MAGICAL ORIGIN OF THE NATURAL NUMBERS
EXTENDED VERSION

Paul Corazza

Abstract. A turning point in the history of mathematics was Cantor's discovery that infinite sets exist. Some time after this discovery, when the foundational axioms for all of mathematics—the ZFC Axioms—were being developed, Cantor's discovery took the form of a fundamental axiom, now known as the Axiom of Infinity. This axiom expresses Cantor's discovery with extreme economy, asserting nothing more than that the natural numbers $1, 2, 3, \ldots$ can be collected together to form a single set (an *infinite* set). Because of this economical formulation, the Axiom of Infinity provides little intuition about the nature of "mathematical infinity." Lacking a sufficiently clear idea about the nature of the infinite, mathematicians have floundered as they have attempted to come to grips with very strong and unusual forms of the infinite, known now as *large cardinals*, which have emerged in research in the past century. These notions of the infinite cannot be proved to correspond to "real" infinite objects in the mathematical universe, but nevertheless seem quite real. A question for which there is, to this day, no universally accepted answer, is, *Do large cardinals exist?*

In this article, we suggest a new form of the Axiom of Infinity, which provides much richer intuition about the mathematical infinite, and which points the way toward an account of large cardinals. This new axiom is based on a deep insight about the true nature of the infinite. This insight is drawn both from the ancient wisdom of several traditions of knowledge, concerning the origin of the natural numbers, and also from the paradigm provided by quantum field theory for understanding the ultimate constituents of the physical universe. Both perpectives suggest to us that a collection of discrete objects, like the set of natural numbers, should be understood as precipitations of the dynamics of an unbounded field. What is important about the set of natural numbers, therefore, is the field that gives rise to them. In this spirit, we show that the sequence of natural numbers "arises from" the transformational dynamics of a *Dedekind self-map*. We show that a deep understanding of Dedekind self-maps suggests that large cardinals themselves arise as "precipitations" of Dedekind self-maps. Following this logic to its natural conclusion, we conclude that, mathematically speaking, "everything" arises from "unmanifest" transformational dynamics that move the totality of the universe within itself.

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Contents

1. Introduction

The set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers[1] has always been recognized as playing a vital role in the development of mathematics. Nineteenth-century German mathematician Leopold Kronecker made the often-quoted statement [71], "God made natural numbers; all else is the work of man." Certainly, the natural numbers are the starting point for the construction of all the other number systems that are used, including the integers, rational numbers, real numbers, and complex numbers. Before formal foundational theories were developed, many believed that the natural numbers were the basis for all of mathematics.

At the end of the 19th century, the mathematics community was confronted with a bold conjecture by a young mathematician, Georg Cantor, who proposed that the natural numbers can be collected together to form a *set*, a single mathematical object. The unquestioned view of nearly all mathematicians of that period was that the natural numbers $1, 2, 3, \ldots$ extend as far as one cares to go, forming a *potential infinity*, but that it was beyond human conception, and perhaps even sacrilegious, to think of them as forming an *actual infinity*, a completed set. One reason had to do with apparent paradoxes that seemed to arise from treating $\mathbb{N}$ as a set. Galileo observed that if we look at the *squares* of the natural numbers, $1, 4, 9, 16, \ldots$, they can be placed in 1-1 corresondence with the natural numbers (matching 1 with 1, 2 with 4, and, in general, n with n^2). This correspondence would suggest that $\mathbb{N}$ (as a set) has *the same size* as the set of squares of natural numbers. To Galileo, and most other mathematicians of those early days, this conclusion was absurd—how could the squares be equinumerous with the natural numbers when *infinitely many* natural numbers are missing from the list of squares? Another issue was theological: The "infinite" in mathematics was linked to beliefs about God. But once we allow $\mathbb{N}$ to be a set, it is possible to perform operations on it that would violate treasured religious beliefs. One such operation, also observed by Cantor, was the *power set operation* $\mathcal{P}$: For any set X, $\mathcal{P}(X)$ denotes the set of all subsets of X. Cantor showed that the size of $\mathcal{P}(X)$ is always greater than the size of X. But then, $\mathcal{P}(\mathbb{N})$ must be a bigger infinity than that represented by $\mathbb{N}$, so from the point of view of the theologies of the time, an entity greater than God was being professed.

Despite these and other strong objections, a compelling practical argument led to eventual acceptance of this new point of view. During that period, there was considerable confusion about how to build a rigorous foundation for analysis, which includes in its fold the subject of *calculus* as well as more advanced areas of research. The difficulty boiled down to the fact that there was no clear conception of how to define a *real number* so that the main theorems of the subject could be proven rigorously. Cantor and others showed that, without actually infinite sets, there would be little hope of solving the problem. Eventually, the mainstream mathematical community agreed.

Some years after Cantor's triumph, the world of mathematics faced another crisis: The somewhat loose definition of "set" that had been used by Cantor, which said, roughly speaking, that any collection of objects one could imagine could be

[1] In this article, when we speak of the natural numbers in a historical context, we will refer to them, as was done in earlier times, as a list $1, 2, 3, \ldots$, with 0 omitted. Outside of that context, we adhere to the modern convention that the natural numbers form a *set*, which includes 0.

collected together to form a *set*, led to paradoxes—inconsistencies in the foundation of mathematics.

This second crisis led to the development of a formal set of axioms, intended to provide a foundation for all of mathematics, known as *ZFC*: the Zermelo-Fraenkel axioms of set theory, including the Axiom of Choice.

ZFC set theory was a great success. Mathematics now had a single foundation that unified all branches of mathematics. Yet, behind the scenes, another challenge was emerging—a challenge that would not be resolved in Cantor's lifetime, and that in fact remains unresolved to this day. Early set theorists, notably Hausdorff, discovered new notions of infinity that were unexpectedly strong. Cantor's work showed that there is an endless hierarchy of infinite sizes, called *infinite cardinals*. Using notation for these that is common today, the list of these infinite cardinals begins like this:

$$\omega_0, \omega_1, \omega_2, \dots, \omega_\omega, \omega_{\omega+1}, \dots, \omega_\alpha, \dots. \tag{1}$$

Usually, ω_0 is simply written ω. The cardinal ω is the size (or *cardinality*) of the set $\mathbb{N}$ of natural numbers. The symbol 'ω' is also used as another name for $\mathbb{N}$; in that case, ω is often called the *set of finite ordinals*.[2]

Early set theorists identified properties that some cardinals had and that others did not. For instance, some cardinals have the property of being *regular*: A regular cardinal λ has the property that, for any set X having size λ, it is not possible to write X as the union of fewer than λ of its subsets, each having size λ. The cardinal ω is regular (no infinite set can be obtained as a union of finitely many of its finite subsets), but ω_ω is not: If X has size ω_ω, for each n, X has a subset X_n of size ω_n and $X = \bigcup_{n\in\omega} X_n$. So X is the union of fewer than ω_ω of its subsets, each of size less than ω_ω.

Early set theorists, working as if in a laboratory to combine infinite cardinal properties to see what could be produced, found that if the property of regularity was combined with the property of being a *fixed point*,[3] one obtains a cardinal that is much bigger than anything that can be built up from below, using any kind of set operations.

Many years later, the logician Kurt Gödel showed that it is impossible to prove from ZFC set theory that regular fixed points exist, though he did *not* prove that such a cardinal could not exist.

Regular fixed points were the first example in the history of mathematics of a *large cardinal*. Large cardinals are "large" because there is no way to arrive at one of these infinite cardinals with operations that can be formalized in ZFC—so if a large cardinal is postulated to exist, it must be much bigger than any of the accessible cardinals used in ordinary mathematics.

Historically, what has made large cardinals problematic is that they cannot simply be ignored. They have played a key role in the solutions to research problems

[2]The concepts of ordinal and cardinal numbers are defined formally on p. 97.

[3]A cardinal ω_α is a fixed point if $\alpha = \omega_\alpha$. This property is not found to hold for any of the cardinals that occur early in the list (1), since $0 \neq \omega_0, 1 \neq \omega_1, 2 \neq \omega_2, \dots$. But fixed points do exist. The smallest one is defined recursively as follows: Let $f_0 = \omega_0$ and define $f_{n+1} = \omega_{f_n}$. Then the cardinal obtained by forming the union $f_0 \cup f_1 \cup \dots \cup f_n \cup \dots$ is the least fixed point.

in many areas of pure mathematics, including analysis, algebra, functional analysis, and topology (see [16]).

Given that large cardinals over time have proven themselves to be vital to the mathematical enterprise, it has become evident that the axioms of ZFC need to be expanded[4] in order to provide a foundation for these exotic mathematical entities. But how is this to be accomplished? Which axiom or axioms should be added? And which large cardinals, among the many that have been discovered, really ought to be derivable? These are questions that form a part of the *Problem of Large Cardinals*, which has no generally agreed upon solution even to this day [44].

A natural place to look for some clue about a solution is in the axioms of ZFC. We can examine all the axioms that talk about infinite sets and try to extract from them a clear intuition about the nature of "infinite sets"—an intuition that could suggest why large cardinals are in reality quite natural. This effort could lead to the kinds of new axioms that need to be added to ZFC and to resolve the Problem of Large Cardinals.

As it happens, the only axiom among the ZFC axioms that talks about infinite sets is Cantor's legacy: the one that says that the natural numbers form a *set*. Stated in another way, this Axiom of Infinity states that "an infinite set exists." Looking closely at the formal statement, one discovers that the axiom provides very little intuition that could be used to understand bigger infinities.

In this paper, we will propose an alternative version of the Axiom of Infinity, one that is rich in intuition about the nature of the infinite, but that has the same mathematical content as the current axiom. We will arrive at this new axiom by looking to the ancient perspectives on the subject of infinity. We will see that, in all the traditions we consider, the natural numbers are seen to emerge from a *source*, and that they, in some way, remain "connected" to their source in their emergence.

We will attempt to formulate a mathematical version of these ancient insights. In the process, we will observe that, for entirely different reasons, modern physics, in particular, quantum field theory, successfully tackled a similar challenge, in the discovery that the underlying reality of *particles* in the physical universe is the dynamics of unbounded *quantum fields*.

Incorporating all of these points, we will formulate a New Axiom of Infinity that captures the idea that the discrete quantities that constitute the set $\mathbb{N}$ of natural numbers arise as "precipitations" of an underlying, unbounded field. In particular we will show how the dynamics of this field, as it interacts with a distinguished point within it called its *critical point*, generate a blueprint from which the sequence of natural numbers may be formally derived.

Using the dynamics suggested by our new axiom, we will conjecture that large cardinals arise in the same basic way as the natural numbers, by way of analogous underlying dynamics. We will study generalizations of our axiom that are naturally suggested, and that accord with more elaborated insights from the ancient wisdom. In the end, we will formulate a new axiom, the Wholeness Axiom, motivated by the intuition suggested by our New Axiom of Infinity, which accounts for virtually all large cardinals, and which, in fact, gives an account of the emergence of all mathematical objects.

[4]See for example [44].

2. The Natural Numbers According to Ancient Wisdom

In this section, we consider viewpoints about the natural numbers from several ancient traditions of knowledge. We will see that the ancients had a more expanded view of what the natural numbers are, where they come from, and what their role is in the unfoldment of the universe.

We first consider the Vedic tradition, represented here by the teachings of Maharishi Mahesh Yogi, which we will refer to as *Maharishi Vedic Science*.[5] One point in Maharishi's approach that stands out immediately is that the emergence of "diversity" is always on the ground of unity, and that as parts emerge from the whole, they remain connected to the whole, so that unity is never lost [48]:

> All fields of creation are the diverse projections of self-referral consciousness, and, as they always maintain connectedness with their source, the entire field of diversity is the field of consciousness. That is why self-referral consciousness administering itself means the entire universe is administered by consciousness (p. 18).

This general principle of unfoldment is quite different from the usual way we conceive of the unfoldment of the natural numbers: Natural numbers are not considered to emerge from any kind of source, and they are conceived of as distinct quantities, not unified in any way. This usual way of understanding the natural numbers represents a second way, which Maharishi has discussed, by which diversification may occur. Maharishi explains that, although remaining connected to the source is a natural occurrence in the process of diversification from the field of pure consciousness, it is nevertheless possible for diversity to *dominate* the process of unfoldment to such an extent that connection to the source is lost. This loss of connection is called in the Vedic Literature *pragya-aparadh*.[6] Quantum physicist John Hagelin describes the emergence of pragya-aparadh in this way [27]:

> Hence the notion of diversity disconnected from unity is a fundamental misconception. This misconception is known as pragya-aparadh or "mistake of the intellect." Pragya-aparadh results when, in the mechanics of creation from the field of consciousness, the intellect loses sight of the essential unity which is the true nature of the self .... The intellect gets caught up in its own creation, i.e., gets overshadowed by the perception of diversity to the exclusion of the unity which is the actual nature of the self being discriminated. According to Maharishi, this mistake of the intellect is so fundamental to the nature of human experience that it is responsible for all problems and suffering in life (p. 284).

[5] Maharishi Vedic Science is Maharishi Mahesh Yogi's systematic presentation, both theoretical and practical, of the Veda and Vedic Literature. An introduction to Maharishi Vedic Science can be found in [7].

[6] Maharishi [49, p. 287] also characterizes this loss of connection as *ignorance*, and characterizes it further in the following remark [47, pp. 200-1]: "When the connectedness of individual life with Cosmic Life is damaged, individual intelligence remains disconnected from its own cosmic value. It remains like a bud without flowering."

Maharishi expresses a similar point, that, in the usual treatment of the natural numbers, there is no mention of a "source" of natural numbers, and diversity dominates. However, viewed in the right way, he explains, the natural numbers are the basis of the unfoldment of the universe itself, so it is important that an account of the natural numbers does not overlook their source, which he calls the *Absolute Number* [45]:

> The ever-expanding value of the universe, in terms of an infinity of numbers, is the natural characteristic feature of the Absolute Number, which enables all numbers to function from their common basis (pp. 614–615).

According to Maharishi, failing to appreciate the unfoldment of the natural numbers in terms of their source—in terms of the Absolute Number—leads to an awareness dominated by diversity[7] and dominated by the intellect, without support from the source of either one. In [45], Maharishi describes this limited value of awareness as the "intellectual level of logic," and as "limited to the mathematics of the natural numbers." He mentions this point in describing the ancient classic dialogues[8] between the great warrior Vishwamitra and the fully enlightened sage Vasishtha, in which Vishwamitra time and again fails to understand the ways of Vasishtha because of his reliance on the intellect alone [45]:

> This means that Vishwamitra was trying to understand the infinite world of wholeness (the Saṁhitā[9] level of reality) on the level of his fully awake intellect, which was held on the intellectual level of logic (limited to the mathematics of the natural numbers $1, 2, 3, \ldots$), and therefore could not fathom the depth of wholeness that transcends all numbers and is the common source of all numbers—the Absolute Number (p. 613).

By restoring to awareness the true source of the natural numbers, he says [45, p. 614], the boundaries that, in an intellect-based approach, keep these numbers strictly disconnected from their source, begin to melt. Each number can then play its role in contributing to the evolution of the universe. Moreover:

> Its [each number's] individual status has become Cosmic—as an individual, it has been elected to be a ruler—the full potential of its creativity has blossomed (p. 614).

Restoring to the natural numbers their source—the Absolute Number—has profound consequences for the entire field of manifest life. Maharishi [45] says, "It is this effect of the Absolute Number on all numbers that actually initiates and maintains order in the ever-evolving infinite diversity of the universe" (p. 615). Indeed, bringing individual awareness to its source in the Absolute Number leads to a problem-free life (p. 615).

Certainly the view that Maharishi warns against here—that the natural numbers are nothing more than conceptual devices, without a common source, only to serve the practical need of counting items in the world around us—is the common view,

[7] See [46, p. 399].

[8] These dialogs can be found in Vālmīki Rāmāyan, for instance I.53.

[9] In other words, the totally *unified* level of reality.

the view that one learns in school, and a view that one would ordinarily have little occasion to reflect upon or call into question.

Yet, if we take seriously the view that the natural numbers have far greater significance than the common view suggests, it is reasonable to wonder how these numbers would be any different if they were to realize their "full potential." One answer comes from several observations Maharishi has made regarding the greater signficance of these numbers, indicating that the true nature of each is more of a *universal principle* than merely a discrete quantity. For instance, he describes the number 1 as *unity*, an *eternal continuum* [45, p. 613], from which all the other natural numbers emerge. He has described the number 2 as the view of wholeness in which wholeness assumes the role of subject and object, infinite silence and infinite dynamism, intelligence and existence, Purusha and Prakriti [45, p. 630]. He has described the number 3 as the fundamental structure of unity, in terms of Rishi, Devata, and Chhandas [45, p. 630]. In these examples we see that each number has its own character but at the same time gives expression to, and remains connected to, fullness.

Moreover, in his discussion about his Vedic Mathematics, Maharishi suggests that the natural numbers, rather than being of a fundamentally finite nature, emerge as a result of one infinity being "broken into pieces of infinity" [45, p. 572]. In fact, Maharishi locates the Sutra in the Vedic Literature that is responsible for breaking the infinite in this way [45, p. 347]:

> *Ekā cha me tisraschcha me ...*
> *One is in me, two is in me, etc.*
>
> – *Yajur-Veda 18.24*

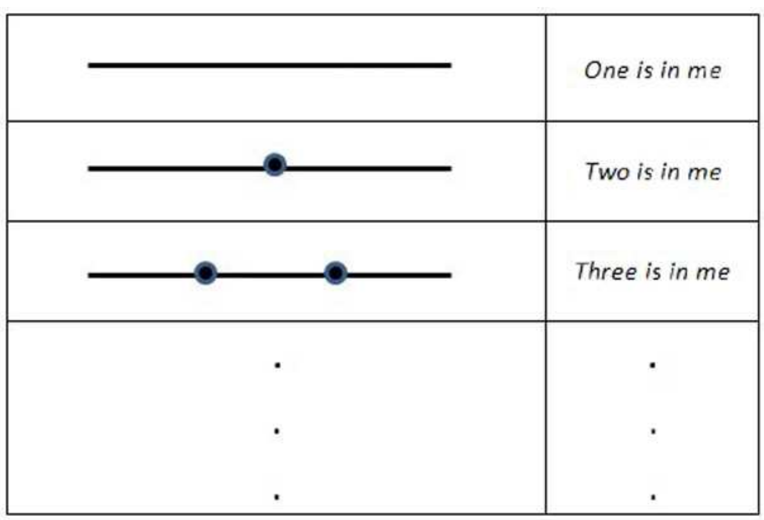


Figure 1. One Appearing as Two, Three, ...

From this perspective, the numbers $1, 2, 3, \ldots$ are different ways of conceiving unity, always remaining unified, always remaining the totality. In Figure 1, we see how a single line segment can also be viewed equally well as partitioned into two pieces, or into three pieces, but in each case, no matter how many segments are conceived, the original line segment remains unchanged.

It is this more expanded view of the nature of the natural numbers, it would seem, that makes it possible for them to truly give rise to everything in the universe.

The view that the natural numbers have a deeper, universal significance has been expressed in other ancient traditions of knowledge. In the West, Pythagoras and his school maintained that all things in the universe are, fundamentally, natural numbers. "All is number" is an expression attributed to this school.[10] Pythagoras also maintained [20, p. 137] that at the basis of all natural numbers is a "Number of numbers," an ultimate source of all numbers, something Divine in nature.

The Neoplatonist Diadochus Proclus[11] (412–485 A.D.), one of the most prolific and profound among the Neoplatonists, also described an ultimate source of number [65]:

> . . . but the cause of all things being unically raised above all motion and division, has established about itself a divine number, and has united it to its own simplicity (p. 177).

Like Maharishi, Proclus maintains that diversification that emerges from the One naturally remains connected to its source:[12]

> . . . that which comes into being, when separated from the cause, is powerless and weak. For, since it is unable to preserve itself and is not maintained by itself, but both the preservation and maintenance are obtained from the cause and are removed if it is deprived of the cause, it is plain that on its own it becomes powerless, and is dispersed into nonexistence. . .

We find similar insights in ancient Chinese philosophy. Here, we also find the view that diversity of manifest existence, embodied in the diversity of the natural numbers, originates from a unified source to which all diversity remains connected. *I Ching* scholar Carol Anthony [1] writes:

> The ancient Chinese, like the ancient Greek Pythagoras, saw numbers as mirroring the order of the universe. The number one represented the undifferentiated whole. . . . Within this whole existed two primary forces, called the Creative and the Receptive. . . that by interacting with each other brought about the creation of all things (p. 1).

The *source* of all number, as explained by Laozi in the ancient classic, the *Tao Te Ching*, is the nameless *Tao* [23]:

> The Tao begot One.
> One begot Two.

[10] A discussion of the Pythagorean school can be found in [6, Part I].

[11] "Neoplatonism" refers to the revival of Plato's teachings and the Platonic Academy for a 300-year period—roughly from 200 A.D. through 529 A.D.—after which the Academy was officially closed. Proclus was the head of the Platonic Academy for nearly fifty years, succeeding Syrianus in 437 A.D. The title "Diadochus," which means "successor," was bestowed upon Proclus when he replaced Syrianus in the Academy.

[12] Quotation from [40, p. 103].

> Two begot Three.
> And Three begot the ten thousand things (v. 42).

Here, Laozi tells us not only that *Tao* is the source of all things, but, he suggests, if we look more closely, *Tao* is in fact the source of One (and Two and Three), which in turn gives rise to all things.

The insight that the parts emerging in this diversification remain connected to their source is expressed in the following passage [23]:

> The beginning of the universe is the mother of all things.
> Knowing the mother, one also knows the sons.
> Knowing the sons, yet remaining in touch with the mother, brings freedom from the fear of death (v. 52).

Tao is therefore seen to play a role similar to that of Maharishi's Absolute Number.

3. The Origin of $\mathbb{N}$ According to Modern Mathematics

It is natural to wonder to what extent the common view of the natural numbers is truly the *mathematical* view. All mathematics arises from set theory, so we can look to the axioms of set theory to see to what extent the commonly held restricted view of the natural numbers is present even in the foundation of mathematics.

Among the axioms of ZFC[13] (Zermelo-Fraenkel set theory with the Axiom of Choice), there is just one axiom that talks about infinite sets; this axiom is called

[13]In this paper, we will often switch between the theories ZFC and ZFC without the Axiom of Infinity: ZFC − Infinity. In the latter case, the theory that actually is needed is ZFC − Infinity + Trans, where Trans is an axiom that asserts that every set is contained as a subset of a *transitive* set (a set X is transitive if, whenever $y \in X$, $y \subseteq X$). Including this axiom is preferable here because of work in [22] where it is shown that the axioms of ZFC − Infinity + ¬Infinity + Trans (where ¬Infinity says that infinite sets *do not* exist) are essentially equivalent to the axioms of arithmetic (in the form of the formal axioms of Peano Arithmetic (PA)). This equivalence will allow us to switch between the set theory perspective and the PA perspective as needed.

Therefore, in this paper we simply assume that Trans is one of the standard axioms of ZFC. With this assumption, ZFC − Infinity automatically also includes Trans. For reference, we list our version of the ZFC axioms here. A *functional formula* is a formula $\phi(x, y, z_1, \ldots, z_k)$ with the property that, for any sets $a_1, \ldots, a_k$, whenever t, u_1, u_2 are sets and both $\phi(t, u_1, a_1, \ldots, a_k)$ and $\phi(t, u_2, a_1, \ldots, a_k)$ hold, then $u_1 = u_2$.

(*Empty Set*) There is a set with no element.

(*Axiom of Infinity*) There is an inductive set.

(*Axiom of Extensionality*) Two sets are equal if and only if they have the same elements.

(*Pairing Axiom*) For any sets x, y, the collection $\{x, y\}$ is also a set. More precisely, for all x, y, there is z such that the only elements of z are x and y.

(*Union Axiom*) The union of any set of sets is again a set. More precisely, for all x, there is z such that $z = \cup x$.

(*Power Set Axiom*) The collection of all subsets of a set is again a set. More precisely, for all x, there is z such that, for all u, $u \in z$ if and only if $u \subseteq x$.

(*Foundation*) Every set has an $\in$-minimal element. In other words, for every x, there is y such that for all $z \in y$, $z \notin x$.

(*Separation*) For every formula $\phi(x, z_1, \ldots, z_k)$, every set A, and all sets $a_1, \ldots, a_k$, the collection $\{y \in A \mid \phi(y, a_1, \ldots, a_k)\}$ is a set.

(*Replacement*) For any *functional* formula $\phi(x, y, z_1, \ldots, z_k)$, any set A, and any sets $a_1, \ldots, a_k$, the collection $\{v \mid \exists u \in A\, \phi(u, v, a_1, \ldots, a_k)\}$ is a set.

(*Choice*) For any set X of nonempty sets, there is another set Y containing an the *Axiom of Infinity*. Historically, this axiom was considered to be essential because of the work of Cantor, who showed that a rigorous formulation of a number system as fundamental as the real number line would not be possible without the concept of infinite sets [28]—in particular, it is necessary to conceive of the natural numbers $0, 1, 2, 3, \ldots$ as elements of a single, completed set, which we denote in this paper, informally by $\mathbb{N}$, and formally by the Greek letter ω.

In coming up with a precise statement of the Axiom of Infinity, the early founders of set theory had to decide how to express the idea that "an infinite set exists" or "there is a set whose elements are the natural numbers" in the language of set theory. In the language of set theory, everything is taken to be a set, but at the time the axioms were being formulated, the natural numbers themselves were not usually thought of in this way. To meet the need of representing natural numbers as sets and asserting that there is a set that contains all natural numbers, the early crafters of the axioms settled on the concept of an *inductive set* [32, 35].

Therefore, the Axiom of Infinity as it is known today asserts the existence of an inductive set. A set S is inductive if it satisfies two properties: (1) S contains the empty set $\emptyset$, and (2) for any set x belonging to S, the set $s(x) = x \cup \{x\}$ also belongs to S. The natural numbers are then defined to be precisely those sets that belong to all inductive sets. This definition established ω, the set of natural numbers, as the *smallest* inductive set. This approach to the natural numbers provides a formal way of declaring that the natural numbers have the following definition:

$$
\begin{array}{rcl}
0 & = & \emptyset \\
1 & = & \{0\} \\
2 & = & \{0, 1\} \\
3 & = & \{0, 1, 2\} \\
\cdot & = & \cdot \\
\cdot & = & \cdot \\
\cdot & = & \cdot \\
n+1 & = & s(n) = n \cup \{n\} = \{0, 1, 2, \ldots, n\} \\
\cdot & = & \cdot \\
\cdot & = & \cdot \\
\cdot & = & \cdot
\end{array}
$$

The function s is fundamental to the definition of the natural numbers; it tells us how to go from any number in the sequence to the *next* number in the sequence. The usual way of defining s is by $s(n) = n + 1$—simply add '1' to n to get the next number in the sequence. But in the context of pure sets, this definition becomes $s(x) = x \cup \{x\}$.[14]

element of each of the elements of X.

(*Trans*) For every set X there is a transitive set Y such that $X \subseteq Y$.

Note that including the Empty Set Axiom in ZFC is redundant, but it is necessary when considering the theory ZFC − Infinity.

[14]Note that s, acting on any input x, has the effect of producing a new set $x \cup \{x\}$ that includes all elements of x together with one additional element, x itself. It should be observed that the

Viewing the natural numbers in this way—as simply the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$, or even as the "smallest inductive set"—provides no clue that there might be, even within the field of mathematics itself, an *origin* to the natural numbers; that the natural numbers might reasonably be seen as *emerging* from some source—a source that could be considered a mathematical analogue to Maharishi's Absolute Number.[15]

4. A Plan for a New Axiom of Infinity

We will argue that failure to recognize a suitable origin of the natural numbers has resulted in an unnecessarily limited view of the nature of the mathematical infinite. One of the ongoing problems in foundations of mathematics has been to discover axioms that could be added to the ZFC axioms to provide an axiomatic foundation for certain extremely large sets that have arisen in mathematical practice, known as *large cardinals*.[16] A natural place to look for answers to questions about the "infinite" in mathematics (such as "Which, if any, large cardinals exist—that is, which should be taken as valid objects in the universe of mathematics?") is the Axiom of Infinity. As we have seen, however, the Axiom of Infinity tells us little more than that an infinite set exists, and that, in particular, there is a set ω consisting precisely of the set-versions of the natural numbers. And, although it can be shown that ω does have some properties which are suggestive of large cardinals, the definition of ω itself reveals very little about the "nature of the infinite," and the Axiom of Infinity itself provides little help in determining which sorts of notions of infinity really belong in the universe.

Our aim, then, is to provide a richer form of the Axiom of Infinity which, though mathematically equivalent to the version that is in common use today, will have a

set $x \cup \{x\}$ always consists of one more set than x itself because x and $\{x\}$ are disjoint. They are disjoint because, in the universe of sets, no set is a member of itself—it is never the case that $x \in x$.

[15]One might argue that the empty set $\emptyset$ could play the role of the "source" of natural numbers. Certainly, $\emptyset$ is the *first* in this sequence of numbers. But it is difficult to support the claim that the numbers that come after 0 emerge from dynamics that are somehow contained in 0. In fact, from what we have seen so far, such dynamics are embodied in the successor function s, rather than in 0 itself. In our view, s is a reasonable candidate for the "source" of natural numbers, except for the fact that we are unable to specify the domain of s without already knowing about ω.

[16]Cantor showed, at the end of the 19th century, that there are different sizes of infinite sets; every infinite set has one of these sizes. But shortly after his discovery, certain types of infinite sets emerged that were so enormous, they were very difficult to classify, and, years later, it was shown that such infinities could not actually be proven to exist at all from ZFC. These notions of infinity are known today as *large cardinals*. What is surprising about these large cardinals is that they have appeared as key elements in the solutions of a wide range of research problems in mathematics; moreover, despite concerted efforts by many early set theorists, no one has ever proved that large cardinals do *not* exist. The so-called "Problem of Large Cardinals" is the problem of adding to the standard axioms of ZFC one or more axioms that could be used to *derive* the known large cardinals. The need here is to find *naturally motivated* axioms that could be truly considered foundational axioms for mathematics, in the same spirit as the ZFC axioms themselves. See [10, 11, 13]. One other aspect of the study of large cardinals that is also somewhat mysterious is the fact that there is no generally agreed upon *definition* of "large cardinal"; in practice, though, large cardinals have the property of being *weakly inaccessible*, so this property can be used as a definition for our purposes here. See page 157 for a definition of *weakly inaccessible*.

formulation that suggests a direction for generalizing, or "scaling." To get started, we consider the possibility that the natural numbers might indeed have a source, as ancient philosophies have suggested, and that this source is something like the "dynamics of an unbounded field."

But the question remains, How can this intuition be implemented in a rigorous mathematical way? Interestingly, modern physics has already taken a similar step in a very different context.

One of the challenges in the history of physics has been to identify the ultimate constituents of the physical universe. For centuries it was believed that the answer had something to do with finding an ultimate particle, or fundamental set of particles, that everything else, including other particles, was made of. However, the answer that was found was not a discovery about extraordinary particles. What was found instead, by physicists in the area of Quantum Field Theory (QFT), was that the source of all particles is *unbounded quantum fields.* Every particle has a corresponding quantum field—for instance, each electron is related to the electron quantum field. And, in fact, each electron is a *precipitation* of this quantum field.

This solution to the problem of finding what is at the "root" of physical reality has been so successful that by now the physics community is in agreement that the *truth* about particles is their underlying fields; the particles themselves are simply *side effects.* Summarizing this insight, Art Hobson [31], in a 2013 *American Journal of Physics* article, "There Are No Particles, There Are Only Fields," writes,

> Quantum foundations are still unsettled, with mixed effects on science and society. By now it should be possible to obtain consensus on at least one issue: Are the fundamental constituents fields or particles? As this paper shows, experiment and theory imply that unbounded fields, not bounded particles, are fundamental.... Particles are epiphenomena arising from fields (p. 211).

In the QFT solution, a class of discrete particles are seen to be a side effect of the dynamics of an underlying field. Considering the fact that the natural numbers are, in a mathematical way, a discrete collection of quantities, we might conjecture that they too are the expression of the dynamics of some sort of unbounded field.[17] We still wonder though, how can these dynamics be expressed mathematically?

A candidate to represent these dynamics has been known for a long time in mathematics and precedes historically the formulation of the Axiom of Infinity that we have today. This candidate is the concept of a *Dedekind self-map*, a special kind of self-map $j : A \to A$, for an arbitrary set A, having the following properties:

(1) j is 1-1: Different elements of A are sent by j to distinct elements.
(2) j has a *critical point*—an element $a \in A$ that is not in the range of j.[18]

The map j can be seen as a kind of "dynamics," and, as can be proved, in order for j to have properties (1) and (2), A must be *unbounded*, that is, *infinite.*

[17] Seeking to apply this QFT solution to account for a possible origin of the natural numbers accords well with Maharishi's perspective, mentioned earlier, since he sees the Absolute Number as being itself a *field* [45]: "Incomplete mathematics, which is modern Mathematics, does not have the insight into the Absolute Number—into the FIELD level of reality, from where all negativity can be eliminated at one time, in one stroke (p. 633)."

[18] In other words, there is an element a of A such that, for each $x \in A$, $j(x) \neq a$.

It can be shown that, on the basis of interaction between j and a, a precursor or blueprint W of the set of natural numbers arises, and through another kind of machinery, called the *Mostowski collapse*, W and j are "collapsed" to the standard set ω of natural numbers, together with its successor function s.

We propose, then, to "rewrite" the Axiom of Infinity to obtain the following:

There is a Dedekind self-map.

Though, as can be demonstrated, this new version adds no new mathematical content to the original Axiom of Infinity, it does suggest a direction for generalization, for scaling to much bigger kinds of infinities, and to move toward a solution to the Problem of Large Cardinals.

The intuition that the new axiom suggests is that, just as the natural numbers themselves should, on the QFT view, be viewed as precipitations of an unbounded field, realized mathematically as a Dedekind self-map interacting with its critical point, so likewise should we expect large cardinals to arise as precipitations of some larger-scale unbounded field, realized once again as the interaction of a generalized Dedekind self-map with its critical point. Since large cardinals in many cases are *global*,[19] we conjecture that our generalized Dedekind self-maps will need to map *the universe V to itself.* Therefore, justifying large cardinals should amount to finding a natural kind of Dedekind self-map from V to V, whose interaction with its critical point ultimately gives rise to particular large cardinals.

There are many ways one might choose to rewrite the Axiom of Infinity to aim for the goals we have described here. We have chosen to do it in a way that realizes some of the vision of the ancients concerning the infinite. Our working hypothesis in this paper is that the ancient vision uncovered deep truths about the nature and dynamics of the Infinite that underlies the unfoldment of the universe, and that the discoveries and insights of the sages of antiquity can be used as intuition to successfully guide mathematical research into the infinite. Examining the ancient view concerning the emergence of the natural numbers from the dynamics of an unbounded source has already led us to the concept of a Dedekind self-map $j : A \to A$, which is a natural realization of this idea. Generalizing to Dedekind self-maps of the form $j : V \to V$, and envisioning that, by means of the interaction between j and its critical point, the full nature of the mathematical infinite (including large cardinals) could unfold, leads us even more deeply into the ancient view of the unfoldment of manifest existence: On this larger scale, $j : V \to V$ can now be seen as an analogy for the fundamental dynamics of the source, of pure consciousness.[20]

[19] Being "global" means that they do not simply exist in isolation in some part of the universe; but rather, their existence has an impact in arbitrarily large stages of the universe.

[20] We quote two descriptions from Maharishi of this underlying flow of life.

> The infinite diversity and dynamism of creation is just the expression of the eternally silent, self-referral, self-sufficient, unbounded field of consciousness—pure wakefulness, unbounded alertness, pure intelligence, pure existence, all knowingness [47, p. 67].

> One individual is nothing but a bundle of waves, nothing but a bundle of energy waves. Where do these waves come from, these waves of energy? They all originate from that one eternal hum and that hum in its exact status is the origin of the Vedas [52, p. 16].

And those dynamics are seen to emerge from the interaction of j with its critical point—and again, this view reflects the insight that the unfoldment of creation begins with the collapse of unboundedness to a point and continues to emerge from the point into infinite expansion.[21]

The purpose of this article is to develop these ideas fully and arrive at a solution to the Problem of Large Cardinals, at each step making use of the rich insights about the Infinite available from the wisdom of the ancients.

5. Dedekind-Infinite Sets and Dedekind Self-Maps

In this section, we look more closely at the concept of a Dedekind self-map and consider several examples. We will observe in some detail how the dynamics of such a self-map give rise to a blueprint for the set of natural numbers. We also look at a "higher order" Dedekind self-map, of type $A^A \to A^A$, and how it generates its own kind of blueprint for ω. We observe in this case a strong analogy between the unfoldment of integer precursors and the sequential unfoldment of the Ṛk Veda, as described by Maharishi's *Apaurusheya Bhashya*.[22]

Among the many early definitions of "infinite set" that were considered as the axioms of set theory were being formulated, a notion of infinity that did not rely on the sequence of natural numbers was *Dedekind-infinite sets*, named after the mathematician, Richard Dedekind, who proposed the idea. We state the formal definition here:

Definition 1. (Dedekind-Infinite Sets) A set A is *Dedekind-infinite* if it can be put in 1-1 correspondence with a proper subset[23] of itself. In other words, A is Dedekind-infinite if there is a 1-1 and onto function $f : A \to B$ where B is a proper subset of A.

An easy example of a Dedekind-infinite set is $A = \{1, 2, 3, \ldots\}$. Here, an example of a proper subset B of A that can be put in 1-1 correspondence with A is given by $B = \{2, 3, 4, \ldots\}$, with correspondence given by $f(n) = n + 1$ (Figure 2).

Associated with every Dedekind-infinite set A is a corresponding self-map $j : A \to A$. For instance, if A is Dedekind-infinite and $B \subsetneq A$ is a proper subset, and $f : A \to B$ is a 1-1 correspondence, the associated self-map $j : A \to A$ is defined just to be f itself, except the codomain[24] is changed from B to A. Now, since j is a function from A to A, j is not itself a 1-1 correspondence: It is 1-1 but not onto.[25] Since j is not onto, there is an element a in the codomain of j that is not

[21] Maharishi gives an overview of this process here:

> The first syllable of Ṛk Ved, AK, expresses the dynamics of akshara—the 'kshara of A' or collapse of infinity to its point value, which is the source of all the mechanics of self-interaction [53, p. 1].

[22] See [45, 495–505] for a definition and full discussion.

[23] A subset B of a set A is a *proper subset* if $B \neq A$.

[24] The *codomain* of a function $h : C \to D$ is D; one writes cod $h = D$.

[25] A function $h : C \to D$ is *1-1* if, whenever x, y are distinct elements of C, $h(x), h(y)$ are distinct elements of D. Also, h is *onto* if, for every $d \in D$ there is an $x \in C$ such that $h(x) = d$.

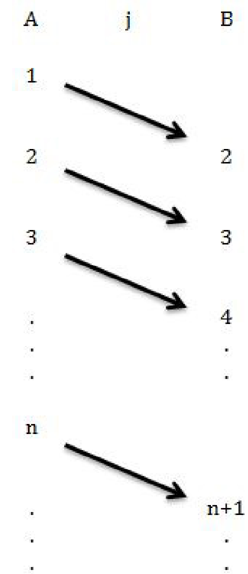


FIGURE 2. Bijection from A to a Proper Subset B

in its range.[26] Any such element a that is in the codomain but not the range will be called a *critical point of* j. We can now state our formal definition of Dedekind self-map:

Definition 2. (Dedekind Self-Maps) A *Dedekind self-map* is a 1-1 function $j : A \to A$ that has a critical point.

The diagram in Figure 3 shows an example of a Dedekind self-map with critical point 1. Here, the function $f : A \to B$ of Figure 2 has been replaced by $j : A \to A$, but acts on elements of A in exactly the same way: For each n, $j(n) = n+1$. Notice that the critical point of j has been circled.

The behavior of a Dedekind self-map $j : A \to A$ has the characteristic of *preserving its own nature*, in the following sense: First, the range of j, which we denote B (as in the examples), is, like A itself, a Dedekind-infinite set, and second, the restriction[27] of j to B, denoted $j \restriction B$, is also a Dedekind self-map, now with critical point $j(a)$.

We spend a moment to verify these details: Let $B = j[A] = \{j(x) \mid x \in A\}$ and let $i = j \restriction B$. We show:

(a) i is a function from B to B.
(b) i is a Dedekind self-map.
(c) B is Dedekind-infinite.

[26]The *range* of a function $h : C \to D$ is the set of all outputs of h: $\operatorname{ran} h = \{h(x) \mid x \in C\}$.

[27]The *restriction* i of a function $h : S \to T$ to a subset R of its domain S, denoted $i = h \restriction R$, has domain R and is defined by $i(x) = h(x)$ for all $x \in R$.

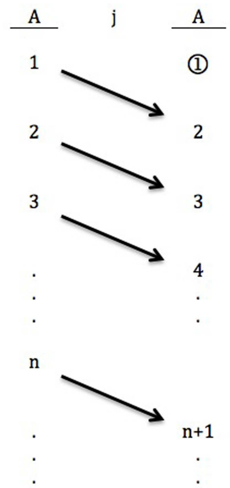


FIGURE 3. Dedekind Self-Map $A \to A$ with Critical Point 1

For (a), if $b \in B$, we can find $x \in A$ such that $b = j(x)$. Since $j(x) \in A$, we have

$$j(b) = j(j(x)) \in j[A] = B.$$

It follows that all values of i lie in B, and (a) is established.

For (b), since i is a restriction of j, i is 1-1. We verify that $j(a)$ is a critical point of i: If $i(b) = j(a)$ for some $b \in B$, let $x \in A$ with $b = j(x)$. Then $i(j(x)) = j(a)$ implies $j(x) = a$, which is impossible since $a \notin \operatorname{ran} j$. It follows therefore that $i = j \restriction B : B \to B$ is a Dedekind self-map with critical point $j(a)$.

For (c), since i is 1-1, it follows that $i : B \to C = i[B]$ is a bijection; but since $j(a) \notin C$, C is a proper subset of B. It follows that B itself is Dedekind-infinite.

It is not difficult to carry the reasoning further and show that C itself is Dedekind-infinite and $j \restriction C : C \to C$ is another Dedekind self-map with critical point $j(j(a))$. This reasoning leads to an infinite chain of Dedekind-infinite sets, Dedekind self-maps, and a sequence of critical points $a, j(a), j(j(a)), \ldots$. These dynamics are pictured in Figure 4.

The reader will notice that the sequence $a, j(a), j(j(a)), \ldots$ closely resembles the sequence of natural numbers. As we will show, it is correct to regard this sequence as a precursor to the "real" natural numbers. Another sequence derived from j, which also represents the natural numbers, but now more abstractly, is $\mathrm{id}_A, j, j \circ j, j \circ j \circ j, \ldots$, where $\mathrm{id}_A : A \to A$ is the identity function (that is, $\mathrm{id}_A(x) = x$ for all $x \in A$).

From this perspective (which, at this point in the discussion, has not yet been fully or rigorously developed) "natural numbers" $a, j(a), j(j(a)), \ldots$ seem to arise as concrete "precipitations" from a self-referral flow ($j : A \to A$), originating from the interaction between j and its critical point a. And, more abstractly, a subtler version of the natural numbers, $\mathrm{id}_A, j, j \circ j, \ldots$, arises simply from the interaction

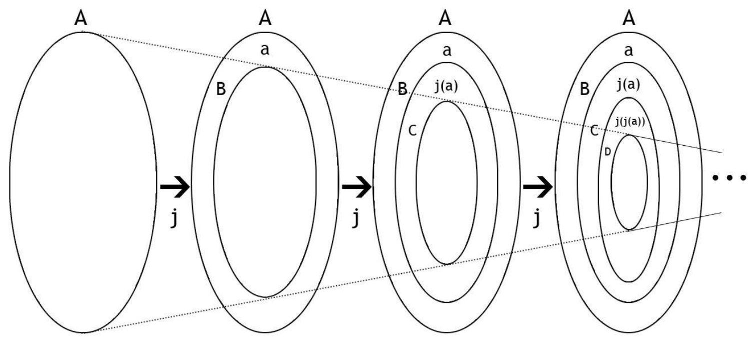


FIGURE 4. Transformational Dynamics of $j : A \to A$

of j with itself. With this subtler approach, each natural number is seen as a *self-referral loop.*

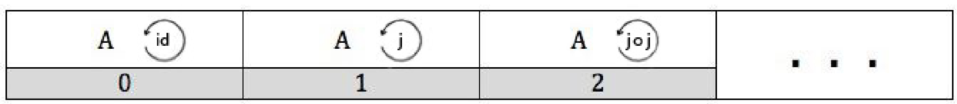


FIGURE 5. Natural Numbers $0, 1, 2, \ldots$ As Self-Referral Loops

In addition, the dynamics expressed by j have an important characteristic: While j does in fact *transform* A (in the sense that j is not simply the identity function; values of A are *moved*), j nevertheless preserves the essential character of A, namely, that of being Dedekind-infinite, since, as we showed earlier, the image B of A under j is itself Dedekind-infinite.

These characteristics of a Dedekind self-map $j : A \to A$ are reminiscent of the transformational dynamics of wholeness, as described in Maharishi Vedic Science, from which the Veda, the blueprint of creation, emerges.[28] In these dynamics, the

[28]For the reader who may be unacquainted with this aspect of Maharishi's work, we offer a quick summary here, and refer the reader to a full treatment of the topic by Maharishi in [45, 495–505]. The Veda describes, in one of its own verses (Ṛk Veda I.164.39), how the Veda itself arises. The verse states, "The verses of the Veda exist in the collapse of fullness (the *kshara* of अ (A)) in the transcendental field, in which reside all the *Devas*, the impulses of Creative Intelligence, the Laws of Nature responsible for the whole manifest universe" [50, pp. 52–53]. Maharishi explains that this collapse of fullness is represented by the very first syllable of Ṛk Veda, *AK*. In the syllable AK, the letter 'A' represents fullness (pronounced without restriction in intonation), while 'K' represents a *stop*, uttered with a closed throat. "The pronunciation of अ (A) requires full opening of the mouth, indicating that अ (A) is the expression of the total value of speech. अ (A) presents unbounded totality, अ (A) is the total potential of speech. Pronunciation of क् (K) requires complete closing of the channels of speech (the throat). अ (A) fully opens the channels of speech; क् (K) closes the channels of speech. Full opening followed by full closing displays the phenomenon of collapse of the unbounded field (of speech) to the point value (of speech). The whole range of speech is contained in this collapse; all sounds are in this collapse, and all the mechanics of transformation of one sound into the other are also contained in this collapse" [48, pp. 171, 354].

first move of wholeness, of "A," gives rise to a sequence of transformations from "AK" to "Agni" to "Agnimile" to the first pada, first richa, first mandala, to the entire Ṛk Veda [45, p. 636]. In these transformations, what is preserved is the essential fullness of "A" even in its collapse to "K" (which Maharishi describes as "fullness of emptiness"). The particulars of manifest existence, like the natural numbers themselves, emerge as a side-effect of the unmanifest dynamics of wholeness. Moreover, Maharishi has described the ultimate nature of these particulars as "self-referral loops" within pure consciousness [47]: "The evolution of consciousness into its object-referral expressions, ever maintaining the memory of its self-referral source—ever evolving structure of consciousness, maintaining the memory of its source—progresses in self-referral loops—every step of progress is in terms of a self-referral loop" (p. 64).

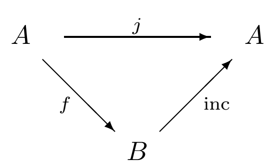
When we use the representation of the natural numbers given by $\mathrm{id}_A, j, j \circ j, j \circ j \circ j, \ldots$, it is possible to see that each successive "natural number"—each successive term of the sequence—is an *elaboration* of the previous term. This viewpoint is expressed in the following diagrams, showing how the move to each successive term elaborates the previous term.

Corresponding to the number 0, and by analogy, the first letter A of Ṛk Veda, we have the identity map $\mathrm{id}_A : A \to A$ that does nothing; it represents complete silence:

(2) $$A \xrightarrow{\mathrm{id}_A} A.$$

Corresponding to the number 1, and by analogy, the first syllable AK of Ṛk Veda, we have the fundamental transformation given by a Dedekind self-map $j : A \to A$, including the "collapse" of A to the critical point a, as in diagram (3).

(3)



In diagram (3), $f : A \to B$ is a 1-1 correspondence from A to the proper subset B of A, and $\mathrm{inc} : B \to A$ is a function that behaves like the identity map, in that $\mathrm{inc}(x) = x$ for all $x \in B$; the only difference between the identity map on B and the function inc is that the codomain of inc is A rather than B. The function inc

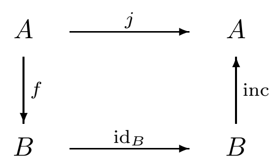
Being the focal point of the collapse of the unbounded totality, 'K' represents a point of infinite dynamism, all possibilities, that can burst forth into the diversity of creation. This "point value," as he explains elsewhere, is represented by 'K': "The total potential of अ (A) is available between the infinity of अ (A) and its point क् (K). The liveliness of the inner structure of अ (A), the liveliness of the Constitution of the Universe, is represented by अक् (Ak). क् (K), the point of the Constitution of the Universe, is the total Constitution of the Universe concentrated at the point of WHOLENESS, अ (A)" [49, p. 454]. Therefore, the syllable AK, he says, embodies in seed form the entire transformational dynamics of the unfoldment of the Veda, which in turn gives rise to manifest life.

is called the *inclusion map from B to A*.[29] The diagram as a whole—known as a *commutative diagram*—indicates that, following the behavior of the map across the top, namely, j, produces the same result as following the two lower arrows: f followed by inc, denoted $\mathrm{inc} \circ f$. Thus, for any $x \in A$, the value of $j(x)$ is the same as $\mathrm{inc}(f(x))$ (this is an easy verification).

The diagram shows that j is formed in two steps: The first step is the 1-1 correspondence between A and the set B, given by f; the second step is the assertion that B itself is in fact a proper subset, expressed by the fact that $\mathrm{inc}(x) = x$ for all $x \in B$ and that inc is *not* onto. In this diagram, the self-map j *does* something; there is a critical point; elements of A are moved. Notice that j is a composition of two factors with very different properties: $j = \mathrm{inc} \circ f$. The bijection f captures the dynamic relationship between A and its subset B, whereas the inclusion map $\mathrm{inc} : B \to A$ is almost the same as id_B—this map is completely silent as it does not move any element of B. Whereas diagram (2) displays pure silence, diagram (3) shows that this "collapse" of A to a arises from a map composed of both silence and dynamism.

As a prelude to the diagram (5), we could re-draw diagram (3) in the following way (diagram (4)):

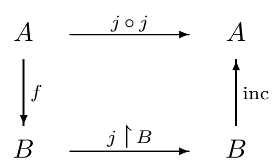
(4)



This version of the diagram makes evident the fundamental pattern of unfoldment, that will repeat as the sequence of compositions of j with itself proceeds: The horizontal maps, as we scan the diagram from bottom to top, display the maps as they unfold from id to j to $j \circ j$ to $j \circ j \circ j$, and so on, whereas the vertical maps display the relationships that connect each step to the next.

Corresponding to the number 2, and by analogy, to the fuller elaboration of AK given by the first word of Ṛk Veda, Agnim, is the composition map $j \circ j$, pictured in diagram (5).

