consciousness; a material universe is only appearance, whose reality is pure consciousness alone. The appearance of pure consciousness as the universe is due to the principle of *vivarta*. The state of consciousness that takes this appearance to be separate from, distinct from, pure consciousness is *pragya-aparadh*, the mistake of the intellect. Observed differences and distinctions are, when seen from the vision of unity, "transparent"; what dominates the enlightened vision is the unity among all objects of perception, and connectedness to their source as pure consciousness.

We discovered that if there could exist some set x that satisfies the equation $x = \{x\}$, such a set could be a good model for (1). This is because, as we showed, once $x = \{x\}$ is known to hold, it also follows that $x = x^x$ as well, and from these we can conclude that x is Rishi, being equal to the totality of all its transformations (namely, x^x); x is Devata, since $x \in x^x$, whence $x : x \to x$; and x is Chhandas, being equal to its own objectification as $\{x\}$. At the same time, being just x in every case, it is the samhita (unity) of Rishi, Devata, and Chhandas.

For (2), we have seen that the transformational dynamics, which begin with the emergence of three from one, arise from the very existence of an x for which $x = \{x\}$. This equation indicates, as was mentioned, that x is equal to its own objectification as a point. From this relationship emerges the division into three, and as we argued before, the further structuring impulses of knowledge and natural law, as embodied in the *evaluation map*: $x = x : x^x \times x \to x$. Such an x plays the role of each structuring impulse of nature, remaining all the while as nothing but x itself.

We observed that, using the usual ZFC axioms for set theory, there can be no such set x since such a set cannot be well-founded. We suggested considering an expansion of the standard well-founded universe V by adjoining to V the ideal element Ω, which does satisfy the equation $x = \{x\}$ (together with many other such non-wellfounded elements). We accomplished this step by replacing the usual ZFC Axiom of Foundation with the Anti-Foundation Axiom, or AFA. Any universe of $\mathrm{ZFC}^- + \mathrm{AFA}$ is, essentially, an expansion of the usual well-founded universe to include Ω and other ideal elements. Indeed, the well-founded part of any such expansion is isomorphic to the well-founded universe we started with.

The AFA guarantees that any graph (apg) has a unique decoration; Ω is the unique decoration for the single-loop graph (Figure 7). This picture of Ω gives a visual illustration of its own self-referral dynamics; it is because of these dynamics that Ω can provide a model of (1) and (2).

For (3), working in a $\mathrm{ZFC}^- + \mathrm{AFA}$ universe, we showed how every well-founded set's essential nature as Ω can be unveiled by introducing a single reconnecting edge to a canonical picture of the set. The reconnecting edge serves to "awaken" every vertex of the apg to its underlying reality as Ω. The unique decoration for this reconnected apg is a graph each of whose vertices is decorated with Ω itself, including the distinguished point. Adding the reconnecting edge is analogous to the delivery from the master of a *mahavakya*, providing the final impulse of knowledge—for the student who is "ripe" for enlightenment—needed to realize the final truth that "all this is Brahman." Conversely, we saw that removing the reconnecting edge from any such apg has the effect of restoring the original set, with all its distinctions within itself and from other sets. In this way, well-founded sets were seen to "emerge from" Ω; each emerges from Ω by removing the reconnecting edge of a suitable picture of Ω. The underlying truth of this "manifested" set is seen in introducing the reconnecting edge once again; this step reveals in the resulting apg the dynamic relationships within the set as a variation of Ω alone. Viewing the manifested set instead as truly disconnected from its source—with the reconnecting edge removed—illustrates the state of *pragya-aparadh*, where differences dominate and individuals appear to be cut off from their source.

This mathematical context—a model $\hat{V}$ of $\mathrm{ZFC}^- + \mathrm{AFA}$—gives us, therefore, two views of the unfoldment of the universe of mathematics. One view is the usual one, in which sets unfold stage by stage from the empty set, each set being different from every other and for the most part disconnected from any kind of source. This view arises from cutting away all ideal elements of $\hat{V}$, leaving only disconnected well-founded sets. The other view is a world of sets that is identical, set for set, to the well-founded universe V itself, except that the distinctions between sets have become transparent; what dominates in this second view is the reality of every set as Ω. This view arises when each well-founded set's canonical picture is reconnected to itself, connecting its unique empty set vertex to itself. Transition from the first to the second view parallels awakening to Brahman, wherein every individual, and the universe V itself, are seen to be nothing but Ω itself. The transition from the second view to the first parallels the emergence of *pragya-aparadh*, creating the appearance of separation of sets from their source, arising from cutting away the connection of each set with itself.

This mathematical vision of creation as being the dynamics of pure consciousness alone illumines the eternal truth from the *Yoga Vasistha* [16]:

> The ignorant regard this samsara as real. In reality it does not exist at all. What does exist is in fact the truth. But it has no name! (p. 528)

6 Appendix: The Bisimulation and Bisimilarity Relations

Earlier in this paper, we gave an overview of the construction of a model $\hat{V}$ of $\mathrm{ZFC}^- + \mathrm{AFA}$. The idea we discussed was that $\hat{V}$ should consist of equivalence classes of apgs, where two apgs are to be considered equivalent if they picture the same set. Two such apgs are said to be *bisimilar*. We pointed out that, for formal correctness, the bisimilarity equivalence relation must not actually be defined in terms of sets that have not yet been constructed. In this Appendix, we describe the right way to define bisimilarity. The first step is to define the *bisimulation* relation on graphs, and then to define bisimilarity as a special kind of bisimulation.