(5)



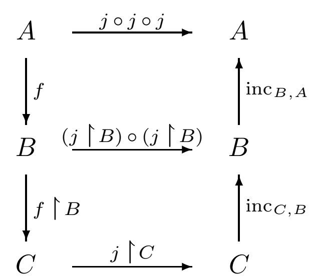
Recall that if $j : A \to A$ is a Dedekind self-map with critical point a and range B, then $j \restriction B : B \to B$ is a Dedekind self-map with critical point $j(a)$. Diagram (5)

[29]Sometimes it is helpful, when specifying inclusion maps, to include the domain and codomain as subscripts. In this example, we could write $\mathrm{inc}_{B,A}$ in place of inc. We will adopt this notation when needed.

shows that the self-map $j \circ j : A \to A$ is obtained by first applying f, then $j \upharpoonright B$, and finally the inclusion map inc from B to A. One can view this suite of maps as a further elaboration of j itself. The visual impact of the diagram itself suggests that the previous diagram is being *enlivened*: The identity map $\mathrm{id}_B : B \to B$ in the bottom row of the previous diagram is now replaced in the present diagram by the Dedekind self-map $j \upharpoonright B$.

For our final example, the number 3, analogous to the next packet of expression in the Veda—the first pada, consisting of 8 syllables—corresponds to $j \circ j \circ j$, shown in diagram (6).

(6)



Recall that $\mathrm{inc}_{B,A} : B \to A$ is the inclusion map from B to A, while $\mathrm{inc}_{C,B}$ is the inclusion map from C to B. Diagram (6) illustrates the more fully elaborated transformational dynamics as $j \circ j$ moves to $j \circ j \circ j$. The dynamics seen in diagram (5) are now recapitulated in the lower square of diagram (6) (replacing A with B and B with C), but diagram (6) as a whole tells a fuller story about how j acts on A in various ways. This makes the analogy to Maharishi's *Apaurusheya Bhashya* more clear: Successive elaborations recapitulate the expressions that have already appeared but develops them further.

In this way, we see that the view that takes $\mathrm{id}_A, j, j \circ j, \ldots$ as the blueprint of the natural numbers illustrates how the emergence of the natural numbers, at its foundation, consists of successive elaborations of the dynamics inherent in $j : A \to A$.

One final observation about this view of the natural numbers as arising from a sequence of Dedekind self-maps is that this sequence itself originates from a higher-order Dedekind self-map in the following way: Let $j : A \to A$ be a Dedekind self-map with critical point a, and let $A^A = \{h \mid h \text{ is a self-map with domain } A\}$.[30] Note that the identity map $\mathrm{id}_A : A \to A$ is one of the elements of A^A. Define a self-map $J_j : A^A \to A^A$ by $J_j(h) = j \circ h$. Notice that id_A is not in the range of J_j and that J_j is 1-1: For $h, k \in A^A$ we have

$$J_j(h) = J_j(k) \Rightarrow j \circ h = j \circ k \Rightarrow h = k,$$

since 1-1 functions are precisely the left-cancellable functions.

We have shown that $J_j : A^A \to A^A$ is a Dedekind self-map with critical point id_A. Now we can observe that, in the same way as $a, j(a), j(j(a)), \ldots$ are obtained by repeated applications of j to its critical point a, so likewise $\mathrm{id}_A, j, j \circ j, \ldots$ is obtained by repeated applications of J_j to its critical point id_A. We will establish

[30] In general, if X, Y are sets, Y^X denotes the set of all maps from X to Y.

in a moment the way in which both of these sequences can be considered *blueprints* for the natural numbers in a more precise way.

6. A New Axiom of Infinity and the Blueprint W

We have seen that the existence of a Dedekind self-map gives rise to a kind of blueprint for the natural numbers, obtained by considering iterations of the map, and we showed a couple of ways at arriving at such a blueprint. The power of a Dedekind self-map to "precipitate" objects will generalize to a larger context and allow us to provide a very natural account for the existence of large cardinals, as we will see in later sections. Because of the importance of this generative power of Dedekind self-maps, we invest some effort in developing the details of this process. In this section, we set the stage for a somewhat long analysis, showing how the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$, and the successor function $s : \omega \to \omega$, are *formally* (and not just *intuitively*) derivable from any Dedekind self-map.

We begin with our formal proposal for a new version of the Axiom of Infinity:

New Axiom of Infinity. *There is a Dedekind self-map.*

Notice how stating the Axiom of Infinity in this way really causes a shift in viewpoint—a shift away from the idea that "infinite" means a vast collection of discrete objects and towards the recognition that the reality of the "infinite" is self-referral dynamics of an unbounded field. The discrete values that are usually taken as the "reality" of infinite sets can now be seen as derivable side-effects of the dynamics of this "field."

It is well-known to set theorists[31] that, from the ZFC axioms, minus the usual Axiom of Infinity (the theory ZFC−Infinity), one can prove that the usual Axiom of Infinity and the New Axiom of Infinity are equivalent; using this known equivalence, we could then use the existence of an inductive set to demonstrate that the sequences $a, j(a), j(j(a)), \ldots$ and $\mathrm{id}_A, j, j \circ j, \ldots$, described in the previous section, do indeed form sets that each represent—in a sense that can be made precise—the set ω of natural numbers.

We do not wish to rely on this standard result, however. The usual proof either already assumes ω exists, or else uses a "proper class" version of the natural numbers to then derive a "set" version of them.[32]

[31]See for example [32, p. 97]. We also establish this result here; see Remark 6 on p. 24.

[32]For the interested reader, we briefly outline this approach here. We will work in a model[33] of ZFC − Infinity and, because the axioms for arithmetic, PA (Peano Arithmetic), are interpretable in that theory, it follows that one can refer to a (possibly proper) *class* of natural numbers for V; we denote this class $\overline{\omega}$. The usual notions of induction and inductive definition can be shown to hold relative to $\overline{\omega}$ in almost exactly the same way they hold for ω even without the Axiom of Infinity. Then, one can, given a Dedekind self-map $j : A \to A$, define another *class* $W = \{a, j(a), j(j(a)), \ldots\}$ in V by proceeding as follows: Define a *class function* $\mathbf{F}$ on $\overline{\omega}$ by

$$\begin{aligned} \mathbf{F}(0) &= a \\ \mathbf{F}(n+1) &= j(\mathbf{F}(n)). \end{aligned}$$

One may then define the class W by $W = \operatorname{ran} \mathbf{F}$. One then shows that W is in fact a *set* using the Separation Axiom as follows:

$$W = \operatorname{ran} \mathbf{F} \cap A.$$

This quick approach to defining the blueprint $W = \{a, j(a), j(j(a)), \ldots\}$, and then producing the actual set ω of natural numbers, though mathematically sound, is somewhat unsatisfying since it makes use of one version of the natural numbers that does not require them to form a set in order to prove that they *do* form a set. It is more revealing, we feel, to see how the dynamics of our Dedekind self-map j give rise to the set of natural numbers without any reliance on the notion of a natural number initially.

We will therefore take a somewhat longer journey[34] than is usually done to arrive at the natural numbers and to establish the equivalence of these two axioms of infinity. We will use the following outline to guide our steps of reasoning:

(A) We will define the concept of a j-inductive subset of A and we will let W denote the *smallest* j-inductive set. (In this step, we do not assume that W contains the values $a, j(a), j(j(a)), \ldots$; this will be proved later on.)

(B) We will define an order relation E on W. Roughly speaking, we will say that $x \mathrel{\mathsf{E}} y$ if and only if one can obtain y from x by applying j at most fintely many times to x: $y = j(j \ldots (j(x)) \ldots)$. This will be done without reliance on any notion of natural number. (Since the usual definition of "finite" involves the natural numbers by definition, some cleverness will be required.)

(C) We will show that E is a wellfounded partial order.[35]

(D) Using slightly different methods from those used in (A)–(C), we will show that E is a total order as well, so that in fact, E is a well-ordering of W.[36]

These four points will provide us with a blueprint W equipped with a natural well-ordering, and will set the stage for the final step, in which we derive the concrete set of natural numbers, and the canonical successor function, by way of the Mostowski Collapsing Map. The details of this final step, which will be the subject of Section 8, are essentially the same, regardless of how we arrive at W, whether we use the "fast" approach mentioned above, or the longer approach that we will develop here.

Therefore the collection $W = \{a, j(a), j(j(a)), \ldots\}$ does indeed form a set, and one may derive the canonical *set* ω of natural numbers from W. The technique for collapsing W to the real set ω of natural numbers is worked out in Section 8.

Regarding this class $\overline{\omega}$, after we have verified that the set of natural numbers can be derived from a Dedekind self-map *without* the help of the natural numbers in any form, we will come back to this class $\overline{\omega}$, as it is extremely useful when working with models of ZFC − Infinity. A proper development of the properties of $\overline{\omega}$ is given in Section 16.

[34]The reader who does not want to take this longer journey and is satisfied with the definition of the blueprint W just described, which relies on a class of natural numbers, can skip to Section 8 where the set of natural numbers is computed.

[35]A relation R defined on a set X is a *partial order* if the following holds true for any x, y, z in X:

(1) (Irreflexive) It is not the case that $x \mathrel{R} x$.
(2) (Antisymmetric) If $x \mathrel{R} y$, then it is not the case that $y \mathrel{R} x$.
(3) (Transitive) If $x \mathrel{R} y$ and $y \mathrel{R} z$, then $x \mathrel{R} z$.

A definition of *wellfounded partial order* is given on p. 25.

[36]A partial order R on a set X is a *total order* (also called a *linear order*) if, for all $x, y \in X$, exactly one of the following holds: $x < y$, $x = y$, $y < x$. A *well-ordering* of X is a total order R with the additional property that every nonempty subset of X has an R-least element. One can show as an exercise that every well-founded total order is a well-ordering.

This derivation will show how the set ω and the successor function arise as the *collapse* of the set $W = \{a, j(a), j(j(a)), \ldots\}$ of the blueprint and of the restriction $j \restriction W$. The logic behind this derivation works because the ordering $\sqsubseteq$ defined on the blueprint W gives us a well-ordering of W that is isomorphic to the natural well-ordering on ω.

Remark 1. Once we have in this way derived ω and $s : \omega \to \omega$ from any given Dedekind self-map, we will have established the equivalence, in ZFC$-$Infinity, of the Axiom of Infinity and the New Axiom of Infinity, since the other direction (Axiom of Infinity implies New Axiom of Infinity) follows from the following observations:

(i) The Axiom of Infinity implies ω and the successor function $s : \omega \to \omega$ exist, and
(ii) The function $s : \omega \to \omega$ is itself a Dedekind self-map.

7. Obtaining the Blueprint W

This section will complete the outline of steps (A)–(D), listed in the last section. We begin by fixing, for this section, a Dedekind self-map $j : A \to A$ with critical point a.

Establishing (A). We define the concept of a j-inductive set: A set $B \subseteq A$ will be called *j-inductive* if

(1) $a \in B$,
(2) whenever $x \in B$, $j(x)$ is also in B.

Notice that A itself is j-inductive. Therefore, the set $\mathcal{I} = \{B \subseteq A \mid B$ is j-inductive$\}$ is nonempty. Let $W = \bigcap \mathcal{I}$.

Remark 2. For later reference, we observe here that W is definable from j and a.

Lemma 1. *W is a j-inductive subset of A.*

Proof. For (1), since a belongs to every j-inductive subset of A, $a \in W$. For (2), assume $x \in W$. Then x belongs to every j-inductive subset of A. For each such j-inductive subset B, since $x \in B$, $j(x) \in B$. Therefore $j(x) \in W$. $\square$

Establishing (B). As mentioned earlier, we wish to define an order relation $\sqsubseteq$ on W by declaring that, for all $x, y \in W$, $x \sqsubseteq y$ if and only if y can be obtained by finitely many applications of j to x: $y = j(j(\ldots(j(x))\ldots))$. The difficulty with this definition in our present context is the word "finite," since a set is ordinarily defined to be finite if it can be put in 1-1 correspondence with one of the elements n of ω (recall that each such n is equal to its set $\{0, 1, 2, \ldots, n-1\}$ of predecessors).

To get around this difficulty, we will reword our requirements on $\sqsubseteq$ so that no mention of "finiteness" is necessary; the properties of the underlying structure W will ensure in the background that the number of applications of j that actually occur to reach from x to y whenever $x \sqsubseteq y$ will always be finite, but we will not need to prove this or make this fact explicit at any point in this early stage of development.

In order to define E properly, we will first define a more elementary relation E_0 on W: We shall say that, for all $x, y \in W$, $x \mathrel{\mathsf{E}_0} y$ if and only if $y = j(x)$. We prove some basic facts about j and E_0.

We define several concepts pertaining to relations. For any relation R on a set X, an *R-minimal element* of X is an $x \in X$ such that for all $y \in X$, it is not the case that $y \mathrel{R} x$. Moreover, R is said to be *wellfounded* if every nonempty subset Y of X has an R-minimal element. A familiar example is the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers with its less than relation $<$ (so here, R is simply $<$); $<$ is wellfounded as every nonempty subset of $\mathbb{N}$ has a smallest element. We will not make use of this example here, of course, since the natural numbers have not yet formally been defined; the example is intended just to give some concrete sense for the new definitions.

Claim 1. j has no fixed points in W; that is, $j(x) \neq x$ for all $x \in W$. Consequently, E_0 is irreflexive.

Proof. The last part follows immediately from the fact that j has no fixed points. We prove the first part: Suppose $B \subseteq A$ is j-inductive and $x \in B$ is a fixed point of j. We observe that $B - \{x\}$ is also j-inductive: First, notice $a \neq x$ since $a \notin \operatorname{ran} j$ but $x = j(x) \in \operatorname{ran} j$. Therefore, $a \in B - \{x\}$. Suppose $y \in B - \{x\}$; we show $j(y) \in B - \{x\}$. Certainly $j(y) \in B$ since B is j-inductive. If $j(y) = x$, then since $j(x) = x$ too, we have $j(x) = j(y)$, and since j is 1-1, $x = y$, which is impossible. Therefore $j(y) \in B - \{x\}$. We have shown $B - \{x\}$ is j-inductive.

To prove the claim, suppose $B \subseteq A$ is a j-inductive set that contains a fixed point x of j. As we have seen, $B - \{x\}$ is j-inductive, and by leastness of W, $W \subseteq B - \{x\}$. Therefore $B \neq W$. This proves the claim. □

Remark 3. An immediate consequence of Claim 1 is that $a \neq j(a)$; this of course also follows from the fact that $a \notin \operatorname{ran} j$. From the latter observation, we also conclude that $a \neq j(j(a))$. And by j's 1-1 property, $j(a) \neq j(j(a))$. Therefore, $a, j(a), j(j(a))$ are distinct.

Claim 2. E_0 is wellfounded.

Proof. Suppose $X \subseteq W$ is a subset having no E_0-minimal element. Let $B \subseteq W$ be defined by

$$B = \{x \in W \mid x \notin X\}.$$

We show that B is j-inductive and therefore equal to W; the conclusion will then be that X itself is empty, as required.

First we show that $a \in B$. If not, $a \in X$, then there is a $b \in X$ such that $b \mathrel{\mathsf{E}_0} a$; that is, $j(b) = a$. But this is impossible since $a \notin \operatorname{ran} j$.

Next, assume $x \in B$; we show $j(x) \in B$. Assume for a contradiction that $j(x) \notin B$, so $j(x) \in X$. Since X has no E_0-minimal element, there is $u \in X$ with $u \mathrel{\mathsf{E}_0} j(x)$, that is, $j(u) = j(x)$. Since j is 1-1, $u = x$. But this is impossible because $u \in X$ but x is not.

This completes the proof that B is j-inductive and that W is empty. □

Claim 3. $\mathtt{E}_0$ is antisymmetric.

Proof. Suppose $x, y \in W$ and $x \mathrel{\mathtt{E}_0} y$. If $y \mathrel{\mathtt{E}_0} x$ also, then the set $\{x, y\}$ is a nonempty subset of W with no $\mathtt{E}_0$-minimal element. □

The relation $\mathtt{E}_0$ is a first try at defining a prototype of the less than relation on the natural numbers. It has many of the characteristics that are needed, but it is not transitive: Whenever x, y, z are distinct elements of W (by Remark 3, such distinct elements exist in W) and we have $x \mathrel{\mathtt{E}_0} y$ and $y \mathrel{\mathtt{E}_0} z$, it is never the case that $x \mathrel{\mathtt{E}_0} z$, since this would imply $j(x) = z = j(y)$, which would in turn imply $x = y$.

To obtain a transitive order relation, we will expand $\mathtt{E}_0$ to a relation $\mathtt{E}$ as follows: For all $x, y \in W$, $x \mathrel{\mathtt{E}} y$ if and only if there is a subset F of W satisfying the following:

(i) $x, y \in F$,
(ii) for some $v \in F$, $y = j(v)$,
(iii) there is no $u \in F$ for which $x = j(u)$,
(iv) if $u \in F$ and $u \neq y$, then $j(u) \in F$,
(v) if $v \in F$ and $v \neq x$, there is $u \in F$ such that $v = j(u)$.

The set F is said to *join x to y*, and is called a *joining set*.

Intuitively speaking, $x \mathrel{\mathtt{E}} y$ if and only if $y = j^n(x)$ for some natural number n. Our formal definition of $\mathtt{E}$ is a way of capturing this idea without the use of natural numbers. As this intuition suggests, any joining set F must be finite, but we do not state this in the definition, nor try to prove it (in fact, at this point, we do not even have a definition of "finite").[37]

Establishing (C).

We begin by proving some facts about $\mathtt{E}$ and joining sets.

Claim 4. $\mathtt{E}_0 \subseteq \mathtt{E}$. That is, whenever $x, y \in W$ and $x \mathrel{\mathtt{E}_0} y$, then $x \mathrel{\mathtt{E}} y$.

Proof. Suppose $x \mathrel{\mathtt{E}_0} y$, so $y = j(x)$. Let $F = \{x, y\}$. We show F joins x to y. Parts (i), (ii), (iv), (v) in the definition of $\mathtt{E}$ are obviously true for F. We prove that (iii) also holds: We must verify that neither of the following holds: (a) $j(x) = x$, (b) $j(y) = x$. (a) fails because j has no fixed points; (b) fails because $\mathtt{E}_0$ is antisymmetric. We have established (iii); the result follows. □

Claim 5. $\mathtt{E}$ is irreflexive.

Proof. Suppose $x \in W$. If $x \mathrel{\mathtt{E}} x$, let F join x to x. By part (ii) in the definition of joining sets, there is $u \in F$ such that $x = j(u)$, but by (iii), there is *no* $u \in F$ for which $x = j(u)$. This contradiction shows that it is never the case that $x \mathrel{\mathtt{E}} x$. □

[37]After the necessary preliminaries have been established, we will give the usual definition of "finite" and prove in Theorem 24 that joining sets are indeed always finite.

Claim 6. E is wellfounded.

Proof. We use the same strategy as was used in the proof of Claim 2. Suppose $X \subseteq W$ is a set with no E-minimal element. Let $B = \{x \in W \mid x \notin X\}$. We show B is j-inductive.

We first prove that $a \in B$. Suppose, for a contradiction, that $a \in X$. Since X has no E-minimal element, there is $x \in X$ with $x \mathsf{E} a$. Let $F \subseteq X$ join x to a. Then there is $u \in F$ such that $j(u) = a$. But this is impossible since $a \notin \operatorname{ran} j$.

Next, suppose $x \in B$; we show $j(x) \in B$. Suppose not; then $j(x) \in X$ and so for some $u \in X$, $u \mathsf{E} j(x)$. Let F join u to $j(x)$. By property (ii) of the definition of E, there is some $v \in F \subseteq X$ such that $v \mathsf{E}_0 j(x)$, that is, $j(v) = j(x)$. Since j is 1-1, $v = x$. But this is impossible since $v \in X$ but x is not. □

We extract an observation made in the proof of Claim 6:

Claim 7. For all $x \in W$, $x \not\mathsf{E} a$. □

Claim 8. E is anti-symmetric; that is, for all $x, y \in W$, if $x \mathsf{E} y$ then it is not the case that $y \mathsf{E} x$.

Proof. Suppose $x, y \in W$ are such that $x \mathsf{E} y$ and also $y \mathsf{E} x$. Then $\{x, y\}$ has no E-minimal element, contradicting Claim 6. □

Claim 9. For all $x \in W$, if $a \neq x$, then $a \mathsf{E} x$.

Proof. Let $B = \{z \in W \mid a = z \text{ or } a \mathsf{E} z\}$. It suffices to show B is a j-inductive set. Since it is obvious that $a \in B$, it suffices to assume $z \in B$ and prove $j(z) \in B$. If $a = z$, then certainly $a \mathsf{E} j(a)$, and so $j(z) \in B$. If $a \neq z$, then, since $z \in B$, we have $a \mathsf{E} z$. Let F join a to z. Let $G = F \cup \{j(z)\}$. We wish to show that G joins a to $j(z)$. We verify that G satisfies (i)–(v) above. Properties (i), (ii), (iv), and (v) are immediate. For (iii), certainly no $u \in F$ has the property that $u \mathsf{E} a$, since F joins a to z (note (iii) already holds for F). But $j(z) \not\mathsf{E} a$ either because of Claim 7. We have shown B is j-inductive, as required. □

Claim 10. Suppose $x, y \in W$.

(1) $x \mathsf{E} y$ if and only if $j(x) \mathsf{E} j(y)$.
(2) $x \mathsf{E} j(j(x))$.
(3) Suppose $x \mathsf{E} y$ and $y \neq j(x)$. Then $j(x) \mathsf{E} y$.
(4) Suppose $x \mathsf{E} j(y)$ and $x \neq y$. Then $x \mathsf{E} y$.
(5) Suppose $x \mathsf{E} y$. Then $j(y) \not\mathsf{E} x$.
(6) If $x \mathsf{E} y$, then $x \mathsf{E} j(y)$.
(7) If $j(x) \mathsf{E} y$, then $x \mathsf{E} y$.
(8) If $x \neq a$, there is $u \in W$ such that $j(u) = x$. Moreover, there is no $v \in W$ such that $u \mathsf{E} v \mathsf{E} x$.

(9) Suppose F joins x to y, $u \in F$, and $u \neq x$. Then $x \sqsubseteq u$.
(10) Suppose F joins x to y and $u \in F$. Then $y \not\sqsubseteq u$.
(11) Suppose F joins x to y, $u \in F$, and $u \neq y$. Then $u \sqsubseteq y$.
(12) Suppose F joins x to y. Then F is *connected*; that is, whenever $u \in W$ and $x \sqsubseteq u \sqsubseteq y$, then $u \in F$.

Regarding part (8) of Claim 10, for a given $x \in W$ different from a, we will call an element $u \in W$ for which $j(u) = x$ an $\sqsubseteq$*-immediate predecessor of* x, or, when the context makes it clear, an *immediate predecessor of* x. Since j is 1-1, if x has an immediate predecessor, it is unique. Part (8) says that every element of W other than a has an immediate predecessor. The notion "immediate predecessor" is to be distinguished from the unqualified "predecessor": Whenever $x \sqsubseteq y$, x is called a *predecessor* of y, but x is not necessarily the immediate predecessor of y.

Proof of (1). For one direction, suppose $x \sqsubseteq y$ and F joins x to y. It is straightforward to verify that if $G = \{j(u) \in W \mid u \in F\}$, then G joins $j(x)$ to $j(y)$. Likewise, for the other direction, assuming $j(x) \sqsubseteq j(y)$ and G joins $j(x)$ to $j(y)$, one easily verifies that $F = \{u \in W \mid j(u) \in G\}$ joins x to y.

Proof of (2). Let $B = \{x \in W \mid x \sqsubseteq j(j(x))\}$. We show B is j-inductive. By Claim 9, $a \in B$. Assume $x \in B$; we show $j(x) \in B$. Since $x \in B$, $x \sqsubseteq j(j(x))$. By Claim 10(1), $j(x) \sqsubseteq j(j(j(x)))$. It follows that $j(x) \in B$, as required.

Proof of (3). Suppose $x \sqsubseteq y$ and $y \neq j(x)$. Let F join x to y. Let $G = F - \{x\}$. By (iv) in the definition of joining sets, $j(x) \in G$. It is straightforward to verify that G joins $j(x)$ to y.

Proof of (4). Suppose $x \sqsubseteq j(y)$ and $x \neq y$. Let F join x to $j(y)$. By (v) in the definition of joining sets, there is $u \in F$ such that $j(u) = j(y)$. Since j is 1-1, $u = y$, so $y \in F$. Let $G = F - \{j(y)\}$. It is straightforward to verify that G joins x to y.

Proof of (5). Fix $x \in W$. Let $B = \{y \in W \mid \text{if } x \sqsubseteq y \text{ then } j(y) \not\sqsubseteq x\}$. We show B is j-inductive. Vacuously, $a \in B$. Assume $y \in B$; we show $j(y) \in B$. Assume $x \sqsubseteq j(y)$. We consider two cases: $x = y$ or $x \neq y$.

If $x = y$, then by Claim 10(2), $y \sqsubseteq j(j(y))$. Since $\sqsubseteq$ is antisymmetric, then $j(j(y)) \not\sqsubseteq y$. Therefore, in this case, $j(y) \in B$.

Now assume $x \neq y$. By Claim 10(4), $x \sqsubseteq y$. If $j(j(y)) \sqsubseteq x$, then, using Claim 10(2) again, we have the following cycle:

$$j(j(y)) \sqsubseteq x \sqsubseteq y \sqsubseteq j(j(y)),$$

and the set $\{x, y, j(j(y))\}$ has no $\sqsubseteq$-minimal element, contradicting wellfoundedness of W. Therefore, $j(j(y)) \not\sqsubseteq x$ and so $j(y) \in B$ in this case as well. This completes the proof that B is j-inductive. We have shown therefore that for all $x, y \in W$, if $x \sqsubseteq y$, then $j(y) \not\sqsubseteq x$.

Proof of (6). Suppose $x \sqsubseteq y$. We show $x \sqsubseteq j(y)$. Let F join x to y. Let $G = F \cup \{j(y)\}$. We show G joins x to $j(y)$. Verification of (i),(ii), (iv), (v) in the definition of $\sqsubseteq$ is straightforward. We check that (iii) holds: Suppose $u \in G$; we must show $j(u) \neq x$. Certainly $j(u) \neq x$ if $u \in F$ (since F joins x to y). We consider the case in which $u = j(y)$. If $j(j(y)) = x$, then since $j(y) \sqsubseteq j(j(y))$, it follows that $j(y) \sqsubseteq x$, and this contradicts (5). Therefore, in this case also, $j(u) \neq x$, and (iii) is established. We have shown G joins x to $j(y)$, and so $x \sqsubseteq j(y)$.

Proof of (7). Suppose $j(x) \sqsubseteq y$. We show $x \sqsubseteq y$. Let F join $j(x)$ to y and let $G = F \cup \{x\}$. We show G joins x to y. Verification of (i),(ii), (iv), (v) in the definition of $\sqsubseteq$ is straightforward. We check that (iii) holds: We must show that $j(u) \neq x$ for any $x \in G$. First we observe that if $u = j(x)$, $j(u) \neq x$: If $j(j(x)) = x$, then $j(x) \sqsubseteq x$, violating antisymmetric property of $\sqsubseteq$. Next, suppose $u \in F$ is such that $j(u) \sqsubseteq x$. By Claim 10(6), $j(u) \sqsubseteq j(x)$, but this contradicts the fact that F joins $j(x)$ to y (in particular, (iii) is violated). Therefore, no such u exists, and (iii) holds for G as well. We have shown G joins x to y, and so $x \sqsubseteq y$.

Proof of (8). For the first part, if $x \neq a$ then, by Claim 9, $a \sqsubseteq x$, and so there is an F that joins a to x. By property (ii) in the definition of joining sets, there is $u \in F$ such that $j(u) = x$, as required.

For the second part, assume, by way of contradiction, that there are $u, v \in W$ with $u \sqsubseteq v \sqsubseteq j(u)$. Since $v \sqsubseteq j(u)$, by Claim 10(4), either $v = u$ or $v \sqsubseteq u$. But if $v = u$, we would have $u \sqsubseteq u$ (violating Claim 5), and if $v \sqsubseteq u$, it would violate the antisymmetric property (violating Claim 8). Therefore, there does not exist a $v \in W$ for which $u \sqsubseteq v \sqsubseteq j(u)$.

Proof of (9). Suppose F joins x to y, $u \in F$, and $u \neq x$. We show $x \sqsubseteq u$. Let $B = \{u \in W \mid \text{if } u \in F \text{ and } x \neq u, \text{ then } x \sqsubseteq u\}$. We show B is j-inductive. To show $a \in B$, there are three possibilities. If $a \notin F$ or if $a \in F$ and $a = x$, then $a \in B$, vacuously. The third possibility—that $a \in F$ and $a \neq x$—is impossible by (v) in the definition of joining sets, since (v) implies there is $z \in F$ such that $j(z) = a$, and this is impossible since a is a critical point of j.

Next, assume $u \in B$; we show $j(u) \in B$. Assume $j(u) \in F$ and $x \neq j(u)$. There are two cases: (i) $x = u$; (ii) $x \neq u$. If $x = u$, then $x \sqsubseteq j(u)$, as required. If $x \neq u$, then since $u \in B$, $x \sqsubseteq u$. It follows by Claim 10(6) that $x \sqsubseteq j(u)$. In each case we have shown that $j(u) \in B$, and so B is j-inductive. The result follows.

Proof of (10). We show that whenever F joins x to y and $u \in F$, then $y \not\sqsubseteq u$. Let $B = \{u \in W \mid \text{if } u \in F, \text{ then } y \not\sqsubseteq u\}$. Since for no $z \in W$ is it true that $z \sqsubseteq a$, we have that $a \in B$. Assume $u \in B$; we show $j(u) \in B$. Assume $j(u) \in F$; we show $y \not\sqsubseteq j(u)$. There are two cases: (i) $u \notin F$ and (ii) $u \in F$.

For (i), using the fact that $u \notin F$, we claim that $j(u) = x$: Suppose not. Then by (v) in the definition of joining sets, there must be $v \in F$ with $j(v) = j(u)$. It follows that $u = v$, which is impossible since $v \in F$ but $u \notin F$. This proves the claim. But now, by antisymmetry of $\sqsubseteq$, $y \not\sqsubseteq x$ (since $x \sqsubseteq y$ by our initial assumption), and so $j(u) \in B$.

For (ii), since $u \in B$, it follows that $y \not\sqsubseteq u$. If $y \sqsubseteq j(u)$, then $y \sqsubseteq u$ (by Claim 10(4)), giving a contradiction. Therefore, $y \not\sqsubseteq j(u)$ and $j(u) \in B$.

We have shown in each case that $u \in B$ implies $j(u) \in B$, and so B is j-inductive.

Proof of (11). Suppose F joins x to y, $u \in F$, and $u \neq y$. We show $u \sqsubseteq y$. Let $B = \{u \in W \mid \text{if } u \in F \text{ and } y \neq u, \text{ then } u \sqsubseteq y\}$. We show B is j-inductive. Vacuously, $a \in B$. Assume $u \in B$; we show $j(u) \in B$. Assume $j(u) \in F$ with $j(u) \neq y$. We show $j(u) \sqsubseteq y$. There are two cases: (i) $u \notin F$, (ii) $u \in F$.

For (i), we use the logic in (10), case (i), to conclude that $j(u) = x$. But by assumption $x \sqsubseteq y$. We have shown $j(u) \in B$ in this case.

For (ii), since $u \in F$, I claim we must have $y \neq u$: If $y = u$, then $y \sqsubseteq j(u)$, but, since $j(u) \in F$, this contradicts the result established in (10). Therefore $y \neq u$. Since $u \in B$, $u \sqsubseteq y$. By Claim 10(3), because we are assuming $j(u) \neq y$, it follows that $j(u) \sqsubseteq y$. Therefore, once again, $j(u) \in B$.

We have shown $j(u) \in B$ in both cases. Therefore, B is j-inductive, as required.

Proof of (12). We show by j-induction that joining sets are always connected. Let $x \in W$. Let

$$B = \{y \in W \mid \text{every } F \text{ that joins } x \text{ to } y \text{ is connected}\}.$$

We prove B is j-inductive. Note that $a \in B$ since no set F joins x to a. Assume $y \in B$; we must show $j(y) \in B$. There are two cases to consider:

Case I. $x \not\sqsubseteq y$. To show $j(y) \in B$, we first consider the possibility that $x = y$. Let F join y to $j(y)$. By Claim 10(8), there is no $v \in W$ for which $y \sqsubseteq v \sqsubseteq j(y)$. Therefore, vacuously, F is connected and $j(y) \in B$.

The other possibility is that $x \neq y$. We show in this case that $x \not\sqsubseteq j(y)$: Assume for a contradiction that $x \sqsubseteq j(y)$. Then by Claim 10(4), it follows that $x \sqsubseteq y$, which is impossible for this case. But now since $x \not\sqsubseteq j(y)$, $j(y) \in B$ (since there is *no* F that joins x to $j(y)$).

Case II. $x \sqsubseteq y$. Suppose F joins x to $j(y)$ and $v \in W$ is such that $x \sqsubseteq v \sqsubseteq j(y)$; we must show $v \in F$. By Claim 10(4), either $v = y$ or $v \sqsubseteq y$. Suppose $v = y$. Recall by the definition of joining set that there must be $z \in F$ with $j(z) = j(y)$; but because j is 1-1, we conclude $z = y$. We have shown $v \in F$.

The other possibility is that $v \sqsubseteq y$. Let $F_0 = F - \{j(y)\}$. We show that F_0 joins x to y. By (ii) of the definition of joining set, it follows (as we argued a moment ago) that $y \in F$; therefore (i) in the definition of joining set is satisfied for F_0. Because $y \in F$, by (v) of this definition, it follows that for some $t \in F_0$, $y = j(t)$, and (ii) is satisfied for F_0. Parts (iii)–(v) of the definition, relative to F_0, follow immediately from the fact that F itself is a joining set. We have shown F_0 joins x to y. Since $y \in B$, it follows that F_0 is connected. Since for this case we have $x \sqsubseteq v \sqsubseteq y$, it follows that $v \in F_0$. Therefore, $v \in F$. We have shown $j(y) \in B$.

Continuing the main proof, we have shown B is j-inductive. Therefore, joining sets are always connected. □

Claim 11. Suppose $x, y, z \in W$ are distinct. Then if F joins x to y and G joins y to z, then $F \cup G$ joins x to z. In particular, E is transitive.

Proof. Verification of properties (i), (ii), (iv), (v) for $F \cup G$ is straightforward. For (iii), suppose $u \in F \cup G$ is such that $j(u) = x$. Since F joins x to y, it follows that $u \in G$ and $u \notin F$. In particular, $u \neq y$. By Claim 10(9), $y \mathsf{E} u$. By Claim 10(6), $y \mathsf{E} j(u)$, and so $y \mathsf{E} x$. But this violates the antisymmetric property of E (since $x \mathsf{E} y$ by assumption). Therefore, for no $u \in F \cup G$ is it the case that $j(u) = x$. Therefore, we have established that all properties (i)–(v) hold for $F \cup G$, as required. □

Theorem 2. (W, E) *is a wellfounded partial order.*

Proof. This follows from Claims 5, 6, 8, and 11. □

Establishing (D).

We are now ready to show that E is a well-ordering. We will make use of the following definition:

Definition 3. (Set of Predecessors) Suppose $x \in W$. The *set of predecessors of* x, denoted W_x, is defined by

$$W_x = \{z \in W \mid z \mathsf{E} x\}.$$

Lemma 3. *Suppose* $x, y \in W$.
(1) *If* $y \in W_x$, *then* $W_y \subseteq W_x$.
(2) *Suppose* $W_x \subseteq W_{j(y)}$, *but* $W_x \not\subseteq W_y$. *Then* $W_x = W_{j(y)}$.

Proof of (1). If $z \mathsf{E} y$, then since $y \mathsf{E} x$, it follows from transitivity of E (Claim 11 above) that $z \mathsf{E} x$ and $z \in W_x$. □

Proof of (2). By the hypotheses, the only element of W_x that is not in W_y is y. Since $y \mathsf{E} x$, it follows that each z for which $z \mathsf{E} y$ satisfies $z \mathsf{E} x$ (by transitivity of E), and so we have both $y \in W_x$ and $W_y \subseteq W_x$, that is, $W_y \cup \{y\} \subseteq W_x$. It follows from Claim 10(4) above that $W_{j(y)} \subseteq W_x$. Since we also have, by hypothesis, that $W_x \subseteq W_{j(y)}$, the result follows. □

The next lemma[38] shows that the relation E is extensional.

Lemma 4. (Extensionality) *Suppose* $x, y \in W$.
(1) *Suppose* $W_x \subseteq W_y$. *Then either* $x = y$ *or* $x \mathsf{E} y$.
(2) *Suppose* $W_x = W_y$. *Then* $x = y$.

Proof of (1). Suppose $W_x \subseteq W_y$. Assume $x \neq y$ and $x \not\mathsf{E} y$; we will arrive at a contradiction. We first observe that $x \neq a$: If $x = a$, then $a \neq y$ (since $x \neq y$). But we also have $a \not\mathsf{E} y$ (since $x \not\mathsf{E} y$), and this contradicts Claim 9. By Claim 10(8), x

[38]In a private communication, Martial Leroy pointed out the need for this lemma (and provided a proof) as a preliminary to Lemma 5.

has an immediate predecessor u and $x = j(u)$. Then we have $u \in W_x \subseteq W_y$ and so $u \sqsubseteq y$. By Claim 10(1), $x = j(u) \sqsubseteq j(y)$. By Claim 10(4), either $x = y$ or $x \sqsubseteq y$, and this contradicts our initial assumption. We have shown that either $x = y$ or $x \sqsubseteq y$. □

Proof of (2). Suppose $W_x = W_y$. By part (1), both of the following disjunctions hold:

(a) $x = y$ or $x \sqsubseteq y$.
(b) $y = x$ or $y \sqsubseteq x$.

By the irreflexive and antisymmetric properties of $\sqsubseteq$, it follows that $x = y$. □

Lemma 5. *Suppose $x, y \in W$. Then either $W_x \subseteq W_y$ or $W_y \subseteq W_x$.*

Proof. Let $x \in W$ and let $A_x = \{y \in W \mid \text{either } W_x \subseteq W_y \text{ or } W_y \subseteq W_x\}$. We use j-induction to show that $A_x = W$, and this will complete the proof.

For the base case, it is clear that $a \in A_x$, since $W_a = \emptyset \subseteq W_x$.

Next, assume $y \in A_x$, so that either $W_x \subseteq W_y$ or $W_y \subseteq W_x$. We show $j(y) \in A_x$.

Case I. $W_x \subseteq W_y$. In that case, since $W_y \subseteq W_{j(y)}$, it follows that $W_x \subseteq W_{j(y)}$, and we have $j(y) \in A_x$.

Case II. $W_y \subseteq W_x$. By Lemma 4(1), either $y = x$ or $y \sqsubseteq x$. If $y = x$, then $W_x = W_y$ and we have (by Claim 10(4)): $W_x \subseteq W_y \subseteq W_y \cup \{y\} = W_{j(y)}$. If $y \sqsubseteq x$, then $W_{j(y)} = W_y \cup \{y\} \subseteq W_x$. Either way, we have shown that $j(y) \in A_x$.

We have shown that A_x is j-inductive and is equal to W, as required. □

For the next theorem, let us say that two elements x, y of W are a *decidable pair* if exactly one of the following holds:

(7) $$x \sqsubseteq y,\ x = y,\ y \sqsubseteq x.$$

Theorem 6. ($\sqsubseteq$ Is a Well-Ordering) *Suppose $x, y \in W$. Then x, y is a decidable pair. Therefore, W is well-ordered by $\sqsubseteq$.*

Proof. We first observe that at most one of the three possibilities listed in display (7) can hold: If any two of the conditions hold, it would violate either the irreflexive or the antisymmetric property of $\sqsubseteq$.

Next, we establish a preliminary result:

Claim. For any $z \in W$, any x, y that are both predecessors of z form a decidable pair.

Proof of Claim. Let

$$B = \{z \in W \mid \text{for all } x, y \in W, \text{ if } x \sqsubseteq z \text{ and } y \sqsubseteq z, \text{ then } x, y \text{ is a decidable pair}\}.$$

We show B is j-inductive.

First note that $a \in B$, vacuously. Assume $u \in B$. We show $j(u) \in B$. Suppose $x \sqsubseteq j(u)$ and $y \sqsubseteq j(u)$. Then one of the following must be true:

(a) $x \sqsubseteq u$ and $y \sqsubseteq u$.
(b) $x \sqsubseteq u$ and $y = u$.
(c) $x = u$ and $y \sqsubseteq u$.
(d) $x = u$ and $y = u$.

If $x = u$ and $y = u$, then $x = y$. If $x \sqsubseteq u$ and $y = u$ or $x = u$ and $y \sqsubseteq u$, then x, y is clearly a decidable pair. And if $x \sqsubseteq u$ and $y \sqsubseteq u$, then x, y is a decidable pair because of the assumption that $u \in B$. This completes the proof of the claim. □

To prove the theorem, suppose $x, y \in W$. We show x, y is a decidable pair. Consider the set $C = W_{j(x)} \cup W_{j(y)}$. Certainly, $x \in C$ and $y \in C$. To complete the proof, it will be sufficient to show there is $z \in W$ for which $C = W_z$, since, having shown this, we can conclude that x, y is a decidable pair because of the claim just proved.

By Lemma 5, either $W_{j(x)} \subseteq W_{j(y)}$ or $W_{j(y)} \subseteq W_{j(x)}$, and so $W_{j(x)} \cup W_{j(y)}$ is equal to one of $W_{j(x)}, W_{j(y)}$. Either way, $C = W_z$ for some $z \in W$. This completes the proof. □

A useful corollary that now follows from Theorem 6 and Claim 10(8) is the following:

Corollary 7. *Let $x \in W$. Then $j(x)$ is the $\sqsubseteq$-least element of $\{z \in W \mid x \sqsubseteq z\}$.* □

Therefore, we may list the elements of W by $a \sqsubseteq j(a) \sqsubseteq j(j(a)) \sqsubseteq \dots$. After we define the natural numbers, we will be able to establish more formally that $W = \{a, j(a), j(j(a)), \dots\}$.[39] For future use, for each $x \in W$, let $W^x = \{z \in W \mid x \sqsubseteq z\}$. Thus, for each $x \in W$, $W = W_x \cup \{x\} \cup W^x$.

For later use, we compute W_x for a few values $x \in W$:

Corollary 8.

(1) $W_a = \emptyset$.
(2) $W_{j(a)} = \{a\}$.
(3) $W_{j(j(a))} = \{a, j(a)\}$.
(4) $W_{j(j(j(a)))} = \{a, j(a), j(j(a))\}$.

Proof of (1). This follows from Claim 7.

Proof of (2). Recall that by Claim 10(4), if $x \sqsubseteq j(a)$, either $x \sqsubseteq a$ or $x = a$. Thus,

$$W_{j(a)} = \{x \in W \mid x \sqsubseteq j(a)\} = \{a\}.$$

Proof of (3). By Claim 10(4) again, if $x \sqsubseteq j(j(a))$, then either $x \sqsubseteq j(a)$ or $x = j(a)$, and the only x satisfying the first of these is a itself. Therefore

$$W_{j(j(a))} = \{x \in W \mid x \sqsubseteq j(j(a))\} = \{a, j(a)\}.$$

[39] This is done in Theorem 21.

Proof of (4). By Claim 10(4) again, if $x \mathrel{\sqsubseteq} j(j(j(a)))$, then either $x \mathrel{\sqsubseteq} j(j(a))$ or $x = j(j(a))$. We have seen in (3) that the x for which $x \mathrel{\sqsubseteq} j(j(a))$ are a and $j(a)$. Therefore,

$$W_{j(j(j(a)))} = \{x \in W \mid x \mathrel{\sqsubseteq} j(j(j(a)))\} = \{a, j(a), j(j(a))\}. \square$$

8. A Derivation of ω and the Successor Function

The previous section, in establishing (A)–(D) of Section 6, demonstrated that $(W, \sqsubseteq)$ is a well-ordered set. In this section, we introduce the *Mostowski Collapsing Map*, which will be used to collapse $(W, \sqsubseteq)$ to $(\omega, \in)$, and to collapse $j \restriction W : W \to W$ to the successor function $s : \omega \to \omega$.

Before describing the Mostowski Collapsing Map, we need to introduce one other concept. A set T is said to be *transitive* if, whenever $z \in T$ and $y \in z$, we have $y \in T$. Transitive sets are the "well-behaved" sets in the universe. It is not hard to verify that $\omega = \{0, 1, 2, \ldots\}$ is transitive, and that each of its elements is transitive.

We now define the *Mostowski Collapsing Map* π on W.[40] The function π will turn out to be a bijection and will have the effect of identifying W with a *transitive* set N whose membership relation $\in$ behaves exactly like the relation $\sqsubseteq$ on W. The Mostowski Collapsing Map will transform W and its internal relationships (specified by $\sqsubseteq$) into a *concrete, well-behaved set*, which lives in the early stages of the universe, but which mirrors the relationships which hold in W under the relation $\sqsubseteq$. We will demonstrate that π collapses the elements of W to corresponding elements of the set $\omega = \{0, 1, 2, 3, \ldots\}$, and collapses W itself to ω. From our initial assessment of W, we would expect that a will be mapped to 0, $j(a)$ to 1, $j(j(a))$ to 2, and so forth; we will verify these expectations in our proof.

Theorem 9. (Mostowski Collapsing Theorem for W) *There is a unique function π defined on W that satisfies the following relation, for every $x \in W$:*

$$\pi(x) = \{\pi(y) \mid y \mathrel{\sqsubseteq} x\}. \tag{8}$$

Proof. Let $B \subseteq W$ be defined by putting $z \in B$ if and only if the formula $\psi(z)$ holds where $\psi(z)$ is the formula $\exists! g\, \phi(z, g)$ and where "$\exists!$" means "there exists exactly one" and $\phi(z, g)$ is the following formula:

$$\text{dom } g = W_z \cup \{z\} \text{ and, for all } x \in W_z \cup \{z\},\ g(x) = \{g(y) \mid y \mathrel{\sqsubseteq} x\}.$$

Whenever there exists a g such that $\phi(z, g)$, we say that g is a *witness for* $\psi(z)$. When such a g defined on $W_z \cup \{z\}$ exists, it will typically be denoted π_z.

We will show that B is j-inductive, and then, from B, obtain the Mostowski Collapsing map. We first observe that $a \in B$: Since there is no $y \in W$ for which $y \mathrel{\sqsubseteq} a$, $\{\pi_a(y) \mid y \mathrel{\sqsubseteq} a\}$ must be empty, no matter how π_a is defined on "predecessors" of a. Therefore, there is one and only one function π_a with domain $W_a \cup \{a\} = \{a\}$ that satisfies $\pi_a(a) = \{\pi_a(y) \mid y \mathrel{\sqsubseteq} a\}$, and that is the function for which $\pi_a(a) = \emptyset$. We have shown that $\psi(a)$ holds, so $a \in B$.

[40] This function can be defined more generally on any wellfounded, extensional relation defined on a set.

Now assume $z \in B$ and let π_z be the unique map defined on $W_z \cup \{z\}$ that is a witness for $\psi(z)$. We prove $j(z) \in B$. We define $\pi_{j(z)}$ on $W_{j(z)} \cup \{j(z)\}$ by

$$\pi_{j(z)}(x) = \begin{cases} \pi_z(x) & \text{if } x \sqsubset j(z), \\ \{\pi_z(y) \mid y \sqsubset j(z)\} & \text{if } x = j(z). \end{cases}$$

Notice that if $y \sqsubset j(z)$, then either $y = z$ or $y \sqsubset z$ (by Claim 10(4)). Therefore, defining $\pi_{j(z)}(x)$ to be $\{\pi_z(y) \mid y \sqsubset j(z)\}$ when $x = j(z)$ makes sense. We verify that $\pi_{j(z)}$ is a witness for $\psi(j(z))$:

If $x \sqsubset j(z)$, then

$$\begin{aligned} \pi_{j(z)}(x) &= \pi_z(x) \\ &= \{\pi_z(y) \mid y \sqsubset x\} \\ &= \{\pi_{j(z)}(y) \mid y \sqsubset x\}. \end{aligned}$$

The last line follows because, by definition of $\pi_{j(z)}$, $\pi_{j(z)}$ agrees with π_z on all y for which $y \sqsubset x$ (since $x \sqsubset j(z)$).

On the other hand, if $x = j(z)$, then

$$\begin{aligned} \pi_{j(z)}(x) &= \{\pi_z(y) \mid y \sqsubset j(z)\} \\ &= \{\pi_{j(z)}(y) \mid y \sqsubset j(z)\}. \end{aligned}$$

Once again, by definition of $\pi_{j(z)}$, $\pi_{j(z)}$ agrees with π_z on y for which $y \sqsubset j(z)$, so the second equality in the display follows from the first.

This shows that a witness $\pi_{j(z)}$ for $\psi(j(z))$ exists; we need to show it is unique. Assume f is defined on $W_{j(z)} \cup \{j(z)\}$ is also a witness for $\psi(j(z))$; in particular, that f satisfies $f(x) = \{f(y) \mid y \sqsubset x\}$. It is not hard to check that $f \upharpoonright W_z$ is a witness for $\psi(z)$, so by uniqueness of π_z as a witness for $\psi(z)$, $f \upharpoonright W_z = \pi_z$. By this observation, we have

$$f(j(z)) = \{f(y) \mid y \sqsubset j(z)\} = \{\pi_z(y) \mid y \sqsubset j(z)\} = \{\pi_{j(z)}(y) \mid y \sqsubset j(z)\} = \pi_{j(z)}(j(z)).$$

Hence $f = \pi_{j(z)}$ and we have established uniqueness. It follows that $j(z) \in B$.

We have shown B is j-inductive, and so $W = B$. We now define the Mostowski Collapsing map π on W as follows: For each $x \in W$,

$$\pi(x) = \pi_x(x).$$

Claim 1. For all $y \in W$,

$$\text{if } x \sqsubset y, \text{ then } \pi_x(x) = \pi_y(x). \tag{9}$$

Proof of Claim 1. Let $B = \{y \in W \mid \text{if } x \sqsubset y \text{ then } \pi_x(x) = \pi_y(x)\}$. We show B is j-inductive. Vacuously, $a \in B$. Suppose $x \in B$; we show $j(x) \in B$. Let y be such that $j(x) \sqsubset y$. Then, using the fact (twice) that $x \in B$, we have

$$\pi_y x = \{\pi_y(u) \mid u \sqsubset x\} = \{\pi_x(u) \mid u \sqsubset x\} = \{\pi_{j(x)}(u) \mid u \sqsubset x\} = \pi_{j(x)}(x).$$

Therefore $j(x) \in B$. Since B is j-inductive, $B = W$ and the result follows. □

We show that π satisfies (8) for each $x \in W$. Using statement (9), we have:

$$\pi(x) = \pi_x(x) = \{\pi_x(y) \mid y \sqsubset x\} = \{\pi(y) \mid y \sqsubset x\},$$

as required.

Finally, we show that π is the unique f satisfying, for all $x \in W$, $f(x) = \{f(y) \mid y \sqsubseteq x\}$. Given any such f, we show $f = \pi$. Let

$$B = \{x \in W \mid \text{for all } y \text{ such that } y = x \text{ or } y \sqsubseteq x,\ f(y) = \pi(y)\}.$$

We show B is j-inductive. The fact that $a \in B$ is immediate; in particular, $\pi(a) = \emptyset = f(a)$. Assume $x \in B$, so that $f(y) = \pi(y)$ for all y for which $y = x$ or $y \sqsubseteq x$. Then since $y \sqsubseteq j(x)$ implies $y = x$ or $y \sqsubseteq x$ (as we observed earlier),

$$f(j(x)) = \{f(y) \mid y \sqsubseteq j(x)\} = \{\pi(y) \mid y \sqsubseteq j(x)\} = \pi(j(x)).$$

To show $j(x) \in B$, we must also show that for $u \sqsubseteq j(x)$, $f(u) = \pi(u)$, but this follows from the fact that $x \in B$. We have shown $j(x) \in B$. Therefore B is j-inductive and $B = W$. It follows that $f = \pi$, as required. □

As we establish properties of the collapsing map π, we will denote its range by N. We establish some important properties of π and N:

Theorem 10. (Properties of π and N)

(1) *N is a transitive set.*
(2) *π is 1-1.*
(3) *For all $x, y \in W$, $x \sqsubseteq y$ if and only if $\pi(x) \in \pi(y)$.*
(4) *$(N, \in)$ is a well-ordered set. In particular, $0 = \emptyset$ is the $\in$-least element of N.*
(5) *Each $n \in N$ is a transitive set. Moreover, $n = \{m \in N \mid m \in n\}$.*

Remark 4. Part (3) of the theorem tells us that the usual membership relation $\in$ on the range N of π exactly parallels the relation $\sqsubseteq$ on W: Whenever x is an $\sqsubseteq$-predecessor of y in W, in the sense of the relation $\sqsubseteq$, the images $\pi(x)$ and $\pi(y)$ have the same relationship to each other, namely, that $\pi(x)$ is an $\in$-predecessor of $\pi(y)$; the converse statement also holds. This property, together with the fact that π is a bijection from W to N, tells us that π is an *order isomorphism* from W to N; this means that $(N, \in)$ is an exact reproduction of $(W, \sqsubseteq)$. Intuitively, we may think of this relationship as indicating that the "blueprint" W has been faithfully reproduced as a concrete (transitive) object in the bottom portion of the universe. As we shall show in a moment, N turns out to be the concrete set ω of natural numbers.

Part (5) tells us that each $n \in N$ consists precisely of the elements of N that precede it in the well-ordering $\in$. Thus, $1 = \{0\}, 2 = \{0, 1\}$, and so forth.

Proof of (1). Suppose $\pi(x) \in N$ and $u \in \pi(x)$. We must show $u \in N$. Since $\pi(x) = \{\pi(y) \mid y \sqsubseteq x\}$, it follows that $u = \pi(y)$ for some $y \in W$. Thus $u \in N$.

Proof of (2). Suppose $\pi(x) = \pi(z)$ but $x \neq z$. Recall by extensionality of $\sqsubseteq$ (Lemma 4) that $W_x \neq W_z$. Without loss of generality, assume there is $u \in W_x - W_z$, so $u \sqsubseteq x$ and $u \not\sqsubseteq z$. Then $\pi(u) \in \pi(x)$. Since $\pi(z) = \pi(x)$, then $\pi(u) \in \pi(z)$, whence $u \sqsubseteq z$, and this is a contradiction. We have shown π is 1-1.

Proof of (3). This follows immediately from the definition of π.

Proof of (4). It is easy to see that the first part follows from (2) and (3); we verify a few of the required points. For the irreflexive property, notice that for $x \in W$, $x \sqsubseteq x$

if and only if $\pi(x) \in \pi(x)$; since the former is false, the latter is also false. Similar reasoning shows $(N, \in)$ is antisymmetric, transitive, and a total order. We verify that $\in$ well-orders N: Suppose $C \subseteq N$ is nonempty. Let B denote the preimages of C in W; that is, $b \in B$ if and only if $\pi(b) \in C$. Let b_0 be the least element of B. Let $c_0 = \pi(b_0)$. We show c_0 is $\in$-least in N: Let $\pi(b) \in C$. Then $b_0 \sqsubseteq b$, and so $\pi(b_0) \in \pi(b)$. Since b was arbitrary, we have shown C has an $\in$-least element.

For the second clause, let $m \in N$. In defining π, we have already seen that $\pi(a) = 0$. Let $x \in W$ be such that $\pi(x) = m$. Since $a \sqsubseteq x$, then by (3), $0 = \pi(a) \in \pi(x) = m$.

Proof of (5). The fact that each $n \in N$ is a transitive set follows from the fact that $\in$ is transitive as an order relation: For all $m, n, r \in N$, if $m \in n$ and $n \in r$, then $m \in r$. To show that $n = \{m \in N \mid m \in n\}$, we perform a computation: Let $x \in W$ be such that $n = \pi(x)$.

$$\begin{aligned} n &= \pi(x) \\ &= \{\pi(y) \mid y \sqsubseteq x\} \\ &= \{\pi(y) \mid \pi(y) \in \pi(x)\} \quad \text{(because } \pi \text{ is an order isomorphism)} \\ &= \{m \in N \mid m \in n\}. \square \end{aligned}$$

Having defined the Mostowski Collapsing Map π on W, we can now demonstrate how π "collapses" the blueprint W to the concrete set ω of natural numbers. We begin with the computation of the first few elements of ω. We will make use of Corollary 9 in which a few values of W_x were computed.

Computation of 0. Here, 0 arises as the collapse of a in W:

$$\pi(a) = \{\pi(y) \mid y \sqsubseteq a\} = \emptyset = 0.$$

Computation of 1. Here, 1 arises as the collapse of $j(a)$ in W. Recall from

Corollary 9 that $W_{j(a)} = \{a\}$.

$$\begin{aligned} \pi(j(a)) &= \{\pi(x) \mid x \sqsubseteq j(a)\} \\ &= \{\pi(x) \mid x \in W_{j(a)}\} \\ &= \{\pi(x) \mid x \in \{a\}\} \\ &= \{\pi(a)\} \\ &= \{0\} \\ &= 1. \end{aligned}$$

Computation of 2. Here, 2 arises as the collapse of $j(j(a))$ in W. Recall from Corollary 9 that $W_{j(j(a))} = \{a, j(a)\}$.

$$\begin{aligned} \pi(j(j(a))) &= \{\pi(x) \mid x \sqsubset j(j(a))\} \\ &= \{\pi(x) \mid x \in W_{j(j(a))}\} \\ &= \{\pi(x) \mid x \in \{a, j(a)\}\} \\ &= \{\pi(a), \pi(j(a))\} \\ &= \{0, 1\} \\ &= 2. \end{aligned}$$

Computation of 3. Here, 3 arises as the collapse of $j(j(j(a)))$ in W. Recall from Corollary 9 that $W_{j(j(j(a)))} = \{a, j(a), j(j(a))\}$.

$$\begin{aligned} \pi(j(j(j(a)))) &= \{\pi(x) \mid x \sqsubset j(j(j(a)))\} \\ &= \{\pi(x) \mid x \in W_{j(j(j(a)))}\} \\ &= \{\pi(x) \mid x \in \{a, j(a), j(j(a))\}\} \\ &= \{\pi(a), \pi(j(a)), \pi(j(j(a))\} \\ &= \{0, 1, 2\} \\ &= 3. \end{aligned}$$

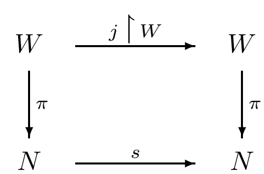
These observations suggest that, at least in some sense, π "gives rise" to the natural numbers. We must be careful not to claim too much here, though, because the numbers $0, 1, 2, 3, \ldots$, with their usual definition as sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$, are already present in the universe, even without the Axiom of Infinity, as part of the way the first stages of the universe are built; these come into being on the basis of the other axioms of ZFC. What the other axioms do *not* tell us, however, is how to form the *set* of natural numbers, and, in fact, without the Axiom of Infinity, it is not possible to prove that such a set even exists. Therefore, the real content of our claim that π "gives rise" to the natural numbers is that it gives rise to the sequence $0, 1, 2, \ldots$ in its entirety, as a completed set—seen as emerging from the self-referral dynamics of the original Dedekind self-map $j : A \to A$.

We accomplish our aim by showing how to derive from π the *successor function* s, which, once defined, specifies the full sequence of natural numbers: $0, s(0), s(s(0)), \ldots$. Recall that the set-based version of the successor function has this form: $s(x) = x \cup \{x\}$. For instance,

$$s(0) = s(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\} = 1.$$

We will see that the values of the successor function s are obtained simultaneously from a certain type of interaction between π and j; to say it another way, s turns out to be the unique map that makes the following diagram commutative:

(10)



Notice that, in order for any map $s : N \to N$ to make diagram (10) commutative, it must be true that $s = \pi \circ (j \restriction W) \circ \pi^{-1}$. (This computation makes sense because π is a bijection.) We can therefore take this to be the definition of s and then verify that s is truly the successor function on the usual set of natural numbers.

Theorem 11. *Define* $s = \pi \circ (j \restriction W) \circ \pi^{-1} : N \to N$. *Then, for all* $n \in N$, $s(n) = n \cup \{n\}$.

Proof. By the definition of s, diagram (10) must be commutative. Let $B = \{x \in W \mid s(\pi(x)) = \pi(x) \cup \{\pi(x)\}\}$. We show B is j-inductive. This is enough because every $n \in N$ is $\pi(x)$ for some $x \in W$, so, assuming $B = W$, we have $s(n) = s(\pi(x)) = \pi(x) \cup \{\pi(x)\} = n \cup \{n\}$.

To prove B is j-inductive, first we show $a \in B$: By commutativity,

$$s(\pi(a)) = \pi(j(a)) = \{\pi(u) \mid u \sqsubseteq j(a)\} = \{\pi(a)\} = \{0\} = 0 \cup \{0\} = \pi(a) \cup \{\pi(a)\}.$$

Next, assume $x \in B$, so $s(\pi(x)) = \pi(x) \cup \{\pi(x)\}$. We show $j(x) \in B$, that is, $s(\pi(j(x))) = \pi(j(x)) \cup \{\pi(j(x))\}$. But

$$\begin{aligned} s(\pi(j(x))) &= \pi(j(j(x))) \\ &= \{\pi(y) \mid y \sqsubseteq j(j(x))\} \\ &= \{\pi(y) \mid y \sqsubseteq j(x) \text{ or } y = j(x)\} \\ &= \{\pi(y) \mid y \sqsubseteq j(x)\} \cup \{\pi(y) \mid y = j(x)\} \\ &= \pi(j(x)) \cup \{\pi(j(x))\}. \ \square \end{aligned}$$

We will now verify in several theorems below that N and s have the expected properties. We will say that a set S is *inductive* if $\emptyset \in S$ and whenever $x \in S$, we have $x \cup \{x\}$ is in S.

Theorem 12. *N is an inductive set. Indeed,* $N = \bigcap\{I \mid I \text{ is inductive}\}$.

Proof. We have seen already that $\emptyset \in N$. Suppose $n \in N$. Then for some $x \in W$, $n = \pi(x)$. But now

$$n \cup \{n\} = s(n) = s(\pi(x)) = \pi(j(x)) \in N.$$

For the second clause, it is sufficient to show that $N \subseteq I$ for every inductive set I. Let I be any inductive set. Let $B = \{x \in W \mid \pi(x) \in I\}$. We show B is j-inductive. By definition $\pi(a) = \emptyset \in I$, so $a \in B$. If $x \in B$, then $n = \pi(x) \in I$. But now

$$j(x) \in B \Leftrightarrow \pi(j(x)) \in I \Leftrightarrow s(\pi(x)) \in I \Leftrightarrow s(n) \in I \Leftrightarrow n \cup \{n\} \in I,$$

and the last of these statements is true by definition of "inductive." Hence B is j-inductive, and so, for every $n \in N$, $n \in I$, as required. $\square$

The property that N has of being an inductive set leads to the Principle of Mathematical Induction:

Principle of Mathematical Induction. Suppose $A \subseteq N$ has the following two properties:

(1) $\emptyset \in A$;
(2) whenever $n \in A$, $s(n) \in A$.
Then $A = N$.

We can now prove the Principle of Mathematical Induction. Before doing so, we state a weaker version that is often also used, based on *formulas*. A formula is simply an expression involving set parameters. Examples include statements like "x is an even natural number" and "x has at least two elements." We have the following weak principle of induction:

Weak Principle of Mathematical Induction. Suppose $\phi(x)$ is a formula[41] whose parameter x represents an element of N. Suppose further that
(1) $\phi(\emptyset)$ holds true;
(2) whenever $\phi(n)$ holds, $\phi(s(n))$ also holds.
Then $\phi(n)$ holds for every $n \in N$.

This principle is called "weak" because the Principle of Induction implies it, but not conversely: Given any formula $\phi(x)$, where x represents elements of N, suppose (1) and (2) of the Weak Principle hold. The Axiom of Separation (a ZFC axiom) implies that the collection $A = \{n \in N \mid \phi(n)\}$ is a set, and so (1) and (2) of the Principle also hold. Since the Principle of Induction holds, the conclusion of the Weak Principle now follows. On the other hand, one cannot prove the Principle of Induction from the Weak Principle because one can find subsets of $\mathbb{N}$ that are not the extension of any formula; this is because there are *more* subsets of $\mathbb{N}$ than there are formulas of this kind.[42]

The Weak Principle is useful because it generalizes to *classes*—a topic we will take up in Section 15.

Theorem 13. *The Principle of Mathematical Induction is correct.*

Proof. Let $A \subseteq N$ satisfy properties (1) and (2). Properties (1) and (2) assert that A is in fact an inductive set. By Theorem 13, $\mathbb{N} \subseteq A$. But since $A \subseteq N$ also, we conclude that $A = N$.

An alternative proof is obtained by using our earlier observation that $(W, \sqsubset)$ and $(N, \in)$ are order-isomorphic, so that, in particular, N is well-ordered under $\in$. Thus, suppose $A \subseteq N$ satisfies (1) and (2). We will use the fact that N is well-ordered to show $A = N$. Certainly $0 \in A$. Assume $A \neq N$. Let $S = \{m \in N \mid m \notin A\}$. $S \neq \emptyset$ by assumption. Let n be the $\in$-least element of S, using the well-ordering; we have seen that $n \neq 0$. Let $y \in W$ with $\pi(y) = n$. Let x be the $\sqsubset$-immediate predecessor of y. Let $m = \pi(x)$. The order isomorphism between $(W, \sqsubset)$ and $(N, \in)$ guarantees that

[41] Formally, formulas are formed from variables and the membership relation $\in$; notions like "natural number" and "function" are defined in terms of these. Formulas are defined inductively (on the length of the expression). If x, y are variables, both $x = y$ and $x \in y$ are formulas. If ϕ, ψ are formulas and x is a variable, so are $\phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi, \neg\phi, \exists x\, \phi$, and $\forall x\, \phi$. Formulas can have any finite number of parameters, so, to be more formal, we could write $\phi(x, y_1, \ldots, y_k)$, where $y_1, \ldots, y_k$ are variables standing for arbitrary sets; for readability, we just display x.

[42] There are only countably many formulas but uncountably many subsets of $\mathbb{N}$.

m is the $\in$-immediate predecessor of n (so that $s(m) = n$, and, moreover, $m \in n$, but for all $r \in N$, it is not true that $m \in r \in n$). Now $m \notin S$ by leastness of n, so $m \in A$, and it follows that $s(m) \in A$. But this contradicts the fact that $n \notin A$. □

Notation. A part of the proof shows that, as we would expect, every nonzero element of N has a unique immediate predecessor. For $n \neq 0$ in N, we denote the immediate predecessor of n by $n - 1$. Similarly, we will adopt the usual convention of writing $n + 1$ for $s(n)$, whenever $n \in N$. Finally, it will be convenient, when emphasizing the use of $\in$ on N as an order relation, to write $<$ in place of $\in$, as is customarily done. Moreover, we shall write $m \leq n$ if and only if either $m \in n$ or $m = n$.

Theorem 14. $0 \notin \operatorname{ran} s$ *and* s *is 1-1. Moreover,* $N = \{0\} \cup \operatorname{ran} s$.

Proof. If $s(m) = 0$ for some $m \in N$, let $x \in W$ be such that $\pi(x) = m$. Then $0 = \pi(a) = s(\pi(x)) = \pi(j(x))$, and so, because π is 1-1, $a = j(x)$, which is impossible by Claim 7.

To prove s is 1-1, suppose $s(m) = s(n)$, where $m, n \in N$. Let $x, y \in W$ with $\pi(x) = m, \pi(y) = n$. Then

$$s(m) = s(n) \Leftrightarrow \pi(j(x)) = \pi(j(y)) \Leftrightarrow j(x) = j(y) \Leftrightarrow x = y \Leftrightarrow m = n.$$

Finally, suppose $n \neq 0$ is in N. Let m be its unique immediate predecessor, so $s(m) = n$. We have shown that $\operatorname{ran} s = N - \{0\}$. It follows that $N = \{0\} \cup \operatorname{ran} s$. □

Now we can define the concrete notion "natural number." A set n is a *natural number* if and only if $n \in N$; equivalently, if and only if n belongs to every inductive set. This concrete definition is the one that allows us to see each number explicitly rendered as a set:

Theorem 15. *For each* $n \in N$*, if* $n \neq 0$*, then* $n = \{0, 1, 2, \ldots, n - 1\}$.

Proof. By Claim 10(8) and the fact that π is an isomorphism, for every $m \in n$, $m \leq n - 1$. Conversely, if $m \leq n - 1$, then either $m \in n - 1$—so by transitivity, $m \in n$—or $m = n - 1$, in which case also $m \in n$. Thus n consists precisely of the elements of N that precede n, the largest of these being $n - 1$. □

In light of Theorem 13, it is clear that N and ω (defined at the beginning of this paper) are the same set. For the rest of the paper, we will use "ω" as the name of this set.

9. Definition by Induction and the Peano Axioms

We have shown how to derive ω and the successor function from an arbitrary Dedekind self-map. Since the collapsing map π is an order-isomorphism, we obtain immediately that, letting $<$ denote the membership relation $\in$, $(\omega, <)$ is a well-ordered set. We will now carry this development one step further, to establish that the usual Peano axioms for arithmetic hold true. To take this step, we need to provide definitions of addition and multiplication, and to do this, we will need a formulation of definition by induction, a simple version of which follows almost immediately from the work we have already done in the previous section. We then state a slightly stronger version that will be more suitable for a definition of addition $(+)$ and multiplication $(\cdot)$. We state these formulations of definition by induction in terms of Dedekind self-maps to illustrate how pervasive Dedekind self-maps are in the fabric of mathematics. These definitions will allow us to rigorously define addition and multiplication on ω, and we will finally be able to state the Peano axioms and indicate why they are satisfied by the structure $(\omega, +, \cdot)$.

To begin, we return to the diagram that showed how $j \restriction W : W \to W$ is collapsed to the successor function $s : \omega \to \omega$:

$$\begin{array}{ccc} W & \xrightarrow{\;j \restriction W\;} & W \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \omega & \xrightarrow{\;s\;} & \omega \end{array} \tag{11}$$

The Mostowski Collapsing Theorem tells us that π is the unique function defined on W satisfying $\pi(x) = \{\pi(y) \mid y \sqsubseteq x\}$. This result allowed us to demonstrate that the set ω and the successor function s *exist* (as a function from ω to ω).

We wish now to look at the diagram (11) in a slightly different way. Initially, we viewed the diagram as starting with the maps $j \restriction W$ and π, which occupied the top and two vertical sides of the square determined by the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{\;j \restriction W\;} & W \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \omega & & \omega \end{array} \tag{12}$$

The bottom edge of diagram (12) was filled in by *defining* s to be a composition of the other maps (and their inverses): $s = \pi \circ (j \restriction W) \circ \pi^{-1}$.

Having established the existence of ω and s, we can view the diagram (11) in another way. We begin as before with $j \restriction W : W \to W$ and now also take as given the map $s : \omega \to \omega$, defined by the formula $s(n) = n \cup \{n\}$.[43] For the moment, we leave the two vertical edges of diagram (13) below unspecified.

$$\begin{array}{ccc} W & \xrightarrow{\;j \restriction W\;} & W \\ & & \\ \omega & \xrightarrow{\;s\;} & \omega \end{array} \tag{13}$$

[43] Technically, this amounts to defining s on ω to be $\bar{s} \restriction \omega$; recall that $\bar{s}$ is defined on *all* sets by $\bar{s}(x) = x \cup \{x\}$. One needs to verify that the range of $\bar{s} \restriction \omega$ is a subset of ω, so that we may write, as usual, $s : \omega \to \omega$. We do this quick verification here: Let $A \subseteq \omega$ be the set $\{n \in \omega \mid s(n) \in \omega\}$. Certainly $0 \in A$ and if $n \in A$, $s(n) \in \omega$, since ω is inductive. Therefore, A is inductive, so $\omega \subseteq A$, yielding that $A = \omega$.

We now define π on W as before, this time as a candidate to fill in the vertical edges of the diagram: $\pi(x) = \{\pi(y) \mid y \in x\}$. As we argued earlier, π is uniquely determined by this relation. Let $A \subseteq \omega$ be defined by

$$A = \{n \in \omega \mid \text{for some } x \in W,\ \pi(x) = n\}.$$

Showing that A is inductive will establish that $\operatorname{ran} \pi = \omega$. Since $\pi(a) = 0$, as we argued before, $0 \in A$. If $n \in A$, then let $x \in W$ with $\pi(x) = n$. We show $s(n) \in A$ by showing $\pi(j(x)) = s(n) = n \cup \{n\}$. But

$$\pi(j(x)) = \{\pi(y) \mid y \in j(x)\} = \{\pi(y) \mid y \in x\} \cup \{\pi(x)\} = \pi(x) \cup \{\pi(x)\} = n \cup \{n\},$$

as required.

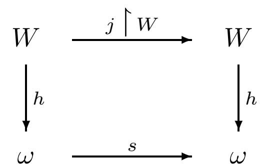
We can now show the following:

Theorem 16. *$\pi : W \to \omega$ is the unique map for which the following two conditions hold:*

(i) $\pi(a) = 0$.
(ii) *Diagram (11) is commutative; that is,*

$$s \circ \pi = \pi \circ (j \restriction W) \tag{14}$$

Proof. We showed in Theorem 12 that equation (14) holds true (we established this using a different definition of s, but we also showed that for each n, $s(n) = n \cup \{n\}$). We verify uniqueness: Suppose $h : W \to \omega$ satisfies (i) and (ii) above; that is, $h(a) = 0$ and $s \circ h = h \circ (j \restriction W)$.



We establish uniqueness by j-induction: Let $B \subseteq W$ be defined by

$$B = \{x \in W \mid h(x) = \pi(x)\}.$$

Since (i) holds for both functions, $a \in B$. Assume $x \in B$. Then since both functions make the diagram commutative, we have

$$\pi(j(x)) = j(s(x)) = h(j(x)),$$

and so $j(x) \in B$, and B is j-inductive, as required. □

We observed earlier that π is an order isomorphism. In the present context, π is another kind of isomorphism, called a *Dedekind self-map isomorphism.* We define this concept now. Suppose $g : B \to B$ is a Dedekind self-map with critical point b and $h : C \to C$ is a Dedekind self-map with critical point c. A *Dedekind self-map morphism from g to h* is a map $\beta : B \to C$ satisfying:

(1) $\beta(b) = c$;

(2) the diagram is commutative; that is, $\beta \circ g = h \circ \beta$.

$$\begin{array}{ccc} B & \xrightarrow{g} & B \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow{h} & C \end{array}$$

Moreover, if β is a bijection, β is a *Dedekind-self map isomorphism.*[44] The intuition behind a Dedekind self-map morphism β from $g : B \to B$ to $h : C \to C$ is that the structural relationships given by g are reflected in h: if g takes x to y in B, then h takes $\beta(x)$ to $\beta(y)$ in C. Moreover, if β is also an isomorphism, then the relationships between the two maps are structurally identical: For all $x, y \in C$, g takes x to y if and only if h takes $\beta(x)$ to $\beta(y)$.

We introduce some notation for Dedekind self-map morphisms: Notice that every Dedekind self-map $g : B \to B$ with critical point b specifies three parameters: B, g, and b. We therefore may specify this Dedekind self-map by specifying the triple (B, g, b). Then, we can indicate that β is a Dedekind self-map morphism from g to h taking b to c by simply saying that $\beta : (B, g, b) \to (C, h, c)$ is a Dedekind self-map morphism.

Our reasoning in the previous paragraphs has demonstrated the following:

Theorem 17. *Let $\pi : W \to \omega$ be defined by $\pi(x) = \{\pi(y) \mid y \mathrel{\mathsf{E}} x\}$. Then π is the unique Dedekind self-map morphism from $(W, j \restriction W, a)$ to $(\omega, s, 0)$. Moreover, π is in fact a Dedekind self-map isomorphism.* □

Theorem 18 tells us that, structurally, W and ω are identical, with "successor" functions that behave in exactly the same way; the structures $(W, j \restriction W, a)$ and $(\omega, s, 0)$ are identical except for notational differences.

It is natural to expect that the isomorphism π is invertible—that π^{-1} is also a Dedekind self-map isomorphism, and that it is the unique morphism from s to $j \restriction W$. We verify this now. Henceforth, we let τ denote π^{-1}.

(15)
$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ W & \xrightarrow{j \restriction W} & W \end{array}$$

[44] As an interesting sideline, we point out that the concept of a Dedekind-self map isomorphism from $j \restriction W$ to $s : \omega \to \omega$ is a slight weakening of "E-order isomorphism" to "E_0-order isomorphism." We can explain this point in the following way. Suppose we are given g and h as in the definition of Dedekind self-map morphisms. Suppose E_g is defined on B by $x\, E_g\, y$ if and only if $y = g(x)$ and E_h is defined on C by $x\, E_h\, y$ if and only if $y = h(x)$, then β is a Dedekind self-map isomorphism if and only if β is an (E_g, E_h)-isomorphism (that is, β is a bijection and $x\, E_g\, y$ if and only if $\beta(x)\, E_h\, \beta(y)$). In particular, to say that π is a Dedekind self-map isomorphism from $j \restriction W$ to s is the same as saying that it is an order isomorphism, relative to the relation E_0.

Theorem 18. *Let $\tau : \omega \to W$ be π^{-1}. Then τ is a bijection and is the unique Dedekind self-map morphism from $(\omega, s, 0)$ to $(W, j \restriction W, a)$.*

Proof. Since $\tau = \pi^{-1}$, τ is a bijection and $\tau(0) = a$. Then

$$\tau \circ s = j \circ \tau \Leftrightarrow s = \pi \circ j \circ \tau \Leftrightarrow s \circ \pi = \pi \circ j.$$

Since π is a Dedekind self-map morphism (diagram (11)), the last of these equations ($s \circ \pi = \pi \circ j$) holds true, and so the first one ($\tau \circ s = j \circ \tau$—see diagram (15)) does as well.

For uniqueness, we first observe that if g is a Dedekind self-map morphism from $(\omega, s, 0)$ to $(W, j \restriction W, a)$, then g must be 1-1 and onto (see diagram (16)).

(16)
$$\begin{array}{ccc} \omega & \xrightarrow{\;s\;} & \omega \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{\;j \restriction W\;} & W \end{array}$$

Let $A = \{n \in \omega \mid g(n) \notin \{g(0), g(1), \ldots, g(n-1)\}\}$. We show A is inductive; this will establish that g is 1-1. Clearly $0 \in A$. If $n \in A$ and $g(s(n)) = g(i)$ for some $i, 0 \le i \le n-1$, notice first that $i \ne 0$ since we would have in that case

$$a = g(0) = g(s(n)) = j(g(n)),$$

which is impossible since, for no $x \in W$ is it true that $x \mathrel{\mathsf{E}} a$. Therefore, $i = s(k)$ for some $k, 0 \le k < n-1$, and we have

$$j(g(n)) = g(s(n)) = g(s(k)) = j(g(k)).$$

Since j is 1-1, $g(n) = g(k)$ which contradicts the fact that $n \in A$. Therefore, $s(n) \in A$ as required.

To see that g is also onto, let $B \subseteq W$ be defined by $B = \{x \in W \mid \text{for some } n \in \omega, g(n) = x\}$. Clearly $a \in B$. If $x \in B$, let $n \in \omega$ with $g(n) = x$. We show $j(x) \in B$. But

$$j(x) = j(g(n)) = g(s(n)),$$

as required. Since B is j-inductive, $B = W$, and g is onto.

To complete the proof, we must show that $\tau = g$. But notice now that g^{-1} makes the following diagram commutative:

(17)
$$\begin{array}{ccc} W & \xrightarrow{\;j \restriction W\;} & W \\ \downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\ \omega & \xrightarrow{\;s\;} & \omega \end{array}$$

By uniqueness of π, $g^{-1} = \pi$, and so $g = (g^{-1})^{-1} = \pi^{-1} = \tau$. $\square$

Generalizing this result slightly provides us with a way to state a weak form of the Definition by Recursion Theorem:

Theorem 19. (Definition by Recursion on ω) *Suppose $j : A \to A$ is a Dedekind self-map with critical point a. Then there is a unique Dedekind self-map morphism $\overline{\tau}$ from $(\omega, s, 0)$ to (A, j, a), as in diagram (18).*

(18)
$$\begin{array}{ccc} \omega & \xrightarrow{\;s\;} & \omega \\ \Big\downarrow{\scriptstyle \overline{\tau}} & & \Big\downarrow{\scriptstyle \overline{\tau}} \\ A & \xrightarrow{\;j\;} & A \end{array}$$

Proof. We have already completed the main steps of the proof; we review these now. We obtain $W \subseteq A$ as the smallest j-inductive set. We have seen that $\pi : W \to \omega$, as defined earlier, is unique such that $\pi(a) = 0$ and $\pi \circ j \restriction W = s \circ \pi$ and that π is a bijection. It follows, as we have shown, that if $\tau = \pi^{-1}$, then τ is a bijection and the unique Dedekind self-map morphism from $(\omega, s, 0)$ to $(W, j \restriction W, a)$. The existence of $\overline{\tau}$ requires one small additional step. Consider the following diagram, recalling that $\mathrm{inc}_{W,A}$ is the inclusion map $W \hookrightarrow A$:

(19)
$$\begin{array}{ccc} \omega & \xrightarrow{\;s\;} & \omega \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau} \\ W & \xrightarrow{\;i\;} & W \\ \Big\downarrow{\scriptstyle \mathrm{inc}_{W,A}} & & \Big\downarrow{\scriptstyle \mathrm{inc}_{W,A}} \\ A & \xrightarrow{\;j\;} & A \end{array}$$

We now define $\overline{\tau} = \mathrm{inc}_{W,A} \circ \tau$. Clearly $\overline{\tau}(0) = a$ and, for all $n \in \omega$,

(20) $$j \circ \mathrm{inc}_{W,A} \circ \tau = \mathrm{inc}_{W,A} \circ i \circ \tau = \mathrm{inc}_{W,A} \circ \tau \circ s,$$

so diagram (19) is commutative. It follows that

$$j \circ \overline{\tau} = \overline{\tau} \circ s,$$

as required.

For uniqueness, suppose $h : \omega \to A$ satisfies the same conditions: $h(0) = a$ and $h \circ s = j \circ h$. We show $h = \overline{\tau}$ by proving by induction that $h(n) = \overline{\tau}(n)$ for all $n \in \omega$. Certainly $h(0) = a = \overline{\tau}(0)$ by assumption. Assuming $h(n) = \overline{\tau}(n)$ we show $h(s(n)) = \overline{\tau}(s(n))$. But

$$h(s(n)) = j(h(n)) = j(\overline{\tau}(n)) = \overline{\tau}(s(n)),$$

as required. This completes the induction and shows that $h = \overline{\tau}$. □

Theorem 20 allows us to perform *inductive definitions* of sequences, indexed by the elements of ω, in the usual way. For instance, suppose we wish to formally define the sequence of powers of 2: $2^0, 2^1, 2^2, \ldots = 1, 2, 4, \ldots$.[45] Theorem 20 guarantees that a single function or sequence exists that will produce precisely these values.

(21)
$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ \omega & \xrightarrow{j} & \omega \end{array}$$

To use the theorem, we need to specify a Dedekind self-map j, as in the bottom row of diagram (21), and the critical point of j that will be used. The map j declares how the next value is computed from the current value as we move through the sequence, and the critical point tells us the value we start with. For this purpose, we define $j : \omega \to \omega$ by $j(m) = 2m$ and, for our critical point, choose the value 1. Now Theorem 20 guarantees there is a unique $\overline{\tau} : \omega \to \omega$ that takes 0 to 1 and makes the diagram commutative.

Now we *define* $2^n = \overline{\tau}(n)$ for each $n \in \omega$ (recall that we are *defining* the exponential function in this example). We can now prove that $\overline{\tau}$ has the expected properties of the exponential function, namely:

$$\begin{array}{rcl} 2^0 & = & 1, \\ 2^{n+1} & = & 2 \cdot 2^n. \end{array}$$

The first clause follows because $2^0 = \overline{\tau}(0) = 1$, by definition of $\overline{\tau}$. For the second clause, we have

$$2^{n+1} = \overline{\tau}(n+1) = \overline{\tau}(s(n)) = j(\overline{\tau}(n)) = j(2^n) = 2 \cdot 2^n.$$

We will now use Theorem 20 to tie up some loose ends from the previous section. Let $j : A \to A$ be a Dedekind self-map with critical point a, and define $W \subseteq A$ as above. Let $i = j \restriction W$. We obtain by definition by recursion the sequence $\langle i^0, i^1, \ldots, i^n, \ldots \rangle$ of iterates of i, where i^0 is by convention id_W:

Let $W^W = \{g \mid g : W \to W\}$ and let $J_i : W^W \to W^W$ be defined by

(22)
$$J_i(g) = i \circ g.$$

Using Theorem 20, let $\overline{\tau} : \omega \to W^W$ be the unique map for which $\overline{\tau}(0) = \mathrm{id}_W$ and diagram (23) is commutative:

(23)
$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ W^W & \xrightarrow{J_i} & W^W \end{array}$$

[45] Note that in the formal development of arithmetic, before defining this exponential function, we would need to define addition and multiplication and establish some of their properties. At this stage, however, we are not attempting to provide a formal definition of exponentiation; our intention is just to illustrate the use of the theorem with a simple example, stepping outside the formal development for a moment.

Define $i^n = \overline{\tau}(n)$ for each $n \in \omega$. We have the following result.

Theorem 20.

(1) $i^0 = \mathrm{id}_W$ *and, for each* $n \in \omega$, $i^{n+1} = i \circ i^n$, *so that* i^n *is the nth iterate of* i.
(2) $W = \{a, j(a), j^2(a), \dots\} = \{a, i(a), i^2(a), \dots\} = \{i^n(a) \mid n \in \omega\}$

Proof of (1). The case $n = 0$ follows by definition of $\overline{\tau}(0)$. Also, for $n \geq 0$, commutativity of diagram (23) gives us the following:

$$\overline{\tau}(n+1) = \overline{\tau}(s(n)) = J_i(\overline{\tau}(n)) = J_i(i^n) = i \circ i^n.$$

Proof of (2). By (1), for all $n \in \omega, i^n(a) \in \omega$. This shows that W contains all the terms $i^n(a)$. We show that these are the *only* elements of W. Let $B \subseteq W$ be defined by

$$B = \{x \in W \mid \text{for some } n \in \omega,\ x = i^n(a)\}.$$

Notice $a \in B$ since $a = i^0(a)$. Assume $x \in B$, so $x = i^n(a)$ for some $n \in \omega$. Then

$$i(x) = i(i^n(a)) = i^{n+1}(a) \in B.$$

We have shown that B is j-inductive, and so $B = W$. Therefore, every element of W is one of the terms $i^n(a)$. This completes the proof of (2). $\square$

A slight generalization of this result allows us to verify a point made at the beginning of the paper, that, whenever $j : A \to A$ is a Dedekind self-map with critical point a, the sequence of maps id_A, j, $j \circ j, \dots$ forms a blueprint for ω just as $W = \{a, j(a), j(j(a)), \dots\}$ does. Let us define the Dedekind self-map $J_j : A^A \to A^A$, as in equation (22), by $J_j(h) = j \circ h$, with critical point id_A. Recall from earlier observations (page 21) that, if j is a Dedekind self-map, so is J_j. Combining our work in Theorems 19 and 21, we get:

Theorem 21.

(1) *The map* $J_j : A^A \to A^A$ *is a Dedekind self-map with critical point* id_A.
(2) *There is a set* $\overline{W} \subseteq A^A$ *whose elements are precisely* $\mathrm{id}_A, j, j \circ j, \dots$. *Moreover, there is a unique* $\hat{\tau} : \omega \to A^A$ *for which* $\hat{\tau}(0) = \mathrm{id}_A$ *and the diagram below is commutative; in particular,* $\overline{W} = \mathrm{ran}\,\hat{\tau}$.

(24)
$$\begin{array}{ccc} \omega & \xrightarrow{\ s\ } & \omega \\ \big\downarrow{\scriptstyle \hat{\tau}} & & \big\downarrow{\scriptstyle \hat{\tau}} \\ A^A & \xrightarrow{\ J_j\ } & A^A \end{array}$$

(3) *The map* $\hat{\tau} : s \to J_j \restriction \overline{W}$ *is a Dedekind self-map isomorphism.*
(4) *For each* $n \in \omega$, $j^n = \hat{\tau}(n)$. $\square$

To define addition and multiplication on ω requires a parametrized form of definition by induction. One way to do this is due to P. Freyd:[46]

[46] His result, proven in [24, Theorem 5.21], is somewhat stronger than the version we state here.

Theorem 22. (Freyd's Recursion Theorem) *Suppose* $j : A \to A$ *is a Dedekind self-map with critical point* a *and* $1_A \times s : A \times \omega \to A \times \omega$ *is defined by* $(1_A \times s)(a, n) = (a, s(n)) = (a, n + 1)$*. Then there is a unique* $\tau : A \times \omega \to A$ *for which* $\tau(a, 0) = a$ *and the following diagram commutes:*

(25)
$$\begin{array}{ccc} A \times \omega & \xrightarrow{1_A \times s} & A \times \omega \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau} \\ A & \xrightarrow{j} & A \end{array}$$

Proof. Given A, an element $a \in A$, and a Dedekind self-map $g : A \to A$ with critical point x, we obtain τ as follows.

Let $J_j : A^A \to A^A$ be the function defined as before by $J_j(f) = j \circ f$. As was previously observed, J_j is a Dedekind self-map with critical point id_A and, as in Theorem 22, we have a unique map $\hat{\tau}$ for which $\hat{\tau}(0) = \mathrm{id}_A$ and the following is commutative:

(26)
$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \Big\downarrow{\scriptstyle \hat{\tau}} & & \Big\downarrow{\scriptstyle \hat{\tau}} \\ A^A & \xrightarrow{J_j} & A^A \end{array}$$

We now pull back to diagram (25) by defining $\tau : A \times \omega \to A$ so that $\hat{\tau}$ is the *exponential adjoint of* τ. In other words, we define τ by

(27)
$$\tau(b, n) = (\hat{\tau}(n))(b).$$

We verify that τ makes diagram (25) commutative:

$$\begin{aligned} j(\tau(b, n)) &= (j \circ \hat{\tau}(n))(b) \\ &= (J_j(\hat{\tau}(n)))(b) \\ &= ((J_j \circ \hat{\tau})(n))(b) \\ &= (\hat{\tau}(s(n)))(b) \\ &= \tau(b, s(n)) \\ &= (\tau \circ (1_A \times s))(b, n). \end{aligned}$$

Uniqueness of τ follows from its adjoint relationship with $\hat{\tau}$ (alternatively, uniqueness can be checked directly). □

We can make use of the theorem by expressing it in the following more familiar form: Given a set A, an element $a \in A$, and a function $g : A \to A$, the theorem says that a function $\tau : A \times \omega \to A$ is uniquely determined by the following data:

$$\begin{aligned} \tau(a, 0) &= a, \\ \tau(a, n + 1) &= g(\tau(a, n)). \end{aligned}$$

Using Freyd's Recursion Theorem, we may define addition on ω as follows: For each $m \in \omega$, define $\tau^+ : \omega \times \omega \to \omega$ by

$$\begin{aligned} \tau^+(m, 0) &= m, \\ \tau^+(m, n+1) &= s(\tau^+(m, n)). \end{aligned}$$

Then, for each $m, n \in \omega$, we define $m + n = \tau^+(m, n)$.

Likewise, for multiplication, we define, for each $m \in \omega$, a function $\tau^* : \omega \times \omega \to \omega$ by

$$\begin{aligned} \tau^*(m, 0) &= 0, \\ \tau^*(m, n+1) &= \tau^*(m, n) + n \end{aligned}$$

Then, for each $m, n \in \omega$, we define $m \cdot n = \tau^*(m, n)$.

It is now a straightforward exercise to verify that the axioms of Peano Arithmetic are satisfied by $(\omega, +, \cdot)$. Here is a version of the Peano axioms:[47]

(i) For each $m \in \omega$, $s(m) \neq 0$.
(ii) For all $m, n \in \omega$, $s(m) = s(n)$ implies $m = n$.
(iii) For all $m \in \omega$, $m + 0 = m$.
(iv) For all $m, n \in \omega$, $m + s(n) = s(m + n)$.
(v) For all $m \in \omega$, $m \cdot 0 = 0$.
(vi) For all $m, n \in \omega$, $m \cdot s(n) = (m \cdot n) + n$.
(vii) Principle of Induction.

We will assume that, on the basis of these axioms, the usual theorems of arithmetic on ω have been established.

We are now in a position to give formal definitions of the concepts "finite" and "infinite," and we can verify that, in our definition of $\sqsubset$, joining sets are always finite.

Definition 4. (Finite and Infinite Sets) *A set X is* finite *if there is $n \in \omega$ for which there is a bijection from n to X. A set is* infinite *if it is not finite.*

Theorem 23. *Suppose $j : A \to A$ is a Dedekind self-map with critical point a. Let $W = \{a, j(a), j(j(a)), \ldots\}$. Suppose F joins x to y in W. Then F is finite.*

Proof. Recall $\sqsubset$ is a well-ordering of W and $(W, \sqsubset)$ isomorphic to $(\omega, \in)$ under the collapsing map $\pi : W \to \omega$. Let $m, n \in \omega$ be such that, $\pi(x) = m$ and $\pi(y) = n$. Then, using familiar properties of the arithmetic of ω, the chain $m < m+1 < \ldots < n$ has $\ell = n - m + 1$ terms. Since π is a bijection, $|F| = \ell$, as required. $\square$

10. Building the First Stages of V

An important application of Theorem 20 is the definition of the first stages $\langle V_0, V_1, V_2, \ldots \rangle$ of the universe V. The stages $V_0, V_1, V_2, \ldots$ of the universe V are obtained by taking repeated power sets, starting with the empty set $\emptyset$, where, by definition, the power set $\mathcal{P}(X)$ of a set X is the set whose elements are the subsets of X.

[47] These are taken from [37].

Speaking less rigorously for the moment, what we wish to do, as a first step, is apply definition by recursion to obtain the sequence $\langle V_0, V_1, V_2, \ldots \rangle$ as follows:

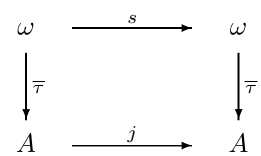
$$\begin{aligned} V_0 &= \emptyset \\ V_{n+1} &= \mathcal{P}(V_n) \end{aligned}$$

Then, for our second step, we wish to define V_ω as the union of these stages:

(28) $$V_\omega = \bigcup_{n \in \omega} V_n.$$

Carrying out these steps more rigorously presents a few obstacles. First of all, if we try to make careful use of our Definition by Recursion Theorem (Theorem 20—see diagram (29) below),

(29)



it is not clear what the value of A should be, which must be big enough to contain all the finite stages V_n, $n \in \omega$. A natural choice for A would be V_ω, but diagram (29) is telling us about how V_ω is to be built, so we may not assume that V_ω already exists as a background set.

A solution is to use the collection **HF** of all *hereditarily finite sets* (defined below), which will indeed contain all the sets in the stages $V_0, V_1, V_2, \ldots$, and which can be defined without referring to ω.

Taking this approach, we then are faced with the theoretical problem that **HF** may not be a *set*; since the Definition by Recursion Theorem requires the entity A in diagram (29) to be a set, use of **HF** in place of A in diagram (29) is not allowed. This difficulty can be solved by introducing a slight generalization of the Definition by Recursion Theorem, which will allow us to use even very large collections (known as *proper classes*) in place of A in diagram (29). This generalization will allow us to use **HF**, or even V itself, in place of A in that diagram. Using this approach, we will be able to describe formally the sequence of stages $\langle V_0, V_1, V_2, \ldots \rangle$ and define the union of the stages in a rigorous way. If we are working in a ZFC universe (in which case we are assured of the existence of ω), we will then be able to define V_ω as the union of the stages $\langle V_0, V_1, V_2, \ldots \rangle$ in a formally correct way. On the other hand, if we are working in a ZFC $-$ Infinity universe V, as will often be the case, these techniques will allow us to conclude, in a rigorous way, that forming the union of these stages produces the entire universe V if ω does *not* exist in the universe.

In the rest of this section, we will develop the details for this more rigorous treatment. The reader who wishes to skip these technical details may safely skip to the next section.

Our first step in this exposition is to treat the issue of replacing the set A in diagram (29) with a larger type of collection. For this purpose, we formulate a *class* version of Theorem 20. We discuss classes in more detail in Section 15. For our purposes in this section, we think of a class **C** as a collection of objects defined by a formula. In other words, a class $\mathbf{C}$ will be defined by $\mathbf{C} = \{x \mid \phi(x)\}$ for some formula ϕ (having possibly finitely many other set parameters not displayed).

Let us first observe that any *set* is a class. For instance, the set $S = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$ can be specified by:

$$\{0, 1\} = \{x \mid \phi(x, S)\},$$

where $\phi(x, y)$ is the formula $x \in y$.

In other words, in the case of an ordinary *set* z, a defining formula can use z itself as one of its parameters.

On the other hand, an example of a class that is *not* a set is $\mathbf{B} = \{\{y\} \mid y \text{ is a set}\}$. Formally, the formula $\phi(x)$ that defines $\mathbf{B}$ states "there is z such that for all u, $u \in x$ if and only if $u = z$": $\mathbf{B} = \{x \mid \phi(x)\}$. (Here, x represents a set of the form $\{z\}$ and the condition states that the only element of x is z.) Intuitively, a collection like $\mathbf{B}$ spans the universe and so is too big to be a set. Such classes are called *proper classes.*

Here, then, is a class version of Theorem 20.

Theorem 24. (Strong Definition by Recursion Theorem for ω) *Suppose* $\mathbf{C}$ *is a class,* $c \in \mathbf{C}$*, and* $j : \mathbf{C} \to \mathbf{C}$ *is 1-1 and has critical point* c*, and is itself a class function. Then there is a unique class map* $\overline{\tau} : \overline{\omega} \to \mathbf{C}$ *such that* $\overline{\tau}(0) = c$ *and the following is commutative:*

(30)
$$\begin{array}{ccc} \overline{\omega} & \xrightarrow{\;s\;} & \overline{\omega} \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ \mathbf{C} & \xrightarrow{\;j\;} & \mathbf{C} \end{array}$$

Remark 5. We have introduced a new symbol $\overline{\omega}$ in the statement of Theorem 25. We provide an explanation of this symbol here. Our intention for the proof is to mimic the proof of the Mostowski Collapsing Theorem (p. 34). If we are certain that we are working in a ZFC universe V, then the proof will work as before, and the notation $\overline{\omega}$ should be understood to mean simply ω. But if we are working in a ZFC $-$ Infinity universe V, then it is uncertain whether ω exists in the universe, and the steps in the proof that make reference to ω will not make sense. In that case, we think of the finite ordinals as collected together to form a *proper class* within V, and we denote this class $\overline{\omega}$. We may still perform the usual inductive arguments on $\overline{\omega}$ as are typically done on ω.[48] Thus, if we are working in a ZFC universe V that contains the set ω, then $\overline{\omega} = \omega$; if we are working in a ZFC $-$ Infinity universe V that does not contain any infinite sets, $\overline{\omega}$ signifies the *proper class* consisting of all the finite ordinals $0, 1, 2, \ldots$.

A second point about the statement of the theorem that should be mentioned is that the phrase "there exists a class map..." that we see there must be interpreted in the appropriate way, depending on whether the underlying theory is ZFC or ZFC $-$ Infinity. In the former case, the "class map" in this case is just an ordinary

[48]This point does require justification, but we postpone this verification until Section 16, where the topic is developed systematically.

(set) function (by the Axiom of Replacement). However, if $\overline{\omega}$ is itself a proper class, then the last line of the theorem statement should be understood to mean "there is a finitary procedure which transforms s and ϕ_j to a formula $\phi_{\overline{\tau}}$, where ϕ_j is the formula that defines j, $\phi_{\overline{\tau}}$ is a functional formula, $V \models \phi_{\overline{\tau}}(0, c)$,[49] and, for all $n \in \omega$, if $x \in \mathbf{C}$ is such that $V \models \phi_{\overline{\tau}}(n, x)$, y is such that $V \models \phi_{\overline{\tau}}(n+1, y)$, and z is such that $V \models \phi_j(x, z)$, then $y = z$."

One fact that we will need in the proof below is that a form of induction can be proved for $\overline{\omega}$, which states that if B is a subclass of $\overline{\omega}$ that contains 0 and that has the property that, for all $n \in \overline{\omega}$, if $n \in B$, then $\overline{s}(n) \in B$, then we may conclude that $B = \overline{\omega}$. This induction principle will be proven rigorously in Section 16; see Theorem 38. Intuitively, this principle follows from the fact that the usual axioms of arithmetic (the Peano Axioms) can be interpreted in the theory ZFC − Infinity; since weak induction is included in those axioms, it holds for $\overline{\omega}$.[50] □

Proof. We follow the proof of the Mostowski Collapsing Theorem given on p. 34. Let $\phi(x, u)$ be the formula "$u \in \overline{\omega}$ and x is a function with domain u so that $x(0) = c$ and for all i, $0 \leq i < u - 1$, $x(s(i)) = j(x(i))$." We use induction on $\overline{\omega}$ (see the preceding remark) to show that $B = \overline{\omega}$, where B is the subclass of $\overline{\omega}$ defined by

$$B = \{n \in \overline{\omega} \mid \text{there is a unique function } t_n \text{ such that } \phi(t_n, n) \text{ holds}\}.$$

(31)
$$\begin{array}{ccc} n & \xrightarrow{s \restriction n} & n+1 \\ \downarrow{\scriptstyle t_n} & & \downarrow{\scriptstyle t_{n+1}} \\ \mathbf{C} & \xrightarrow{j} & \mathbf{C} \end{array}$$

Commutative Diagram Showing Behavior of t_n and t_{n+1}

Certainly $0 \in B$; here, the unique map t_0 is the empty function. Also, $1 \in B$; in this case, t_1 is defined on $1 = \{0\}$ and $t_1(0) = c$. Note that this value for t_1 is determined by the formula ϕ. Next, we observe $2 \in B$. Here, t_2 is defined on $\{0, 1\}$ and $t_2(0) = c$. The value for $t_2(1)$ is also determined, in this case by the commutativity requirement of the diagram: $t_2(1) = t_2(s(0)) = j(t_2(0)) = j(c)$.