Suppose $G = (M_G, E_G), H = (M_H, E_H)$ are graphs. A *bisimulation* for G, H is any relation $R \subseteq M_G \times M_H$ having the following properties: There is a relation R^+ with $R \subseteq R^+ \subseteq M_G \times M_H$ satisfying the following: For each $a \in M_G$ and $b \in M_H$, aR^+b if and only if both of the following hold:

(i) whenever $a \to x$ there is $y \in M_H$ with $b \to y$ and xRy

(ii) whenever $b \to y$ there is $x \in M_G$ with $a \to x$ and xRy

The equivalence relation that we will need is a *maximum* bisimulation; we will discuss this concept after giving an example.

Example 1. Consider the two apgs mentioned earlier that picture the set $\{0, 1\}$; for this example, we will call them G and H.

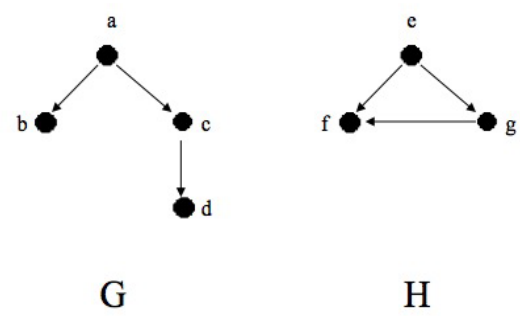

Figure 13. Bisimulations and Bisimilar Graphs

We have labelled these graphs differently here to emphasize the fact that bisimulation relations make sense for any kind of directed graph, not just apgs. We define three bisimulations for G, H. The first is a trivial bisimulation, which is obtained by declaring that childless vertices are related to each other:

$$R_1 = \{(b, f), (d, f)\}.$$

The conditions for a bisimulation are satisfied vacuously because none of the vertices in the relation have children.

The second bisimulation includes vertices that do have children:

$$R_2 = \{(c, g), (d, f), (b, f)\}.$$

The new pair (c, g) can be included because the respective children of c and g occur as pairs also. Notice that $R_2' = \{(c, f), (d, f), (b, f)\}$ is *not* a bisimulation since, although $c \to d$, there is no d' in H such that $f \to d'$. Also, $R_2'' = \{(c, g), (b, f)\}$ is not a bisimulation because, though $g \to f$ in H, there is no corresponding child of c that is paired with f in the relation.

The third bisimulation includes all the vertices.

$$R_3 = \{(a, e), (c, g), (d, f), (b, f)\}.$$

It is not hard to see that there is only one bisimulation that includes all the vertices; it is called the *total* bisimulation, or the *maximum* bisimulation. Two graphs that admit such a bisimulation are called *bisimilar*. Considering these graphs as apgs (with respective designated points a and e), it is apparent in this example that, although the graphs are not isomorphic, the membership structures that they specify are the same; it is clear in this case that these apgs must picture the same set. In this example, we write $G \equiv H$ to indicate that the graphs are bisimilar.■

Bisimilarity can be shown to be an equivalence relation on directed graphs. Moreover, the discussion in the example gives some idea about why this equivalence relation is the one we are seeking: When we consider apgs as displays of potential membership structure, it seems intuitively clear that whenever apgs are bisimilar, the membership structure of both apgs is the same, so they should picture the same set.

Using this equivalence relation $\equiv$ on apgs, we wish to form the collection of resulting equivalence classes. Since typically each equivalence class will itself be a proper class, we use a familiar technique (known as *Scott's trick*) to reduce their size: we take a representative r from each such class having least rank and we let $[r]$ denote the set of all apgs equivalent to r and having the same rank as r; and finally, we let $\hat{V}$ denote the collection of all such reduced equivalence classes $[r]$.

We have described the "sets" of our new universe. We also need to specify the "membership relation." Suppose $[r], [s] \in \hat{V}$. Since r, s are pointed graphs, we may write $r = (G, p_G)$ and $s = (H, p_H)$, where p_G and p_H are the designated points of the apgs. We declare that $[r]$ is a "member of" $[s]$ if there is a vertex q in H with $p_H \to q$ such that the sub-apg Hq of H determined by q (defined as: $Hq = \{h \in M_H \mid \text{there is a path in } H \text{ from } q \text{ to } h\}$) is bisimilar to G:

$$\exists q \in M_H \, Hq \equiv G.$$

We give a simple example:

Example 2. Consider the following apgs:

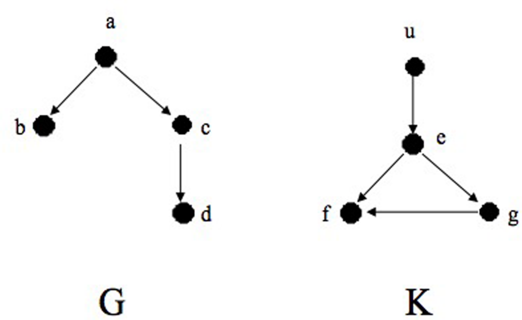

Figure 14. The Membership Relation Between Equivalence Classes of Bisimilar Graphs

We indicate here why the equivalence class $[G]$ is an "element" of $[K]$, according to our new definition. We show that G is equivalent to a sub-apg of K whose designated point is a child of u. Here, there is only one way for this to happen since u has only one child, namely, e. Clearly, the sub-apg of K whose designated point is e is precisely the apg H of Example 1, which, as we have seen, is indeed equivalent to G. Therefore, $[G]$"$\in$"$[K]$. Of course, this is what we expect since G is a picture of $\{0, 1\}$ and K is a picture of $\{\{0, 1\}\}$. ∎

It can be shown that the equivalence classes belonging to $\hat{V}$ that contain well-founded trees correspond exactly to the sets in the original universe V; more precisely, if we let $\mathrm{WF} = \{[r] \in \hat{V} \mid r \text{ is well-founded}\}$, then $V \cong \mathrm{WF}$.

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