For the induction step, assume $n \in B$ and $n \geq 2$. In particular, there is a unique t_n defined as in the definition of B, so that for $0 \leq i < n-1$, $t_n(s(i)) = j(t_n(i))$. Define $t_{n+1} = t_n \cup \{(n, j(t_n(s(n-2))))\}$. Note that $j(t_n(s(n-2))) \in \mathbf{C}$ since $t_n(s(n-2)) \in \mathbf{C}$. For $0 \leq i < n-1$, we have, by the induction hypothesis,

$$t_{n+1}(s(i)) = t_n(s(i)) = j(t_n(i)) = j(t_{n+1}(i)).$$

[49] Note that the terminology $V \models \phi$ means that the formula ϕ holds in the model/universe V. Intuitively, $V \models \phi_{\overline{\tau}}(0, c)$ means $\overline{\tau}(0) = c$; $V \models \phi_{\overline{\tau}}(n, x)$ means $\overline{\tau}(n) = x$; and $V \models \phi_j(x, z)$ means $j(x) = z$.

[50] The Peano Axioms are listed on p. 50. The connection between the Peano Axioms and ZFC − Infinity is discussed in the footnote on p. 22.

Also, for $i = n-1$, because $t_{n+1} = t_n \cup \{(n, j(t_n(s(n-2))))\}$, we have:

$$\begin{aligned} t_{n+1}(s(i)) &= t_{n+1}(s(n-1)) \\ &= t_{n+1}(n) \\ &= j(t_n(s(n-2))) \\ &= j(t_{n+1}(s(n-2))) \\ &= j(t_{n+1}(i)). \end{aligned}$$

To see that t_{n+1} is unique, suppose r is also defined on $n+1$ and satisfies $r(s(i)) = j(r(i))$ whenever $0 \leq r \leq n-1$. Certainly $r \restriction n = t_n$, by uniqueness of t_n. But now

$$r(s(n-1)) = j(r(n-1)) = j(t_n(n-1)) = j(t_{n+1}(n-1)) = t_{n+1}(s(n-1)),$$

and so $r = t_{n+1}$. We have shown $n+1 \in B$.

This completes the induction argument and shows $B = \overline{\omega}$, so for each n, we have a uniquely defined t_n as described in the definition of B. We define $\overline{\tau}$ on $\overline{\omega}$ by: $\overline{\tau}(n) = t_{n+1}(n)$. As in the proof of Claim 1 of the Mostowski Collapsing Theorem, it follows that $\overline{\tau}(n) = t_m(n)$ for all $m \geq n+1$. Verification that $\overline{\tau}$ has the required properties follows easily. □

Theorem 25 allows us to replace the Dedekind self-map $j : A \to A$ from diagram (29) with a Dedekind self-map defined on a class. For the purpose of defining the sequence $V_0, V_1, V_2, \ldots$ and V_ω, one could use a map $V \to V$, but a somewhat more elegant way is to use the class **HF** of hereditarily finite sets, mentioned earlier.

We define the class **HF** as follows: We place $x \in \mathbf{HF}$ if the smallest transitive set that contains x as a subset is finite.[51] We define $j : \mathbf{HF} \to \mathbf{HF}$ by $j(x) = \mathcal{P}(x)$, where $\mathcal{P}$ denotes the power set operator and $\mathcal{P}(x)$ is the set of all subsets of x. We verify that $\operatorname{ran} j \subseteq \mathbf{HF}$: Notice that if y is finite and transitive, so is $\mathcal{P}(y)$. Therefore, suppose $x \in \mathbf{HF}$ and y is the smallest transitive set containing x (which, in particular, must be finite). Then, since $\mathcal{P}(y)$ is finite and transitive and $\mathcal{P}(x) \subseteq \mathcal{P}(y)$, it follows that $\mathcal{P}(x) \in \mathbf{HF}$. Therefore, $\operatorname{ran} j \subseteq \mathbf{HF}$. Notice that $\emptyset$ is a critical point for j and that j is 1-1. Therefore, j is a Dedekind self-map on the class **HF**.

Theorem 25 now guarantees there is a unique $\overline{\tau} : \overline{\omega} \to \mathbf{HF}$ taking 0 to $\emptyset$ and making the following commutative:

(32)

$$\begin{array}{ccc} \overline{\omega} & \xrightarrow{\ s\ } & \overline{\omega} \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ \mathbf{HF} & \xrightarrow{\ j\ } & \mathbf{HF} \end{array}$$

In particular, $\overline{\tau}$ satisfies the following:

(a) $\overline{\tau}(0) = \emptyset$;
(b) for all $n \in \omega$, $\overline{\tau}(s(n)) = j(\overline{\tau}(n))$.

[51] Recall that the axiom Trans (which asserts that every set is included in a transitive set) has been included in ZFC; see the footnote on p. 10. The "smallest" such transitive set is found by forming the intersection of them all.

If we write $V_n = \overline{\tau}(n)$, then these clauses become
(a′) $V_0 = \emptyset$;
(b′) for all $n \in \omega$, $V_{n+1} = \mathcal{P}(V_n)$.

It follows that $\bigcup_{n\in\overline{\omega}} V_n \subseteq \mathbf{HF}$. Working in ZFC − Infinity, it is possible to show that in fact $\bigcup_{n\in\overline{\omega}} V_n = \mathbf{HF}$.[52]

In the presence of the Axiom of Infinity—in particular, if ω is present in the universe—the sequence $\langle V_0, V_1, V_2, \ldots\rangle$ can be seen to be a *set*; indeed, this sequence is just another name for the function $\overline{\tau}$, which in this case is defined on ω. The Union Axiom allows us now to form the union, which we denote V_ω:

$$V_\omega = \bigcup \operatorname{ran} \overline{\tau} = \bigcup_{n\in\omega} V_n. \tag{33}$$

Moreover, since $\mathbf{HF} = V_\omega$, it follows that $\mathbf{HF}$ is a *set*.

In the absence of the Axiom of Infinity—in particular, when we assume that no infinite set exists (that is, working in ZFC − Infinity + ¬Infinity)—**HF** is defined in the same way, but in that context, it is a proper class, though as we mentioned earlier, it is still the case that $\mathbf{HF} = \bigcup_{n\in\overline{\omega}} V_n$.

For the ZFC context, we complete the construction of the stages of V beyond V_ω in Section 15.

11. Initial Dedekind Self-Maps

Starting from an arbitrary Dedekind self-map, we have derived the set ω of natural numbers together with its usual successor function $s : \omega \to \omega$, which is itself a Dedekind self-map. Intuitively, one considers the set of natural numbers as the smallest type of infinite set. In the usual development of ZFC set theory, for example, one proves that the size $\aleph_0$ of ω is the smallest infinite cardinal, and one can also show that ω is faithfully embedded in every infinite set; that is, for any infinite S, there is a 1-1 function $f : \omega \to S$.

In Theorem 20, we proved something that is apparently even stronger: that the successor function $s : \omega \to \omega$ is embedded in every Dedekind self-map, in a unique way, so that not only is it true that every infinite set contains a *copy* of ω, but in fact every Dedekind self-map has within it the dynamics of the successor function s.

In this section, we show that $s : \omega \to \omega$ is not the only such function. Just as there are many infinite sets that have the same size as ω, and that therefore can lay claim to this property that every infinite set must contain a copy of *them* as well, so likewise are there many Dedekind self-maps that are "just like" s, and that are likewise embedded in every other Dedekind self-map. These "smallest" Dedekind self-maps are called *initial*; this terminology originates from the field of *category theory*, and will be explained as this section develops. As we now show, a Dedekind self-map $j : A \to A$ has this special "leastness" property if and only if j is Dedekind self-map isomorphic to $s : \omega \to \omega$.

[52]This is shown in Theorem 87 in the Appendix; in particular, Corollary 88 shows that from ZFC − Infinity + ¬Infinity, we have $V = \bigcup_{n\in\overline{\omega}} V_n$. A direct proof from ZFC − Infinity + ¬Infinity that $V = \mathbf{HF}$ can be given as follows: Given any set x, let y be the smallest transitive set that contains x. Since ¬Infinity holds, y is finite, and so $x \in \mathbf{HF}$.

Theorem 25.

(1) *Suppose $k : U \to U$ is a Dedekind self map, with critical point u such that (U, k, u) is Dedekind self-map isomorphic to $(\omega, s, 0)$. Suppose $j : A \to A$ is any Dedekind self-map with critical point a. Then there is a unique Dedekind self-map morphism σ from (U, k, u) to (A, j, a).*

$$\begin{array}{ccc} U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow{j} & A \end{array} \tag{34}$$

(2) *Suppose $k : U \to U$ is a Dedekind self-map with critical point u. Suppose (U, k, u) has the property that for every Dedekind self-map $j : A \to A$ with critical point a, there is a unique Dedekind self-map morphism σ from (U, k, u) to (A, j, a). Then there is a Dedekind self-map isomorphism from $(\omega, s, 0)$ to (U, k, u).*

$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ U & \xrightarrow{k} & U \end{array} \tag{35}$$

Remark 6. Part (2) of Theorem 26 is the converse to Part (1). These two results tell us that the structure of the Dedekind self-map $(\omega, s, 0)$—and therefore the fundamental structure of the set of natural numbers—is completely characterized by the universal property stated in the hypothesis of Part (2).

Proof of (1). Let β be a Dedekind self-map isomorphism from (U, k, u) to $(\omega, s, 0)$. Let $\overline{\tau}$ be the Dedekind self-map morphism from $(\omega, s, 0)$ to (A, j, a), as defined in Theorem 20.

$$\begin{array}{ccc} U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ A & \xrightarrow{j} & A \end{array} \tag{36}$$

Define σ from k to j by $\sigma = \overline{\tau} \circ \beta$. The proof that $\sigma(u) = a$ and $j \circ \sigma = \sigma \circ k$ is essentially identical to the proof of equation (20) in Theorem 20. The proof of uniqueness is also essentially the same as the logic used in the proof of Theorem 20.

Proof of (2). The idea here is that both $(\omega, s, 0)$ and (U, k, u) have the universal property described in the hypothesis of Part (2) of the theorem. Diagram (37) captures the relationships involved.

(37)
$$\begin{array}{ccc} U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \omega & \xrightarrow{s} & \omega \end{array}$$

We let $\beta : (U, k, u) \to (\omega, s, 0)$ be the unique Dedekind self-map guaranteed to exist because of the hypothesis, and we let $\gamma : (\omega, s, 0) \to (U, k, u)$ be the unique Dedekind self-map guaranteed to exist by Theorem 20. Notice $\gamma \circ \beta$ takes u to u and makes the following diagram commutative:

(38)
$$\begin{array}{ccc} U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle\gamma\circ\beta} & & \downarrow{\scriptstyle\gamma\circ\beta} \\ U & \xrightarrow{k} & U \end{array}$$

but $\mathrm{id}_U : U \to U$ does the same: $\mathrm{id}_U(u) = u$ and the following is commutative:

(39)
$$\begin{array}{ccc} U & \xrightarrow{k} & U \\ \downarrow{\scriptstyle\mathrm{id}_U} & & \downarrow{\scriptstyle\mathrm{id}_U} \\ U & \xrightarrow{k} & U \end{array}$$

By the uniqueness guaranteed by the hypothesis,

(40)
$$\mathrm{id}_U = \gamma \circ \beta.$$

Likewise, we have $\beta(\gamma(0)) = 0$ and the following diagram is commutative:

$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \beta\circ\gamma} & & \downarrow{\scriptstyle \beta\circ\gamma} \\ \omega & \xrightarrow{s} & \omega \end{array} \tag{41}$$

but $\mathrm{id}_\omega : \omega \to \omega$ does the same: $\mathrm{id}_\omega(0) = 0$ and the following is commutative:

$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \mathrm{id}_\omega} & & \downarrow{\scriptstyle \mathrm{id}_\omega} \\ \omega & \xrightarrow{s} & \omega \end{array} \tag{42}$$

Therefore, we also have

$$\mathrm{id}_\omega = \beta \circ \gamma. \tag{43}$$

It is straightforward to verify that equations (40) and (43) together imply that both β and γ are bijections. Hence, in particular, $\gamma : (\omega, s, 0) \to (U, k, u)$ is a Dedekind self-map isomorphism. □

We recall that Theorem 20 showed that the Dedekind self-map $(\omega, s, 0)$ is "less than or equal to" all other Dedekind self-maps—that is, $(\omega, s, 0)$ is an *initial* Dedekind self-map—by analogy with the leastness of ω among all infinite sets. Now, Theorem 26 shows that every Dedekind self-map that is Dedekind self-map isomorphic to $(\omega, s, 0)$ is initial as well, and, moreover, the *only* initial Dedekind self-maps are those that are Dedekind self-map isomorphic to $(\omega, s, 0)$.

This idea can be expressed more simply using the concept of a *category*.[53] A category is a pair $(\mathcal{O}, \mathcal{M})$, where $\mathcal{O}$ is a collection of objects and $\mathcal{M}$ is a collection of morphisms, satisfying the following:

(1) Each morphism $f \in \mathcal{M}$ has a domain and a codomain, written dom f, cod f, respectively, both belonging to $\mathcal{O}$; in the familiar way, if $A = \text{dom } f$ and $B = \text{cod } f$, we write $f : A \to B$.
(2) Morphisms can be *composed*: If $f : A \to B$ and $g : B \to C$ both belong to $\mathcal{M}$, then there is another morphism $g \circ f : A \to C$ that also belongs to $\mathcal{M}$; moreover, *composition is associative*: $h \circ (g \circ f) = (h \circ g) \circ f$.
(3) For each object $A \in \mathcal{O}$, there is an *identity morphism* $1_A : A \to A$, which has the following two properties: For all $f : X \to Y$ in $\mathcal{M}$, $1_Y \circ f = f$ and $f \circ 1_X = f$. We often denote 1_A by id_A.

A simple example of a category is **Set**, which has all sets as its objects, and all functions between sets as its morphisms. If we denote the collection of all functions

[53] See [2] and [25] for good introductions to category theory.

defined on a set in V by $V^{<V}$, then[54] $\mathbf{Set} = (V, V^{<V} \cap V)$. Recall that it is convenient at times to let V denote the universe of all sets for the theory ZFC − Infinity—a universe in which existence of an infinite set is not postulated. Whether we are letting V denote the full universe of sets modeling all of ZFC set theory, or letting V be a universe for ZFC − Infinity, we think of the category **Set** to be defined in this same way, with objects being the sets in V and morphisms, the functions in V.

Another very different example is the category **Nat**, whose objects are the natural numbers $0, 1, 2, \ldots$ and whose morphisms are pairs (m, n) of natural numbers for which $m \leq n$. If $S = \{(m, n) \in \omega \times \omega \mid m \leq n\}$, then $\mathbf{Nat} = (\omega, S)$. Notice that in this case, "morphisms" are nothing like the usual concept of functions, but they do satisfy the requirements mentioned in the definition of categories.[55] Finally, a category of primary interest to us in this paper is **SelfMap**, whose objects are Dedekind self-maps and whose morphisms are Dedekind self-map morphisms.

Checking that the requirements for a category have been met for each of our examples is straightforward, and we shall assume that this verification has been done.

A concept that we have already defined in particular cases, but which is best formulated in the language of category theory, is *isomorphism*. In any category $\mathcal{C} = (\mathcal{O}, \mathcal{M})$, if $A, B \in \mathcal{O}$, an *isomorphism from A to B* is a morphism $f \in \mathcal{M}$ with the property that there is another $g \in \mathcal{M}$, $g : B \to A$, with $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. In **Set**, the isomorphisms are the bijections. In **Nat**, two objects are isomorphic if and only if they are equal. And in **SelfMap**, the isomorphisms are precisely the Dedekind self-map isomorphisms.

The concept we wish to introduce at this point is that of an *initial object*. An initial object in a category is the "smallest" object of the category. We give the definition: An object I in a category $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ is *initial* if there is exactly one morphism from I to each object X in $\mathcal{O}$. In **Set**, $\emptyset$ is initial because there is just one map (the empty map) from $\emptyset$ to any other set. In **Nat**, 0 is initial because $0 \leq m$ for all $m \in \omega$. Now, with this new concept in hand, it is easy to see that $(\omega, s, 0)$ is an initial object for **SelfMap**. In fact, Theorems 20 and 26 can be summarized as follows, using this new concept:

Theorem 26.

(1) $(\omega, s, 0)$ *is initial in the category* **SelfMap**.

(2) *Any object* (U, k, u) *of* **SelfMap** *that is initial is Dedekind self-map isomorphic (hence* **SelfMap**-*isomorphic) to* $(\omega, s, 0)$.

[54] The curious reader may wonder why the collection of morphisms for **Set** is $V^{<V} \cap V$ rather than simply $V^{<V}$. The reason is that it is possible to devise a function f defined on some set $X \in V$ that is not itself a member of V. We will see an example at the end of this paper. Such functions are necessarily not definable in the universe, nor derivable from the axioms of set theory, and hence do not properly belong to **Set**, which must be viewed as a (definable) class. Moreover, the range of such functions is not itself a set, so it does not make sense to include them as morphisms. Nevertheless, such undefinable functions play an important role in the theory of sets (but not so much in the category of sets); see the footnote on p. 183 for an example.

[55] It is helpful to think of a morphism $f : A \to B$ in a category as a *directed edge* in a directed graph. One could have directed graphs in which such edges are indeed functions, and other graphs in which they are not. It can be shown that any category is a directed graph with additional properties.

(3) *Any object* (U, k, u) *of* **SelfMap** *that is Dedekind self-map isomorphic (hence* **SelfMap***-isomorphic) to* $(\omega, s, 0)$ *is initial.*

Proof. Part (1) is a restatement of Theorem 20. Part (2) is a restatement of Theorem 26(2). Part (3) is a restatement of Theorem 26(1). □

Our reasoning so far has shown that, given a Dedekind self-map $j : A \to A$ with critical point a, if we form in A the smallest j-inductive set W, then $W = \{j^n(a) \mid n \in \omega\}$. We have also just seen that $(W, j \restriction W, a)$ is initial—a structural duplicate of $(\omega, s, 0)$. If W is defined as the smallest j-inductive set, then we will call $(W, j \restriction W, a)$ (or, by an abuse of notation, W itself) the *initial object generated by* j.

We have also shown that $\tau = \pi^{-1} : \omega \to W$ is a map that lists the elements of W: For all $n \in \omega$, $\tau(n) = j^n(a) = j(j(\ldots((a))\ldots))$ (where the right hand side of the expression consists of n applications of j to a). We will call τ the *canonical enumeration of* W. Recall that τ is itself a Dedekind self-map isomorphism from $(\omega, s, 0)$ to $(W, j \restriction W, a)$.

Our results in this section lead to important characterizations of the notions of "infinite set" and "set of natural numbers." We began our discussion with an effort to show that the Axiom of Infinity could be formulated as the assertion that a Dedekind self-map exists. We showed that, without reliance on the natural numbers (defined in the usual way), the set ω of natural numbers could be derived, using just a Dedekind self-map in conjunction with the other axioms of ZFC. These efforts led not only to a derivation of ω but also a characterization of those Dedekind self-maps that are in every respect, up to notational differences, equivalent to ω together with its successor function.

Speaking philosophically, we have given evidence for the following two conclusions:

(1) The "underlying reality" of infinite sets is the concept of a Dedekind self-map.
(2) The "underlying reality" of the set of natural numbers is the concept of an *initial* Dedekind self-map.

In particular, every Dedekind self-map $j : A \to A$ with critical point a generates a *blueprint* for the set of natural numbers, in the form

$$W = \{a, j(a), j(j(a)), \ldots\} \subseteq A.$$

Indeed, $j \restriction W : W \to W$ is itself an *initial* Dedekind self-map, isomorphic to $s : \omega \to \omega$ and naturally embedded in every other Dedekind self-map.

We consider next the *dual* of a Dedekind self-map. Dedekind/co-Dedekind pairs will be central to our formal definition of a *blueprint*.

12. Dedekind/co-Dedekind Duality

In our effort to capture a deeper meaning of the "mathematical infinite" in our Axiom of Infinity, we isolated the concept of a Dedekind self-map. This concept gives mathematical expression to one end of a polarity that characterizes the Infinite, according to the ancient wisdom, namely, the dynamics of expansion to the infinite from a singularity. We have seen that this expansion takes place, for a given

Dedekind self-map $j : A \to A$ with critical point a, by repeated applications of j to its critical point, producing the infinite sequence $a, j(a), j(j(a)), \ldots$.

The other half of this polarity, is, according to the ancient view,[56] the dynamics of collapse, by which the vast diversity returns to the point from which it originates.[57]

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[56]This theme is prevalent in the eternal wisdom found in ancient texts. Maharishi [47] remarks:

> The Vedic theme of education cherishes this aspect of gaining knowledge in the word *Nivartadhwam*, which means 'return.' From point to infinity and from infinity back to the point is the path of gaining knowledge (p. 42).

Maharishi explains that the unfoldment of diversity from within pure consciousness simultaneously involves a return to pure consciousness through "self-referral loops"; this is the mechanism by which expressed values remain connected to their source. As T. Nader [57] explains:

> This is how Ṛk Veda and the whole Vedic Literature emerge within the pure Self, Ātmā, in its self-referral quality, expressing, transforming, expanding, silence and dynamism, sounds and the gaps between sounds; always coming back to the source via the loops at the basis of the structuring dynamics of pure knowingness (p. 42).

Indeed, according to Maharishi [57, p. 25], there is a part of the Vedic Literature that is responsible for expansion and another that is responsible for return. In "Fundamental Principles of Maharishi Vedic Science," P. Oates [58] remarks

> In addition to the balance maintained through the unfolding of opposite qualities, the structure of the self-referral loops of Vedic Literature reveals that the first three qualities of each loop of Vedic Literature in effect emerge from Ātmā, from Unity, and unfold through Ṛishi, Devatā, and Chhandas into its diverse expressions, while the second set of three aspects of Vedic Literature reveals the process which connects the unfoldment of qualities with its source, through the return path, or self-referral feedback loop, from Chhandas, Devatā, to Ṛishi and back to Ātmā, or Unity (p. 127).

[57]Continuing the previous footnote, we mention here that this theme of return is an essential characteristic of the dynamics of the source according to Laozi. In the *Tao Te Ching* [23], we read:

> Returning is the motion of the Tao (v. 40).

and

> Something mysteriously formed,
> Existing before heaven and Earth.
> In the silence and the void,
> Standing alone and unchanging,
> Ever present and in motion.
> It is the mother of ten thousand things.
> I do not know its name,
> Call it Tao.
> For lack of a better word, I call it great.
> Being great, it flows.
> It flows far away.
> Having gone far, it returns (v. 25).

Finally, we mention the fact that Plato and the Neoplatonists recognized these fundamental dual tendencies of the One. For instance, Plotinus, the founder of the Neoplatonic school, writes [64]:

> By a natural necessity does everything proceed from, and return to unity; thus creatures which are different, or even opposed, are not any the less co-ordinated in the same system, and that because they proceed from the same principle (p. 1077).

In a rather natural way, these dynamics are expressed mathematically using the dual notion of a *co-Dedekind self-map*.

Let us say that an onto self-map $h : A \to A$ is *co-Dedekind* if the preimage $h^{-1}(a)$, for some $a \in A$, has two or more elements. Whenever $a \in A$ is such that $|h^{-1}(a)| \geq 2$, a will be called a *co-critical point* of h.

Recall that, whenever $h : A \to A$ is onto, there is, by the Axiom of Choice, a function $s : A \to A$ whose range contains exactly one element from each preimage $h^{-1}(a)$, for $a \in A$; such an s is called a *section of* h. Clearly, any section of an onto map must be 1-1. Moreover, we have the following:

Theorem 27. (ZFC – Infinity) *Suppose A is a set. Then the following are equivalent:*

(1) *There is a Dedekind self-map on A.*
(2) *There is a co-Dedekind self-map on A.*

Proof. Suppose there is a Dedekind self-map $j : A \to A$ with critical point a. Let $a_0 \in \operatorname{ran} j$. Define $h : A \to A$ by

$$h(x) = \begin{cases} a_0 & \text{if } x \notin \operatorname{ran} j, \\ y & \text{otherwise, where } y \in A \text{ is unique such that } j(y) = x. \end{cases}$$

Under the definition, we have that $h(a) = a_0$ since a is a critical point of j. It is obvious that $h : A \to A$ is onto since even $h \restriction \operatorname{ran} j$ is onto. It follows that some $b = j(x) \in \operatorname{ran} j$ is also mapped by h to a_0, since $h \restriction \operatorname{ran} j$ is onto. Certainly $b \neq a$ since $a \notin \operatorname{ran} j$. Therefore, $|h^{-1}(a_0)| \geq 2$, so h is co-Dedekind.

Conversely, suppose $h : A \to A$ is co-Dedekind, and suppose $s : A \to A$ is a section of h. We show s itself is Dedekind. We have already observed that s is 1-1. Let $x \in A$ be such that $|h^{-1}(x)| \geq 2$, and let $u \neq v \in A$ be elements of $h^{-1}(x)$. Then one of u, v does not belong to the range of s; in particular, one of u, v is a critical point of s. □

The argument shows that any section s of a co-Dedekind self-map $h : A \to A$ is itself a Dedekind self-map; moreover, for any co-critical point x of h, some element of $h^{-1}(x)$ is a critical point of s.

We give an example to illustrate the "collapsing" effect that co-Dedekind self-maps often have. Let us say that a set A is *closed under singletons* if, for all $x \in A$, we have $\{x\} \in A$; more generally,[58] A is *closed under pairs* if, whenever $x, y \in A$, $\{x, y\} \in A$. As we now show, in studying co-Dedekind self-maps $A \to A$, there is nothing lost if we assume A is a transitive set closed under pairs:

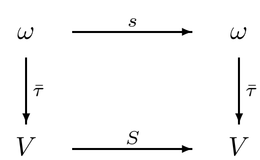
Proposition 28. *There is a co-Dedekind self-map on a set if and only if there is a co-Dedekind self-map on a transitive set that is closed under pairs. Moreover, any set that is closed under pairs admits a co-Dedekind self-map.*

Proof. Suppose $h : A \to A$ is a co-Dedekind self-map on A. Let $t : A \to A$ be a section of h that is a Dedekind self-map with critical point a. We will lift t to a

[58]If a set is closed under pairs, it is closed under singletons, since every singleton set $\{x\}$ is itself the pair $\{x, x\}$.

Dedekind self-map $\hat{t} : B \to B$, where B is a transitive set that is closed under pairs and that includes A.

Let $S : V \to V$ be defined by $S(x) = \{x\}$. By Theorem 25, there is a unique $\bar{\tau} : \omega \to V$ such that $\bar{\tau}(0) = a$ and $\bar{\tau} \circ S = S \circ \bar{\tau}$.



In particular, there is a set W whose elements are precisely $a, \{a\}, \{\{a\}\}, \ldots$. In other words,

$$W = \{a, \{a\}, \{\{a\}\}, \ldots\},$$

and $S \restriction W : W \to W$ is initial. Moreover, $(\omega, s, 0)$ and $(W, S \restriction W, a)$ are Dedekind self-map isomorphic.

Claim. There is a Dedekind self-map $\hat{t} : B \to B$, where B is a transitive set closed under pairs, that includes A and such that $\hat{t} \restriction A = t$.

Proof of Claim. We first observe that for any set C, we can obtain a set $\mathcal{T}(C) \supseteq C$ that is closed under pairs: Define $C = C_0 \subseteq C_1 \subseteq \cdots$ as follows: Let $C_1 = [C_0]^2$ (where, for any set D, $[D]^2$ denotes the set of all unordered pairs from D), and, in general,[59] $C_{n+1} = [C_n]^2$. Let $\mathcal{T}(C) = \bigcup_{n \in \omega} C_n$. If $u, v \in \mathcal{T}(C)$, then for some $n \in \omega$, $u, v \in C_n$, and so $\{u, v\} \in C_{n+1} \subseteq \mathcal{T}(C)$.

Now we build a set B as the union of the following chain:

$$A = A_0 \subseteq B_0 \subseteq A_1 \subseteq B_1 \subseteq \cdots,$$

where, for each $i \in \omega$, $B_i = \mathcal{T}(A_i)$ and A_{i+1} is a transitive set that contains B_i. If $u, v \in B$, then $u, v \in B_i$ for some i, and so, because $B_i = \mathcal{T}(A_i)$, $\{u, v\} \in B_i \subseteq B$, and so B is closed under pairs. Also, if $u \in v \in B$, then $v \in A_i$ for some $i > 0$, and so, by transitivity of A_i, $u \in A_i \subseteq B$; this shows B is transitive, as required.

We obtain a Dedekind self-map $\hat{t} : B \to B$ as follows:

$$\hat{t}(b) = \begin{cases} b & \text{if } b \notin A, \\ t(b) & \text{if } b \in A. \end{cases}$$

It is easy to see that $\hat{t}$ is a Dedekind self-map. $\square$

To complete the proof of the main clause of Proposition 29, we simply recall that, by Theorem 28, whenever there is a Dedekind self-map $B \to B$, there is also a co-Dedekind self-map $B \to B$.

[59]Formally, we are using Theorem 25 here. Define $j : V \to V$ by $j(x) = \{\{u, v\} \mid u \in x \text{ and } v \in x\}$; j is a Dedekind self-map. As in the Theorem, there is a unique $\overline{\tau} : \omega \to V$ satisfying $\overline{\tau}(0) = C_0$ and $j\overline{\tau} = \overline{\tau}s$, where $s : \omega \to \omega$ is the successor function. Then define $C_n = \overline{\tau}(n)$ for each $n \in \omega$. This gives us the sequence $\langle C_0, C_1, C_2, \ldots \rangle$ as in the main text, and one may then form the union, as described there.

Finally, for the "moreover" clause, notice that if B is closed under pairs, it is closed under singletons. If $x \in B$, then $W = \{x, \{x\}, \{\{x\}\} \ldots\}$ is a copy of ω in B, and so B is infinite, and therefore must admit a co-Dedekind self-map (by Theorem 28 again together with the first half of Proposition 29). □

Example 1. (Generate/Collapse Duality) Let A be a transitive nonempty set that is closed under pairs. We assume that A is a set "in the universe" in the sense that it is governed by the usual axioms of set theory. (In particular, no element of A is an element of itself.) Consider the self-map $F : A \to A$, defined by[60]

$$F(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ y & \text{where } y \text{ is any } \in\text{-minimal element of } x. \end{cases}$$

Because A is transitive, $\operatorname{ran} F \subseteq A$.[61] We also observe that F is onto: Suppose $y \in A$. Then $\{y\} \in A$, and clearly $F(\{y\}) = y$. Finally, suppose $z \in A$ and consider the sets $x = \{z\}$ and $y = \{z, \{z\}\}$. The fact that no set is an element of itself ensures that $z, \{z\}$ are disjoint, and so $F(y) = z = F(x)$. Thus, $|F^{-1}(z)| > 1$. We have shown F is a co-Dedekind self-map.

Let $S_A = S \restriction A : A \to A$, where, we recall, $S(x) = \{x\}$ for all $x \in A$. We show that S_A is a section of F by showing $F \circ S = \mathrm{id}_A$:

$$F(S(x)) = F(\{x\}) = x.$$

The dual notions of Dedekind self-map and co-Dedekind self-map are expressed in the self-maps S_A and F. Certainly, S_A plays the role of *generating* a blueprint for the natural numbers: Given a, $S_A \restriction \{a, S_A(a), S_A^2(a), \ldots\}$ is an initial Dedekind self-map. We wish to show that, conversely, F plays the role of *collapsing* the values of A to their point of origin.

We show that for every $x \in A$, there is $n \in \omega$ such that $F^n(x) = \emptyset$. Suppose not. Then for each $n \in \omega$, $F^n(x) \neq \emptyset$. It follows that the following is an infinite descending $\in$-chain:

$$\cdots \in F^n(x) \in F^{n-1}(x) \in \cdots \in F(x) \in x.$$

Such chains cannot exist in the presence of the Axiom of Foundation. The result follows.

Let $\mathcal{E} = \{i_n \mid n \in \omega\}$, where, for each n, $i_n : A^A \to A^A$ is defined by

$$i_n(f) = \begin{cases} f^n & \text{if } n > 0 \\ \mathrm{id}_A & \text{if } n = 0 \end{cases}. \tag{44}$$

Here, f^n denotes the nth iterate of f. We have the following:

[60]The definition of $\in$-*minimal element* of a set is given on p. 25. Note that the definition of F relies on the Axiom of Choice: For each nonempty $x \in A$, let $A_x \subseteq A$ be the set of all $\in$-minimal elements of x. Let $C : \{A_x \mid x \in A, x \neq \emptyset\} \to A$ be a choice function; that is, $C(A_x) \in A_x$ for each nonempty set x in A. Then, define F as follows: Whenever $x \neq \emptyset$, $F(x) = C(A_x)$.

[61]Certainly any $\in$-minimal element of an element of A also belongs to A, by transitivity. We verify here that $\emptyset \in A$: Let $a \in A$ be $\in$-minimal in A (recall that A is nonempty). If $a = \emptyset$, we are done, so assume $a \neq \emptyset$ and let $x \in a$. Since A is transitive, $x \in A$, but this contradicts the fact that a is $\in$-minimal in A. Therefore, the only possibility is that $a = \emptyset$, as required.

Proposition 29. *Suppose A is a transitive set that is closed under pairs. Let $W = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\} \subseteq A$.*

(1) *For every $x \in A$—in particular, for every $x \in W$—there is $i \in \mathcal{E}$ such that $i(F)(x) = \emptyset$.*

(2) *For every $x \in W$, there is $i \in \mathcal{E}$ such that $i(S_A)(\emptyset) = x$.*

The proposition indicates how every element of A is "returned to its source" via the interplay of F and a naturally occurring set $\mathcal{E}$ of functionals defined over A. Likewise, through the interplay of S_A and $\mathcal{E}$, we also see once again in the present context how a blueprint for the natural numbers is generated. In summary, the dual self-maps S_A and F play the roles, respectively, of "generating a blueprint" and "returning elements to their source."

The functionals belonging to the class $\mathcal{E}$ in this example have a number of nice properties that will be useful for us to consider when we formulate a more precise version of the notion of a "blueprint."[62]

Definition 6. (Weakly Elementary Functionals) Let $B^B = \{f \mid f : B \to B\}$ and suppose S is a collection of functions. A functional $i : B^B \to S$ is said to be *weakly elementary relative to B* if it has the following properties (a)–(g):

(a) If g is 1-1, then $i(g)$ is also 1-1; in fact, if g is a Dedekind self-map, then $i(g)$ is also a Dedekind self-map.

(b) If g is onto, then $i(g)$ is also onto; in fact, if h is a co-Dedekind self-map, then $i(g)$ is also a co-Dedekind self-map.

(c) If g preserves the empty set (terminal objects), then $i(g)$ preserves the empty set (terminal objects).

(d) If g preserves disjoint unions, then so does $i(g)$.

(e) If g preserves intersections, then so does $i(g)$.

(f) If g *preserves membership*—that is, $x \in y$ implies $g(x) \in g(y)$ for x, y in the domain of g—then so does $i(g)$.

(g) If g *reflects membership*—that is, whenever $x, y \in \mathrm{dom}\ g$ and $g(x) \in g(y)$, then $x \in y$—then so does $i(g)$.

It is a straightforward exercise to show that each $i_n \in \mathcal{E}$, as in definition (44), is weakly elementary.

[62] This notion of "weakly elementary functional" is a simplification of the more technical concept that is needed in a more rigorous treatment. The concept that is needed is Σ_0-*preserving* (see [17]):

Definition 5. Suppose C, D are sets, each equipped with binary relations E, R, respectively. A function $f : (C, E) \to (D, R)$ will be called Σ_0-*elementary* if, for every Σ_0^{ZFC} $\in$-formula $\phi(x_1, \ldots, x_m)$ and all $c_1, \ldots, c_m \in C$, $\phi^{(C,E)}(c_1, \ldots, c_m) \Leftrightarrow \phi^{(D,R)}(f(c_1), \ldots, f(c_m))$ (as usual, it is understood that E interprets $\in$ in C and R interprets $\in$ in D). Let S be a nonempty collection of functions $C \to D$. Suppose $i : S \to T$ is a functional defined so that for each $f \in S$, $i(f) : C' \to D'$. Suppose also that C', D' are equipped with binary relations E', R', respectively. We shall say that i is Σ_0-*preserving* if, whenever $f \in S$ is Σ_0-elementary, then $i(f)$ is also Σ_0-elementary.

To see the connection to the previous example, let $S_W = S_A \restriction W : W \to W$. We can define a binary relation $\sqsubseteq$ on W as we did earlier that satisfies $x \sqsubseteq y$ if and only if one can obtain y from x by applying S_W at most finitely many times to x: $y = S_W(S_W \ldots (S_W(x)) \ldots)$. Then it is easy to verify that each of the functionals $i_n \in \mathcal{E}$ is Σ_0-preserving.

As we shall see, the concept of a "blueprint" appears naturally in the context of large cardinals, and weakly elementary functionals play an important role in that context. In the next section, we make the notion of a blueprint, suggested by our results here, more precise.

13. Blueprints Generated by a Dedekind Self-Map

In this section, we look more deeply into the concept of a blueprint and give a more formal treatment of blueprints that arise from Dedekind self-maps. This formal treatment will generalize well to broader contexts and will provide further evidence of the rich foreshadowing of large cardinals suggested by the concept of a Dedekind self-map.

Before diving into the formal treatment, we discuss the reason for going to the trouble of providing a rather technical account of the idea of a "blueprint."

We began our study of Dedekind self-maps with the intention of finding a way to express the existence of an infinite set in the universe in a way that provided a fuller intuition about the nature of the infinite in mathematics. The hope was that such an intuition could provide the sort of insight that would suggest solutions to the Problem of Large Cardinals: Is there a "right way" to think about the Infinite that suggests that large cardinals *really do* exist?

Ancient traditions of knowledge suggest that the sequence of natural numbers—and indeed, the unfoldment of any multiplicity—has a *source*. In Maharishi Vedic Science, that source is Maharishi's Absolute Number; for Pythagoras, it was the Number of numbers; for Proclus, it was a *divine number united with the One*; and for Laozi, it was *Tao*. The QFT perspective for dealing with classes of discrete particles suggests that this source is an *unbounded quantum field* and that particles are precipitations of the field.

Applying these perspectives to mathematical foundations and the quest for a new angle on the Axiom of Infinity, we considered the possibility that the discrete values that make up infinite sets—particularly, the set of natural numbers—should be considered to be "precipitations" of some kind of "field." A realization of this intuition is the concept of a Dedekind self-map $j : A \to A$ with critical point a: The set A represents the unbounded "field" whose dynamics are represented by j and its interaction with a. Moreover, its "precipitations" $a, j(a), j(j(a)), \ldots$, arising from repeated application of j to its critical point, turned out to form, in a precise sense, a *blueprint* for the set of natural numbers (Theorems 11– 14).

These discoveries provide a degree of confirmation that the intuitions obtained from ancient texts, and also from quantum field theory, truly can be realized in a mathematical context, and do indeed bear fruit.

Motivated by this success, we seek to develop a more precise mathematical definition of "blueprint" so that, as we explore generalizations of the concept of a Dedekind self-map, we will be able to accurately identify blueprints if and when they arise. We would expect, based on what we have seen so far, that a characteristic of scaled versions of Dedekind self-maps—which would presumably generate larger types of infinities—would, like the Dedekind self-maps we have seen so far, give rise to blueprints for some sort of interesting sets.

As we formulate a mathematical definition of "blueprint," we will continue to be guided by the insights of ancient traditions of knowledge. To this end, we will look more closely at conceptions of blueprints described by these traditions and attempt to catalog characteristics common to all approaches. We will use these results—just as we used ancient insights about a possible "source" of the natural numbers—to guide our mathematical formulation. Having obtained a satisfactory definition of "blueprint," we will then examine Dedekind self-maps in a broader context, and check whether the self-maps we encounter do indeed generate this kind of blueprint. Our hypothesis will be that those that do are the ones that will yield the results we are seeking: an account of large cardinals and, possibly, even an account of *all sets* as well.

We turn now, therefore, to a brief account of "blueprints for creation," according to several ancient traditions of knowledge. We also say a few words about how the QFT world view is related to our formulation of blueprints.

13.1. **Blueprints According to the Ancients.** We examine the concepts of blueprint described by the Vedic tradition of knowledge, represented by Maharishi Vedic Science; by ancient Chinese philosophy, represented by Laozi and the *I Ching*; and by the Platonic tradition in the West.

Maharishi Vedic Science

In Maharishi Vedic Science, the blueprint of creation is the Veda. For instance, in [58], we read:

> The totality of all the laws is the Veda; or, expressed from another perspective, Veda is the "root of all laws." Veda is referred to as a blueprint of creation, but Veda is not merely a description of the mechanics of intelligence in motion within itself; the self-interacting dynamics of consciousness generate Veda and therefore may be seen as the essence—the source of the laws which give rise to the infinite diversity of creation (p. 122).

In another passage, Maharishi further describes the role of Veda as a blueprint. Citing Maharishi, R.K. Wallace [70] writes,

> Maharishi describes the four Vedas as "a beautiful, sequentially available script of nature in its own unmanifest state, eternally functioning within itself, and, on that basis of self-interaction, creating the whole universe and governing it" (p. 218).

As mentioned earlier, Maharishi [50, pp. 52–53] explains that there is one verse in the Ṛk Veda (I.164.39) that describes the way in which the Veda is built up.

> *Richo akshare parame vyoman*
> *yasmin deva adhivishve nisheduh*
>
> The verses of the Veda exist in the collapse of fullness (the *kshara* of अ (A)) in the transcendental field, in which reside all the *Devas*, the impulses of Creative Intelligence, the Laws of Nature responsible for the whole manifest universe.

The hymns of the Veda arise in the collapse of wholeness, of totality—represented by the first letter of Ṛk Veda, 'A'—to its own point value—represented by the second letter of Ṛk Veda, 'K.' Within AK, therefore, is contained all the structuring dynamics underlying the full unfoldment of Ṛk Veda and all the Vedic Literature, and from these, the entire manifest universe.

The successive unfoldment of the hymns of the Ṛk Veda proceeds according to Maharishi's *Apaurusheya Bhashya*: From the first letter 'A' emerges the first syllable "AK," which is a fuller elaboration of the dynamics within 'A.' From "AK" emerges "Agnim," a still fuller elaboration. And then emerges "Agnimile," then the first pada, first richa, first mandala, the entire Ṛk Veda, and the entire Veda and Vedic Literature [45, p. 636].

As we mentioned before, the Veda expands in terms of self-referral loops so that diversification always remains connected to its source. This means that expansion and collapse (or return) are always occurring. Likewise, on a different scale, the Veda as a blueprint for creation is equally responsible for the return of manifest existence to its source:

> Over and above this, the proof of the practical effect of the Absolute Number in maintaining mathematical precision in the orderly evolution of the individual and the universe is the discovery of the building blocks of the Absolute Number—the Veda and Vedic Literature—in the human physiology, which has given us the complete, sequential development of the unmanifest into the manifestation of the whole universe, and has completed the cycle of the return of the manifest universe to the state of the unmanifest Absolute—the Absolute Number [45, pp. 617-8].

How, then, does this blueprint actually give rise to manifest existence? Maharishi [45, p. 589] explains that this final step is due to a principle contained within the Veda itself: *Vivart.* Maharishi defines *Vivart* as the principle that causes one thing to appear to be something else [45, p. 589].

> The principle of *Vivart* makes the unmanifest quality of self-referral consciousness appear as the Veda and Vedic Literature, and makes the Veda and Vedic Literature appear as Vishwa (pp. 377, 589).

Ancient Chinese Philosophy

Having reviewed points from Maharishi Vedic Science about the characteristics of the Veda as a blueprint, we turn to the ancient wisdom of China, represented primarily by the *I Ching* and the work of Laozi. In the *Tao Te Ching*, Laozi describes the unfoldment of manifest existence from *Tao*; repeating a citation mentioned earlier ([23]), we find the following passage:

> The Tao begot One.
> One begot Two.
> Two begot Three.
> And Three begot the ten thousand things (v. 42).

Each level of existence unfolds according to its own laws, but all depend on *Tao*, which governs itself according to its own nature [67]:

Mankind depends on the laws of Earth
Earth depends on the laws of Heaven
Heaven depends on the laws of *Tao*
But *Tao* depends on itself alone.
Supremely free, self-so, it rests in its own nature (v. 25).

The *Tao Te Ching* discusses the full range of unfoldment of *Tao*. Contained within the wholeness that is *Tao*, there is *wuji* (Limitless) and *youji* (Limited);[63] both exist as possibilities, and together represent a dynamic wholeness, *tai chi*, represented by the familiar *tai chi* symbol that displays these two principles joined as one (Figure 6).

FIGURE 6. *Tai Chi*, One, Wholeness

The wholeness *tai chi*—this *unity*—is what is intended by *One* in verse 42 mentioned above. The *I Ching* declares [1, p. 14] that all things emerge from this primal unity. One can say that all things arise from *Tao*; one can also say that *tai chi* arises from *Tao* and all things arise from *tai chi*. Indeed, it is likely that this is the distinction that is being made in the following verse in the *Tao Te Ching*:

Tao is both Named and Nameless.
As Nameless, it is the origin of all things.
As Named, it is the mother of all things ([67, v. 1]).

The two possibilities that form this primal unity are sometimes referred to as "primal mother" and "primal father"; indeed, Laozi refers to "primal mother" in several passages of *Tao Te Ching*. In the following passage, Laozi expresses the idea that the One is barely distinguishable from *Tao* itself, using "primal mother" to name the primal unity [23]:

It is the woman, primal mother.
Her gateway is the root of heaven and Earth.
It is like a veil barely seen (v. 6).

The potential for *Two* inherent in *tai chi* starts to manifest as an actual two, yin and yang, named in the *I Ching* [1, p. 2] the "Receptive" and the "Creative," respectively. Because Two is to be seen as remaining unified, yin and yang are to be appreciated as a single principle, called the *liang yi* ("two as one") [8, p. 21].

Two is the starting point for the emergence of the field of change that arises from the interaction of yin and yang. In the *Yellow Emperor's Internal Classic* (*Huángdì Nèijīng*) one reads, "The entire universe is an oscillation of the forces of Yin and Yang" [8, p. 19].

[63]See for example the Wikipedia article `http://en.wikipedia.org/wiki/Bagua`.

Therefore, in this ancient wisdom, the first stage of manifestation is the One, *tai chi*, which, as it begins to "sprout," becomes the *liang yi*, the principle of yin and yang, and this principle serves as the beginning of the unfoldment of all diversity.

The field of change itself, which emerges from *liang yi*, and which is the domain of the *I Ching*, is seen as the expression of unchanging archetypes, variously described as *forms*, *images*, or *ideas* [72]:

> [In Laozi's and Confucian teachings] every event in the visible world is the effect of an "image," that is, of an idea in the unseen world. Accordingly, everything that happens on earth is only a reproduction, as it were, of an event in a world beyond our sense perception.... The holy men and sages, who are in contact with those higher spheres, have access to these ideas through direct intuition and are therefore able to intervene decisively in events in the world.

The *blueprint* of creation, therefore, from the perspective of *I Ching*, is this higher world of images. These images originate from the Creative principle (yang) and are nurtured into being by the Receptive principle (yin) [1]:

> In the Cosmic Mind, the image arises. The arising of the image was seen by the Chinese as the action of Yang; therefore, in the *I Ching*, Yang is called the Creative. Still, it is only half of the complementary whole. Its other half is Yin, its opposite and complementary force, that in the *I Ching* is called the Receptive. The image offered by Yang is received and nurtured by Yin, bringing it into being. The spin-off of this interaction was seen as an ongoing Creation, and the ever-moving Wheel of Change (p. 15).

Moreover, the first of these images within Cosmic Mind—the first impulse—is *self-awareness*, awareness of its own nature [8, p. 19].

Therefore, the structure of *I Ching*, unfolding from One, and giving full expression to Two, may be understood to be the structure and design of the first principles of the universe.

FIGURE 7. Gua #63: Water Over Fire

The *I Ching* is composed of 64 *gua* (referred to in the West as *hexagrams*). An example of one such *gua* is Figure 7.

Each *gua* is built as a combination of two primary *gua* (called *trigrams* in the West), stacked one upon another; there are 8 primary gua (Figure 8).

The primary *gua* are built from two fundamental components (Figure 9), a broken line and an unbroken line. The broken line represents yin and the unbroken line represents yang. The 8 primary *gua* give expression to all possible ways yin and yang may interact, through three steps. The 64 *gua* give fuller elaboration of these 8. The

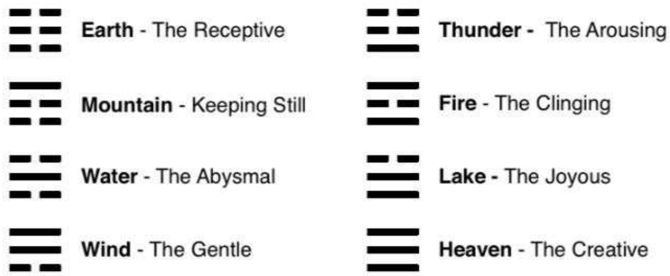


FIGURE 8. Primary Gua

FIGURE 9. Yin and Yang in *I Ching*

64 *gua*, representing the fundamental set of archetypes within the higher realm—the "unseen world"—constitute a map or blueprint of manifest existence. Taoist scholar Stephen Chang, discussing the 64 *gua* as an "evolutionary" unfoldment of the fundamental *liang yi*, explains this point as follows [8, p. 21]:

> Taoists have traced the "evolution" of the Liang Yi into a "blueprint" of the universe, describing all levels of transmutation in the universe; from creation through growth, maturity, decline, dissolution, and re-creation (p. 21).

The Tradition of Plato and the Neoplatonists

As our final example of an ancient teaching about the blueprint of manifest existence, we consider the work of Plato, supplemented by commentaries from the Neoplatonic tradition. We begin with a quick overview.

In Plato's philosophy, the ultimate reality, the Absolute, is designated "the One" and also "the Good." The One contains the potential for Two; this potential was called by Plato in his lectures in the Academy the *Indefinite Dyad* [59]. Through interaction between the One and the Indefinite Dyad, Two emerges, remaining unified through the mechanism of *continuous geometric proportion.* From Two, the first Triad emerges (described by one Neoplatonist as *intellect, intellection, intelligible*), from which emerge multiple triads, and, ultimately, the intelligible world of eternal, unchanging *forms*; these provide a template for the creation of the sensible world. Finally, a primary force, called the *Demiurge* in Plato's *Timaeus*, uses the forms to produce the objects of the material world. In Plato's philosophy therefore, the *blueprint* of manifest existence is the world of forms, spawned by the ultimate principle, the One. The rest of this subsection is devoted to an elaboration of these points.

To introduce Plato's Good and his world of forms, we begin with Plato's own description of the Good, taken from his classic, *The Republic* [30]:

> The sun, I presume you will say, not only furnishes to visibles the power of visibility, but it also provides for their generation and growth and nurture, though it is not itself generation.
>
> Of course not.
>
> In like manner, then, you are to say that the objects of knowledge not only receive from the presence of the Good their being known, but their very existence and essence is derived to them from it, though the Good itself is not essence but still transcends essence in dignity and surpassing power (509b, p. 744).

According to Plato, then, the Good gives rise to the intelligible world of forms, but the Good itself is not a form, but beyond all forms, beyond even all essence. What then are the forms, and how are they related to the visible world?

Plato's forms are fundamental archetypes or templates on the basis of which the changeable manifest world is constructed.[64] Plato cites as examples of forms the form of virtue and geometric forms, such as a circle. The form of virtue is to be understood as the archetype by which we recognize a great variety of behaviors, arising in an endless variety of contexts, as instances of virtue. Likewise, geometric forms, like a circle, represent archetypes of another kind: For instance, though no one has ever seen in the physical world a perfect circle, it is by virtue of the form of a circle that the round objects that we encounter in physical experience are recognized as approximations of a *circle*. In Plato's dialogue, *The Phaedo* [30], he makes it clear that the forms are unchangeable, eternal, beyond sense perception, of divine nature, and the cause of the multiplicity of beings, each shaped according to the parent form's nature.

In the *Republic*, Plato provides a diagram, called the *Divided Line*, to show the relationships of these different levels of life [30]:

> Suppose you have a line divided into two unequal parts, to represent the visible and intelligible orders, and then divide the two parts again in the same ratio. . . in terms of comparative clarity and obscurity (509d).

In Figure 10, the union $C \cup D$ represents the intelligible world, and the union $A \cup B$ represents the visible world in Plato's Divide Line analogy. While section C represents more concrete forms, for which we can form images, like geometric objects, section D represents more abstract forms, not representable by images, and standing for first principles, like Beauty, Justice, and Truth. Section B, the upper portion of the visible world, consists of the objects and beings of the physical world that ordinarily occupy our attention, while section A are the shadowy elements of manifest existence, which he calls *images*; Plato gives as examples shadows and reflections on various types of surfaces [30, p. 745].

In this overview of Plato's philosophy, we see that Plato views the source of all things, which he calls the One or the Good, as a pure unity, beyond all difference and diversity. The One gives rise to the multiplicity of forms, which are eternal unchanging patterns, templates, or archetypes, and constitute the *intelligible world*. The manifest or "visible" world is then a realization of the forms in terms of physical

[64] A basic introduction to Plato's theory of forms can be found in [6] and also in the online article `http://en.wikipedia.org/wiki/Theory_of_Forms`.

FIGURE 10. Plato's Divided Line

existence. The world of forms provides us with Plato's concept of a *blueprint* for material existence.

We consider several other points about this blueprint. We first ask how Plato understands that diversity can arise from the pure unity of the One. Like the sages of ancient China, who saw within *Tao* the potential for two—Limitless and Limited—so likewise did Plato see within the One, which he also termed on some occasions the *Equal*, a second principle, the *Unequal*. The Unequal was also called by him the Greater and the Lesser, and also the Indefinite Dyad. The Unequal had these other appellations since Unequal implies a two, one bigger and the other smaller. These are "indefinite" because we cannot answer exactly *how big* or *how small*. This aspect of Plato's philosophy is expressed by one of the students of the Academy, named Alexander, who makes this remark in his *Commentary of (Aristotle's) Metaphysics*, quoted in [59]:

> Thinking to prove that the Equal and Unequal [other names for One and Indefinite Dyad] are first Principles of all things, both of things that exist in their own right and of opposites. . . he assigned equality to the monad, and inequality to excess and defect: for inequality involves two things, a great and a small, which are excessive and defective. This is why he called it an Indefinite Dyad—because neither the excessive nor the exceeded is, as such, definite.

How then does a more concrete Two arise from the Indefinite Dyad, so that diversity can emerge? Continuing Alexander's quote, we find a major clue:

> But, when limited by the One, the Indefinite Dyad, he says, becomes the Numerical Dyad.

Recent research by Plato scholar Scott Olsen [59] suggests a concrete way to understand these dynamics—in particular, a way to understand these words of Alexander.

The first step of the analysis brings us back to Plato's Divided Line (Figure 10). In the initial partition of the line, how is the dividing point to be chosen? What should be the ratio of the larger segment, representing the intelligible world, to the smaller, representing the visible world? Olsen makes a very strong case that Plato had in mind the most "sublime" of ratios, the Golden Ratio,[65] denoted ϕ, whose value can be found by computing the unique positive root of $x^2 - x - 1$, resulting in

$$\phi = \frac{1+\sqrt{5}}{2}.$$

This means that, in Figure 10, we have the following proportion:

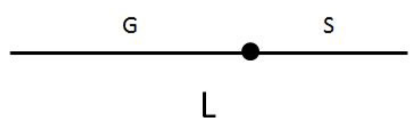
$$L : \ell(C \cup D) :: \ell(C \cup D) : \ell(A \cup B), \tag{45}$$

where L denotes the length of the whole line, and other lengths are specified using the notation ℓ; so, for example, $\ell(C \cup D)$ denotes the length of the segment $C \cup D$. In particular, since we are assuming these ratios are equal to the Golden Ratio, we have:

$$\frac{\ell(D)}{\ell(C)} = \frac{\ell(C)}{\ell(A)} = \phi.$$

[65]The Golden Ratio has been of great interest to mathematicians, scientists, artists, architects, philosophers, and others for many centuries both because of its unique mathematical properties and because of its perhaps unexpected appearance in such diverse areas as geometry, biology, art and architecture, and even stock market analysis. Pythagoras considered this ratio to be a divine proportion and argued that it can be found everywhere in nature, including the structure of the human body. The proportion was used in Greek and Egyptian architecture, making its appearance in the design of the Greek Parthenon and the Egyptian pyramids. During the Renaissance, the author Fra Pacioli wrote a book, *The Divine Proportion*, that made a case for the belief that this ratio was fundamentally divine. Leonardo da Vinci attached great significance to the Golden Ratio, which he called the golden section, and he used this ratio explicitly in many of his most famous paintings. Some pieces of classical music (for instance, by Bartok and Debussy) made explicit use of the ratio. Johannes Kepler remarked: "Geometry has two great treasures: one is the theorem of Pythagoras; the other the division of a line into extreme and mean ratio [golden cut]. The first we may compare to a measure of gold; the second we may name a precious jewel" (quoted in [29]). An introduction to some of the history and mathematics of the Golden Ratio can be found in the Wikipedia article at `http://en.wikipedia.org/wiki/Golden ratio`.

The Golden Ratio arises by considering a line L, partitioned into a greater piece (G) and a smaller piece (S):



If the pieces of the line bear the relationship

$$\text{whole} : \text{longer} :: \text{longer} : \text{shorter},$$

in other words:

$$L : G :: G : S,$$

then the ratio $\frac{G}{S}$ is, by definition, the Golden Ratio, and a direct computation yields:

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Plato does not explicitly mention the Golden Ratio in his dialogues, but does strongly hint at it; in his treatment of the five Platonic solids, viewed as the five elements at the basis of material existence, he invites the reader to discover this ratio himself. See the discussion in [59]. It is a mathematical fact that once one has in hand a ruler and compass construction of ϕ, one can construct all five Platonic solids. The students in Plato's Academy seemed to be aware of this fact, and also of Plato's apparent need to avoid explicit discussion about it [59].

Also, returning to Plato's specifications of ratios given in the *Republic* (509d) (see quotation above), we also have the following proportions:[66]

(46) $$\ell(C \cup D) : \ell(A \cup B) :: \ell(D) : \ell(C) :: \ell(B) : \ell(A).$$

Returning to our question about how the Indefinite Dyad could be transformed into Two—the Numerical Dyad mentioned by Alexander—we study more closely Alexander's remark, cited earlier:

> *But, when limited by the One, the Indefinite Dyad, he says, becomes the Numerical Dyad, mentioned earlier.*

The meaning here (echoed by the work of Olsen) is based on another point raised in the *Timaeus*. Plato remarks that, when there is a need to bring unity to differences, the technique is to find the *mean between extremes*; moreover, this technique has a mathematical realization: The mean between extremes of two natural numbers $a < c$ is the *geometric mean* of these numbers. Before elaborating further, we examine Plato's remarks on this point from the *Timaeus* [63]:

> But two things cannot be rightly put together without a third; there must be some bond of union between them. And the fairest bond is that which makes the most complete fusion of itself and the things which it combines, and proportion is best adapted to effect such a union. For whenever in any three numbers ... there is a mean, which is to the last term what the first term is to it, and again, when the mean is to the first term as the last term is to the mean—then the mean, becoming first and last, and the first and last both becoming means, they will all of them of necessity come to be the same, and having become the same with one another will be all one (31b–32a, p. 1163).

Elaborating further, the geometric mean of $a < c$ is a third number b, with $a < b < c$, for which the following proportion holds: $a : b :: b : c$—which may also be written $c : b :: b : a$. This proportion is called a *geometric proportion*, and the number b is called the *geometric mean* of a and c. Such a proportion may also be written in the form $a : b : c$ (or $c : b : a$); in that case it is called a *continuous geometric proportion.* Some easily verified continuous geometric proportions are $2 : 4 : 8$ and $3 : 9 : 27$. Given positive natural numbers[67] $a < c$, the geometric mean of a, c is always equal to $\sqrt{a \cdot c}$.

Therefore, in the Divided Line, equation (45) tells us that the geometric mean of $\ell(D)$ and $\ell(A)$ is $\ell(C)$, and recall that $\ell(B) = \ell(C)$.

Since Plato is giving an account of the emergence of multiplicity from unity, and since we have already seen from the quote from Alexander that the *Numerical* Dyad arises from the Indefinite Dyad by virtue of the presence of One (being "limited"

[66] The fact that the lengths of B and C appear to be equal in Figure 10 is not accidental, but rather a mathematical fact. It is not hard to show that whenever the proportions indicated in (46) hold, it must be true that $\ell(B) = \ell(C)$; the Golden Ratio plays no part in this calculation.

[67] It is important for this discussion that the two numbers *are* positive. For instance, there is no geometric mean between 0 and 2 (the only candidate would be 0, but it is not the case that $0 < 0 < 2$), and, if we try to compute the geometric mean between -1 and 2, we get the complex number $i\sqrt{2}$!

by the One), it is natural to identify this geometric mean $\ell(C)$ in the diagram with One, so that we have $\ell(C) = \ell(B) = 1$.

Doing so completely determines the values of $\ell(D)$ and $\ell(A)$: Once we know $\ell(C) = 1$, it follows that

$$\begin{aligned} \ell(D) &= \phi; \\ \ell(A) &= \frac{1}{\phi}. \end{aligned}$$

and we have $\phi : 1 : \frac{1}{\phi}$. See Figure 11.

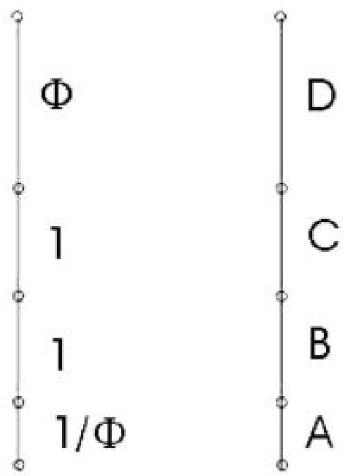


FIGURE 11. Plato's Divided Line Mediated by ϕ

Now One is the mean between the extremes of the Greater (ϕ) and the Lesser ($\frac{1}{\phi}$). These two values, though distinct, are as mathematically tied to unity as possible because, not only is it true that $\phi \times \frac{1}{\phi} = 1$, as is always the case with a pair of reciprocals, but also $\phi - \frac{1}{\phi} = 1$. It is easy to check that ϕ is the only positive real number that has this second property.[68]. And now, this Indefinite Dyad $(\phi, \frac{1}{\phi})$, "limited" in this way by 1, gives rise to the definite Dyad 2: $2 = \phi + 1 - \frac{1}{\phi}$.

Summing up, Plato's answer to the question of how diversity emerges from the One is this: First, the One contains within it the *possibility* of two, since the Unequal can be located as a principle secondary to the Equal (recall that the Equal is another name for the One), and the Unequal is to be understood as the Indefinite Dyad—the possibility of "greater" and "lesser." For the Indefinite Dyad to then manifest as Two requires a dynamic relationship between the One and the Indefinite Dyad. Study of the work of Plato and the extant documents from the Platonic Academy suggests that the ratios in Plato's Divided Line are all the Golden Ratio ϕ; letting the One play the role of the geometric mean of the two extreme sections A and D

[68]The computation follows from the fact that ϕ is the unique positive solution to $x^2 - x - 1 = 0$.

of the diagram leads to the mathematical conclusion that the top section has length ϕ, the bottom section has length $\frac{1}{\phi}$ and the Divided Line reveals the continuous geometric proportion $\phi : 1 : \frac{1}{\phi}$. Once it is seen that the Indefinite Dyad is $(\phi, \frac{1}{\phi})$, the concrete computation of Two follows immediately from the unique mathematical properties of ϕ.

These points suggest that, in a way, the Golden Ratio represents the dynamism inherent in the One. Striking mathematical evidence for this point of view, beyond what we have seen so far, is the following pair of mathematical equations:[69]

$$\begin{aligned} \phi &= \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}; \\ \phi &= 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}. \end{aligned}$$

Here we see that the Golden Ratio ϕ is the result of infinitary dynamics of 1 as it interacts with itself.[70]

So far, we have given an elaborate discussion of how Plato conceives of the One unfolding into Two. What happens after that?

In Plato's dialogue *Philebus*, the Indefinite Dyad is referred to as *bound, infinite, mixed*, and also as *symmetry, truth, beauty*;[71] and is described by the Neoplatonist Simplicius [62], as *intellect, intellection, intelligible.*

This emergence of three within the Indefinite Dyad is, according to Neoplatonic doctrines, further elaborated in groups of three, forming a vast hierarchy of intelligence, remaining all the while unified with its source [56]. In this way, the entire realm of forms arises.

Finally, we ask, what is the process by which the forms, the blueprint, take shape as manifest existence? Plato addresses this point in the *Timaeus*, where he explains [6, Part I] how the *Demiurge*, a fundamental principle of the intelligible

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[69]Proofs can be found in [15].

[70]It is reasonable to identify the infinitary dynamics represented by ϕ with some kind of *mapping*. Define a map ϕ_{int} (whose domain and codomain we describe in a moment), which transforms any number x to the number $\text{round}(\phi \cdot x)$:

$$\phi_{\text{int}}(x) = \text{round}(\phi \cdot x).$$

Computing a few values, we see that

$$(47) \qquad \phi_{\text{int}}(1) = 2, \quad \phi_{\text{int}}(2) = 3, \quad \phi_{\text{int}}(3) = 5, \ldots$$

Let $\mathcal{F} = \{1, 2, 3, 5, 8, \ldots\}$, the *positive distinct Fibonacci numbers.* Recall that the Fibonacci sequence is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-2} + F_{n-1}$, and the first values are $0, 1, 1, 2, 3, 5, 8$. We may write $\mathcal{F} = \{F_k \mid k \geq 2\}$.) The outputs of ϕ_{int}, shown in the display (47), are the elements of $\mathcal{F}$, with its first element 1 omitted.

We now specify the domain and codomain of ϕ_{int} to be precisely $\mathcal{F}$. Clearly ϕ_{int} is 1-1 and the number 1 is not in its range. So ϕ_{int} is a Dedekind self-map; indeed, as is easily shown, it is an *initial* Dedekind self-map. Our earlier work shows, therefore, that $\phi_{\text{int}} : \mathcal{F} \to \mathcal{F}$ is Dedekind self-map isomorphic to $s : \omega \to \omega$. In other words, ϕ_{int} can be seen as a blueprint for the natural numbers. This result suggests in another way that the Golden Ratio ϕ has the "organizing power" to transform 1 into the full, infinite sequence of natural numbers.

[71]See Thomas Taylor's discussion of this division in his notes on the *Parmenides* [62].

world,[72]molds the *Receptacle*—which may be understood as pure emptiness or pure, unformed matter[73]—using the forms as a template. In the process, the Demiurge intends that "all things should come as near as possible to being like himself" [6, Part I, p. 217].

13.2. **Blueprints and the QFT Perspective.** Adopting the simplified view that the manifest world is made of particles, we can say that the manifest world arises from quantum fields since each type of particle arises as a precipitation of its own type of quantum field [26, p. 31].

As a first try at framing the world of quantum fields as a "blueprint" in the sense that we are discussing here, we can simply view the entire collection of quantum fields—electron fields, quark fields, etc.—to be the constituents of the blueprint, just as the entire range of forms in Plato's philosophy was seen to be the blueprint in that context. Pursuing the parallel between these approaches a bit further, we propose that, as in Plato's philosophy in which there is a natural hierarchy of forms ranging from most abstract and universal to most concrete and specialized, so likewise there is a kind of hierarchy of quantum fields.

The most "expressed" level of this hierarchy is the fundamental force and matter fields: the electromagnetic, weak, strong, and gravitational force fields, on the one hand, and the various lepton and quark matter fields, on the other hand. In the 1960s, the work by Glashow, Weinberg, and Salam demonstrated that, while the electromagnetic and weak forces behave, at classical time and distance scales, very differently (with very different gauge bosons), at a distance scale of 10^{-16} cm, the forces are identical; as a result of this discovery, the force is now referred to as the *electroweak force*. Moreover, their work showed that it is the result of "spontaneous symmetry breaking" that, at larger distance scales, these fields appear to be distinct. Symmetries within the electroweak field—which are present only at a scale of 10^{-16} cm or smaller—are not maintainable at larger distance scales where the available energy is considerably less. Their work also resulted in a unification of apparently distinct matter fields—the charged lepton fields and the neutrino fields.[74]

The next level of abstraction in this hierarchy aimed at a theory that could unify the electroweak force with the strong force. The most successful theory of this kind known today is referred to as the *Standard Model*, though there appear to be a number of points about this theory that still need clarification.[75] At a distance scale of 10^{-29} cm, the Standard Model achieves unification of the electroweak and strong

[72]Thomas Taylor suggests, in his introductory remarks to *Proclus' Commentary on Euclid* [68, p. 5] that, in Plato's theology, the Demiurge represents one of the ultimate *forms*—"the idea of all things."

[73]The Stanford Encyclopedia of Philosophy gives this account:

> [The Receptacle is] a totally characterless subject that temporarily in its various parts gets characterized in various ways. This is the receptacle—an enduring substratum, neutral in itself but temporarily taking on the various characterizations. The observed particulars just are parts of that receptacle so characterized.

See `http://plato.stanford.edu/entries/plato-timaeus/#6`.

[74]See [26], as well as the Wikipedia article `http://en.wikipedia.org/wiki/Electroweak_interaction`.

[75]See [26] and the Wikipedia article on the Standard Model: `http://en.wikipedia.org/wiki/Standard_Model`.

forces, as well as of many more matter fields, most notably the quark and lepton fields. As with the electroweak unification, the observed differences between the electroweak and strong forces also arise from spontaneous symmetry breaking [26].

The ultimate completion of this direction of unification would be a super-unified theory, which would account for a single unified field unifying all four forces and all matter fields. Over the past half century, many proposals for such a theory have emerged. The most widely accepted approach replaces the zero-dimensional point particles of quantum field theory with one-dimensional (or possibly higher dimensional) *strings*, which "act like" particles. There are a variety of so-called *superstring* theories, based on this concept; the "super" prefix indicates that the theories exhibit a special kind of symmetry called *supersymmetry*. To understand supersymmetry, we first note that there are two basic classes of elementary particles: *bosons*, which have an integer-valued spin, and *fermions*, which have a half-integer spin. Each particle from one group is associated with a particle from the other, known as its *superpartner*. In a theory with perfectly "unbroken" supersymmetry, each pair of superpartners would share the same mass and internal quantum numbers besides spin. For example, there would be a "selectron" (superpartner electron), a bosonic version of the electron, with the same mass as the electron. At ordinary time and distance scales, supersymmetry does not exist; this implies that, assuming there is an underlying superfield composed of superstrings, at a certain distance scale (namely, the *Planck* scale, 10^{-33} cm), the supersymmetry must be broken.[76]

In the 1990s, it was demonstrated that there are only five viable superstring theories. Then E. Witten made the remarkable discovery, in his development of *M Theory*, that all five of these superstring theories are equivalent.[77]

We see that the pattern for unfolding this hierarchy of quantum fields, ranging from the super-unified level to the level of classical time and distance scales, is in each case *spontaneous symmetry breaking*. In each case, deep symmetries of natural law that are lively at one scale are lost because of one of these transitions.

Finally, we ask, what is the mechanism by which quantum fields give rise to, and destroy, the particles of the material world? The appearance of particles, according to QFT, is due to the phenomenon of *field collapse*, by which infinitely extended quantum fields appear as *quanta*—precipitates of the field—and by which quanta "disappear." The mechanism by which field collapse occurs is still not known, but the fact that it does occur is an experimentally verified fact [5, p. 52ff]. In this sense then, QFT shares even this feature with the ancient perspective regarding blueprints, though details about the underlying mechanism are still being researched.

13.3. **Common Features of Blueprints.** Our aim in this section so far has been to give an account of the role of blueprints in the unfoldment of manifest life, as described in several ancient traditions of knowledge, including in addition a few points from QFT. These accounts are, in each case, an extension of the world views discussed earlier concerning the *source* of natural numbers—world views that suggested a direction as we sought an alternative, intuitively-rich formulation for

[76] See `http://en.wikipedia.org/wiki/Supersymmetry`.

[77] See the Wikipedia article `http://en.wikipedia.org/wiki/M-theory`.

the Axiom of Infinity. As we seek to refine our New Axiom of Infinity further—to find the "right" generalization of the concept of a Dedekind self-map—we wish to cull from the ancient wisdom insights that could have a bearing on, and provide direction for, this program of generalization.

To conclude these preliminary discussions, then, we catalogue some of the main characteristics of blueprints for the universe that are common to each of the traditions we have considered.

(1) *Blueprint Arises from the Dynamics of One*: The blueprint emerges sequentially from the internal dynamics of the One interacting with itself. In Maharishi Vedic Science, this point is addressed by Maharishi's *Apaurusheya Bhashya*: Starting from the first letter A of Ṛk Veda, the Veda unfolds as successive elaborations of all that is contained in A, from A to AK—representing a collapse of the unbounded value of wholeness to its own point—to Agnim to Agnimile, and so on. In Chinese philosophy, within *Tao* is seen the potential for two—the Unlimited and the Limited—and *Tao*, appreciated from the point of view of this possibility for two is *tai chi*. Then *tai chi* gives rise to the yin-yang principle *liang yi*, from which the images of the cosmic mind are woven and, ultimately, nurtured into being by the primal mother. For Plato, within the One, the Equal, is seen the opposite principle, the Unequal, which represents the potential for Two, in the form of the Greater and Lesser also known as the Indefinite Dyad, represented mathematically by the Golden Ratio ϕ and its reciprocal. Interaction between the One and the Indefinite Dyad produce Two. Since the Indefinite Dyad is itself a three-in-one (described for instance as the unity of intellect, intellection, intelligible), it achieves fuller expression, beyond Two, as Three, which in turn unfolds into multiple triads and ultimately the intelligible world of forms. Finally, in the world of quantum field theory, the hierarchy of quantum fields mentioned earlier unfold from the super-unified level at the Planck scale (10^{-33} cm) into more and more diverse force and matter fields at ever larger time and distance scales. This diversification is driven by the process of *spontaneous symmetry breaking*.

(2) *Unity Preserved in the Emergence of the Blueprint*: The One moves toward diversification in such a way that parts remain connected to their source, and unity is never lost. In the case of Maharishi Vedic Science, diversity emerges in *self-referral loops*, so parts remain connected to the whole. In Chinese philosophy, Two arises first as just a potential for two, in *tai chi*, then as a principle of two, *liang yi*. Duality is understood to remain unified since it is the expression of the one *liang yi*.

For Plato, diversity remains connected by applying the principle of finding the mean between extremes, which is, mathematically, the geometric mean. The One is found to be the mean between the extremes ϕ and $\frac{1}{\phi}$, which together represent the Indefinite Dyad, and which together remain unified with One, by the formulas $\phi \cdot \frac{1}{\phi} = 1$ and $\phi - \frac{1}{\phi} = 1$. Two arises from the involvement of One in its own dynamics, represented by the Indefinite

Dyad $(\phi, \frac{1}{\phi})$, by the formula $\phi + 1 - \frac{1}{\phi} = 2$. Here, 2 is seen to be nothing other than a combination of One with its inner dynamics (represented by the Indefinite Dyad), so is itself still fundamentally One. Plato and the Neoplatonists viewed the triad, and multiple subsequent triads, emerging from One and the Indefinite Dyad to remain unified, and likewise for the entire realm of eternal forms.

Finally, in the case of the emergence of everything from a superstring field, preservation of the one superfield, with its supersymmetry and self-interacting dynamics, occurs for a very different reason: The supersymmetry breaking that appears to take place, resulting in a multiplicity of lower-energy quantum fields and an apparent world of classical physics, never really happens! Professor John Hagelin explains this point [27]:

> Every stage in the sequential unfoldment of the laws of nature from the unified field is an automatic consequence of the detailed structure of the unified field and its self-interacting dynamics. At no stage in this sequential unfoldment is it necessary to introduce additional *ad hoc* postulates and assumptions: the creative process occurs entirely by itself in a self-sufficient manner as a spontaneous and inevitable consequence of the unified field itself.... the transition from quantum gravity to classical gravity and from string dynamics to field theory are "transformations in appearance" only... neither of these transformations are genuine (pp. 282–283).

Therefore, physics itself tells us that the unity of the unified field remains forever unperturbed.

(3) *The Blueprint Gives Rise to All Manifest Existence*: Every aspect of manifest existence is the image of the blueprint. In Maharishi Vedic Science, the Veda and Vedic Literature give rise to *Vishwa*, the manifest universe. In Chinese philosophy, the two-as-one primary unity, as the Cosmic Mind, as primal father, gives rise to eternal images, and as primal mother, nurtures them into being. This is how everything in the manifest world arises. For Plato, the forms provide the template for the creation of all manifest existence.

In the QFT world, we have seen that the ultimate quantum field is some kind of superstring field. The dynamics of the lower-energy quantum fields that arise from spontaneous symmetry breaking represent portions of the dynamics of the unified field; these collectively give rise to our physical universe. But the full dynamics of the QFT blueprint are to be found at the super-unification level. Indeed, Hagelin describes the super-unified level as the "blueprint" for the physical universe [27]:

> The precise mathematical structure of the unified field serves as an unmanifest blueprint for the entire creation: all the laws of nature governing physics at every scale are just partial reflections or derivatives of this basic mathematical structure (p. 185).

(4) *A Fundamental Power of the One Is Responsible for Transforming the Blueprint to Manifest Existence*: There is a principle or power associated with the blueprint that causes items in the manifest universe to be created from the blueprint. In the case of Maharishi Vedic Science, the principle of tranformation is *Vivart*, which is responsible for the appearance of Veda and the Vedic Literature as the manifest world. In the case of Chinese philosophy, the forms arise and are nurtured into existence by the work of the primal father / primal mother. In Plato's philosophy, the Demiurge, a power belonging to the intelligible world, is responsible for crafting objects of the visible world by imprinting forms onto the Receptacle of pure matter. In physics, this fundamental power can be viewed in one of two ways. From one perspective, spontaneous symmetry breaking is a characteristic of each of the unifying quantum field theories we have looked at: the electroweak theory degrades to the separate electromagnetic and weak fields—with a corresponding diversification of matter fields—because of broken symmetry; the same is true for the diversification into separate fields from the Standard Model, and again, for any of the superstring models. An alternative perspective [27, pp. 282–283] is like the perspective from Maharishi Vedic Science—a point that was made in (2): The reality is that the unified field never does undergo symmetry breaking; classical physics never does emerge from quantum physics. In this case, these transformations are only appearances and could be described as taking place, as in the Maharishi Vedic Science case, by the power of *Vivart*.

(5) *The Blueprint Is Responsible for Both Generation and Return*: There is likewise a principle or power associated with the blueprint that causes everything manifest to return to its source. The way in which each of the ancient traditions of knowledge that we have been considering views the motion of *return* as a fundamental dynamic of the source was addressed in the footnotes on p. 61. For QFT, part (1) indicates the generative aspect. But particles can be both created and destroyed, so the same blueprint that generates particles also destroys particles.

13.4. **A Formal Treatment of Blueprints.** Having surveyed the insights from the ancient texts—and to some extent, the features of quantum field theory—regarding the emergence and dynamics of a blueprint of creation from the source, we turn now to a mathematical account of blueprints that naturally arises in a careful analysis of Dedekind self-maps. Linking the characteristics discovered by the ancients to those we find connected with the dynamics of a Dedekind self-map will provide us with material for conjecture about the sort of behaviors and dynamics we should expect to find as we start to examine generalized Dedekind self-maps. In this subsection, then, we give a detailed account of blueprints, and how the behavior of a Dedekind self-map produces a blueprint, in the formal sense, of the set ω of natural numbers.

Starting with a Dedekind self-map $j : A \to A$ with critical point a and a set $X \subseteq A$, our goal is to state as precisely as possible what we mean by a *blueprint for X*. The intention is that, first of all, we have a class $\mathcal{E}$ of weakly elementary functionals on B^B, for some $B \subseteq A$, and, through the interaction of j, a, and $\mathcal{E}$, we obtain a dual pair of self-maps f, g (one of which is Dedekind, the other, co-Dedekind), each defined on B. The map f will *encode* the set X. Moreover, f will provide a way of *generating* the elements of X. Dually, g will provide a way to *return* elements of X to their source, a. We think of f as containing all the information about the elements of X in its "seed" (or encoded) form; in this sense, f may be thought of as a *substrate* for X. We consider the "blueprint" for X to consist not only of f, but also of the mechanism by which elements of X are obtained from f; this mechanism includes $\mathcal{E}$ and the critical point a.

One other aspect of our definition of blueprint is that we require that $f, g, \mathcal{E}$, and a all "come from" the underlying self-map j. The reason for this requirement is that we wish to think of $f, g, \mathcal{E}, a$ as arising from the dynamics of j, just as, for example, Plato's forms arise from the dynamics of the One.

In the formal definition, we first consider a simpler case in which elements of X are generated, but in which we do not necessarily have a mechanism for returning elements to their source. We will call the machinery by which elements of X are generated a *blueprint*. Then we consider the "ideal" case, in which we have both generation and collapse of elements of X from the blueprint; the resulting strengthened form of a blueprint will be called a *strong* blueprint.

We need the following definition: Suppose B is a set and $\mathcal{E}$ is a collection of functionals such that, for each $i \in \mathcal{E}$, dom $i \supseteq B^B$. Let $\mathcal{E}_0$ be defined by $\mathcal{E}_0 = \{i \upharpoonright B^B \mid i \in \mathcal{E}\}$. Then $\mathcal{E}_0$ is called the *restriction of* $\mathcal{E}$ *to* B^B and we denote this fact by writing $\mathcal{E}_0 \sqsubseteq_{r,B} \mathcal{E}$, or simply $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ when the meaning is clear from the context.

Definition 7. (Blueprints) Suppose $j : A \to A$ is a Dedekind self-map with critical point a, and suppose $X \subseteq A$. A j-*blueprint* (or simply a *blueprint*) for X is a triple $(f, a, \mathcal{E})$ having the following properties:

(1) For some set B, $f : B \to B$ is either a Dedekind self-map or a co-Dedekind self-map. (Note: The critical (co-critical) point for the self-map may or may not be equal to a.)
(2) The class $\mathcal{E}$ is compatible with j and each of its elements is weakly elementary relative to B.[78] Moreover, if $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ is the restriction of $\mathcal{E}$ to B^B, then, for each $i \in \mathcal{E}_0$:
   (a) there exist C_i, D_i so that $i : B^B \to {D_i}^{C_i}$;
   (b) $C_i \supseteq B$;
   (c) if $C_i \neq D_i$, then there is a bijection $\pi_i : D_i \to C_i$ that is definable from j and a;
   (d) $a \in \text{dom } i(f)$.
(3) (*Encoding*) The self-map f is *definable from* $\mathcal{E}, j, a$.
(4) (*Decoding*) f generates X in the following sense:

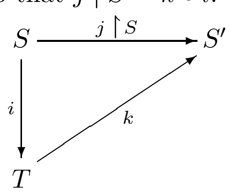
$$\text{For every } x \in X, \text{ there is } i \in \mathcal{E}_0 \text{ such that } i(f)(a) = x.$$

Remark 7.

[78]As mentioned in an earlier footnote, the concept that is needed here is Σ_0-preserving, but to avoid introducing unnecessary complexities, we have made use of this approximation to Σ_0-preserving, which will suffice for this paper.

(A) In the example given earlier, the generating function $x \mapsto \{x\}$ turned out to be a Dedekind self-map, but in some contexts, the generating function will be a co-Dedekind self-map. Condition (1) leaves room for either possibility.

(B) One point in the definition that remains vague is the requirement that the elements of $\mathcal{E}$ should be "compatible with j." For the example that we know about so far, and others we will see that belong to a relatively simple context, to say that the elements of $\mathcal{E} = \{i_0, i_1, i_2, \ldots, i_n, \ldots\}$ are "compatible with j" simply means they are definable from j and its critical point. In Example 1, no ambient self-map $j : A \to A$ was specified; however, since any set A that is closed under pairs is infinite, we can certainly find a Dedekind self-map on A, and the elements of $\mathcal{E}$, which are functionals that specify various iterations, can be shown to be definable from A itself (which is in turn defined from j by $A = \text{dom } j$).[79] When we expand to a more general context, the requirement that each $i \in \mathcal{E}$ is definable from j and its critical point will be too strong. In that context, we will make use of a weaker notion of compatibility, derived from the work in [10]. We outline the idea here, which will be applicable in contexts in which dom $i \subseteq$ dom j for each $i \in \mathcal{E}$; under this condition, we will say that $\mathcal{E}$ is compatible with j if some $i : S \to T \in \mathcal{E}$ (where $i \upharpoonright B^B : B^B \to C_i^{C_i}$) is a *right factor* of $j \upharpoonright S$; more precisely, for some $i \in \mathcal{E}$, there is a weakly elementary k (relative to C_i), defined on T, so that $j \upharpoonright S = k \circ i$.



This requirement, together with the requirement that elements of $\mathcal{E}$ are weakly elementary, captures reasonably well the intuition that elements of $\mathcal{E}$ "arise from" j.

(C) In some contexts, the set $\mathcal{E}_0$ of maps $i : B^B \to {D_i}^{C_i}$ arise as restrictions of maps from a broader, naturally defined class $\mathcal{E}$. In such cases, for "decoding" purposes, $\mathcal{E}_0$ suffices, but for "encoding" purposes, the broader class $\mathcal{E}$ is needed.

We observe that generating the elements of a set X requires only one self-map f, but one can always obtain a dual for the function f. If $f : B \to B$ is a Dedekind self-map with critical point a, there is in fact a *canonical* co-Dedekind self-map g, with co-critical point a, that is dual to f—in other words, f is a section of g. Define $g : B \to B$ as follows.

$$g(x) = \begin{cases} y & \text{if } f(y) = x, \\ a & \text{if } x \notin \text{ran } f. \end{cases}$$

On the other hand, if $f : B \to B$ is a co-Dedekind self-map with co-critical point a, there is no canonical choice for a dual to f, but any section g of f must have a critical point that belongs to $f^{-1}(a)$.

[79] These points are verified below; see Remark 8.

Therefore, even in the absence of an explicit definition of both parts of a Dedekind/co-Dedekind pair in a blueprint, the "other half" of the pair is present implicitly.

The requirements on elements of $\mathcal{E}$ may appear needlessly general. Based on the example we have considered so far, it would be reasonable to expect that each $i \in \mathcal{E}_0$ would have type $i : B^B \to B^B$. Later, however, we will see examples in which it is natural for functions in B^B to be taken to functions in C^C, where $C \supseteq B$, or even (though rarely) D^C, where D is a bijective image of C, under a bijection π that is definable from j. This latter situation can arise when $D = X$, where X is the set that is being generated, but will not arise when $B^B \subseteq \text{dom } j$. It will also happen sometimes that the codomain C^C of the a functional i on B^B may vary depending on i, as condition (2)(a) indicates. Nevertheless, for any such C, we always have $B \subseteq C$.

As we move toward a definition for *strong blueprint*, in which elements of X are also returned to their "source" element a, an obstacle needs to be addressed in the case that functionals $i \in \mathcal{E}_0$ are of the form $i : B^B \to D^C$, where $D \neq C$. In that case it is not clear how to meet the requirement of obtaining a dyad (f, g) for which f generates X and g returns elements of X to a.

For concreteness, we consider an example. Suppose we have a class $\mathcal{E}_0$ of weakly elementary functionals of the form $i : B^B \to D^C$, and $\pi : C \to D$ is a bijection definable from j and a. Suppose also we have obtained a generating Dedekind self-map f so that for every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(f)(a) = x$. Now the type of $i(f)$ must be $i(f) : C \to D$. The type presents no problem since $a \in B \subseteq C$. Now, to return elements of X back to a, we will need a co-Dedekind self-map $g : B \to B$ with the property that, for each $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(g)(x) = a$. Here again, the type of $i(g)$ must be $i(g) : C \to D$. This means that a must belong to D, but since D is only an image of C under π, it will not generally be possible for this requirement to be met.

To overcome this obstacle, we will introduce the concept of a *conjugate class* $\mathcal{E}_0^*$. Given $\mathcal{E}_0$ containing functionals of type $B^B \to D^C$, as we have been discussing, and given $i \in \mathcal{E}_0$, we define $i^* : B^B \to C^D$ by $i^*(h) = \pi^{-1} \circ i(h) \circ \pi^{-1} : D \to C$. Then we let $\mathcal{E}_0^* = \{i^* \mid i \in \mathcal{E}\}$. Now $i^*(h)$ has the right type. So now it does make sense to require that for each $x \in X$, there is $i \in \mathcal{E}_0$ so that $i^*(g)(x) = a$.

In the more typical context in which $\mathcal{E}_0$ contains functionals of type $B^B \to C^C$ or $B^B \to C_i^{C_i}$, $i \in \mathcal{E}_0$, the map π is taken to be id_{C_i}, so that $i^* = i$ in such cases.

Using this device, we can give a satisfactory definition of strong blueprints:

Definition 8. (Strong Blueprints) Suppose $j : A \to A$ is a Dedekind self-map with critical point a, and suppose $X \subseteq A$. A *strong j-blueprint* (or simply a *strong blueprint*) for X is a quadruple $(f, g, a, \mathcal{E})$ having the following properties:

(1) For some set B, f and g are functions $B \to B$, and one of these is a Dedekind self-map, the other, a co-Dedekind self-map. (The values of the critical and co-critical points of the self-maps may or may not be equal to a). The pair (f, g) is called the *blueprint dyad* and satisfies one of the following: $g \circ f = \text{id}_B$ or $f \circ g = \text{id}_B$.

(2) The class $\mathcal{E}$ is compatible with j and each of its elements is weakly elementary relative to B. Moreover, if $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ is the restriction of $\mathcal{E}$ to B^B, then, for each $i \in \mathcal{E}_0$:
  (a) there exist C_i, D_i so that $i : B^B \to {D_i}^{C_i}$;
  (b) $C_i \supseteq B$;
  (c) if $C_i \neq D_i$, then there is a bijection $\pi : D_i \to C_i$ that is definable from j and a;
  (d) $a \in \text{dom } i(f)$;
  (e) $\text{dom } i(f) = \text{dom } i(g)$.

(3) (*Encoding*) The self-maps f and g are *definable from* $\mathcal{E}, j, a$.

(4) (*Decoding*)
  (a) The self-map f generates X in the following sense:

  For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(f)(a) = x$.

  (b) The self-map g collapses elements of X in the following sense:

  For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i^*(g)(x) = a$.

Note that in condition (4)(b), we have used the conjugate of i so that the functional type is correct. In most respects, the definition of strong blueprint is the same as the definition of blueprint, except that we have also required the existence of a dual to the generating function, which returns values in X to a.

In the sequel, we will make use of both concepts—blueprint and strong blueprint—as we consider more examples. One situation that arises is that, for a particular set X we may have a blueprint $(f, a, \mathcal{E})$, but not a strong blueprint, but, for an important subset Y of X, we are able to obtain a dual g of f so that $(f, g, a, \mathcal{E})$ is a strong blueprint for Y.

Remark 8. We now rework Example 1 to indicate how the maps defined in the example give rise to a formal blueprint, and also a formal strong blueprint. In that setting, A was a transitive set, closed under pairs, and we discussed a blueprint for the set $W = \{\emptyset, \{\emptyset\}, \ldots\} \subseteq A$. We did not specify a Dedekind self-map on A in Example 1, but *any* Dedekind self-map $j : A \to A$ can be used here. In the example, we defined a function F by

$$F(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ y & \text{where } y \text{ is any } \in\text{-minimal element of } x. \end{cases}$$

Formally, the function $F \restriction W$ corresponds to g in the strong blueprint definition since it "returns" elements of W to their source. Likewise, S_A was defined by $x \mapsto \{x\}$, and so the function $S_A \restriction W$ corresponds to f since it serves to generate elements. Also $\mathcal{E}$, the collection $\{i_0, i_1, i_2, \ldots\}$ of iteration maps, is indeed a collection of weakly elementary functionals. Therefore, a blueprint for W is given by $(S_A \restriction W, a, \mathcal{E})$, and a strong blueprint, by $(S_A \restriction W, F \restriction W, a, \mathcal{E})$. In this case, the restriction $\mathcal{E}_0$ mentioned in the definitions is simply $\mathcal{E}$. Notice that since each $i \in \mathcal{E}$ is, in this formal context, of type $W^W \to W^W$, each element of $\mathcal{E}$ is definable from W, which, being defined from $S_A : A \to A$, is definable from j (since $A = \text{dom } j$). Likewise, $F \restriction W$ can be shown to be definable from j. $\square$

With our formal definition, we can now substantiate the claim, made near the beginning of the paper, that if $j : A \to A$ is a Dedekind self-map with critical point a, the iterates $a, j(a), j(j(a)), \ldots$ form a "blueprint" for ω and the successor $s : \omega \to \omega$. We formulate this statement precisely as a theorem, using our new definition of blueprint, and then give a careful proof.

Theorem 30. (Blueprint for ω Theorem) *Suppose $j : A \to A$ is a Dedekind self-map with critical point a. Then there exist h, W, and $\mathcal{E}$ satisfying the following:*

(1) *$h : A \to A$ is a co-Dedekind self-map with co-critical point a,*
(2) *$W \subseteq A$ and $a \in W$,*
(3) *$\mathcal{E}$ is a set of functionals $W^W \to W^W$,*
(4) *$(j \restriction W, h \restriction W, a, \mathcal{E})$ is a strong blueprint for W.*

In particular, $j \restriction W$ is initial and is the unique section of $h \restriction W$.

Proof. Let $j : A \to A$ be a Dedekind self-map with critical point a and let $\mathcal{E} = \mathcal{E}_0 = \{i_0, i_1, i_2, \ldots\}$ be the set of iteration maps from W^W to W^W; that is, $i_n(g) = g^n$ for $n \geq 1$, and $i_0(g) = \mathrm{id}_W$. We let $h : A \to A$ be the canonical dual for j; recall that h is defined as follows:

$$h(x) = \begin{cases} b & \text{if } j(b) = x, \\ a & \text{if } x \notin \operatorname{ran} j. \end{cases}$$

Note that when $x \in \operatorname{ran} j$, the value of $h(x)$ is uniquely determined because of the fact that j is 1-1.

We claim that h is a co-Dedekind self-map. To see h is onto, let $y \in A$ and let $x = j(y)$. Then $h(x) = y$. Also notice that at least two elements of A are mapped by h to a: Certainly $h(a) = a$. But also, if $j(a) = b$, then, since $a \notin \operatorname{ran} j$, $b \neq a$, and so we have a second element that maps to a, namely, $h(b) = a$. Therefore h is a co-Dedekind self-map with co-critical point a.

Next, we verify that j is a section of h: Given $x \in A$, $h(j(x)) = x = \mathrm{id}_A(x)$. Note also, by induction, that $h^n \circ j^n = \mathrm{id}_A$ for $n \geq 0$.

Now let $W = \{j^n(a) \mid n \in \omega\}$, recalling that j^0 signifies the identity map. Clearly $\operatorname{ran} j \restriction W \subseteq W$, and so we conclude that $j \restriction W : W \to W$ is a Dedekind self-map with critical point a. Likewise, $h \restriction W : W \to W$ is a co-Dedekind self-map with co-critical point a. Since j is a section of h, it follows $j \restriction W$ is a section of $h \restriction W$, and so $(j \restriction W, h \restriction W)$ forms a blueprint dyad. This takes care of part (1) of the definition of strong blueprint.

The requirements of (2) in the definition of strong blueprint follow from our previous discussion. In particular, $\mathcal{E}$ is compatible with j (recall here that, in the present context, this means that each element of $\mathcal{E}$ is definable from j and a) since each $i : W^W \to W^W$ is definable from W, which in turn is definable from j, a, and A, and A itself is definable from j (since $A = \mathrm{dom}\ j$).[80] For (3), we must verify that $j \restriction W$ and $h \restriction W$ are defined from j. A review of the definition of h shows that

[80]We indicate how the weaker notion of compatibility, discussed in Remark 7(B), can also be satisfied here. Our formulation will be somewhat artificial since $W^W \not\subseteq \mathrm{dom}\ j$; in order to meet the requirements of this version of compatibility, we make use of the fact that $j \restriction W$ has a higher-order cousin J_j. In particular, recall that $J_j : A^A \to A^A$ is defined by $f \mapsto j \circ f$ and that

$$\overline{W} = \{\mathrm{id}_A, J_j(\mathrm{id}_A), J_j(J_j(\mathrm{id}_A)), \ldots\} = \{\mathrm{id}_A, j, j \circ j, \ldots\}.$$

it is indeed definable from j. It is therefore clear that both $j \restriction W$ and $h \restriction W$ are defined from j using W as a *parameter*. But W itself is defined as $\{i(j)(a) \mid i \in \mathcal{E}_0\}$, and so W itself is definable from $j, a, \mathcal{E}$ without any additional parameter.

Finally, for (4), it is clear by the definition of W that, for every $x \in W$, there is $i \in \mathcal{E}_0$ such that $i(j \restriction W)(a) = x$. We prove the parallel result for $h \restriction W$: Let $y \in W$ and let $n \in \omega$ be such that $y = j^n(a)$. By our earlier observation $h^n \circ j^n = \mathrm{id}_A$, and so $h^n(y) = h^n(j^n(a)) = a$.

Our work so far shows that $(j \restriction W, h \restriction W, a, \mathcal{E})$ is a strong blueprint for W. To finish the proof, we need to show that $j \restriction W$ is the unique section of $h \restriction W$. By earlier work, $j \restriction W$ is an initial Dedekind self-map. Using the easily proved fact that every initial Dedekind self-map has exactly one critical point, it follows that a is the only critical point of j, and so the only element $y \in W$ for which the set $H_y = \{x \in W \mid h(x) = y\}$ has more than one element is a. We claim that H_a contains exactly *two* elements: a and $j(a)$. To prove this claim, suppose $x \in H_a$ and $x \neq a$. Since the only critical point of $j \restriction W$ is a, there is $c \in W$ so that $j(c) = x$, from which it follows that $h(x) = c$. Since we are assuming $h(x) = a$, it follows that $j(a) = x$, and our claim is proved.

Finally, to complete the proof that $j \restriction W$ is the only section of $h \restriction W$, let $j' : W \to W$ be another section of h. Then if $x \in W$ and $x \neq a$, since $h(j(x)) = x = h(j'(x))$, it follows that $j(x) = j'(x)$. We check that $j(a) = j'(a)$: Recall that, if $y \notin \operatorname{ran} j$,

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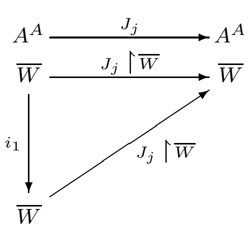
We have already shown that $j \restriction W$ and $J_j \restriction \overline{W}$ are Dedekind self-map isomorphic (Theorem 22). Using this isomorphism, we identify $j \restriction W$ with $J_j \restriction \overline{W}$. In order for the example to work, elements of $\mathcal{E}$ need to be of type $\overline{W}^{\overline{W}} \to \overline{W}^{\overline{W}}$ in order to guarantee the decoding requirement, but any right factor of $J_j \restriction \overline{W}$ will have to be of type $\overline{W} \to \overline{W}$. Because of the definability and generic nature of the functions i_n defined in the proof of Theorem 31, this requirement can be handled in a natural way and will allow us to illustrate the method. For readability, we will denote elements of $\mathcal{E}$ as $i_0, i_1, i_2, \ldots$ as in the main text, regardless of the required type; this is a reasonable convention since the definitions for both types will be the same. In particular, for $C \in \{\overline{W}, \overline{W}^{\overline{W}}\}$, we define $i_n : C \to C$ by $i_n(f) = f^n$ and let $\mathcal{E} = \{i_0, i_1, i_2, \ldots\}$. Using the isomorphism between $j \restriction W$ and $J_j \restriction \overline{W}$, we can show, as in the proof of Theorem 31, that there is a function $H_j : \overline{W} \to \overline{W}$ (defined whenever possible as the inverse of $J_j \restriction \overline{W}$) such that $(J_j \restriction \overline{W}, H_j, \mathrm{id}_A, \mathcal{E})$ is a strong blueprint for $\overline{W}$. For decoding, we have, for any $j^n \in \overline{W}$,

$$i_n(J_j \restriction \overline{W})(\mathrm{id}_A) = J_j^n \restriction \overline{W}(\mathrm{id}_A) = j^n,$$

and

$$i_n(H_j)(j^n) = H_j^n(j^n) = \left(J_j^n\right)^{-1}(j^n) = \mathrm{id}_A.$$

To establish the weak compatibility requirement, we can simply use $i_1 : \overline{W} \to \overline{W}$ as a right factor of $J_j \restriction \overline{W}$. Then, trivially, the following diagram is commutative, and we have $J_j \restriction \overline{W} = (J_j \restriction \overline{W}) \circ i_1$ (the role of k is played by $J_j \restriction \overline{W}$ which is certainly weakly elementary relative to $\overline{W}$).



$h(y) = a$. Therefore, suppose $j(a) = b$ and $j'(a) = c$. Then $h(b) = a = h(c)$. By the claim just proved, since $b \neq a$ and $c \neq a$, it follows that $b = j(a) = c$, and so $j(a) = j'(a)$. We have shown that $j \restriction W = j'$, and so $j \restriction W$ is the only section of $h \restriction W$. □

We have exhibited a strong blueprint for W. Note that since $j \restriction W$ is *initial*, we have, in effect, located a blueprint for ω from j, since initiality implies that $(W, j \restriction W, a)$ is Dedekind self-map isomorphic to $(\omega, s, 0)$ by a unique isomorphism.

It is possible, however, to use our new blueprint framework to demonstrate in greater detail how ω and its successor map are generated, starting with any Dedekind self-map. The proof develops the ideas in the proof of the Blueprint for ω Theorem further. We outline the argument here.

In order to get a blueprint that in actual fact generates the elements of ω and the successor function s, we will revisit the proof of Theorem 12; the proof of the theorem shows that the successor function $s : \omega \to \omega$ arises as the Mostowski collapse of $j \restriction W : W \to W$, where $W = \{a, j(a), j(j(a)), \ldots\}$ and $j : A \to A$ is a Dedekind self-map with critical point a. Diagram (48) summarizes the results of Theorem 12.

(48)
$$\begin{array}{ccc} W & \xrightarrow{\;j_W\;} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \omega & \xrightarrow{\;s\;} & \omega \end{array}$$

Here, $j_W = j \restriction W$, π is the Mostowski collapsing isomorphism, and $s : \omega \to \omega$ is the usual successor function $n \mapsto n \cup \{n\}$.

We now state a more direct version of the Blueprint Theorem for ω.

Theorem 31. (Blueprint for ω Theorem, Reformulated) *Suppose $j : A \to A$ is a Dedekind self-map with critical point a. Let $W = \{a, j(a), j^2(a), \ldots\}$. Then there are a set $\mathcal{E} = \mathcal{E}_0$ of weakly elementary functionals compatible with j and a co-Dedekind self-map $h : W \to W$ with co-critical point a such that $(j \restriction W, h, a, \mathcal{E})$ is a strong blueprint for ω.*

Remark 9. Here, we have not mentioned anything about initiality of $j \restriction W$ because our blueprint will literally generate ω and s. To get this sharper result, we will need to define the elements of $\mathcal{E} = \mathcal{E}_0$ somewhat differently; they will now be of type $W^W \to \omega^W$. This asymmetry will be handled by making use of *conjugate elements* of $\mathcal{E}_0$, mentioned earlier. Note that, once again, the restriction $\mathcal{E}_0$ mentioned in the definitions of blueprint and strong blueprint will, in the present context, simply be $\mathcal{E}$.

Proof. Let $j_W = j \restriction W$. First, we want to generate ω using j_W. From the commutative diagram, we see that to arrive at elements of ω, we will need to compose with π. We can derive the following facts from diagram (48):

$$\begin{aligned}\pi(a) &= 0;\\ \pi(j_W(a)) &= 1;\\ \pi(j_W^2(a)) &= 2.\end{aligned}$$

Thus, for each $n \in \omega$, it follows that $(\pi \circ j^n)(a) = n$. Now this formula suggests how to define elements of $\mathcal{E} = \mathcal{E}_0 = \{i_0, i_1, i_2, \ldots\}$. Instead of requiring i_n to be the functional that produces nth iterations, we will require i_n to produce nth iterations *composed with* π. Therefore, we define i_n as follows.

(49) For each $n \in \mathbb{N}$ and each $g : W \to W$, $i_n(g) = \pi \circ g^n$.

Notice that i_n takes elements of W^W to elements of ω^W. With this definition, we have obtained a blueprint for ω: For each $n \in \omega$, there is $i \in \mathcal{E}_0$ such that $i(j_W)(a) = n$, since

$$i_n(j_W)(a) = \pi(j_W^n(a)) = n.$$

Next, we obtain the dual h for j_W, which is also a map from W to W. Moreover, it must be the case that j_W is a section of h; that is, for each $j_W^n(a) \in W$, we should have $(h \circ j_W)(j_W^n(a)) = j_W^n(a)$. Since $j_W(j_W^n(a)) = j_W^{n+1}(a)$, we need to send $n+1$ back to n, and this can be done in the obvious way using a "predecessor" function *pred*. It is easy to see that the predecessor function is the dual for the successor function. We formally define a co-Dedekind self-map pred $: \omega \to \omega$:

$$\text{pred}(n) = \begin{cases} n-1 & \text{if } n \geq 1,\\ 0 & \text{if } n = 0.\end{cases}$$

We have $\text{pred}(s(n)) = n$ for all $n \in \omega$. This function will help us define the dual h for j_W. Consider the following diagram.

(50)

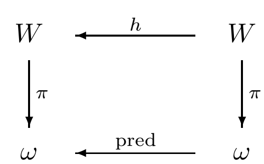


Diagram (50) suggests to us how h must be defined: $h = \pi^{-1} \circ \text{pred} \circ \pi$. We can now check that h is indeed a dual for j_W: For any $j_W^n(a) \in W$, we have:

$$\begin{aligned}(h \circ j_W)(j_W^n(a)) &= h(j_W(j_W^n(a)))\\ &= h(j_W^{n+1}(a))\\ &= \pi^{-1}(\text{pred}(\pi(j_W^{n+1}(a))))\\ &= \pi^{-1}(\text{pred}(n+1))\\ &= \pi^{-1}(n)\\ &= j_W^n(a)\\ &= \text{id}_W(j_W^n(a)).\end{aligned}$$

We now verify that h is a co-Dedekind self-map. First we show h is onto: For each $j_W^n(a) \in W$, we have $h(\pi^{-1}(s(\pi(j_W^n(a))))) = j_W^n(a)$, since

$$\begin{aligned} h(\pi^{-1}(s(\pi(j_W^n(a))))) &= \pi^{-1}(\mathrm{pred}(\pi(\pi^{-1}(s(\pi(j_W^n(a)))))) \\ &= \pi^{-1}(\mathrm{pred}(s(\pi(j_W^n(a))) \\ &= \pi^{-1}(\pi(j_W^n(a))) \\ &= j_W^n(a). \end{aligned}$$

Moreover, h has co-critical point a since $h(a) = a = h(j_W(a))$. We have shown h is a co-Dedekind self-map.

Next, we verify that, for every $n \in \omega$, there is $i \in \mathcal{E}_0$ such that $i^*(h)(n) = a$. Recall that, because our functionals i are of type $i : W^W \to \omega^W$, to get the collapsing step to work out, we need to use a conjugate $i^* : W^W \to W^\omega$ of i; in that case, the type of $i^*(h)$ is $i^*(h) : \omega \to W$, exactly as needed. Here is the required verification:

$$\begin{aligned} i_n^*(h)(n) &= i_n^*(\pi^{-1} \circ \mathrm{pred} \circ \pi)(n) \\ &= \big(\pi^{-1} \circ i_n(\pi^{-1} \circ \mathrm{pred} \circ \pi) \circ \pi^{-1}\big)(n) \\ &= \big(\pi^{-1} \circ \pi \circ \pi^{-1} \circ \mathrm{pred}^n \circ \pi \circ \pi^{-1}\big)(n) \\ &= (\pi^{-1} \circ \mathrm{pred}^n)(n) \\ &= \pi^{-1}(0) \\ &= a. \end{aligned}$$

To complete the proof, a few details need to be checked. We refer to Definition 8 where strong blueprints are defined. For (2)(c), we need to verify that π is defined from j and a; a review of the definition of π as the Mostowski collapsing map given in equation 8 (Theorem 10) shows that this is indeed the case. It follows that $\mathcal{E}$ is compatible with j, since each $i \in \mathcal{E}_0$ is a composition of an iteration function with π.[81]

For (3), we need to verify that j_W and h are also defined from $j, a, \mathcal{E}$. Certainly $j_W = j \restriction W$ is defined from j and W, but $W = \bigcap \mathcal{I}$ where $\mathcal{I} = \{B \subseteq A \mid B \text{ is } j\text{-inductive}\}$, and the j-inductive property is defined in terms of a and j. So W is defined from j and a. Also, h is defined from the successor $s : \omega \to \omega$, and, in our treatment, s is defined by $s = \pi \circ (j \restriction W) \circ \pi^{-1}$, the factors of which, as has already been indicated, are defined from j and a.

[81]We indicate how the weaker notion of compatibility, discussed in Remark 7(B), can also be satisfied here.

As in the example developed in the footnote on p. 87, we make use of the Dedekind self-map $J_j : A^A \to A^A$ defined from j. Recall $J_j(f) = j \circ f$ and

$$\overline{W} = \{\mathrm{id}_A, J_j(\mathrm{id}_A), J_j(J_j(\mathrm{id}_A)), \ldots\} = \{\mathrm{id}_A, j, j \circ j, \ldots\}.$$

By Theorem 22, there is a Dedekind self-map isomorphism $\tau : J_j \restriction \overline{W} \to j \restriction W$; from previous work it follows that $\tau(j^n) = j^n(a)$. There is also a Mostowski collapsing isomorphism $\overline{\pi} : \overline{W} \to \omega$ that makes the upper square in the diagram below commutative. Chasing the diagram shows that $\overline{\pi}$ is defined by

$$\overline{\pi}(j^n) = \pi(\tau(j^n)) = \pi(j^n(a)) = n,$$

where $\pi : W \to \omega$ is the Mostowski collapse of W to ω.

The remaining verifications are straightforward. □

Having provided a detailed account of the concept of "blueprint" from the mathematical point of view, we step back for a moment and see to what extent our mathematical blueprint reflects the characteristics of blueprints that we catalogued earlier in this section. Let us recall our summary of these characteristics (starting on page 80). As we list the points from this summary, we describe how our mathematical concept of blueprint exhibits the same kinds of characteristics. As background, we assume we have a Dedekind self-map $j : A \to A$ with critical point a.

(1) *Blueprint Arises from the Dynamics of One*: "Dynamics of One," in this context, are the dynamics represented by j—this was the intuition that inspired our New Axiom of Infinity. In our definition of a strong blueprint $(f, g, a, \mathcal{E})$ for X, the blueprint as generator, namely, f, is definable from $j, a, \mathcal{E}$; but $\mathcal{E}$ itself is "compatible with" j, so in a sense, it also arises from

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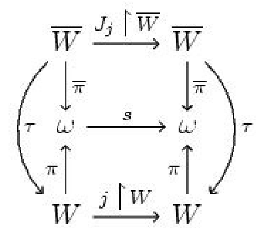


We define $\mathcal{E} = \{i_0, i_1, i_2, \ldots\}$ uniformly across two types—following the approach in the previous example (p. 87)—in essentially the same way these functions were defined in the main text in the proof of Theorem 32.

The first of these types is $\overline{W} \to \omega$; this type will be needed for the compatibility requirement, when we seek a right factor of $J_j \restriction \overline{W}$ that belongs to $\mathcal{E}$. The definition of i_n in this case is

$$i_n(g) = \overline{\pi}(g^n).$$

The second of these types is $\overline{W}^{\overline{W}} \to \omega^{\overline{W}}$; this type will be used for verification of decoding in the blueprint. The definition in this case is

$$i_n(F)(g) = \overline{\pi}\big(F^n(g)\big).$$

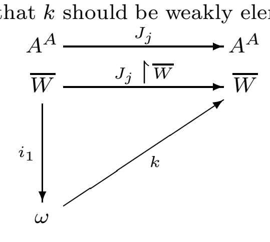
We verify the decoding property: Suppose $n \in \omega$. We find $i \in \mathcal{E}$ with $i(J_j \restriction \overline{W})(\mathrm{id}_A) = n$: Pick $i = i_n$. Then

$$i_n(J_j \restriction \overline{W})(\mathrm{id}_A) = \overline{\pi}\big(J_j^n \restriction \overline{W}(\mathrm{id}_A)\big) = \overline{\pi}\big(j^n\big) = n.$$

Finally, we verify compatibility using the first of the two types mentioned above. We show $i_1 : \overline{W} \to \omega$ is a right factor for $J_j \restriction \overline{W}$. We define $k : \omega \to \overline{W}$ by $k(n) = j \circ (\overline{\pi}^{-1}(n))$. Then

$$(k \circ i_1)(j^n) = k(i_1(j^n)) = k(\overline{\pi}(j^n)) = k(n) = j \circ (\overline{\pi}^{-1}(n)) = j \circ j^n = J_j(j^n).$$

and the diagram below is commutative. (Note that, because the definition of the i_n involved mixing of types, the requirement that k should be weakly elementary is not meaningful here.)



j. Even the critical point of j is itself a property of j. So, in a rather precise sense, the generating blueprint f "arises from the dynamics of One"—the dynamics of j.

(2) *Unity Preserved in the Emergence of the Blueprint*: What we observed from the beginning about j is that, in its dynamics, it preserves its own nature and the nature of its domain: A, as a Dedekind-infinite set, is transformed by j to another Dedekind-infinite set $B = j[A]$. And restricting j to its image, $j \upharpoonright B$, produces a new Dedekind self-map. Therefore, as j unfolds its "precipitations" $a, j(a), j(j(a)), \ldots$, differences arise on the ground of the sameness across the transformations $j, j \upharpoonright j[A], j \upharpoonright j[j[A]], \ldots$.

(3) *The Blueprint Gives Rise to All Manifest Existence*: So far, we have seen that a Dedekind self-map from a *set* to itself induces a blueprint of ω—not all of existence. We treat the present characteristic as a *prediction*—in our quest for better and better generalizations of an ordinary Dedekind self-map, we will look for versions having blueprints that do give rise to "all existence"—that is, all sets, all mathematical objects.

(4) *A Fundamental Power of the One Is Responsible for Transforming the Blueprint to Manifest Existence*: In the present context, this "fundamental power" is captured in the class $\mathcal{E}$ of weakly elementary functionals. For each $x \in X$, it is by virtue of the "power" of a properly selected functional $i \in \mathcal{E}_0$ that it becomes possible to locate x through the computation $i(f)(a)$.

(5) *The Blueprint Is Responsible for Both Generation and Return*: In the case of a strong blueprint for X, the blueprint for collapse, denoted g, returns each element of X to its starting point a, by virtue of the same power we found in (4): For each $x \in X$, it is by virtue of a properly selected $i \in \mathcal{E}_0$ that it is possible to "return x to a," through the computation $i(g)(x) = a$.

14. A First Look at Dedekind Self-Maps of the Universe

One of our motivations for introducing the New Axiom of Infinity is to wrap within the formulation of the set theory axiom "there exists an infinite set" some intuition about the *nature* of "the infinite." The intention is that, by including an intuition of this kind, the axiom can suggest a direction for answering questions about the mathematical infinite that cannot be resolved by the ZFC axioms alone. The intuition that we have wished to include is the idea that the "essence" of infinite collections consists of underlying self-referral dynamics of an unbounded field, represented mathematically as a Dedekind self-map. To express Cantor's original insight that "infinity exists" by saying "a Dedekind self-map exists" is to say that it is enough that the axiom should assert the existence of the *source* of infinite collections; then one *derives* the existence of concrete infinite collections.

One fundamental question, which has been of interest to set theorists for many decades, is the Problem of Large Cardinals: Is there a natural axiom that can be added to the axioms of ZFC on the basis of which the known large cardinals can be derived?[82] Since this is a question about the mathematical infinite, it is reasonable to consult the Axiom of Infinity to get some hint about how we might strengthen the axiom in a natural way to produce a new axiom that could account for large cardinals. But the usual Axiom of Infinity tells us very little.

On the other hand, our New Axiom of Infinity suggests a direction that one would be unlikely to consider on the basis of the usual Axiom of Infinity. We put together two clues as we seek new axioms to justify large cardinals:

(1) Many of the known large cardinals have a *global* character, asserting things about the universe V as a whole.
(2) The New Axiom of Infinity (which has "wrapped within it" some intuition about the origin of infinite sets) asserts the existence of a Dedekind self-map.

Based on these clues, we conjecture that the axiom for large cardinals that we are seeking is of the form "There is a Dedekind self-map defined on the universe of all sets." We therefore are interested in investigating Dedekind self-maps of the form $j : V \to V$.

Based on our work so far, we can describe properties we would expect such a $j : V \to V$ to have. Properties of a Dedekind self-map $j : A \to A$ with critical point a that we have discovered so far are:

(A) j preserves essential properties of its domain (A is Dedekind-infinite, and so is the image $j[A]$ of j) and of itself (the property of being a Dedekind self-map propagates to the restriction $j \upharpoonright j[A]$).
(B) The definition of j entails a *critical point* that plays a key role in its dynamics. There are several points to observe here.
  (i) The *critical sequence* $\langle a, j(a), j(j(a)), \ldots \rangle$ is a precursor to the set ω of finite ordinals. The emergence of this critical sequence provides a strong analogy to the ancient and quantum field theoretic perspective that "particles arise from the dynamics of an unbounded field," where, in this context, particular finite ordinals are viewed as "particles."
  (ii) The "most important" part of the unfoldment of the critical sequence is the transformation from a to $j(a)$. This is where it is demonstrated that *j moves a*, and sets in motion the rest of the critical sequence. Since a and $j(a)$ are the precursors to 0 and 1, respectively, the dynamics by which j moves a to $j(a)$ contain in seed form the dynamics of the unfoldment of everything else. Indeed, the "story of creation" resides in those dynamics. In more concrete terms, these dynamics tell us how 1 emerges from 0, how "something" arises from "nothing."[83]
  (iii) One aspect of these dynamics of j is that repeated restrictions of j to successive images can be seen to give rise to the critical sequence. The sequence of restrictions $j_0, j_1, j_2, \ldots$ of j that produce the critical sequence is defined as follows, where for any n, $\mathrm{crit}(j_n)$ denotes a critical

[82]Large cardinals will be introduced more systematically in Section 22.

[83]This particular theme is elaborated considerably toward the end of this article, starting on p. 197.

point of j_n:

$$\begin{array}{rcl} A_0 & = & A; \\ j_0 & = & j : A \to A; \\ \mathrm{crit}(j_0) & = & a; \\ A_1 & = & j[A_0]; \\ j_1 & = & j \restriction A_1; \\ \mathrm{crit}(j_1) & = & j(a); \\ A_{n+1} & = & j[A_n]; \\ j_{n+1} & = & j \restriction A_{n+1}; \\ \mathrm{crit}(j_{n+1}) & = & j^{n+1}(a). \end{array}$$

Therefore, $\mathrm{crit}(j_n) = j^n(a)$ is a critical point of j_n.

(C) Through the interplay between j and its critical point a, a *blueprint* $(j \restriction W, a, \mathcal{E})$ for ω arises, where $W = \{a, j(a), j(j(a)), \ldots\}$. In particular, $j \restriction W$ is a Dedekind self-map, and for every $n \in \omega$, there is $i \in \mathcal{E}$ such that $i(j \restriction W)(a) = n$.

(D) There is also a *strong blueprint* $(j \restriction W, h, a, \mathcal{E})$ for ω. In particular, $(j \restriction W, h)$ is a blueprint dyad and, for every $n \in \omega$, there is $i \in \mathcal{E}$ such that $i^*(h)(n) = a$.

Using (A)–(D) to guide intuition about the "nature of the infinite," as we expand our consideration from local to global, it is natural to search for Dedekind self-maps $j : V \to V$ that exhibit similar properties. We will seek Dedekind self-maps that

(A) preserve essential properties of their domain;
(B) have a critical point that plays a key role in the dynamics of j;[84]
(C) give rise to a blueprint for some set that plays a significant role in the structure of V;
(D) give rise to a strong blueprint for some, possibly different, set, which is in some way also significant.

As we begin studying Dedekind self-maps defined on the universe V, an issue comes into view that needs to be resolved. Consider the *global successor function* $\overline{s}$ defined by $\overline{s}(x) = x \cup \{x\}$. Certainly this is an example of a Dedekind self-map defined on the universe of sets, having critical point $\emptyset$ (just like the successor function on ω). However, the presence of this function in the universe[85] does not imply the existence of an infinite set. The fact is that the definition of $\overline{s}$ is just

[84] In particular, the critical sequence, or possibly other sets that arise from the interaction between j and its critical point, will have considerable significance. And the action of j on its critical point—the first "sprouting" of the critical point emerging from j—will tell much about the sort of "infinity" that j is capable of producing. Moreover, restrictions of j should give rise to a sequence of critical points and reveal further dynamics of j.

[85] As a proper class function.

as valid[86] in ZFC − Infinity as it is in ZFC.[87] This means that the existence of a Dedekind self-map on V cannot properly be viewed as an axiom of infinity, even though existence of a *set* Dedekind self-map can.

Nevertheless, it is reasonable to expect that one could introduce slight strengthenings of the properties of a self-map $j : V \to V$ that would imply the Axiom of Infinity (or the New Axiom of Infinity). Our next step, therefore, is to formulate several strengthenings of this kind, which we can then use for further intuition concerning how to proceed in the direction of justifying not just infinite sets, but large cardinals as well. To take this step, we will need a deeper understanding of the universe V and of the concept of a *class*.

15. The Classes ON and V

The collection ON of ordinal numbers is to the universe V as the set of natural numbers is to the "universe" V_ω. They allow us to measure the sizes of sets and arrange elements in (long) sequences. Both ON and V are examples of *proper classes*—collections that are too big to be sets, but that we can speak about because they are definable by a formula. In this section, we give precise definitions of these ideas.

We begin with the construction of V. The universe V is built in stages, and, as we have already seen, the first few stages of V can be defined by the following inductive definition:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{n+1} &= \mathcal{P}(V_n). \end{aligned}$$

We also observed before that, assuming the Axiom of Infinity, we can obtain the set

$$V_\omega = \bigcup_{n\in\omega} V_n.$$

To continue the construction further, we need to define the *ordinal numbers*.

[86] In fact, $\overline{s}$ is definable on any transitive class that is closed under formation of pairs and unions; in other words, $\overline{s}$ is definable in the theory {Pairing, Union}, which is just a tiny fragment of ZFC.

[87] One consequence of this observation is the fact that the existence of a Dedekind self-map on a *set* is *stronger than* the existence of such a map on the universe V. We mention briefly here the precise sense in which this is true. Using only ZFC − Infinity, we can demonstrate the existence of a Dedekind self-map from the universe to itself (the generalized successor $\bar{s} : V \to V$ is an example). However, the existence of a Dedekind self-map on a *set*, in conjunction with the other axioms of ZFC − Infinity, is sufficient to construct a *model* of the theory ZFC − Infinity (for instance, V_ω), and so, by Gödel's Second Incompleteness Theorem, assuming ZFC − Infinity is consistent, one cannot prove from ZFC − Infinity the existence of a Dedekind self-map on a set. Indeed, one cannot even prove, from ZFC − Infinity, the *consistency* of existence of a Dedekind self-map on a set with the theory ZFC − Infinity. These results show that the existence of a Dedekind self-map on a set bears the same relationship to the theory ZFC − Infinity as the existence of large cardinals bears to the theory ZFC: One cannot prove from ZFC − Infinity that a *set* Dedekind self-map (or any kind of infinite set) exists; likewise, one cannot prove from ZFC, for the same reason, that any type of large cardinal exists. Gödel's Second Incompleteness Theorem is discussed on p. 157. A rigorous proof of each of these consequences of Gödel's Second Incompleteness Theorem is given in the Appendix; see Theorem 96.

Definition 9. (Ordinals and Cardinals) An *ordinal number* is any transitive set X with the property that $(X, \in)$ is a well-ordered set.[88] The collection of ordinal numbers is denoted ON. A *cardinal number* is an ordinal number which does not have the same size as any of its elements.

The list of all ordinals begins with the *finite ordinals* $0, 1, 2, \ldots$ (defined, as we described earlier, as sets, with the property that each is the set comprised of all finite ordinals that precede it). The first infinite ordinal is ω, which, like the finite ordinals, is the set consisting of all the ordinals that precede it. Adopting the convention that, for any ordinal α, $\alpha + 1 = \alpha \cup \{\alpha\}$, the next few ordinals can be listed as follows: ω, $\omega + 1$, $\omega + 2$, $\omega + 3$, $\ldots$ (note that by $\omega + 2$ we mean $(\omega + 1) + 1$, and so forth). Note that among these first infinite ordinals, only ω is a *cardinal*; certainly, ω does not have the same size as any of the finite ordinals that belong to it. On the other hand, for any $n \geq 1$, there is a 1-1 correspondence $f : \omega + n \to \omega$ (note that ω is one of the elements of $\omega + n$), defined as follows:

$$f(x) = \begin{cases} n + m & \text{if } x = m \in \omega, \\ m & \text{if } x = \omega + m, \text{ where } 0 \leq m < n. \end{cases}$$

The order relation (which is simply the membership relation $\in$) on the ordinal numbers closely resembles that defined on the natural numbers because of the following fact, which we do not prove here:

Proposition 32. *Every nonempty subset of ON is well-ordered (by $\in$).*

Because the ordinals are well-ordered, they can be used for inductively defining long sequences in exactly the same way as is done with the natural numbers. The theorem that makes this assertion is given below, Theorem 35. We begin with two more definitions.

A *successor ordinal* is an ordinal β for which, for some ordinal α, $\beta = \alpha + 1$ (recall $\alpha + 1$ is shorthand for $\alpha \cup \{\alpha\}$). All the positive natural numbers are successor ordinals: For instance, $3 = 2 + 1$. A *limit ordinal* is an ordinal that is not a successor ordinal; equivalently, an ordinal that has no immediate predecessor. The easiest example is 0. Whether or not any other limit ordinals exist depends on whether the Axiom of Infinity holds. If not, then 0 is the only limit ordinal. If it does hold, then ω exists and it is the smallest nonzero limit ordinal.

The ordinals can be used as indices to continue the construction of the stages of the universe. The proof that this is so—as was the case for inductive definitions over ω—requires a principle of induction. We state this principle first, and then show how inductive definitions are done over ON.

Theorem 33. (Transfinite Induction) (ZFC − Infinity) *Suppose $\phi(x)$ is a formula with ordinal parameter x and possibly other set parameters. Suppose the following three conditions hold:*

(1) (Basis) $\phi(0)$ *holds.*
(2) (Successor) *If α is a successor ordinal, $\alpha = \beta + 1$, and $\phi(\beta)$ holds, then $\phi(\alpha)$ also holds.*

[88] Technically, we should say $(X, \in \restriction X \times X)$, since $\in$ is a relation defined on all of V. A definition of *well-ordering* is given on page 23.

(3) (Limit) *If α is a limit ordinal and $\phi(\beta)$ holds for every $\beta < \alpha$, then $\phi(\alpha)$ holds.*
Then $\phi(\alpha)$ holds for every $\alpha \in ON$. □

As the definition indicates, for transfinite induction, the induction step involves two sub-steps—the Successor step and the Limit step. Likewise, inductive definitions over ON take into account successor and limit ordinals separately. Here is a simplified version of this inductive definition principle, which is a generalization of the Strong Definition by Recursion Theorem for ω (Theorem 25) to ON:

Theorem 34. (Definition by Recursion Theorem on ON) *A sequence $\langle x_0, x_1, x_2, \ldots, x_\beta, \ldots\rangle$ indexed by all the ordinals can be specified by providing the following:*
(1) (Basis) *The value of x_0.*
(2) (Successor) *A formula for obtaining the value of $x_{\beta+1}$ from x_β.*
(3) (Limit) *A formula for obtaining the value of x_β from $\langle x_0, x_1, x_2, \ldots, x_\gamma, \ldots\rangle_{\gamma<\beta}$, whenever β is a limit ordinal.*

We can use the theorem to build the long sequence of stages $V_0, V_1, \ldots, V_\alpha, \ldots$ of the universe V:

$$\begin{aligned} V_0 &= \emptyset, \\ V_\alpha &= \mathcal{P}(V_\beta) \quad \text{if } \alpha = \beta + 1, \\ V_\alpha &= \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal,} \\ V &= \bigcup_{\alpha \in \mathrm{ON}} V_\alpha. \end{aligned}$$

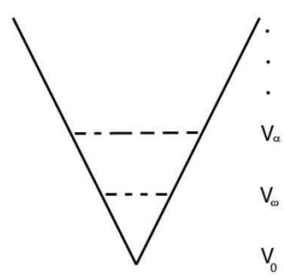


FIGURE 12. The Universe of Sets

The definition of the long sequence $\langle V_0, V_1, \ldots, V_\alpha, \ldots\rangle_{\alpha\in\mathrm{ON}}$ involves two "induction" steps, both of which are concerned with how to define the αth stage on the basis of stages V_β for $\beta < \alpha$. When α is a successor $\beta+1$, V_α is computed to be the power set of the previous stage; when α is a limit, V_α is computed to be the union of all previous stages.

Formally, the mechanism for justifying the existence of such a long sequence is a generalization of the mechanism given in Theorem 25.[89]

[89] See [35] for a rigorous treatment.

A *set* is understood to be any element of V. A *class* is any subcollection of V that one can specify using a formula with set parameters. As we mentioned before, any set x is, trivially, a class since x is specified by the formula (with set parameters x, y) "$y \in x$": $x = \{y \mid y \in x\}$.

A *proper class* is a class that is not a set. An example we have just seen is the class ON of ordinal numbers. Another example is $K = \{x \mid x \text{ is finite}\}$. It is easy to see that a collection C defined by a formula is a proper class if and only if, for every ordinal α, there is an element of C that does not belong to V_α; in other words, there are elements of C arbitrarily high up in V, so $C \not\subseteq V_\alpha$ for any ordinal α. So, our two examples ON and K are proper classes because, for every α,

(i) $\alpha \in \text{ON}$ and $\alpha \notin V_\alpha$; and
(ii) $\{\alpha\} \in K$ and $\{\alpha\} \notin V_\alpha$.

A map $F : A \to B$ is a *set* if both A and B are sets. In some contexts, even if B is a proper class, F may be considered to be a set as long the collection of ordered pairs that determine F forms a set. (In that case, the range R of F certainly is a set, so one could, for example, consider F to be $F : A \to R$, now presented as a set function.) On the other hand, if the collection of ordered pairs that determine F are definable from some formula, F is a *class map*. If, in addition, these ordered pairs form a *proper class*, then F is a *proper class map*; in such cases, the domain of F must be a proper class. An example of a proper class map that was just introduced above is the map $\alpha \mapsto V_\alpha$, for ordinals α; therefore, the sense in which the long sequence $\langle V_\alpha \mid \alpha \in \text{ON}\rangle$ "exists" is as a proper class map. We will see later that it is (meaningfully) possible for a map F to be neither a set nor a class.[90]

We now state a slight strengthening of Proposition 33:

Proposition 35. (ZFC − Infinity) *Every nonempty subclass of ON has an $\in$-least element.*[91]

16. The Class of Natural Numbers

We mentioned at the beginning of this paper[92] that one could define, within any model of ZFC − Infinity, the *class* of natural numbers. As remarked before, our proof that the set ω of natural numbers can be derived from a Dedekind self-map could have been simplified greatly by making use of this class, but we chose not to do so in order to emphasize that it is possible to view the natural numbers as emerging from the dynamics of a Dedekind self-map without any "help" from the natural numbers themselves.

At this point, since our discipline to avoid the use of the class of natural numbers in our derivation of ω has already borne the intended fruit, we will no longer attempt to avoid using this class. We repeat the definition of $\overline{\omega}$ given in Remark 5, now in

[90]See the footnote on p. 183 for an example of such a map.

[91]This follows from the other ZFC axioms in conjunction with Proposition 33: Given a class $\mathbf{C}$ of ordinals, let $\alpha \in \mathbf{C}$. By Separation, $E = \{\beta \in \alpha \mid \beta \in \mathbf{C}\}$ is a set, as is $\{\alpha\}$; therefore, their union is a set of ordinals having a least element γ. Now, for any $\delta \in \mathbf{C}$, either $\delta < \alpha$ or $\delta \geq \alpha$. In the first case, $\delta \in E$, so $\gamma \leq \delta$. In the second case, $\gamma \leq \alpha \leq \delta$.

[92]See p. 22.

a slightly different, but equivalent, guise:

$$\overline{\omega} = \begin{cases} \{0\} \cup \{\overline{s}(\alpha) \mid \alpha \in \text{ON and } \alpha < \gamma\} & \text{if } \gamma \text{ is the least nonzero limit ordinal,} \\ \{0\} \cup \{\overline{s}(\alpha) \mid \alpha \in \text{ON}\} & \text{if ON has no nonzero limit ordinal.} \end{cases}$$

Isolating the concept $\overline{\omega}$ is very useful when working with the theory ZFC − Infinity. The following is an application.

Theorem 36. (ZFC − Infinity) *The following are equivalent:*

(1) *The Axiom of Infinity (there is an inductive set).*
(2) $\omega = \overline{\omega}$.
(3) *There is a nonzero limit ordinal.*

Proof. For (1) ⇒ (2), assuming there is an inductive set, we can form (in the usual way) the intersection ω of all inductive sets. It is easy to verify that $\overline{\omega}$ is an inductive set, so $\omega \subseteq \overline{\omega}$. The usual principle of induction and definition by induction theorems follow. By induction, one verifies that $s = \overline{s} \restriction \omega : \omega \to \omega$ and that $\omega = \{0\} \cup \{s(n) \mid n \in \omega\}$. Since $\overline{\omega}$ is defined using the first clause in the definition in this case, the result follows. For (2) ⇒ (3), note that ω itself is a nonzero limit ordinal. For (3) ⇒ (1), let γ be a nonzero limit ordinal. It is easy to see that γ is inductive, so the Axiom of Infinity holds. □

Whether or not any form of the Axiom of Infinity holds, $\overline{\omega}$ still satisfies a form of the Principle of Induction since it is an initial segment of ON and therefore well-ordered:

Theorem 37. (Class Induction Over $\overline{\omega}$) (ZFC − Infinity) *Suppose* $\mathbf{C}$ *is a subclass of* $\overline{\omega}$ *with the following properties:*

(1) $0 \in \mathbf{C}$;
(2) *whenever* $n \in \mathbf{C}$, $\overline{s}(n) \in \mathbf{C}$.

Then $\mathbf{C} = \overline{\omega}$.

Proof. Let $\mathbf{B} = \overline{\omega} - \mathbf{C}$. Assume $\mathbf{B} \neq \emptyset$, and let n be its least element (using Proposition 36). Since $0 \in \mathbf{C}$, $n > 0$. By the definition of $\overline{\omega}$, n has an immediate predecessor $n-1$, which must be in $\mathbf{C}$. By (2), the successor of $n-1$ must therefore be in $\mathbf{C}$, contradicting the fact that $n \in \mathbf{B}$. Therefore, $\mathbf{B}$ is empty, and the result follows. □

A special case of Theorem 35 allows us to recursively define sequences with indices in $\overline{\omega}$.

Theorem 38. (Class Recursion Over $\overline{\omega}$) (ZFC − Infinity) *A class sequence* $\langle x_0, x_1, x_2, \ldots, x_n, \ldots \rangle$ *indexed by the elements of* $\overline{\omega}$ *can be specified by providing the following:*

(1) (Basis) *The value of* x_0.
(2) (Induction Step) *A formula for obtaining the value* x_{n+1} *from* x_n *for each* $n \in \overline{\omega}$.

In the context of the theory ZFC − Infinity, a set X is said to be *finite* if $|X| = n$ for some $n \in \overline{\omega}$.[93] When $\overline{\omega} = \omega$, of course, this new definition of "finite" coincides with the one given earlier,[94] but this slightly more general definition gives meaning to the term "finite" even in the absence of infinite sets. Likewise, we define a set to be *infinite* if it is not finite. We will use these slightly more general definitions of "finite" and "infinite" for the rest of the paper.

We now verify that existence of an infinite set is equivalent to the usual Axiom of Infinity. This will show that existence of an infinite set is equivalent to each of the clauses given in Theorem 37. Since we have already shown that existence of a Dedekind self-map is equivalent to the Axiom of Infinity (p. 24), it will be sufficient to prove that existence of an infinite set is equivalent to existence of a Dedekind self-map (on a *set*). We make use of the easily proven fact that a finite union of finite sets is finite. We begin with a convenient lemma.

Lemma 39. (ZFC) *There is no 1-1 function from ω into a finite set.*

Proof. Suppose A is finite and suppose there is a 1-1 function $\tau : \omega \to A$. Let $n \in \omega$ be such that there is a bijection $f : A \to n$. Let $h = f \circ \tau : \omega \to n$. Since h is 1-1, so is $h \restriction n+1 : n+1 \to n$, and so the range of h is a subset of n but must contain $n+1$ elements. We have arrived at a contradiction; we conclude that, assuming A is finite, there is no 1-1 function $\omega \to A$. □

Theorem 40. (ZFC − Infinity) *The following are equivalent:*

(1) *There is a Dedekind self-map.*
(2) *There is an infinite set (that is, a set that is not finite).*

Proof of (1) ⇒ (2). Suppose $j : A \to A$ is a Dedekind self-map with critical point a. As in our arguments in Section 7, let $W = \{a, j(a), j(j(a)), \ldots\}$ and let $\pi : W \to \omega$ be the Mostowski collapsing bijection. Let $\tau = \pi^{-1} : \omega \to W \subseteq A$. By Lemma 40, it follows that A is infinite. □

Proof of (2) ⇒ (1). Suppose X is infinite. We prove by induction that for any $n \in \overline{\omega}$, there is a 1-1 function $f : n \to X$. This is clear for $n = 0, 1$. Assuming $f : n \to X$ is 1-1, since $|\text{ran} f| = n$, there must be an $x \in X$ not in the range of f. Define $g : n+1 \to X$ by $g = f \cup \{(n, x)\}$. Clearly g is also 1-1. This completes the induction. We next inductively define a sequence $S = \langle S_n \mid n \in \overline{\omega}, n > 0\rangle$ of disjoint, nonempty subsets of X so that, for each $n > 0$, $|S_n| = n$. Let $x_1 \in X$; we let $S_1 = \{x_1\}$. Assume we have defined disjoint, nonempty sets $S_1, S_2, \ldots, S_n$ so that for each i, $|S_i| = i$. Let $A = S_1 \cup S_2 \cup \ldots \cup S_n$. Since $|A| = 1+2+\cdots+n$ is finite, the set $X - A$ cannot be finite—otherwise $X = (X - A) \cup A$ would be finite. By our earlier claim, we can obtain $f : n+1 \to X - A$ that is 1-1. Let $S_{n+1} = \text{ran}\, f$. Now $S_1, S_2, \ldots, S_{n+1}$ satisfies the necessary requirements, and so the inductive definition is complete, and we have a sequence $S = \langle S_n \mid n \in \overline{\omega}, n > 0\rangle$, with the properties described above. To complete the proof of (2) ⇒ (1), define $Y \subseteq X$ by $Y = \bigcup_n S_n$. Obtain a set $W \subseteq Y$ using the Axiom of Choice by picking one element s_n from each S_n. Define $w : W \to W$ by $w(s_n) = s_{n+1}$. Clearly w is a Dedekind self-map

[93]The expression "$|X| = n$" means there is a bijection $f : X \to n$.
[94]See Definition 4 on p. 50.

with critical point s_1. □

The reasoning given here and in earlier arguments can be used to establish a more fine-grained result about the equivalence of the various notions of "infinite set" that we have discussed so far.

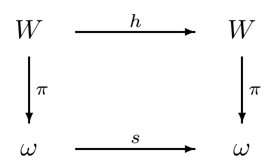
Theorem 41. (Equivalent Notions of Infinite Set) (ZFC – Infinity) *Suppose A is a set. The following are equivalent:*

(1) *A is Dedekind infinite (and hence there is a Dedekind self-map defined on A).*
(2) *There is a 1-1 function from an inductive set into A.*
(3) *A is infinite (that is, not finite).*

Proof of (1) ⇒ (2). As observed earlier, a bijection from A to one of its proper subsets can be viewed as a Dedekind self-map $j : A \to A$; let a be a critical point of j. As in our arguments in Section 7, let $W = \{a, j(a), j(j(a)), \ldots\}$ and let $\pi : W \to \omega$ be the Mostowski collapsing bijection. Let $\tau = \pi^{-1} : \omega \to W \subseteq A$. Since ω is inductive and τ is 1-1 (by Theorem 19), the result follows. □

Proof of (2) ⇒ (1). If $\tau : I \to A$ is 1-1 with I inductive, let $f = \tau \restriction \omega : \omega \to A$ (recall $\omega \subseteq I$). Let $W = \operatorname{ran} f$ and let $a = f(0)$. Let $\pi : W \to \omega$ be $f^{-1} \restriction W$. Define $h : W \to W$ by $h(b) = c$ if and only if $s(\pi(b)) = \pi(c)$, as in Diagram (51).

(51)



Now define $g : A \to A - \{a\}$ by

$$g(x) = \begin{cases} x & \text{if } x \notin W \\ h(x) & \text{otherwise.} \end{cases}$$

Clearly, g is a bijection and so A is Dedekind-infinite. □

Proof of (2) ⇒ (3). Suppose A is finite. It suffices to show there is no 1-1 function $\omega \to A$. But this was proved in Lemma 40. □

Proof of (3) ⇒ (2). Suppose A is infinite. As in the proof of (2) ⇒ (1) above in Theorem 41, obtain a set $W \subseteq A$ and a Dedekind self-map $w : W \to W$ with critical point s_1. Define $f : A \to A - \{s_1\}$ by

$$f(x) = \begin{cases} x & \text{if } x \notin W \\ w(x) & \text{otherwise} \end{cases}$$

Clearly f is a bijection and A is Dedekind-infinite. □

To close this section, we restate Theorem 25 in the language of categories: $(\overline{\omega}, \overline{s}, 0)$ is initial in the category of *class* Dedekind self-maps.[95]

Theorem 42. (Metatheorem) (ZFC − Infinity) *Let* **ClassSelfMap** *be the category of all class Dedekind self-maps, together with Dedekind self-map morphisms. Then* $s : \omega \to \omega$ *with critical point 0 is initial in* **ClassSelfMap***; that is, whenever* $j : \mathbf{C} \to \mathbf{C}$ *is an object in* **ClassSelfMap** *with critical point* c*, there is a unique* **ClassSelfMap***-morphism* $\mathbf{F}$*, definable from* j *and* $\overline{s}$*, for which* $\mathbf{F}(0) = c$ *and the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\omega} & \xrightarrow{\;\overline{s}\;} & \overline{\omega} \\ \Big\downarrow \mathbf{F} & & \Big\downarrow \mathbf{F} \\ \mathbf{C} & \xrightarrow{\;j\;} & \mathbf{C} \end{array}$$

Proof. Theorem 25. □

The work in this section conveniently integrates results on induction and recursive definitions that we established earlier for ω, and then later for ON. These results simplify proofs of results requiring these tools in the context of ZFC − Infinity.

17. Strengthening Proper Class Dedekind Self-Maps

In this section, we discuss methods of strengthening a Dedekind self-map of the form $j : V \to V$ in the theory ZFC − Infinity so that some form of the Axiom of Infinity is derivable. We adhere to the discipline that our methods must be *natural*. We will consider our methods "natural" if they conform to the underlying intuition that guides our search for the right formulation, namely, principles from the ancient texts.

We introduce two methods for strengthening a $j : V \to V$:

(1) require j to satisfy certain *preservation properties*;
(2) seek a version of j for which an infinite set arises through the interaction of j with its critical point.

Regarding approach (1), recall that we are considering Dedekind self-maps $j : V \to V$ to be an analogy for the fundamental dynamics of the source, of pure consciousness, at the basis of all diversity.[96] A primary characteristic of these dynamics is that, as pure consciousness transforms itself, it remains unchanged by these transformations; it preserves its essential nature.[97] By analogy, self-maps

[95] Since the theory ZFC − Infinity cannot in general accommodate such a large category without some adjustments, we have formulated the statement as a metatheorem. The version given in Theorem 25 has the same mathematical content and can be stated formally in ZFC − Infinity.

[96] See the discussion on page 14.

[97] Maharishi explains this idea in *The Science of Being* [51]:

> This absolute state of pure consciousness is of unmanifested nature which is ever maintained as that by virtue of the never-changing cosmic law. Pure consciousness, pure Being, is maintained as pure consciousness and pure Being all

$j : V \to V$ that we are seeking must also preserve the structure of their domain as much as possible; little steps in this direction are captured by the idea that j preserves particular *properties* of its domain.

As for approach (2), we recall that, according to the ancient perspective, *everything* arises from the interaction between the unbounded source and the point it locates within itself.[98] It is natural then that, within a ZFC − Infinity universe, an infinite set would be expected to arise from the dynamics of j as it interacts with its critical point.

We begin with several ways of adding extra preservation properties to j. We define two new concepts: *terminal objects* and *strong critical points*.

A *terminal object* in V is any set that has just one element.[99] Next, suppose $j : V \to V$ is a 1-1 class function. A set $X \in V$ is said to be a *strong critical point* if $|X| \neq |j(X)|$. It is possible that even if j has a strong critical point, j may not be a Dedekind self-map. The relationship between strong critical points and critical points is addressed in Proposition 49 and Theorem 52, below.

Definition 10. Suppose $j : V \to V$ is any function.

(1) j is said to *preserve disjoint unions* if, whenever $X, Y \in V$ are disjoint, $j(X), j(Y)$ are also disjoint and $j(X \cup Y) = j(X) \cup j(Y)$.
(2) j *preserves singletons* if, for any X, $j(\{X\}) = \{j(X)\}$.
(3) j is said to *preserve coproducts*[100] if, whenever X, Y are disjoint, $|j(X \cup Y)| = |j(X) \cup j(Y)|$. Moreover, j *preserves finite coproducts* if, whenever $n \in \varpi$, $n \geq 2$, and $X_1, X_2, \ldots, X_n$ are disjoint, $|j(X_1 \cup X_2 \cup \cdots \cup X_n)| = |j(X_1) \cup j(X_2) \cup \cdots \cup j(X_n)|$.

> the time, and yet it is transformed into all the different forms and phenomena. Here is the cosmic law, one law which never changes and which never allows absolute Being to change. Absolute Being remains absolute Being throughout, although it is found in changed qualities here and there in all the different strata (p. 12).

The great sage Vasishtha makes a similar point in the *Yoga Vasishtha* [73]:

> Thus the pure consciousness brings into being this diversity with all its names and forms, without ever abandoning its indivisibility, just as you create a world in your dream (p. 638).

and also:

> What is seen here as the world is but the supreme self which appears as the world without undergoing any change in its own true nature (p. 501).

[98] See the quote in the footnote on p. 15. The sage Vasishtha makes a similar point [73]:

> The seed or the sole cause for this world-appearance is but the arising of a thought in the infinite consciousness (p. 214).

Here, the point that it locates within itself is "a thought."

[99] The terminology comes from category theory. A terminal object in a category is an object T with the property that for any other object X, there is exactly one morphism $X \to T$. In the category of sets, the terminal objects are the singleton sets.

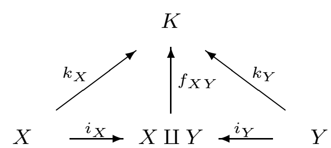
[100] The term *coproduct* comes from category theory. Our definition of preservation of coproducts is somewhat weaker than the category-theoretic definition, but will be more useful for our purposes. In category theory, working in the category of sets, whenever X and Y are sets, not necessarily disjoint, the coproduct $X \amalg Y$ of X and Y is a set with the property that there are maps $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$ such that, for any set K with maps $k_X : X \to K$, $k_Y : Y \to K$, there is a unique map $f_{XY} = [k_X, k_Y] : X \amalg Y \to K$, called the *coproduct map*, that makes the

(4) j *preserves terminal objects* if, whenever T is terminal, $j(T)$ is terminal. In particular, j maps singleton sets to singleton sets since, in **Set**, the terminal objects are precisely the singleton sets.
(5) j *preserves the empty set* (equivalently, j preserves *initial objects*[101]) if, $j(\emptyset) = \emptyset$.
(6) j is *cofinal* if, for every $a \in V$, there is $A \in V$ with $a \in j(A)$.
(7) j *preserves subsets* if, whenever $X \subseteq Y$, we have $j(X) \subseteq j(Y)$.

Remark 10. Note that preservation of disjoint unions is different from the property that every 1-1 function f has, namely, that whenever A, B are disjoint, the *images* $f[A], f[B]$ are disjoint, and $f[A \cup B] = f[A] \cup f[B]$. The example $f : V \to V$ defined by $f(x) = \{x\}$ highlights the distinction since

$$f(\{1\} \cup \{2\}) = f(\{1, 2\}) = \{\{1, 2\}\} \neq \{\{1\}, \{2\}\} = f(\{1\}) \cup f(\{2\}),$$

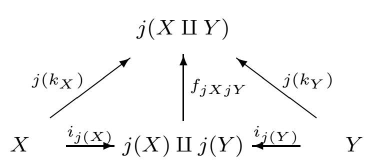
diagram below commutative.



In the category of sets, $X \amalg Y$ can be defined as the set $\{(x, 0) \mid x \in X\} \cup \{(y, 1) \mid y \in Y\}$ with i_X defined by $x \mapsto (x, 0)$ and i_Y defined by $y \mapsto (y, 1)$. Then, given K, k_X, k_Y as above, we may define $[k_X, k_Y]$ by

$$[k_X, k_Y](z, i) = \begin{cases} k_X(z) & \text{if } i = 0 \\ k_Y(z) & \text{if } i = 1 \end{cases}$$

In the usual treatment, a functor (*functors* are defined in this paper in Section 19) j on the category of sets is said to *preserve coproducts* if, for any sets X, Y and maps $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$, not only do we have $j(X \amalg Y) \cong j(X) \amalg j(Y)$ (that is, $|j(X \amalg Y)| = |j(X) \amalg j(Y)|$— as in the definition given in this paper), but this isomorphism is *canonical* in the sense that the bijection is unique and is in fact the coproduct map $f_{jXjY} = [j(i_X), j(i_Y)]$, making the diagram below commutative.



The fact that there is a canonical bijection often leads authors to identify $j(X) \amalg j(Y)$ with $j(X \amalg Y)$ (for example, see [3]), just as the distinct sets $(A \times B) \times C$ and $A \times (B \times C)$ are sometimes identified with each other because of the canonical bijection between them. We do not wish to make this identification between $j(X) \amalg j(Y)$ and $j(X \amalg Y)$ in this paper since some of our arguments will depend on having exact equality between the disjoint unions in question and others will require only identical cardinalities. We have therefore distinguished between preservation of disjoint unions and preservation of coproducts.

[101] An object I in a category is *initial* if, for every object X in the category, there is a unique morphism $I \to X$. In the category of sets, the empty set is the unique initial object

and yet

$$f[\{1\} \cup \{2\}] = f[\{1,2\}] = \{f(1), f(2)\} = f[\{1\}] \cup f[\{2\}].$$

We also observe here that whenever j preserves disjoint unions, j preserves subsets: Given sets A, B with $A \subseteq B$, note that $B = A \cup (B - A)$, which is a disjoint union, and so we may write $j(B) = j(A) \cup j(B - A)$, so that $j(A) \subseteq j(B)$.

Another point we observe for later use is that if j has critical point a and j preserves singletons and disjoint unions, then $\{a\}$ is also a critical point of j: Suppose $\{a\} = j(z)$. If z is not a singleton, let $z_0 \in z$ and, because $\{z_0\}, z - \{z_0\}$ is a partition of z, by preservation of singletons and disjoint unions, $j(z) = \{j(z_0)\} \cup j(z - \{z_0\})$, and so $|j(z)| > 1$, which is impossible. Therefore, z is a singleton set $\{y\}$. We have $\{a\} = j(z) = j(\{y\}) = \{j(y)\}$, and so $a = j(y)$, contradicting the fact that a is a critical point of j.

Finally, we give some explanation of the *cofinal* property that a map $j : V \to V$ may have. If j is cofinal, it means that every set x in the universe V belongs to some set of the form $j(A)$; equivalently, every set belongs to $\bigcup \operatorname{ran} j$. Another perspective is that the image $j[V] = \{j(x) \mid x \in V\}$ *covers* V—every set x belongs to an element of $j[V]$. Using more intuitive language, to say j is cofinal is to say that each set *belongs to the world of* j.

Cofinality may also be understood in terms of *internal sets*. The idea of internal sets is familiar in nonstandard analysis. In that context, the standard elements of some domain of discourse, like the real number line or the universe of sets, are embedded in an expanded universe by a transfer mapping that is usually denoted *. The sets in the expanded universe that are of primary interest are the internal sets—those sets that belong to $^*A$ for some set A; *external* (not internal) sets play a less significant role. Working in the expanded universe, focused on the internal sets, it is often possible to discover truths about the original standard universe that would have been difficult to discover working only within the standard universe.[102] In our present context, the internal sets are those that belong to $j(A)$ for some A; j is cofinal if and only if all sets are internal. Thus, the sets in V that belong to the world of j are the internal sets, the sets that we can talk about relative to j.

Since we are thinking of the self-maps j as representatives of some underlying field, our intuitive framework suggests to us that j "should be" cofinal—"everything" should belong to the world of j. As we shall see, this property will prove to be a key ingredient in the structure of a blueprint of the universe. We consider a couple of examples, foreshadowing some of the results that will be developed in the course of the paper.

One example arises when a Dedekind self-map $j : V \to V$ with critical point a is cofinal, and also, for some set A, $a \in j(A)$ is *weakly universal* (defined on p. 130). In that case we will have:

(52) $$\forall x \, \exists f \, j(f)(a) = x$$

(see p. 130). In other words, every set in the universe is expressible in the form $j(f)(a)$ for some function f. The function f may be thought of as an *approximation* to a blueprint for sets in V. In a typical context, wherein j preserves the domain and codomain of functions (that is, if $g : C \to D$ is a function, then so is $j(f)$:

[102] A good reference for nonstandard analysis is [33].

$j(C) \to j(D)$), the property (52) on its own *implies* that j is cofinal: Assuming (52) holds, let b be a set; we show $b \in j(B)$ for some B. Let $f : A \to B$ be such that $j(f)(a) = b$. Since $j(f) : j(A) \to j(B)$, it follows that $b \in j(B)$.[103]

A second example of the role of the cofinal property, which we will discuss toward the end of the paper (Theorem 81), is the case in which a Dedekind self-map $j : V \to V$ is a strong kind of *elementary embedding*—called a *WA_0-embedding*—having as a canonical critical point the least ordinal moved by j (denoted κ). Any WA_0-embedding $j : V \to V$ satisfies *amenability*, that is, for every set x there is a set y with $y = j \restriction x$. In order for j to be both elementary and amenable, it must be cofinal. Moreover, we will be able to show the following:

(53) $$\exists f\, \forall x\, \exists i \in \mathcal{E}\, (i(f)(\kappa) = x),$$

where elements of $\mathcal{E}$ are elementary embeddings (compatible with j) having critical point κ. It turns out that (53) is a stronger property than (52); indeed, the function f given in (53) will be shown to satisfy the requirements of a blueprint for all sets in V.[104] $\Box$

Theorem 43. (ZFC − Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map. Suppose also that j preserves disjoint unions, the empty set, and singletons. Then there is an infinite set.*

Proof. We show by induction on $n \in \overline{\omega}$ that j preserves finite disjoint unions.[105] The result is clear for $n = 2$ since j preserves disjoint unions. Assume that $n \geq 2$ and whenever $\{X_1, \ldots, X_n\}$ is a collection of n disjoint sets, then the sets $j(X_1), \ldots, j(X_n)$ are also disjoint, and

$$j(X_1 \cup \cdots \cup X_n) = j(X_1) \cup \cdots \cup j(X_n).$$

Given disjoint sets $Y_1, \ldots, Y_n, Y_{n+1}$, certainly $j(Y_1), \ldots, j(Y_n)$ are disjoint. Let $Y = Y_1 \cup \cdots \cup Y_n$. By induction hypothesis, $j(Y) = j(Y_1) \cup \cdots \cup j(Y_n)$. Since Y and Y_{n+1} are disjoint, so are $j(Y)$ and $j(Y_{n+1})$, and

$$j(Y_1 \cup \cdots \cup Y_n \cup Y_{n+1}) = j(Y \cup Y_{n+1}) = j(Y) \cup j(Y_{n+1}) = j(Y_1) \cup \cdots \cup j(Y_n) \cup j(Y_{n+1}),$$

as required. This completes the induction and proves the claim.

We proceed by induction to show that $j(n) = n$ for all $n \in \overline{\omega}$. By assumption on j, $j(\emptyset) = \emptyset$. Assume $j(n) = n$. Note that for all $n \in \overline{\omega}$, $n, \{n\}$ are disjoint because $(n, \in)$ is a well-ordered set (so, in particular, $n \notin n$). We complete the induction step and the proof that $j(n) = n$ for all $n \in \overline{\omega}$ with the following:

$$j(n+1) = j(n \cup \{n\}) = j(n) \cup j(\{n\}) = j(n) \cup \{j(n)\} = n \cup \{n\} = n + 1.$$

Next we show that, for all $x \in \mathbf{HF}$, $j(x) = x$. As we observed earlier,[106] $\mathbf{HF} = \bigcup_{n \in \overline{\omega}} V_n$. For each $x \in \mathbf{HF}$, let $\operatorname{rank} x$ denote the least $n \in \overline{\omega}$ for which $x \subseteq V_n$.[107]

[103] An example of such a j is mentioned briefly in the footnote on p. 161 and developed more fully in the footnote on p. 171.

[104] More details can be found in Theorem 81.

[105] The arguments by induction that are given here rely on the fact that j is definable by a formula; that is, j is a *class* function.

[106] A proof is given in the Appendix, Theorem 87. For a definition of **HF**, see p. 54.

[107] More generally, for any $x \in V$, $\operatorname{rank} x$ is defined to be the least ordinal α for which $x \subseteq V_\alpha$.

We proceed by induction on $n \in \overline{\omega}$ to show that, for all x of rank $\leq n$, $j(x) = x$. This is clear for $n = 0$, since $j(\emptyset) = \emptyset$. Assume the result for n. For the induction step, it suffices to show that if $\operatorname{rank} x = n + 1$, then $j(x) = x$. Given such an x, certainly x is finite; write $x = \{y_1, \ldots, y_k\}$. For each i, $y_i \in x \subseteq V_{n+1}$. By the induction hypothesis, for each i, $j(y_i) = y_i$. We have

$$\begin{aligned} j(x) &= j\left(\{y_1, \ldots, y_k\}\right) \\ &= j\left(\bigcup_{1 \leq i \leq k} \{y_i\}\right) \\ &= \bigcup_{1 \leq i \leq k} j(\{y_i\}) \\ &= \bigcup_{1 \leq i \leq k} \{j(y_i)\} \\ &= \bigcup_{1 \leq i \leq k} \{y_i\} \\ &= \{y_1, \ldots, y_k\} \\ &= x. \end{aligned}$$

This completes the induction and proves that, for all $x \in \mathbf{HF}$, $j(x) = x$.

Finally, we complete the proof of the theorem: Let a be a critical point of j. Since $j \upharpoonright \mathbf{HF}$ is just the identity map on $\mathbf{HF}$, which is onto $\mathbf{HF}$, $a \notin \mathbf{HF}$. Therefore, every transitive set that contains a is infinite; since the Trans axiom holds, some transitive set must include a. Therefore, there is an infinite set. □

Theorem 44 is our first example showing how an infinite set is produced in the presence of a Dedekind self-map $j : V \to V$ that has been equipped with an appropriate set of preservation properties. We proceed to a second example, which makes use of the concept of an *ultrafilter*. We begin with a few definitions.

A *filter* F on a set A is a collection of subsets of A that is closed under intersections and supersets, and for which $A \in F$ and $\emptyset \notin F$ (sometimes such an F is called a *proper* filter). If there is $X \subseteq A$ such that $F = \{Y \mid X \subseteq Y\}$, F is a *principal filter with generator* X. If there is $x \in A$ such that $\{x\} \in F$, then F is a *trivial filter*. Every trivial filter is principal.

A filter F is an *ultrafilter* if, for every $X \subseteq A$, either X or $A - X$ belongs to F. If F is an ultrafilter, F is nontrivial if and only if F is nonprincipal. If F is a nonprincipal ultrafilter on A, F contains no finite sets: If $Y \subseteq A$ is finite, $Y = \{y_1, \ldots, y_k\}$, then by closure under intersections, since $\{y_i\} \notin F$ for $1 \leq i \leq k$, $A - \{y_i\} \in F$, and so

$$A - F = \bigcap_{1 \leq i \leq k} (A - \{y_i\}) \in F.$$

It is possible to construct nontrivial filters on a finite set, and also to construct an ultrafilter on a finite set. But it is not possible to construct a nontrivial (nonprincipal) ultrafilter on a finite set, as is shown in the previous paragraph. These observations give us another equivalent form of the Axiom of Infinity:

Ultrafilter Criterion for Infinite Sets. There is an infinite set if and only if there is a nonprincipal ultrafilter.[108].

Using the Axiom of Choice, one can construct a nonprincipal ultrafilter on any infinite set. Ultrafilters (including nonprincipal ultrafilters) also arise from Dedekind self-maps $j : V \to V$, equipped with adequate preservation properties, as we discuss next in Theorem 45.

Theorem 44. *Suppose $j : V \to V$ is a class Dedekind self-map with critical point a and there is a set A such that $a \in j(A)$. Let $D = \{X \subseteq A \mid a \in j(X)\}$.*[109]

(1) *If j preserves subsets, intersections, and the empty set, then D is a filter.*
(2) *If, in addition to the conditions in (1), one of the following conditions holds, then D is a nontrivial filter:*
   (A) *j preserves singletons;*
   (B) *j preserves terminals and $\{a\}$ is a second critical point of j.*
(3) *If, in addition to the conditions in (1), j preserves disjoint unions, then D is an ultrafilter. Moreover, if one of the conditions in (2) also holds, D is a nonprincipal ultrafilter (and A is infinite).*

Motivation for the conditions in part (2)(B) of the theorem, that $\{a\}$ is a second critical point, are given in Remark 10. We note here that this requirement is not trivial: There are Dedekind self-maps $j : V \to V$ with the property that for *no* critical point a of j is it the case that $\{a\}$ is a second critical point. For instance, consider $S : V \to V$ defined by $S(x) = \{x\}$. The map S is a Dedekind self-map with a proper class of critical points. Moreover, for any critical point a of S, $\{a\} \in \operatorname{ran} S$ and is therefore not a critical point of S.

Remark 11. In the absence of (3), properties (1) and (2) are rather weak. We give an example to illustrate this point, but also to motivate a more complex example for which (3) does hold.

Suppose A is a set with two or more elements, and define $j : V \to V$ by $j(X) = X^A = \{f \mid f : A \to X\}$. Let $\mathrm{id}_A : A \to A$ denote the identity map. Then $\mathrm{id}_A \in j(A)$. Clearly id_A and $\{\mathrm{id}_A\}$ are critical points of j. Also, for any sets X, Y, $X \neq Y \Rightarrow X^A \neq Y^A$, and so j is 1-1; indeed, j is a Dedekind self-map. Since $\emptyset^A = \emptyset$, j preserves the empty set. For any set X, $j(\{X\}) = \{X\}^A = \{t\}$, where $t : A \to \{X\}$ is the unique function from A to $\{X\}$; therefore, j preserves terminals.

Finally, we observe that j preserves intersections. It suffices to show that $(X \cap Y)^A = X^A \cap Y^A$. If $f \in (X \cap Y)^A$, it follows easily that $f \in X^A \cap Y^A$. Conversely, suppose $f \in X^A \cap Y^A$. Then $\operatorname{ran} f \subseteq X$ and $\operatorname{ran} f \subseteq Y$. It follows that $\operatorname{ran} f \subseteq X \cap Y$, and so $f \in (X \cap Y)^A$.

We have shown that this particular $j : V \to V$ satisfies properties (1) and (2) of Theorem 45, and so we conclude that D is a nontrivial filter. However, j does not satisfy property (3): Given two nonempty disjoint subsets B and C of A with $B \cup C = A$, while it is true that B^A and C^A are disjoint, it is not the case that

[108] Note that in the definition of an ultrafilter U, it is not necessary to presuppose existence of a base set X; X is uniquely determined by U by $X = \bigcup U$.

[109] The fact that D, defined in this way, is a *set* depends on the fact that there is a formula that defines j; that is, that j is a *class* function.

$B^A \cup C^A = (B \cup C)^A$; indeed, id_A belongs to the set on the right-hand side, but not to the set on the left-hand side.

In fact, D is rather uninteresting: Although it is true that $A \in D$ (since $\mathrm{id}_A \in j(A)$), D contains no other set! If $X \subsetneq A$, then it is not the case that $\mathrm{id}_A \in X^A$.

To make D more interesting, and to ensure that j preserves disjoint unions, it is necessary to "blur" the sharp differences between sets of the form Z^A, and this can be done with an appropriate choice of equivalence relation; we show how this can be done in Example 12. □

Proof of (1). $A \in D$ by the definition of D and $\emptyset \notin D$ since $j(\emptyset) = \emptyset$. D is closed under intersections since j preserves intersections. D is also closed under supersets since j preserves subsets. We have shown that D is a filter.

Proof of (2). We verify D is nontrivial assuming either (2)(A) or (2)(B). For this, it suffices to show that D has no element of the form $\{z\}$. Suppose for some $z \in A$, $\{z\} \in D$.

If (2)(A) holds, then $a \in j(\{z\}) = \{j(z)\}$, since j preserves singletons in this case. It follows $a = j(z)$, which is impossible since a is a critical point of j. If, instead, (2)(B) holds, then $a \in j(\{z\}) = \{y\}$ for some $y \in j(A)$, since j preserves terminals. It follows that $a = y$, and so $\{a\} = j(\{z\})$; in other words, $\{a\} \in \operatorname{ran} j$ contradicting the assumption that $\{a\}$ is a second critical point for j.

We have shown that D is a nontrivial filter.

Proof of (3). Assuming j preserves disjoint unions, we show that D is an ultrafilter: Suppose $X \subseteq A$ and $X \notin D$. Let $Y = A - X$. We show $a \in j(Y)$. Since j preserves disjoint unions, $j(A) = j(X) \cup j(Y)$; since $a \notin j(X)$, then $a \in j(Y)$, as required. If one of the conditions in (2) also holds, then, by the argument establishing (2), D is a nonprincipal ultrafilter. It follows that A itself is infinite. □

Remark 12. We modify the example described in Remark 11 in a way that leads to the conclusion that the filter D derived from j is a nonprincipal ultrafilter. As we mentioned in that remark, we need to modify the map $X \mapsto X^A$ that was used there by introducing an appropriate equivalence relation. A key obstacle that we encountered in that example is that the only element of D is A. This limitation occurs because the only way the function $\mathrm{id}_A : A \to A$ could be a member of X^A, for $X \subseteq A$, is if $X = A$. This requirement can be relaxed if we allow the possibility that id_A "almost" belongs to X^A. This would mean that for some $f : A \to X$, f agrees with id_A "almost everywhere." Certainly f and id_A cannot agree on $A - X$, but we may require that $f \upharpoonright X = \mathrm{id}_X$.

Of course, our term "almost everywhere" needs further clarification; if "almost everywhere" is not inclusive enough, other problems will arise. To see the issue, suppose $b, c \in A$ and $X = \{b\}$ and $Y = \{c\}$. Let $f \in X^A$ and $g \in Y^A$. (Since each of X and Y is terminal, there is only one choice for f and one choice for g.) Certainly $f(b) = b$ and $g(c) = c$, so we have $f \upharpoonright X = \mathrm{id}_X$ and $g \upharpoonright Y = \mathrm{id}_Y$. If we now conclude that $X \in D$ and $Y \in D$, then D is no longer a (proper) filter since it contains disjoint sets.

This example suggests the need for two requirements for a set X to belong to D:

(1) id_A "almost" belongs to X^A, implemented by requiring that, for some $f \in X^A$, $f \upharpoonright X = \mathrm{id}_X$;
(2) X is a "large" subset of A.

A natural way to fulfill requirement (2) is to require X to belong to some fixed filter U over A, since, intuitively speaking, a filter over A contains the "large" subsets of A.

We describe now an improved version of the example in Remark 11. We first define the concept of a *partial function* on A: A *partial function* on A is a function whose domain is a (possibly proper) subset of A. A partial function that is defined on all of A is sometimes called *total*. Our plan is to re-define classes of the form X^A so that they consist of *partial* functions $f : A \to X$ having domain that belongs to the fixed filter U; and we will say that such an f is "equivalent to" id_A if f agrees with id_A on a set B in U. Then since $\mathrm{id}_A \upharpoonright B = f \upharpoonright B$, it will follow that $B \subseteq X$, and so X itself belongs to U. Since, under this new definition, we have $\mathrm{id}_A \in j(X)$, we will conclude that $X \in D$. We turn to the details.

We begin with a fixed (proper) filter U on A. We do not require U to be a nonprincipal ultrafilter, so it is perfectly possible for A to be finite at this stage. If $f : A \to Z$ is a partial function, we call f *U-good* if $\{x \in A \mid f(x) \text{ is defined}\} \in U$. For this example only, we re-define the sets of the form Z^A in the following way:

$$Z^A = \{f \mid f : A \to Z \text{ is a } U\text{-good partial function}\}.$$

For $f, g : A \to Z$ that are U-good, we define $\sim$ (more formally, $\sim_{A,U,Z}$) by

$$f \sim g \text{ iff } \{x \in A \mid f(x) = g(x)\} \in U.$$

We note that, for any $f \in Z^A$ for which $|Z| \geq 1$, there is a total function $f' : A \to Z$ such that $f \sim f'$: Let $z \in Z$. Define f' by

$$f'(x) = \begin{cases} f(x) \text{ if } x \in \mathrm{dom}\ f, \\ z \text{ otherwise.} \end{cases}$$

Clearly $f' \sim f$.

Denote the equivalence class that contains f by $[f]_U$, or simply $[f]$. For each set Z, we define

$$Z^A/U = \{[f]_U \mid f : A \to Z \text{ is } U\text{-good}\}.$$

Finally, define $j : V \to V$ by $j(Z) = Z^A/U$. We first show, in Claims (1)–(5) below, that j has property (1), mentioned in Theorem 45:

Claim 1. $j(\emptyset) = \emptyset$.

Proof. This follows because $\emptyset^A = \emptyset$. □

Claim 2. j is 1-1.

Proof. Suppose $Y \neq Z$ are sets; without loss of generality, let $y \in Y - Z$. Consider the constant function $f^y : A \to Y$ defined by $f^y(x) = y$. Clearly f^y agrees nowhere with any partial function $A \to Z$. Therefore $f^y \in Y^\omega - Z^\omega$, and so $j(Y) \neq j(Z)$. □

Claim 3. j preserves terminal objects.

Proof. We show j takes singleton sets to singleton sets. Given a singleton set $\{z\}$, let f_z be the unique function $A \to \{z\}$. Notice that if $g : A \to \{z\}$ is U-good, then $g \sim f_z$. Therefore, we have

$$\begin{aligned} j(\{z\}) &= \{z\}^A/U \\ &= \{[g] \mid g \text{ is a } U\text{-good partial function } A \to \{z\}\} \\ &= \{[f_z]\}. \square \end{aligned}$$

Claim 4. $[\mathrm{id}_A]$ is a critical point of j. Therefore, j is a Dedekind self-map. Moreover, $[\mathrm{id}_A] \in j(A)$.

Proof. Certainly, $[\mathrm{id}_A]$ is not itself of the form Z^A/U, so $[\mathrm{id}_A]$ is a critical point. Since $\mathrm{id}_A \in A^A$, certainly $[\mathrm{id}_A] \in A^A/U = j(A)$. $\square$

Claim 5. j preserves intersections.

Proof. Suppose X, Y are sets and $Z = X \cap Y$. Suppose $[f] \in X^A/U \cap Y^A/U$. Let $B, C \in U$ be such that for all $x \in B$, $f(x) \in X$ and for all $x \in C$, $f(x) \in Y$. Since U is closed under instersections, $B \cap C \in U$, and we have that for all $x \in B \cap C$, $f(x) \in X \cap Y$, so $[f] \in (X \cap Y)^A/U$. For the converse, if $E \in U$ is such that for all $x \in E$, $f(x) \in X \cap Y$, then it follows easily, using the fact that U is closed under supersets, that $[f] \in X^A/U$ and $[f] \in Y^A/U$. $\square$

We have shown property (1) of Theorem 45 holds for j. Therefore, if $D = \{X \subseteq A \mid [\mathrm{id}_A] \in j(X)\}$, then D is a filter. In the present context, this is not surprising in light of the next claim:

Claim 6. $D = U$.

Proof. Suppose $X \subseteq A$. We show $X \in D$ if and only if $X \in U$. For one direction, we have:

$$\begin{aligned} X \in D &\Rightarrow [\mathrm{id}_A] \in j(X) = X^A \\ &\Rightarrow \exists f \in X^A\, (f \sim \mathrm{id}_A) \\ &\Rightarrow \exists B \in U\, (f \restriction B = \mathrm{id}_B \text{ and } B \subseteq X). \end{aligned}$$

Since $B \in U$ and $B \subseteq X$, it follows that $X \in U$. For the other direction, suppose $X \in U$. Define a U-good partial function $f \in X^A$ in the following way: Let $\mathrm{dom}\ f = X$ and define f on elements by $f(x) = x$ for all $x \in X$. In other words, $f = \mathrm{id}_X$. Clearly, $\mathrm{id}_A \sim f$ and so $[\mathrm{id}_A] \in X^A = j(X)$. It follows that $X \in D$. $\square$

We next show that j preserves disjoint unions only if our starting filter U was already an ultrafilter:

Claim 7. The following are equivalent:

(A) j preserves disjoint unions.

(B) U is an ultrafilter (equivalently, D is an ultrafilter).

Proof. First, observe that $(A) \Rightarrow (B)$ follows from Theorem 45, part (3), using the fact that $D = U$. For the converse, suppose X, Y are disjoint and let $Z = X \cup Y$; we show $j(X)$ and $j(Y)$ are disjoint, and that $j(X) \cup j(Y) = j(X \cup Y)$.

Disjointness of $j(X)$ and $j(Y)$ follows from the easy calculation $X^A \cap Y^A = \emptyset$. It is obvious that $j(X) \cup j(Y) \subseteq j(Z)$. To prove $j(Z) \subseteq j(X) \cup j(Y)$, let $[f] \in Z^A/D$. Since f is U-good, $S \in U$, where $S = \{x \in A \mid f(x) \in Z\}$. Let $S_X = \{x \in A \mid f(x) \in X\}$ and $S_Y = \{x \in A \mid f(x) \in Y\}$. Since $S = S_X \cup S_Y \in U$ and U is an ultrafilter, one of S_X, S_Y belongs to U, say S_X. Then $[f] = [f \restriction S_X] \in X^A/U$. We have shown that each $[f]$ in Z^A/U belongs to $\big(X^A/U\big) \cup \big(Y^A/U\big)$. $\square$

Assuming from the beginning that U is an ultrafilter, we have established that j preserves disjoint unions and that D is therefore, by Theorem 45, also an ultrafilter. None of the arguments so far require A to be infinite or D to be nonprincipal. In the present setting, the only way D could turn out to be a nonprincipal ultrafilter, given the assumptions we have made so far on U, is if $\{[\mathrm{id}_A]\}$ is a critical point of j, which could happen only if U itself was initially assumed to be nonprincipal. Unfortunately, therefore, tracing through this example does not reveal the mechanics by which a nonprincipal ultrafilter comes into existence; instead, it shows that existence of a nonprincipal ultrafilter is equivalent to existence of a Dedekind self-map having properties (1), (2)(B), and (3) in Theorem 45. The next claim establishes the remaining details.

Claim 8. Assume U is an ultrafilter. Then the following are equivalent:

(A) $\{[\mathrm{id}_A]\}$ is a critical point of j.

(B) U (equivalently, D) is a nonprincipal ultrafilter on A, whence A is infinite.

Proof. The fact that $(A) \Rightarrow (B)$ follows from Theorem 45(2)(B), using the fact that $D = U$. For the converse, assume U is nonprincipal but $\{[\mathrm{id}_A]\}$ is in the range of j, so that $\{[\mathrm{id}_A]\} = Z^A/U$, for some set Z; we will arrive at a contradiction.

First, we show that Z itself must be a singleton set: If Z has at least two distinct elements y, z, the constant functions $f^y : A \to Z : x \mapsto y$ and $f^z : A \to Z : x \mapsto z$ agree nowhere, and so $[f^y] \neq [f^z]$; hence, $|j(Z)| = |Z^A/U| > 1$, contradicting our assumption that $\{[\mathrm{id}_A]\} = Z^A/U$. Therefore, $Z = \{z\}$ for some z. Let f be the unique function from A to $\{z\}$. Then $Z^A/U = \{[f]\} = \{[\mathrm{id}_A]\}$; in other words, $f \sim \mathrm{id}_A$. It follows that $\{x \in A \mid f(x) = \mathrm{id}_A(x)\} \in U$; that is, $\{z\} = \{x \in A \mid z = x\} \in U$. Since U is nonprincipal, this is impossible. We have shown therefore that $\{[\mathrm{id}_A]\}$ is a critical point of j. $\square$

Claims 7 and 8 could have been presented and proven in reverse order, with slight modifications; the only change in the proofs is that "nonprincipal" must be replaced with "nontrivial" in the new version of Claim 8 (which we will now call Claim 7′). We give the restatements here:

Claim 7′. The following are equivalent for j, U as originally defined in this example:

(A) $\{[\mathrm{id}_A]\}$ is a critical point of j.

(B) U (equivalently, D) is a nontrivial filter on A.

Claim 8′. Assume U is a nontrivial filter on A. Then the following are equivalent:

(A) j preserves disjoint unions.
(B) U (equivalently, D) is a nonprincipal ultrafilter, whence A is infinite. □

The condition in Theorem 45 that the critical point a must belong to a set of the form $j(A)$ for some set A is necessary in order to be able to conclude that an infinite set exists; indeed, Dedekind self-maps can be built in the theory ZFC − Infinity that satisfy parts (1), (2)(B), and (3) of the theorem; for such Dedekind self-maps, the theorem tells us that for *no* critical point a of j (for which $\{a\}$ is a second critical point) is it possible to find a set A for which $a \in j(A)$. We give such an example next.

Example 2. (The Role of Cofinality in the Emergence of Infinite Sets) This example shows it is possible for parts (1), (2)(B), and (3) of Theorem 45 to hold and yet for no critical point a of j for which $\{a\}$ is also a critical point is it the case that $a \in j(A)$ for any A; moreover, the example can be built in a universe in which no infinite set exists. Define $j : V \to V$ by

$$j(X) = \overline{s}[X] = \{\overline{s}(x) \mid x \in X\},$$

where $\overline{s}$ is the global successor function. Since $\overline{s}$ is 1-1, so is j. It is straightforward to verify that j preserves disjoint unions, intersections, and the empty set, and that both $\{\{1\}\}$ and $\{\{\{1\}\}\}$ are critical points of j (note though that $\{1\} \in \operatorname{ran} j$ since $j(1) = \{1\}$). We verify that j preserves terminal objects: We compute $j(\{x\})$ for any x:

$$j(\{x\}) = \overline{s}[\{x\}] = \{\overline{s}(y) \mid y \in \{x\}\} = \{\overline{s}(x)\},$$

which is also a singleton.

Finally, we show that if a is any critical point of j for which $\{a\}$ is also a critical point, there is no set A for which $a \in j(A)$: Suppose for a contradiction that there are a, A so that both a and $\{a\}$ are critical points of j and $a \in j(A) = \overline{s}[A]$. It follows from $a \in j(A)$ that

$$a \in \operatorname{ran} \overline{s}. \tag{54}$$

We complete the proof by showing that $\{a\}$ is a critical point of j if and only if $a \notin \operatorname{ran} \overline{s}$, contradicting (54). This final part of the proof follows from the following chain of equivalences:

$$\begin{aligned} \{a\} \in \operatorname{ran} f \quad &\Leftrightarrow \quad \exists X\,(\{a\} = \overline{s}[X]) \\ &\Leftrightarrow \quad \exists x, X\,(x \in X \;\wedge\; \{a\} = \{\overline{s}(x)\}) \\ &\Leftrightarrow \quad \exists x, X\,(x \in X \;\wedge\; a = \overline{s}(x)) \\ &\Leftrightarrow \quad a \in \operatorname{ran} \overline{s}. \end{aligned}$$

Notice that the Dedekind self-map $j : V \to V$ defined in this example can be defined in the theory ZFC − Infinity; its properties—namely, (1), (2)(B), and (3) of Theorem 45—are not strong enough to imply the existence of an infinite set. □

We list several sufficient conditions here for the critical point a to belong to a set of the form $j(A)$, but most of these involve concepts that will be introduced later.

(a) j is cofinal (p. 105);
(b) j strongly preserves $\in$, preserves rank, and both preserves and reflects ordinals (definitions on p. 120) (in that case, for at least one critical point a of j, $a \in j(a)$);
(c) a is a *universal element* for j (defined on p. 130).

A proof of (b) can be found on p. 123. The example given in Remark 12 is an example of (c), as we will see later (p. 130).

Combining Theorem 45 with condition (a) leads to a nice sufficient condition for the existence of a nonprincipal ultrafilter:

Proposition 45. (ZFC − Infinity) *Suppose $j : V \to V$ is a cofinal class Dedekind self-map with critical points a and $\{a\}$. Suppose also that j preserves disjoint unions, intersections, terminal objects, and the empty set. Then there is a nonprincipal ultrafilter over some (infinite) set A.*[110]

Proof. By cofinality of j, there is a set A such that $a \in j(A)$. Define D by $D = \{X \subseteq A \mid a \in j(X)\}$. Then by the proof of Theorem 45, D is a nonprincipal ultrafilter on A. □

We turn now to a third set of preservation properties that a Dedekind self-map may exhibit and which imply existence of an infinite set.

Theorem 46. (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 self-map with a strong critical point. Suppose j preserves finite coproducts and terminal objects. Then there is an infinite set.*

Proof. We first show that

(55) for every finite set X, $|j(X)| = |X|$.

First, suppose $X \neq \emptyset$. Write $X = \{x_1, \ldots, x_n\}$. Because j is 1-1 and preserves finite coproducts and terminal objects,

$$|j(X)| = |j(\{x_1\}) \cup \cdots \cup j(\{x_n\})| = |\{y_1, \ldots, y_n\}| = |X|,$$

where, for each i, $\{y_i\} = j(\{x_i\})$.

We show that $j(\emptyset) = \emptyset$, so, in particular, $|j(\emptyset)| = |\emptyset|$. Suppose not. Then $|j(\emptyset)| \geq 1$ and so, since terminal objects are preserved, for some set y,

$$|j(\emptyset) \cup j(\{\emptyset\})| = |j(\emptyset)| + |j(\{\emptyset\})| = |j(\emptyset)| + |\{y\}| \geq 2.$$

But this contradicts the fact that coproducts are preserved, since, for this same set y,

$$|j(\emptyset \cup \{\emptyset\})| = |j(\{\emptyset\})| = |\{y\}| = 1.$$

Finally, we show there is an infinite set: Let Z be a strong critical point of j, so that $|j(Z)| \neq |Z|$. But now statement (55) implies that Z is not finite. Therefore,

[110] These conditions on a Dedekind self-map $j : V \to V$ are satisfied when j is a WA_0-embedding: Suppose $\mathrm{crit}(j) = \kappa$ and $\kappa \in j(A)$. Let $D = \{X \subseteq A \mid \kappa \in j(X)\}$. Then, D is not only a nonprincipal ultrafilter but also has the property of being *κ-complete*, and this fact implies that κ is a measurable cardinal. (One subtlety here is that in this case, j is *not* definable by a formula—not a class function. As will be explained later, j has strong enough properties to ensure that D is a *set* and to guarantee the truth of Theorem 45.) Measurable cardinals are defined in Definition 19; WA_0 embeddings are discussed in Section 25.

Z is infinite. □

Remark 13. The proofs of Theorems 44, 45, and 47 show that any Dedekind self-map on V that exhibits the specified preservation properties produces an infinite set, and, in the case of Theorem 45, a nonprincipal ultrafilter. Self-maps with such properties cannot be proven to exist in the theory ZFC − Infinity by Gödel Incompleteness.[111] However, for the most part, such examples can be found if we work in ZFC. As an example of a $j : V \to V$ that exhibits the properties specified in the hypotheses of Theorems 45 (using property (2)(B)) and 47, we summarize (with a more concrete example) the points made in Remark 12. We follow this with a simpler example that satisfies the properties of Theorem 47 only.

We do not have a ZFC example of a $j : V \to V$ with precisely the properties of Theorem 44 or of Theorem 45 (using property (2)(A)); in particular, the examples that we know do not preserve singletons (only terminal objects). Nevertheless, it can be shown [17] that it is consistent with ZFC for there to be a $j : V \to V$ satisfying the hypotheses of Theorem 44. The model given in this reference still does not satisfy the hypotheses of Theorem 45 (using property (2)(A)). Assuming modest large cardinals, however, one may build an example in which these hypotheses are satisfied; such an example at least provides reasonable evidence that the combination of preservation properties mentioned in Theorem 45 (using (2)(A)) are consistent. □

Example 3. (Reduced Product Construction) We revisit the example given in Remark 12, specializing to the case in which $A = \omega$. This example will provide us with a class Dedekind self-map with the properties listed in Theorem 45(2) and Theorem 47. It will also set the stage for a generalization in which $A = \kappa$, where κ is some uncountable cardinal.[112]

We begin by fixing a nonprincipal ultrafilter D on ω. We recall several definitions from Remark 12. Given any set X, if f, g are both D-good partial functions from ω to X, we declare f, g are equivalent, and write $f \sim g$, if the set of $n \in \omega$ at which f, g are both defined and equal belongs to D. Let $[f]$ denote the $\sim$-equivalence class containing f and let

$$X^\omega/D = \{[f] \mid f \text{ is a } D\text{-good partial function from } \omega \text{ to } X\}.$$

Define $j_D : V \to V$ by

$$j_D(X) = X^\omega/D.$$

We note as before that, for any $f \in X^\omega$ for which $|X| \geq 1$, there is a total function $f' : \omega \to X$ such that $f \sim f'$: Let $x_0 \in X$. Define f' by

$$f'(n) = \begin{cases} f(n) \text{ if } n \in \text{dom } f, \\ x_0 \text{ otherwise.} \end{cases}$$

Clearly $f' \sim f$.

[111] The statement of the Incompleteness Theorem is given below in Theorem 22.

[112] This generalization is given in Example 5, starting on page 160.

The following claims are proved in Remark 12 (setting $A = \omega$).

Claim 1. $j_D(\emptyset) = \emptyset$.

Claim 2. j_D is 1-1.

Claim 3. j_D preserves terminal objects.

Claim 4. Both $[\mathrm{id}_\omega]$ and $\{[\mathrm{id}_\omega]\}$ are critical points of j_D. Also,

$$D = \{X \subseteq A \mid [\mathrm{id}_\omega] \in j_D(X)\}.$$

Claim 5. j_D preserves disjoint unions. Consequently, j_D preserves all finite coproducts.

Claim 6. j_D preserves intersections.

The following claim shows that our example also illustrates Theorem 47:

Claim 7. ω is a strong critical point of j_D.

Proof. Define, for each $n \in \omega$, the function $f_n : \omega \to \omega$ by $f_n(i) = n + i$. Then, whenever $m \neq n$, f_m, f_n disagree everywhere. Therefore $\{[f_n] \mid n \in \omega\}$ is an infinite subset of $j_D(\omega) = \omega^\omega/D$. We show that $j_D(\omega)$ is in fact uncountable. Suppose $\{[g_n] \mid n \in \omega\}$ is an infinite subset of ω^ω/D. Define $h : \omega \to \omega$ by

$$h(n) = \text{least element of } \omega - \{g_i(n) \mid i < n\}.$$

Then for all $i \in \omega$ and all $n > i$, $h(n) \neq g_i(n)$; in particular $[h] \neq [g_i]$ for all $i \in \omega$. We have shown $j_D(\omega)$ is uncountable, so $\omega < |j_D(\omega)|$. Therefore, ω is a strong critical point. □

Example 4. (Simple Model of Theorem 47) Assuming that ω does exist, we construct a Dedekind self-map $j : V \to V$ having the properties mentioned in the hypothesis of Theorem 47—namely, that j has a strong critical point and preserves terminal objects and finite coproducts.

$$j(x) = \begin{cases} x & \text{if } x \text{ is finite,} \\ \mathcal{P}(x) & \text{if } x \text{ is infinite.} \end{cases}$$

The first clause of the definition ensures that j preserves terminal objects, since such objects must always be finite. Since the powerset operator is 1-1, so is j. Notice ω is a strong critical point by Cantor's Theorem; it is easy to see that ω is a critical point as well. We show that j preserves coproducts. Suppose X and Y are disjoint sets. The cases in which one of X and Y is finite are straightforward;

assume both are infinite. We have:

$$\begin{aligned} |j(X \cup Y)| &= |\mathcal{P}(X \cup Y)| \\ &= 2^{\max\{|X|,|Y|\}} \\ &= \max\{2^{|X|}, 2^{|Y|}\} \\ &= |\mathcal{P}(X) \cup \mathcal{P}(Y)| \\ &= |j(X) \cup j(Y)|. \end{aligned}$$

The proof that finite coproducts are preserved is a straightforward induction. □

Remark 14. We review the logic of Theorems 44, 45, and 47, along with Examples 3 and 4. Theorems 44, 45, and 47 guarantee the existence of an infinite set from the existence of a class Dedekind self-map with sufficiently strong preservation properties. Theorem 45 guarantees, in addition, existence of a nonprincipal ultrafilter, also derivable from a class Dedekind self-map with sufficiently strong preservation properties.

Conversely, Example 4 shows that, assuming the existence of an infinite set, one may recover a Dedekind self-map with the properties listed in Theorem 47. And Example 3 shows how, assuming existence of a nonprincipal ultrafilter on an infinite set, one can define a Dedekind self-map with the properties listed in Theorem 45. These observations result in a characterization of the Axiom of Infinity in terms of Dedekind self-maps $j : V \to V$, as in the next theorem. □

Theorem 47. (ZFC − Infinity) *The following statements are equivalent.*

(1) *There is an infinite set.*
(2) *There is a class Dedekind self-map $j : V \to V$ with a strong critical point that preserves finite coproducts and terminal objects.*
(3) *There is a class Dedekind self-map $j : V \to V$ with critical point a such that*
  (i) *the map j preserves disjoint unions, intersections, terminals, and the empty set;*
  (ii) *there is a set A such that $a \in j(A)$;*
  (iii) *the set $\{a\}$ is a second critical point of j.*[113]

[113]One issue that should be addressed here is that (2) and (3) of Theorem 48 appear to be asserting the existence of proper class functions; but assertions of this kind are not allowed in ZFC since ZFC talks just about *sets*, not about proper classes. This sort of situation arises often in set theory; we handle it in the usual way. The approach we describe here is applicable to the other statements in this paper in which existence of a class Dedekind self-map is asserted. To state Theorem 48 in a formally correct way, we re-state it as a *schema* of theorems, one for each formula of set theory. In other words, for each formula $\phi(x, y, z_1, \ldots, z_k)$, whenever ϕ is a functional formula, the following is a theorem.

Theorem 48$_\phi$. *For all sets $Z_1, \ldots, Z_k$, if $j : V \to V$ is the class function defined by $\phi(x, y, Z_1, \ldots, Z_k)$, then the following statements are equivalent.*

(1) *There is an infinite set.*
(2) *There is a class Dedekind self-map $j : V \to V$ with a strong critical point that preserves finite coproducts and terminal objects.*
(3) *There is a Dedekind self-map $j : V \to V$ with critical point a such that*
  (i) *the map j preserves disjoint unions, intersections, terminals, and the empty set;*
  (ii) *there is a set A such that $a \in j(A)$.*
  (iii) *the set $\{a\}$ is a second critical point of j.*

Proof. The implication (2) $\Rightarrow$ (1) is established by Theorem 47, and the implication (1) $\Rightarrow$ (2) follows from Example 4. The implication (3) $\Rightarrow$ (1) is established by Theorem 45. The implication (1) $\Rightarrow$ (3) is proved in the following way: Given an infinite set A, obtain in the usual way a nonprincipal ultrafilter D on A, and define $j : V \to V$ by $j(X) = X^A/D$. Example 3 shows that j satisfies the properties listed in (3). $\square$

Remark 15. Theorem 48 confirms our original hypothesis, that a $j : V \to V$ equipped with modest preservation properties should produce an infinite set. Certainly, parts (1) and (2) of Theorem 48 demonstrate the truth of this hypothesis. However, part (3) adds a new kind of hypothesis, that the critical point of j must belong to some set of the form $j(A)$. This requirement is certainly not a preservation property. Nevertheless, we may draw upon our original intuition, which led to the quest for preservation properties in the first place, to motivate this hypothesis in (3).

Recall that a $j : V \to V$ is, in our approach, a realization of the idea of the dynamics of an underlying field which produce many things—perhaps everything—by analogy with the notion from the wisdom of sages of the past that all things arise from the internal dynamics of *Tao* or *One* or *pure consciousness*. In the ancient view, those dynamics involve the maximum possible preservation—*Tao*, pure consciousness, is untouched and unchanged by its own dynamics. We therefore seek Dedekind self-maps $j : V \to V$ which exhibit preservation properties. Likewise, as we have discussed, the dynamics of the source are considered to be all-encompassing; nothing lies outside of those dynamics. For this reason—as discussed briefly in Remark 10—we likewise seek Dedekind self-maps $j : V \to V$ that are "as cofinal as possible," since we expect that as many sets as possible belong to the world of j, to $\bigcup \operatorname{ran} j$. At the very least, the starting point of the dynamics of j, represented by its critical point, should belong to this world. For this reason, we consider this requirement, expressed in part (3) of Theorem 48, to be just as naturally motivated as the preservation properties that are listed there.

18. The Relationship Between the Different Notions of Critical Point

In this section, we clarify the relationship between critical points and strong critical points. We introduce one other related concept: Suppose $j : V \to V$ is a 1-1 class function. A set X in the domain of j is a *weak critical point* if $j(X) \neq X$. In general, the relationships between these three notions of critical point are summarized in the following proposition:

Proposition 48. (ZFC − Infinity)

(1) *Suppose $j : V \to V$ is a 1-1 function. If X is a strong critical point or a critical point, then X is also a weak critical point.*

(2) *There is a 1-1 class map $j : V \to V$ having a weak critical point that has no strong critical point.*

(3) *There is a 1-1 class map $j : V \to V$ having a weak critical point but no critical point.*
(4) *There is a 1-1 class map $j : V \to V$ that has a strong critical point but no critical point.*
(5) *There is a 1-1 class map $j : V \to V$ that has a critical point but not a strong critical point.*

Proof of (1). If X is a strong critical point, certainly $j(X) \neq X$, so X is also a weak critical point. If X is a critical point, then since X is not in the range of j, $j(X) \neq X$, so X is a weak critical point.

Proof of (2). Obtain $j : V \to V$ as follows.

$$j(x) = \begin{cases} x & \text{if } x \neq 1 \text{ and } x \neq \{1\}, \\ \{1\} & \text{if } x = 1, \\ 1 & \text{if } x = \{1\}. \end{cases}$$

Here, $j(1) \neq 1$, so 1 is a weak critical point. However, $|j(x)| = |x|$ for all $x \in V$, so j has no strong critical point.

Proof of (3) and (4). Obtain $j : V \to V$ as follows.

$$j(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } x \neq 1, \\ 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0. \end{cases}$$

Now $j(0) \neq 0$ so 0 is a weak critical point. However, $\operatorname{ran} j = \operatorname{dom}\ j$, so j has no critical point. Also, notice that $|j(0)| \neq 0$, so 0 is a strong critical point.

Proof of (5). Define $j : V \to V$ by

$$j(x) = \begin{cases} \{n+1\} & \text{if } x = \{n\} \text{ for some } n \in \overline{\omega}, \\ x & \text{otherwise.} \end{cases}$$

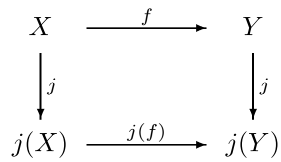
Certainly j is 1-1 and has critical point $\{0\}$. However, for each $n \in \overline{\omega}$, $|\{n+1\}| = |j(\{n\})| = |\{n\}| = 1$, so j has no strong critical point. □

Despite these differences, when j satisfies certain additional preservation properties, these three notions of critical point coincide. Moreover, we will be especially interested in maps that satisfy these properties; we give the definitions below.

Definition 11. Suppose $j : V \to V$ is a 1-1 class function.

(1) *j preserves $\in$* if, whenever $x, y \in \operatorname{dom}\ j$ and $x \in y$, then $j(x) \in j(y)$; *j strongly preserves $\in$* if j preserves $\in$ and for all $x, y \in \operatorname{dom}\ j$, if $j(x) \in j(y)$, then $x \in y$.
(2) *j preserves cardinals* if, whenever $\gamma \in \mathrm{ON}$ is a cardinal, $j(\gamma)$ is also a cardinal.
(3) *j preserves functions* if, for any $f : X \to Y$ and $X, Y \in \operatorname{dom}\ j$, $j(f)$ is a function from $j(X)$ to $j(Y)$ and, for all $x \in X$, $j(f(x)) = j(f)(j(x))$. This

property of j can be represented by a commutative diagram:



(4) *j preserves images* if, whenever $f : X \to Y$ is a function, $j(f)$ is a function and $j(\operatorname{ran} f) = \operatorname{ran} j(f)$.
(5) *j preserves ordinals* if, whenever α is an ordinal, $j(\alpha)$ is also an ordinal.
(6) *j reflects ordinals* if, whenever $j(x)$ is an ordinal, x is also an ordinal.[114]
(7) *j preserves rank* if, for every set x, $j(\operatorname{rank}(x)) = \operatorname{rank}(j(x))$.

Any 1-1 class function $j : V \to V$ that satisfies both parts of (1), as well as (2)–(4), will be called *basic structure-preserving*, or *BSP*.

Theorem 49. (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 class function.*

(1) *If j preserves $\in$, then, for all ordinals α, β, if $j(\alpha) \in j(\beta)$, then $\alpha \in \beta$.*
(2) *If j preserves ordinals and $\in$, then, for all ordinals α, $j(\alpha) \geq \alpha$.*
(3) *Suppose j preserves ordinals and strongly preserves $\in$. Suppose $x \in V$ has the following two properties:*
  (a) *x and $j(x)$ have the same rank*
  (b) *for all y for which $\operatorname{rank}(y) < \operatorname{rank}(x)$, we have $j(y) = y$.*
  Then $j(x) = x$.
(4) *If j preserves ordinals and rank and strongly preserves $\in$, and has a weak critical point, then there is an ordinal α such that $j(\alpha) \neq \alpha$. In particular, $j \restriction ON : ON \to ON$ has a weak critical point.*
(5) *Suppose j preserves ordinals and rank, is BSP, and has a weak critical point. Then j has a weak critical point that is a cardinal.*

Proof of (1). Suppose $j(\alpha) \in j(\beta)$. If $\alpha \notin \beta$, then either $\alpha = \beta$ or $\beta \in \alpha$ (since $\in$ is a total ordering on ON). If $\alpha = \beta$, then we have $j(\beta) = j(\alpha) \in j(\beta)$ which is impossible by irreflexivity of $\in$. If $\beta \in \alpha$, then $j(\beta) \in j(\alpha) \in j(\beta)$, and this contradicts the fact that $\in$ is both irreflexive and transitive. The result follows.

Proof of (2). Suppose not; let α be least such that $j(\alpha) < \alpha$. Since j preserves ordinals and $\in$, $j(j(\alpha)) < j(\alpha)$. But now $j(\alpha)$ is a smaller ordinal β with the property that $j(\beta) < \beta$, contradicting leastness of α.

Proof of (3). We first show $x \subseteq j(x)$: Let $y \in x$. Since j preserves $\in$, $j(y) \in j(x)$. Since $\operatorname{rank}(y) < \operatorname{rank}(x)$, $y = j(y)$. It follows $y \in j(x)$. Conversely, if $y \in j(x)$, then y is of lower rank, so $y = j(y)$. Since $j(y) = y \in j(x)$ and j strongly preserves $\in$, it

[114]We observe here that reflecting ordinals is also a *preservation property*. This follows from the fact that j reflects ordinals if and only if j preserves *non*-ordinals.

follows that $y \in x$, as required.

Proof of (4). Suppose $j(\alpha) = \alpha$ for all ordinals α. Let x be a weak critical point for j; that is, $j(x) \neq x$. Let $\alpha = \text{rank}(x) + 1$, and let $X = V_\alpha$. Let $M = \{x \in X \mid j(x) \neq x\}$. The fact that M is a set follows from an application of Separation. Let $B = \{\text{rank}(x) \mid x \in M\}$. B is a set by Replacement. Also, $B \neq \emptyset$ since $M \neq \emptyset$. Let $\gamma = \inf B$ and let $y \in M$ be such that $\text{rank}(y) = \gamma$. Since $y \in M$, $j(y) \neq y$, but, using (3), we can also show that $j(y) = y$, yielding the needed contradiction. To apply (3), and complete the proof of (4), it suffices to establish condition (3)(a); note that (3)(b) already holds by the leastness of rank y. But (3)(a) holds because j preserves rank and because of the assumption that j is the identity on ON:

$$\text{rank}(j(y)) = j(\text{rank}(y)) = \text{rank}(y).$$

We have shown, therefore, that $j \restriction \text{ON} : \text{ON} \to \text{ON}$ has a weak critical point.

Proof of (5). Let κ be the least ordinal moved by j (which must exist by (4)). By (2), $j(\kappa) > \kappa$. Suppose $\alpha < \kappa$ and $f : \alpha \to \kappa$ is an onto function. By leastness of κ, $j(\alpha) = \alpha$. Since j preserves functions, $j(f) : j(\alpha) \to j(\kappa)$ is also a function, and since j preserves images,

$$\text{ran}\, j(f) = j(\text{ran}\, f) = j(\kappa). \qquad (56)$$

Since $j(\alpha) = \alpha$, $j(f) : \alpha \to j(\kappa)$. We show $f = j(f)$: For any $\beta \in \alpha$, because j preserves functions and $j(\beta) = \beta$, we have

$$j(f)(\beta) = j(f)(j(\beta)) = j(f(\beta)) = f(\beta).$$

The last step follows because $f(\beta) \in \kappa$ and κ is the least ordinal moved by j. Therefore $j(f) = f$ and so $\text{ran}\, j(f) \subseteq \kappa$, which contradicts (56). Therefore, no such onto function exists, and κ is a cardinal. □

Corollary 50. (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 class function having a weak critical point. If j strongly preserves $\in$ and preserves rank and ordinals, then there is a least ordinal α moved by j; moreover, $\alpha < j(\alpha)$.*

Proof. By Theorem 50(4), there is an ordinal β with $j(\beta) \neq \beta$; it follows that there is a least ordinal α with this property.[115] By Theorem 50(2), it follows that $\alpha < j(\alpha)$, as required. □

Notation. Whenever we are working with a self-map $j : V \to V$ that has a weak critical point and that strongly preserves $\in$ and preserves rank and ordinals, we will let $\text{crit}(j)$ denote[116] the least ordinal moved by j. The corollary tells us that if j has these properties and $\alpha = \text{crit}(j)$, then $\alpha < j(\alpha)$.

With these preservation properties in mind, we can return to the task of verifying a point mentioned earlier, regarding Theorem 45. In that theorem, one of the

[115] The existence of such an ordinal α requires that j be a *class* function—that is, j must be definable by a formula. The reason is that, in the argument α is defined to be $\min Y$ where $Y = \{\beta \mid \beta \neq j(\beta)\}$; however, Y is not guaranteed to be a set unless j is definable.

[116] When j does not satisfy all these properties, we will continue to use the notation $\text{crit}(j)$ to denote *any* critical point of j, as has been done up to this point in the text.

hypotheses concerning $j : V \to V$ was that one of its critical points a belongs to a set of the form $j(A)$. We described earlier (p. 114) several sufficient conditions for this hypothesis to hold true. One of the conditions described there is the following:

j strongly preserves $\in$, preserves rank, and also preserves and reflects ordinals.

We explain why this condition suffices: Theorem 49 shows that if $j : V \to V$ is a Dedekind self-map, j has a weak critical point; moreover, since we are assuming j strongly preserves $\in$ and j preserves ordinals and rank, there is, by Corollary 51, a least ordinal α_0 such that $\alpha_0 < j(\alpha_0)$. This ordinal α_0 must be a critical point of j because (a) for all ordinals α, $\alpha \leq j(\alpha)$ (Theorem 50(2)), and (b) by the ordinal reflecting property, no non-ordinal is mapped to α_0. It follows therefore that α_0 itself is the required set A; that is, $\alpha_0 \in j(\alpha_0)$. As we consider Dedekind self-maps that give rise to the bigger large cardinals, it will be typical for j to have such a critical point.

We show now that when a self-map $j : V \to V$ satisfies a modest subset of the preservation properties described so far, the three notions of critical point coincide.

Theorem 51. (ZFC – Infinity) *Suppose $j : V \to V$ is a 1-1 class function that preserves and reflects ordinals, preserves rank, and also is BSP. Then the following are equivalent.*

(1) *j has a weak critical point.*
(2) *j has a critical point.*
(3) *j has a strong critical point.*

Proof. We have already seen that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). It suffices to prove (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

Assume j has a weak critical point. Using Corollary 51 and Theorem 50(5), there is a cardinal κ that is moved by j and that is the least ordinal moved by j. Assume for a contradiction that $\kappa \in \operatorname{ran} j$, and let $j(x) = \kappa$. Because j reflects ordinals, x must also be an ordinal. Because $\alpha \leq j(\alpha)$ for all ordinals α, it follows that $x < \kappa$. However, existence of such an ordinal x contradicts the leastness of κ. It follows that κ is a critical point of j. Also, since κ is a cardinal and j preserves cardinals, $j(\kappa)$ is also a cardinal. Since $|\kappa| = \kappa < j(\kappa) = |j(\kappa)|$, we conclude that κ is also a strong critical point of j. □

Remark 16. The property of j that it *reflects* ordinals is used only in the proof that a weak critical point is also a critical point; all other implications in the theorem are provable without assuming that j reflects ordinals.

19. Class Dedekind Self-Maps and Functors

In the previous few sections, our goal has been to find natural ways to strengthen the characteristics of a bare Dedekind self-map $j : V \to V$ so that it would give rise to an infinite set, where V is a model of ZFC – Infinity. Our first avenue for achieving this has been to identify preservation properties that j could have, which would lead to the conclusion that an infinite set exists, and we have discussed three ways of doing this. In this section, we consider a second method: showing how an infinite set can arise in the interaction between j and its (least) critical point.[117]

[117] These two avenues were introduced on p. 103.

For this discussion, our techniques acquire a category-theoretic flavor. We will lay the groundwork in this section by defining the concept of a *functor*. A functor is a structure-preserving transformation from one category to another. It is possible for a functor to also be a Dedekind self-map $V \to V$, as we will see. In fact, the result we are moving toward is the construction of such a Dedekind self-map functor j with the property that the image of j's critical point, under j, is an infinite set.

Definition 12. (Functors) Suppose $\mathcal{C} = (\mathcal{O}_C, \mathcal{M}_C)$ and $\mathcal{D} = (\mathcal{O}_D, \mathcal{M}_D)$ are categories. A *functor* $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a transformation that maps $\mathcal{O}_C$ to $\mathcal{O}_D$ and $\mathcal{M}_C$ to $\mathcal{M}_D$, satisfying the following properties:

(1) Whenever $f : A \to B$ belongs to $\mathcal{M}_C$, $\mathbf{F}(f) : \mathbf{F}(A) \to \mathbf{F}(B)$ belongs to $\mathcal{M}_D$.
(2) Whenever $f : A \to B$ and $g : B \to C$ belong to $\mathcal{M}_C$, we have $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$.
(3) For any object $A \in \mathcal{O}_C$, $\mathbf{F}(1_A) = 1_{\mathbf{F}(A)}$.

Notice that if $\mathbf{F} : \mathbf{Set} \to \mathbf{Set}$ is a 1-1 functor (1-1 on *objects*) and there is some $Y \in V = \mathcal{O}_{\mathbf{Set}}$ not in the range of $\mathbf{F}$, then $\mathbf{F}$ can be seen as a class Dedekind self-map from V to V.

As we will show now, the concept of a functor is related to that of a Dedekind self-map that preserves functions. To explain this relationship, we introduce the concept of a *natural transformation* from one functor to another. If $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbf{G} : \mathcal{C} \to \mathcal{D}$ are both functors, one can define a certain type of transformation—called a *natural transformation*—from $\mathbf{F}$ to $\mathbf{G}$ that preserves the behavior of $\mathbf{F}$ within the context of $\mathbf{G}$. Specifically, we have the following definition:

Definition 13. (Natural Transformations) Suppose $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbf{G} : \mathcal{C} \to \mathcal{D}$ are functors. A *natural transformation* η *from* $\mathbf{F}$ *to* $\mathbf{G}$ associates to each object X in $\mathcal{C}$ a map $\eta_X : \mathbf{F}(X) \to \mathbf{G}(X)$ so that for any $\mathcal{C}$-morphism $f : X \to Y$, the following is commutative:

$$
\begin{array}{ccc}
\mathbf{F}(X) & \xrightarrow{\mathbf{F}(f)} & \mathbf{F}(Y) \\
\big\downarrow{\scriptstyle \eta_X} & & \big\downarrow{\scriptstyle \eta_Y} \\
\mathbf{G}(X) & \xrightarrow{\mathbf{G}(f)} & \mathbf{G}(Y)
\end{array}
\tag{57}
$$

Proposition 52. *Suppose $j : V \to V$ is a 1-1 functor that preserves $\in$. Define a mapping ζ that associates to each set X the map $\zeta_X = j \restriction X : X \to j(X)$, and suppose that ζ is a natural transformation from 1_V to j (where 1_V denotes the identity functor from V to V). Then j preserves functions.*

Proof. We first observe that, because j preserves $\in$, it follows that for every X, $j[X] \subseteq j(X)$, so $j \restriction X : X \to j(X)$. Suppose $f : X \to Y$ is a function. By naturality of ζ, the following is commutative:

(58)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \zeta_X} & & \downarrow{\scriptstyle \zeta_Y} \\ j(X) & \xrightarrow{j(f)} & j(Y) \end{array}$$

But now, tracing through the diagram beginning with any $x \in X$, we obtain

$$j(f(x)) = (j(f))(j(x)),$$

as required. □

The proposition says that the behavior of the identity functor is preserved within the structure of any function $j(f)$. This preservation of "silence" in dynamism is a characteristic of a class Dedekind self-map that is a BSP functor, but is not a characteristic of class Dedekind self-maps in general—for instance the global successor function $\overline{s} : V \to V : x \mapsto x \cup \{x\}$ does not have this property. The result shows that deeper aspects of the self-referral flow of consciousness, as understood in the ancient wisdom of life, begin to be reflected more profoundly in Dedekind self-maps equipped with stronger preservation properties. We will see this theme develop more fully as we introduce still stronger preservation properties.

Having introduced functors and natural transformations, we are ready to discuss one final way in which *set* Dedekind self-maps emerge from the dynamics of a *class* Dedekind self-map. This final example does not make direct use of preservation properties[118] as a way to enhance the class map. Rather, we follow the *second* avenue for producing an infinite set (described on page 103): We produce a Dedekind self-map $j : V \to V$ with least critical point the number 1 such that $j(1)$ is an infinite set—mirroring the theme that the universe itself unfolds from the dynamics of the unbounded value of pure consciousness interacting with its point.

Since we wish to "create" a Dedekind self-map—in effect, create something infinite—we will work, as in previous sections, in the theory ZFC − Infinity. At the same time, we will need an ambient category in which our new Dedekind self-map can appear. But because Dedekind self-maps may not exist in the absence of an Axiom of Infinity, the category **SelfMap** may be empty, and therefore is not a suitable choice.

We therefore introduce a slightly more general category **SM** whose objects are *arbitrary* self-maps—functions from a set to itself—and whose morphisms are defined as in **SelfMap**. More precisely, **SM** $= (\mathcal{O}, \mathcal{M})$ where $\mathcal{O} = \{f : A \to A \mid A \text{ is a set}\}$, and an element $\alpha : f \to g$ of $\mathcal{M}$, where $f : A \to A$, $g : B \to B$ belong to $\mathcal{O}$, is a

[118] In fact, preservation properties are involved, but the self-map that we end up defining does not actually exhibit the interesting properties that its *factors* do.

function $\alpha : A \to B$ making the following diagram commutative:

(59)
$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{g} & B \end{array}$$

There is a naturally defined functor $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$—known as the *forgetful functor*—that has the effect of collapsing the structure of any self-map by outputting just the domain of the self-map. Thus, for any self-map $g : B \to B$, $\mathbf{G}(g) = B$.

As we will see, existence of a (set) Dedekind self-map is equivalent to the existence of a kind of counterbalancing functor $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ to $\mathbf{G}$, called a *left adjoint of* $\mathbf{G}$. Such a functor $\mathbf{F}$ would produce, from any given set A, a self-map $f_A : X_A \to X_A$; that is, $\mathbf{F}(A) = f_A$. Moreover, $\mathbf{F}$ must also satisfy the following "balance condition": For any self-map $g : B \to B$ and any set A, the number of $\mathbf{Set}$ morphisms from A to $\mathbf{G}(g) = B$ (that is, the number of ordinary functions from A to B) precisely equals the number of $\mathbf{SM}$ morphisms α from $\mathbf{F}(A) = f_A$ to g.

As a matter of notation, in general, for any sets X, Y, we let $\mathbf{Set}(X, Y)$ denote the set of all functions from X to Y, and for any self-maps $u : X \to X$ and $v : Y \to Y$ in $\mathbf{SM}$, we let $\mathbf{SM}(u, v)$ denote the set of all $\mathbf{SM}$-morphisms from u to v. Recall that a morphism $\alpha : u \to v$ is a function $\alpha : X \to Y$ so that the following is commutative:

(60)
$$\begin{array}{ccc} X & \xrightarrow{u} & X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{v} & Y \end{array}$$

With this notation, we give a precise definition of "adjoint." Given functors $\mathbf{F}$ and $\mathbf{G}$ as above, with $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ and $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$,[119] we say $\mathbf{F}$ *is a left adjoint of* $\mathbf{G}$ if, for any A in $\mathbf{Set}$ and $g : B \to B$ in $\mathbf{SM}$, there is a bijection $\Theta_{A,g} : \mathbf{SM}(\mathbf{F}(A), g) \to \mathbf{Set}(A, \mathbf{G}(g))$. Moreover, the transformation $\Theta : (A, g) \mapsto \Theta_{A,g}$ is *natural in A and g* (in the same sense as in Definition 13, for appropriately defined functors). This naturalness requirement means, intuitively speaking, that the way the component bijections $\Theta_{A,g}$ are defined is "the same" for all choices of A, g.[120] We express the fact that $\mathbf{F}$ is left adjoint to $\mathbf{G}$ by writing $\mathbf{F} \dashv \mathbf{G}$, and we call the triple $(\mathbf{F}, \mathbf{G}, \Theta)$ an *adjunction*.

Assuming such a functor $\mathbf{F}$ and corresponding natural bijections $\Theta_{A,g}$ can be found, we show how a (set) Dedekind self-map and its properties emerge from $\mathbf{F}$ and the adjunction between $\mathbf{F}$ and $\mathbf{G}$. Our candidate for such a self-map is $\mathbf{F}(1) =$

[119] For this definition of "adjoint," $\mathbf{G}$ could be *any* functor $\mathbf{SM} \to \mathbf{Set}$, though our main interest here is in the forgetful functor $\mathbf{G}$.

[120] A precise definition of "naturalness" in this context can be found in the Appendix; see p. 201.

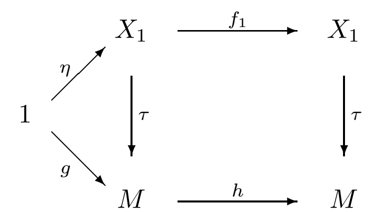
$f_1 : X_1 \to X_1$. Using the adjunction between **F** and **G**, we will be able to show that f_1 is 1-1 and has a critical point, and is in fact an initial Dedekind self-map.

We first show how Θ locates a critical point for f_1. Recall that Θ_{1,f_1} specifies a bijection from $\mathbf{SM}(f_1, f_1)$ to $\mathbf{Set}(1, \mathbf{G}(\mathbf{F}(1)) = \mathbf{Set}(1, X_1)$. Let $\eta = \Theta_{1,f_1}(1_{f_1})$ (where 1_{f_1} is the identity morphism in **SM** from f_1 to itself). The function η picks out one of the maps $1 \to X_1$ from the collection $\mathbf{Set}(1, X_1)$. Noting that $\mathrm{dom}\, \eta = 1 = \{0\}$, we shall show that $\eta(0)$ is a critical point for f_1. We write $\eta_0 = \eta(0)$.

To verify that this value η_0 can be obtained at all, we need to check that X_1 is nonempty. But nonemptiness of X_1 follows from the adjunction: Assume $X_1 = \emptyset$ and consider the identity map $\mathrm{id}_2 : 2 \to 2$ on the set $\{0, 1\}$. Θ_{1,id_2} is a bijection from $\mathbf{SM}(\mathbf{F}(1), \mathrm{id}_2)$ to $\mathbf{Set}(1, 2)$. Certainly $\mathbf{Set}(1, 2)$ has exactly two elements, but, since we have assumed $X_1 = \emptyset$, it then follows that $\mathbf{F}(1) : X_1 \to X_1$ is the *empty map* $\emptyset$, and $|\mathbf{SM}(\mathbf{F}(1), \mathrm{id}_2)| = 1$, contradicting the fact that $|\mathbf{SM}(\mathbf{F}(1), \mathrm{id}_2)| = |\mathbf{Set}(1, 2)|$.

The next point, which follows from the fact that Θ is a natural transformation, is that, for any other map g from 1 to a set of the form $\mathbf{G}(h : M \to M) = M$, we can find a unique **SM**-map $\tau : f_1 \to h$ that makes the following diagram commutative:[121]

(61)



(NNO)

We call this the *NNO property of* f_1. (The acronym NNO stands for *natural numbers object*. See [25, Section 12.2].)

We verify in several steps that $f_1 : X_1 \to X_1$ is a Dedekind self-map with critical point η_0. We begin by defining a class sequence $s = \langle \eta_0, f_1(\eta_0), f_1(f_1(\eta_0)), \ldots \rangle$, using Theorem 39.

$$\begin{aligned} s_0 &= \eta_0, \\ s_{n+1} &= f_1(s_n). \end{aligned}$$

One may use the Axiom of Separation now to conclude that $W = \mathrm{ran}\, s$ is a set:

$$W = \mathrm{ran}\, s \cap X_1.$$

Claim A. The sequence s has no repeated terms.

Proof. We show by (class) induction that, for all n, the terms $s_0, s_1, \ldots, s_n$ are distinct. Let $A \subseteq \overline{\omega}$ be the subclass of $\overline{\omega}$ defined by

$$A = \{n \in \overline{\omega} \mid s_0, s_1, \ldots, s_n \text{ are distinct}\}.$$

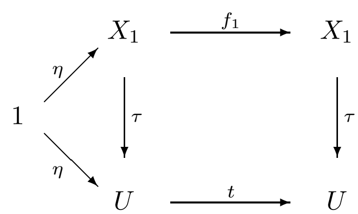
[121]In the Appendix, Theorem 89, we prove a more general fact and indicate how the result mentioned here follows; see p. 203.

Certainly $0 \in A$. Assume $n \in A$ and hence that $s_0, s_1, s_2, \ldots, s_n$ are distinct. Define a set $U = W \cup \{u\}$ where $u \notin W$. (For example, we could let $u = W$, since we have already established that $W \in V$.) Define $t : U \to U$ by

$$t(x) = \begin{cases} s_{i+1} & \text{if } x = s_i \text{ and } 0 \le i < n, \\ u & \text{if } x = s_n, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Now, letting $t^i(x)$ denote the ith iterate of t at x, with $t^0(x) = x$, it follows that for $0 \le i \le n$, $t^i(s_0) = s_i = f_1^i(s_0)$, but $f_1^{n+1}(s_0) \neq u = t^{n+1}(s_0)$. By the NNO property of f_1, there is a unique $\tau : X_1 \to U$ making the following diagram commutative.

(62)



To show that $s_0, s_1, \ldots, s_n, s_{n+1}$ are distinct, assume instead that $s_{n+1} = s_i$ for some $i, 0 \le i \le n$; in other words, $f_1^{n+1}(s_0) = f_1^i(s_0)$ for some i, $0 \le i \le n$. Tracing through diagram (62), we have

$$\begin{aligned} s_i &= t^i(s_0) \\ &= t^i(\tau(s_0)) \\ &= \tau(f_1^i(s_0)) \\ &= \tau(f_1^{n+1}(s_0)) \\ &= t^{n+1}(\tau(s_0)) \\ &= t^{n+1}(s_0) \\ &= u, \end{aligned}$$

which is impossible. Therefore, $s_0, s_1, \ldots, s_n, s_{n+1}$ are indeed distinct. This completes the induction. We have shown that there are no repeated terms in s. □

Claim B. The map $f_1 \restriction W : W \to W$ is 1-1. Indeed, $f_1 \restriction W$ is an initial Dedekind self-map with critical point η_0.

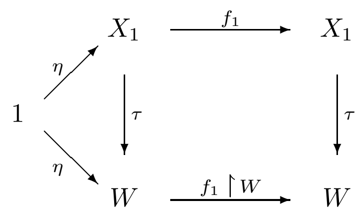
Proof. Assume $f_1 \restriction W$ is not 1-1. Since all elements of W are of the form $f_1^n(\eta_0)$, it follows that $f_1(f_1^n(\eta_0)) = f_1(f_1^m(\eta_0))$ for some $m \neq n$. This means $f_1^{n+1}(\eta_0) = f_1^{m+1}(\eta_0)$, and so the sequence s contains repeated elements, contradicting Claim A.

To see η_0 is a critical point of $f_1 \restriction W$, assume not; then $f_1(s_n) = \eta_0 = s_0$ for some n, violating Claim A.

Finally, we may use Theorem 26(2) to conclude that $f_1 \restriction W$ is initial: Given any Dedekind self-map $g : B \to B$ with critical point b, we can map s_0 to b and, for each $n > 0$, s_n to $g^n(b)$. We have shown that $f_1 \restriction W$ is an *initial* Dedekind self-map. □

We recall that, by the NNO property of $f_1 : X_1 \to X_1$, there is a unique $\tau : X_1 \to W$ making the following diagram commutative:

(63)

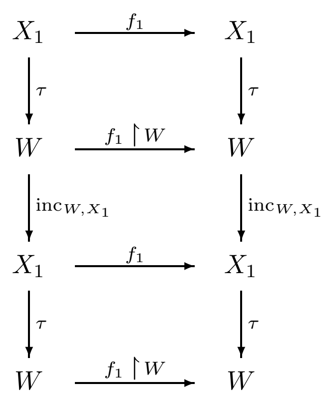


We use this observation in the proof of the next claim.

Claim C. $X_1 = W$. Therefore $f_1 = f_1 \upharpoonright W$ is 1-1 and (X_1, f_1, η_0) is an initial Dedekind self-map.

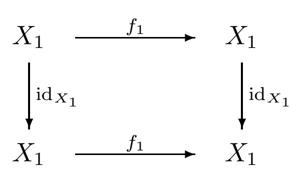
Proof. The second clause follows immediately from the fact that $X_1 = W$, which we prove now. Consider the following diagram:

(64)



It is easy to see that the middle square is commutative, using the inclusion map inc_{W,X_1}. By the NNO property of f_1, id_{X_1} is the unique map taking η_0 to η_0 for which the following is commutative:

(65)



Therefore, $\mathrm{inc}_{W,X_1} \circ \tau = \mathrm{id}_{X_1}$. Likewise, since $(W, f \upharpoonright W, \eta_0)$ is an initial object in **SelfMap**, id_W is the unique self-map taking η_0 to η_0 and making the following diagram commutative:

$$\begin{array}{ccc} W & \xrightarrow{f_1 \restriction W} & W \\ \downarrow{\scriptstyle \mathrm{id}_W} & & \downarrow{\scriptstyle \mathrm{id}_W} \\ W & \xrightarrow{f_1 \restriction W} & W \end{array} \tag{66}$$

Therefore $\tau \circ \mathrm{inc}_{W,X_1} = \mathrm{id}_W$. It follows that both τ and inc_{W,X_1} are bijections. Since one of these is an inclusion map, we conclude $X_1 = W$. $\square$

Remark 17. We note that the entire proof that $f_1 : X_1 \to X_1$ is an initial Dedekind self-map was derived from the NNO property of f_1 (p. 127). $\square$

We make one final observation about the adjunction $\mathbf{F} \dashv \mathbf{G}$ and the resulting Dedekind self-map (X_1, f_1, η_0): The element $\eta_0 \in \mathbf{G}(f_1)$ is a *universal element* for $\mathbf{G}$. We define this concept next.

In general, if $\mathbf{U} : \mathcal{C} \to \mathbf{Set}$ is a functor from some category $\mathcal{C}$ to the category of sets, an element a of a set $\mathbf{U}(A)$ is a *weakly universal element* for $\mathbf{U}$ if, for every $B \in \mathcal{C}$ and every $b \in \mathbf{U}(B)$, there is a $g : A \to B \in \mathcal{C}$ so that $\mathbf{U}(g)(a) = b$. Moreover, a is called a *universal element* for $\mathbf{U}$ if the map g is *unique*. We can picture this universal property using the following diagram, identifying an element x of a set X with the map $1 \xrightarrow{x} X$:

$$\begin{array}{cccc} 1 & \xrightarrow{a} & \mathbf{U}(A) & A \\ & \searrow^{b} & \downarrow{\scriptstyle \mathbf{U}(g)} & \downarrow{\scriptstyle g} \\ & & \mathbf{U}(B) & B \end{array} \tag{67}$$

We recall[122] that a functor $\mathbf{U} : \mathcal{C} \to \mathbf{Set}$ is *cofinal* if, for every set y, there is $c \in \mathcal{C}$ such that $y \in \mathbf{U}(c)$. In this situation, when $\mathbf{U}$ has a weakly universal element a, and $\mathbf{U}$ is cofinal, it follows that

$$V = \{\mathbf{U}(h)(a) \mid h \text{ is a morphism of } \mathcal{C}\}.^{123} \tag{68}$$

We verify this point: Given a set b, we show $b = \mathbf{U}(h)(a)$ for some h: Using cofinality of $\mathbf{U}$, we let $B \in \mathcal{C}$ be such that $b \in \mathbf{U}(B)$, and let $h : A \to B$ be a morphism of $\mathcal{C}$ for which $\mathbf{U}(h)(a) = b$, as in the definition of a weakly universal element.

An important example of a weakly universal element that we encountered earlier arose in the reduced product construction obtained from a nonprincipal ultrafilter D on ω (Example 3, p. 116). Recall that, given such a D, $j_D : V \to V$ is defined

[122] The property of being *cofinal* was defined in exactly the same way for the special case of a class function $j : V \to V$ on p. 105.

[123] In the collection on the right hand side, we ignore morphisms h for which a is not in the domain of $\mathbf{U}(h)$.

by $j_D(X) = X^\omega/D$. The self-map j_D can be turned into a functor $\mathbf{Set} \to \mathbf{Set}$ by defining its values on **Set**-morphisms:

(69) $$j_D(f) : X^\omega/D \to Y^\omega/D : [g] \mapsto [f \circ g],$$

for any $f : X \to Y$. Then $[1_\omega] \in j_D(\omega)$ is a weakly universal element for j_D:

(70) $$\begin{array}{ccccc} 1 & \xrightarrow{[1_\omega]} & j_D(\omega) = \omega^\omega/D & & \omega \\ & \searrow^{[f]} & \Big\downarrow{\scriptstyle [g]\mapsto[f\circ g]} & & \Big\downarrow{\scriptstyle g} \\ & & j_D(X) = X^\omega/D & & X \end{array}$$

Given $[f] \in j_D(X) = X^\omega/D$, with $f : \omega \to X$, f itself is the required function. We show that $j_D(f)([1_\omega]) = [f]$:

(71) $$j_D(f)([1_\omega]) = [f \circ 1_\omega] = [f].$$

Returning to the adjunction $\mathbf{F} \dashv \mathbf{G}$, it is easy to see from previous work that, by the NNO property of f_1 (p. 127), if $\beta : B \to B \in \mathbf{SM}$ and $b \in \mathbf{G}(\beta) = B$, there is a unique morphism $\tau : f_1 \to \beta$ in **SM** so that $\mathbf{G}(\tau)(\eta_0) = b$.

(72) $$\begin{array}{ccccccc} 1 & \xrightarrow{\eta_0} & \mathbf{GF}(1) & & X_1 & \xrightarrow{f_1} & X_1 \\ & \searrow^{b} & \Big\downarrow{\scriptstyle \mathbf{G}(\tau)} & & \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau} \\ & & \mathbf{G}(\beta) & & B & \xrightarrow{\beta} & B \end{array}$$

Therefore η_0 is a universal element for **G**. Moreover, since **G** is cofinal (as is easily checked), we have

(73) $$V = \{\mathbf{G}(\tau)(\eta_0) \mid \tau \text{ is a morphism of } \mathbf{SM}\}.$$

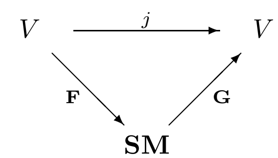
It can be shown that a converse is also true: Existence of a universal element for **G** suffices to guarantee the existence of a Dedekind self-map on a set, and hence also existence of a left adjoint of **G**.[124]

We summarize what we have accomplished so far. We showed that, in obtaining a functor $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ that is *left adjoint* to the *forgetful functor* $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ (so that, in a sense, the "collapsing" effect of **G** is counterbalanced by the "generating" effect of **F**), we were able to locate a seed $\Theta_{1,f_1}(1_{f_1})(0) = \eta_0$ to generate a Dedekind self-map $F(1) = f_1 : X_1 \to X_1 = \langle \eta_0, f_1(\eta_0), f_1^2(\eta_0), \ldots \rangle$. Moreover, the set X_1 is itself obtained by the computation $X_1 = \mathbf{G}(\mathbf{F}(1))$. We can compose the functors **F** and **G** to obtain $j = \mathbf{G} \circ \mathbf{F}$; composing in this way has the effect of joining into a

[124]A proof can be found in the Appendix, Theorem 93(3).

single functor the "collapsing" and "generating" influences of **F** and **G**.

(74)



As we have just seen, j has a strong critical point 1, since $j(1) = \mathbf{G}(\mathbf{F}(1)) = X_1$ (and recall that X_1 is itself infinite). We can also show that 1 is both a critical point and a *weak* critical point (being the least ordinal moved by j), because of the following two facts:

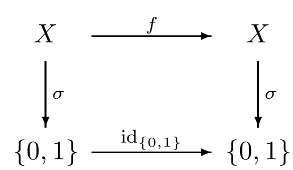
(1) $j(0) = 0$;
(2) $j(A)$ is infinite, for all $A \neq 0$.

We prove (2) in the Appendix, Theorem 93(2). For (1), recall that 0 is just the empty set $\emptyset$. The first thing we will verify is that $F(\emptyset)$ is the unique function e—the *empty function*—from $\emptyset$ to $\emptyset$. If not, then there is some nonempty set X and some function f so that $F(\emptyset) = X \xrightarrow{f} X$. Let $B = \{0, 1\}$ and let $\beta = \mathrm{id}_B$. Then by the adjunction from F to G, we have

(75) $$|\mathbf{Set}(\emptyset, G(\beta))| = |\mathbf{SM}(F(\emptyset), \beta)|.$$

A typical element of $\mathbf{SM}(F(\emptyset), \beta)$ is a function $\sigma : X \to \{0, 1\}$ for which the following is commutative:

(76)



Because X is nonempty, there are at least two possible values of σ: One possibility is that $\sigma(x) = 0$ for all $x \in X$; another is that $\sigma(x) = 1$ for all $x \in X$. Both of these make the diagram commutative. This means that $|\mathbf{SM}(F(\emptyset), \beta)| \geq 2$. However, the only function having domain $\emptyset$ and codomain $G(\beta)$ is the empty function; in other words $|\mathbf{Set}(\emptyset, G(\beta))| = 1$. This contradicts (75). Therefore, the set X must be empty.

We have shown $F(\emptyset) = \emptyset \xrightarrow{e} \emptyset$, so clearly $j(0) = G(F(\emptyset)) = \emptyset = 0$, as required.

Returning to a list of results about j obtained so far, recall that $\eta_0 \in \mathbf{GF}(1)$ is a universal element for **G**, "generating" all of V in the sense that $V = \{\mathbf{G}(\tau)(\eta_0) \mid \tau \text{ is a morphism of } \mathbf{SM}\}$.

Although $j : V \to V$ has a critical point, we cannot quite guarantee that it is 1-1. Let us call a functor $H : V \to V$ *essentially Dedekind* if it is "naturally isomorphic" to a functor $K : V \to V$ which has a critical point and *is* 1-1 (on objects). It turns out that, if we obtain $j = \mathbf{G} \circ \mathbf{F}$ where **F** is *any* left adjoint to **G**, as we have done, though we cannot guarantee that j is a Dedekind self-map,[125] it can be shown that j is essentially Dedekind.[126]

Here, then, we have an example in which a *class* Dedekind self-map $j : V \to V$ (in this case, *essentially* Dedekind) gives rise to a *set* Dedekind self-map.[127] In fact, working in ZFC − Infinity, one can show that existence of the adjoint functor $\mathbf{F}$, and hence, existence of j itself, is equivalent to the Axiom of Infinity.[128]

Diagram (74) shows that j is the composition of two functors, $\mathbf{F}$ and $\mathbf{G}$; we say that $\mathbf{F}$ and $\mathbf{G}$ are the *factors* of j. Because $\mathbf{F}$ is left adjoint to $\mathbf{G}$, it turns out that both $\mathbf{F}$ and $\mathbf{G}$ exhibit strong preservation properties. For instance, it can be shown that $\mathbf{F}$ preserves coproducts; in fact, if $\{X_i \mid i \in I\}$ is a collection of disjoint sets, no matter how big I may be, the functor $\mathbf{F}$ has the property that

$$\left|\mathbf{F}\left(\bigcup_{i\in I} X_i\right)\right| = \left|\bigcup_{i\in I} \mathbf{F}(X_i)\right|.$$

Recall that when we defined "preservation of coproducts" for functors, we considered only the case in which $|I| = 2$; the preservation property that $\mathbf{F}$ exhibits is much stronger.

The functor $\mathbf{G}$ exhibits the "dual" preservation property of preserving *products*. Let us say that a class function $j : V \to V$ *preserves products* if, for any sets A, B, $|j(A \times B)| = |j(A) \times j(B)|$. It can be shown that, if $\{X_i \mid i \in I\}$ is a collection of sets, no matter how big I may be,

$$\left|\mathbf{G}\left(\prod_{i\in I} X_i\right)\right| = \left|\prod_{i\in I} \mathbf{G}(X_i)\right|,$$

where $\prod$ denotes the product operator. Once again, the definition of "preserving products" requires only that $|I| = 2$; the property exhibited by $\mathbf{G}$ is much stronger.

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[125] It can be shown that j is *faithful*. This means that, for any sets A, B, the restriction $j \restriction \mathbf{Set}(A,B) : \mathbf{Set}(A,B) \to \mathbf{Set}(j(A), j(B))$ is 1-1. But this fact does not imply that j is 1-1 on objects. See `http://mathoverflow.net/questions/55182/what-is-known-about-the-category-of-monads-on-set` for a proof of faithfulness.

[126] In the Appendix, Theorem 93, we establish the fact that j is essentially Dedekind together with other related results, summarized in the theorem below.

Theorem 93.
(1) If $\mathbf{F}$ is a left adjoint of $\mathbf{G}$, then $\mathbf{G} \circ \mathbf{F}$ is essentially Dedekind.
(2) Whenever $\mathbf{F} \dashv \mathbf{G}$ and $|A| > 0$, $j(A) = \mathbf{G}(\mathbf{F}(A))$ is infinite.
(3) Whenever $\mathbf{G}$ has a universal element, there is a naturally defined initial Dedekind self-map. Moreover, $\mathbf{G}$ has a left adjoint.

We remark that being essentially Dedekind is preserved by natural isomorphisms even though being Dedekind is not.

[127] Speaking precisely, what we mean here is that $j(1)$ is an infinite set upon which we may define a Dedekind self-map. Note that we cannot fully recover from j the adjunction $\mathbf{F} \dashv \mathbf{G}$ that gave rise to it, so we cannot obtain from j the Dedekind self-map $\mathbf{F}(1) = f_1 : X_1 \to X_1$, even though $\mathbf{F}$ is a factor of j. As we show in the next section, it is nevertheless true that $j : V \to V$ induces a Dedekind self-map—in fact, a Dedekind self-map $X_1 \to X_1$—in a more direct way.

[128] Assuming the adjoint $\mathbf{F}$ exists, we have seen that $\mathbf{F}(1)$ is a Dedekind self-map, with $j(1) = X_1$ an infinite set. Conversely, assuming some form of the Axiom of Infinity, we conclude that ω exists; as in Theorem 93 in the Appendix, $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ defined on objects by $\mathbf{F}(A) = 1_A \times s : A \times \omega \to A \times \omega$ is left adjoint to $\mathbf{G}$.

The concepts of product and coproduct have natural generalizations in category theory to the concepts of *limit* and *colimit*, respectively. These more general definitions allow us to see that products—along with many other well-known constructions in mathematics—are a special type of limit, and likewise, coproducts, and many other constructions, are special instances of the colimit construction. We will not define these more general concepts here, but to give a feeling of concreteness, we will at least introduce some notation. Suppose $\{X_i \mid i \in I\}$ is a collection of sets and suppose $\mathbf{H}$ is a functor. We will say that $\mathbf{H}$ *preserves set-indexed limits* if

$$\mathbf{H}\left(\mathbf{Lim}_{i\in I}\left(X_i\right)\right) = \mathbf{Lim}_{i\in I}\left(\mathbf{H}(X_i)\right),$$

where **Lim** denotes the limit construction. Likewise, $\mathbf{H}$ *preserves set-indexed colimits* if

$$\mathbf{H}\left(\mathbf{Colim}_{i\in I}\left(X_i\right)\right) = \mathbf{Colim}_{i\in I}\left(\mathbf{H}(X_i)\right),$$

where **Colim** denotes the colimit construction.

Now we are in a position to state the properties of $\mathbf{F}$ and $\mathbf{G}$. Because $\mathbf{F}$ is left adjoint to $\mathbf{G}$, it can be shown that $\mathbf{F}$ preserves all set-indexed colimits and $\mathbf{G}$ preserves all set-indexed limits.[129] However, the composition $j = \mathbf{G} \circ \mathbf{F}$ exhibits, by comparison, very little in the way of preservation properties. We will discuss this limitation further in the next section.

We collect these results together in the following theorem. We have labeled this theorem the *Lawvere Construction* since most of the contents of the theorem are due to Lawvere [42]:

Theorem 53. (The Lawvere Construction) (ZFC − Infinity) *Let $\mathbf{G}$ denote the forgetful functor from $\mathbf{SM}$ to $\mathbf{Set}$; that is, for every $f : A \to A$ for which $f \in \mathbf{SM}$, $\mathbf{G}(f) = A$, and for every $\mathbf{SM}$-arrow $\alpha : f \to g$, where $f : A \to A$ and $g : B \to B$, $\mathbf{G}(\alpha)$ is equal to $\alpha : A \to B$. Suppose $\mathbf{G}$ has a left adjoint $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$. Let $\Theta_{A,\beta}$ denote the natural bijection from $\mathbf{SM}(\mathbf{F}(A), \beta)$ to $\mathbf{Set}(A, \mathbf{G}(\beta))$. Write $\mathbf{F}(A) = f_A : X_A \to X_A$. Let $\eta_0 = \Theta_{1,\mathbf{F}(1)}(1_{\mathbf{F}(1)})(0)$. Let $j = \mathbf{G} \circ \mathbf{F}$. Then*

(1) *$j : V \to V$ is an essentially Dedekind self-map for which 1 is both a critical point and a strong critical point of j.*

(2) *$j(1)$ is itself a Dedekind-infinite set X_1 and $\mathbf{F}(1)$ is the corresponding Dedekind self-map $f_1 : X_1 \to X_1$ with critical point η_0. Moreover, f_1 is initial in $\mathbf{SM}$ and $X_1 = \{\eta_0, f_1(\eta_0), f_1^2(\eta_0), \ldots\}$.*

(3) *$\eta_0 \in \mathbf{G}(\mathbf{F}(1))$ is a universal element of $\mathbf{G}$. Moreover,*

$$V = \{\mathbf{G}(f)(\eta_0) \mid f \text{ is a morphism in } \mathbf{SM}\}.$$

(4) *$\mathbf{F}$ preserves set-indexed colimits and $\mathbf{G}$ preserves set-indexed limits.*

We recall that the emergence of an infinite set from the Lawvere Construction validates one of the patterns we mentioned earlier,[130] that a Dedekind-infinite set occurs as the image of the critical point of the class map $j : V \to V$; that is, an infinite set emerges from the action of j on its critical point 1.[131]

[129] Proofs of these preservation properties of $\mathbf{F}$ and $\mathbf{G}$ may be found in [43].

[130] On page 95, point (B), and also on page 103.

[131] One subtle point should be mentioned here. Based on insights from ancient wisdom, we have embraced the view that emergence of an infinite set from the action of a Dedekind self-map on its critical point is "natural" (p. 103). And, although it is indeed a consequence of the Lawvere

In the next section, we show how this result can be improved in a couple of ways. First, as we discussed earlier in passing, it is possible to obtain from the self-map $j : V \to V$ of the Lawvere construction not only an infinite set but also a naturally defined Dedekind self-map $k : X_1 \to X_1$. Moreover, one can show that any self-map $i : V \to V$ that possesses several of the key properties of this j will also give rise to a Dedekind self-map in essentially the same way. The next section is devoted to proofs of these claims.

Construction that an infinite set emerges from the computation $j(1)$—and 1 is indeed the least critical point of j—the existence of such a self-map j has not been postulated on the basis of "naturalness" (by contrast, "preservation properties" that we added to j in our earlier discussion *were* postulated on the ground of naturalness); rather, j arises as a consequence of a different assumption, namely, that the forgetful functor $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ has a left adjoint. Moreover, we show in Section 20 that, more generally, any j that is the functor part of a *Dedekind monad* directly gives rise to a set Dedekind self-map by way of interaction with one of its critical points. But we can now ask whether existence of a left adjoint to $\mathbf{G}$ or existence of a Dedekind monad is justified by appealing to the perspective of the ancients.

We offer a possible answer by recalling a fundamental principle, prevalent in ancient wisdom: that the diversity of existence arises from unity through the interaction and integration of opposing forces. Maharishi describes this phenomenon in his commentary to the *Bhagavad-Gita* [49]:

> Life is a battlefield of opposing forces (p. 26).

Elsewhere, he writes [45]:

> This ability of the field of self-referral consciousness to spontaneously maintain itself, to maintain its own identity in the opposite qualities of silence and dynamism simultaneously, is the seat of invincibility. This absolute structure of invincibility commands the two opposite values of silence and dynamism to co-exist at all times, and neither of the two can annihilate the other (pp. 219–220).

Maharishi here describes these opposing forces as dynamism and silence, but also, elsewhere, as expansion and collapse (recall [53, p. 1], cited on p. 15).

Likewise, as we have seen, Laozi describes the nature of *Tao*, as it unfolds into diversity, as emerging through the dual dynamics of proceeding and returning: "Having gone far, it returns" (*Tao Te Ching*, v. 25).

Plotinus makes a similar point [64]: "By a natural necessity does everything proceed from, and return to unity" (p. 1077).

These fundamental opposing forces at the basis of creation seem to find natural expression in the concept of *adjoints*. When functors form an adjunction, they behave like two opposite forces bound together to produce a new effect. We have already seen how the forgetful functor $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ collapses the structure of $\mathbf{SM}$-morphisms to their domains, while a left adjoint $\mathbf{F}$ to $\mathbf{G}$ serves to *create* $\mathbf{SM}$-morphisms from sets. When combined, in the form $j = \mathbf{G} \circ \mathbf{F}$, an infinite set emerges.

Category theorists have maintained that the adjoint relationship is fundamental to the structure of mathematics itself. In the preface to his classic text on category theory, S. Mac Lane mentions [43, p. vii] a slogan that captures an intuition shared by experts in the field: "Adjoint functors arise everywhere." In the same spirit, S. Awodey mentions [2, p. 231] in his text *Category Theory*, "...in a sense, every functor has an adjoint." It is obvious to those who have sought to detect the presence of adjunctions that they pervade mathematics.

In the Appendix, Section 27.3, we offer some evidence, often cited by category theorists, that adjoint relationships can be seen to structure everything in the mathematical universe. We therefore propose to justify the existence of a left adjoint to $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ and the existence of Dedekind monads on the basis of this eternal wisdom that diversity arises from the integration of opposing forces on the ground of unity.

20. Dedekind Monads

In this section, we show how the functor $j : V \to V$ obtained in the Lawvere construction produces, in a natural way, a *set* Dedekind self-map. We then develop sufficient conditions on a functor $j : V \to V$—internal to the structure of j—for producing a set Dedekind self-map. We begin with some definitions and background results.

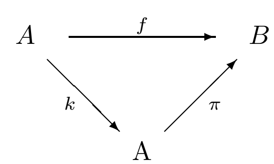
Definition 14. (Dedekind Maps) Let us say that a function $f : A \to B$ is a *Dedekind map* if

(1) $|A| = |B|$
(2) f is 1-1 but not onto.

We will call any element $b \in B$ not in the range of f a *critical point of* f.

The concept of a Dedekind map is a generalization of the concept of a Dedekind self-map. As we now show, Dedekind maps always factor as a composition of a bijection and a Dedekind self-map.

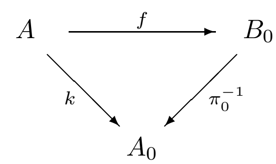
Proposition 54. *Suppose $f : A \to B$ and $b \in B$. Then f is a Dedekind map with critical point b if and only if there exist functions π, k so that $f = \pi \circ k$, where $\pi : A \to B$ is a bijection and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$.*



Proof. For one direction, suppose A, B are sets, $b \in B$, and $f : A \to B$ can be factored as $f = \pi \circ k$, where $\pi : A \to B$ is a bijection and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$. We show f is a Dedekind map. Certainly, f is 1-1. We show b is a critical point of f. Since π is onto, there is $a \in A$ with $\pi(a) = b$. Note that $a = \pi^{-1}(b)$ is, by assumption, a critical point of k. If $b \in \operatorname{ran} f$, then let $x \in A$ be such that $f(x) = b$. Then by commutativity of the diagram, $\pi(k(x)) = b$, and so $k(x) = \pi^{-1}(b) = a$, which is impossible since $a \notin \operatorname{ran} k$. We have shown b is a critical point of f, f is 1-1, and $|A| = |B|$ (by way of π), as required.

For the other direction, suppose $f : A \to B$ is a Dedekind map with critical point b. Since $|A| = |B|$, there is a bijection $\pi : A \to B$. Let $B_0 = f[A] \subseteq B$ and let $A_0 = \pi^{-1}[B_0] \subseteq A$. Let $\pi_0 = \pi \restriction A_0$. We show $\operatorname{ran} \pi_0 = B_0$: Suppose $y \in A_0$. Then $y = \pi^{-1}(x)$ for some $x \in B_0$; that is, for some $x \in B_0$, $\pi(y) = x$. This shows $\operatorname{ran} \pi_0 \subseteq B_0$. If $x \in B_0$, then let $y \in A_0$ with $\pi^{-1}(x) = y$. Then $x = \pi(y) \in \operatorname{ran} \pi$. We have shown therefore that π_0 is onto. Since π_0 is a restriction of the bijection π, π_0 is also 1-1. Therefore, $\pi_0 : A_0 \to B_0$ is a bijection.

Note that $f : A \to B_0$ is a bijection. Let $k = \pi_0^{-1} \circ f : A \to A_0$. Then k is a bijection, and so, viewed as a map $k : A \to A$, k is a Dedekind self-map. The following diagram is commutative:



We claim that $\pi^{-1}(b)$ is a critical point for $k : A \to A$: Suppose $j(x) = \pi^{-1}(b)$ for some $x \in A$. Then

$$b = \pi(k(x)) = \pi(\pi^{-1}(f(x))) = f(x),$$

which is impossible because b is a critical point of f.

Now notice that, for any $x \in A$, since $k(x) = \pi_0^{-1}(f(x))$, then $\pi(k(x)) = \pi_0(k(x)) = f(x)$. We have shown that $f = \pi \circ k$, π is a bijection, and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$, as required. $\square$

As we attempt to locate within the dynamics of the functor $j : V \to V$ obtained from Lawvere's construction a naturally defined Dedekind self-map, we now note that the proposition just proved shows that it will be enough to obtain a Dedekind *map*. To make further progress in this direction, we need to go more deeply into the structure of this self-map j, and, for this endeavor, we need another new concept.

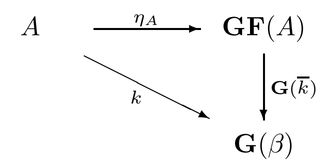
Abstracting to a more general context,[132] suppose $\mathcal{C}, \mathcal{D}$ are categories and $\mathbf{F} : \mathcal{C} \to \mathcal{D}$, $\mathbf{G} : \mathcal{D} \to \mathcal{C}$ are adjoint functors. Let $\Theta_{A,B}$ denote the natural bijection from $\mathcal{D}(\mathbf{F}(A), B)$ to $\mathcal{C}(A, \mathbf{G}(B))$. Recall that, in the case in which $\mathcal{C} = V$, $\mathcal{D} = \mathbf{SM}$, and $\mathbf{F}$ is a left adjoint to the forgetful functor $\mathbf{G} : \mathbf{SM} \to V$, in order to verify that $f_1 = \mathbf{F}(1)$ is a Dedekind self-map, we located a map $\eta_1 : 1 \to X_1 = j(1)$—written before as simply η—defined by $\eta_1 = \Theta_{1,f_1}(1_{f_1})$ (where 1_{f_1} is the $\mathbf{SM}$-identity map on f_1) (see p. 127). In the study of adjunctions in general, one considers a collection of such maps η_A. We repeat from the Appendix the formal definition and important properties of these maps here.

Definition 15. (The Unit of an Adjunction) Given adjoint functors $\mathbf{F} \dashv \mathbf{G}$, with bijections $\Theta_{A,B}$, as described above, define, for each object A in $\mathcal{C}$ a function $\eta_A : A \to \mathbf{GF}(A)$ by $\eta_A = \Theta_{A,\mathbf{F}(A)}(1_{\mathbf{F}(A)})$. η is called the *unit of the adjunction $F \dashv G$*.

In the Appendix (Theorem 89), we prove that the η_A have the following properties:

[132]See the Appendix, starting on p. 204, for definitions and basic results about adjunctions in this general context.

(1) (Universal Mapping Property) Given any $k : A \to \mathbf{G}(\beta)$, there is a unique $\overline{k} : \mathbf{F}(A) \to \beta$ such that the following diagram commutes:



(2) The transformation $\eta : \mathbf{1}_{\mathcal{C}} \to \mathbf{G} \circ \mathbf{F}$ is a *natural transformation* (see Definition 13).

Every such adjunction has not only a unit η, but also a *co-unit* $\varepsilon : \mathbf{F} \circ \mathbf{G} \to \mathbf{1}_{\mathcal{C}}$, defined in a dual manner as follows: For each $\mathcal{D}$-object B,

$$\varepsilon_B : \mathbf{F}(\mathbf{G}(B)) \to B = \Theta^{-1}_{\mathbf{G}(B),B}(1_{\mathbf{G}(B)}).$$

One can show [43] that ε is also a natural transformation and that it satisfies a dual form of the Universal Mapping Property stated in (1) above. The unit and co-unit of an adjunction represent keys for understanding the adjunction; indeed, the mappings $\Theta_{A,B}$ may be derived from η and ε. Therefore, the adjunction $(\mathbf{F}, \mathbf{G}, \Theta)$ is often referred to as the adjunction $(\mathbf{F}, \mathbf{G}, \eta, \varepsilon)$.

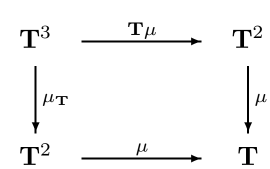
As we have already observed in the case of the categories **SM** and **Set**, any adjunction $(\mathbf{F}, \mathbf{G}, \eta, \varepsilon)$ determines another functor $\mathbf{T} : \mathcal{C} \to \mathcal{C}$ by composition: $\mathbf{T} = \mathbf{G} \circ \mathbf{F}$. By virtue of the properties of the adjunction, $\mathbf{T}$ forms a central part of another structure, called a *monad*. We introduce some of the basic results about monads here as a preliminary to our discussion below about *Dedekind monads*. See [43] for more on monads.

Definition 16. (Monads) Given a category $\mathcal{C}$, a *monad* is a triple $(\mathbf{T}, \eta, \mu)$ for which $\mathbf{T} : \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta : 1 \to \mathbf{T}$ and $\mu : \mathbf{T}^2 \to \mathbf{T}$ are natural transformations, so that, as in the commutative diagrams below,

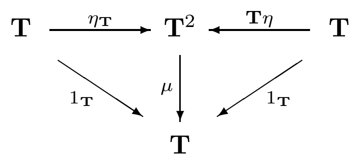
(i) for any object A in $\mathcal{C}$ and $x \in \mathbf{T}^3(A)$, $\mu_A(\mu_{\mathbf{T}(A)}(x)) = \mu_A\left(\mathbf{T}(\mu_A)(x)\right)$.
(ii) for any object A in $\mathcal{C}$ and $x \in \mathbf{T}(A)$, $\mu_A(\eta_{\mathbf{T}(A)}(x)) = x = \mu_A(\mathbf{T}(\eta_A)(x))$.

Note that the maps $\mathbf{T}(\eta_A) : \mathbf{T}(A) \to \mathbf{T}^2(A)$, for $A \in \mathcal{C}$, are themselves the components of a natural transformation.

(77)



(78)



We shall often refer to a functor $\mathbf{T}$ as a "monad" when we mean that $\mathbf{T}$ is the functor part of a monad $(\mathbf{T}, \eta, \mu)$. The transformation η is called the *unit* of the monad, and (for historical reasons) the transformation μ is called the *multiplication* operation for the monad.

Any adjunction $(\mathbf{F}, \mathbf{G}, \eta, \varepsilon)$ gives rise to a monad $(\mathbf{T}, \eta, \mu)$ by way of the following definitions:

(i) $\mathbf{T} = \mathbf{G} \circ \mathbf{F}$
(ii) η is the same in both structures
(iii) for all objects A in $\mathcal{C}$ and $x \in \mathbf{T}^2(A)$, $\mu_A(x) = \big(\mathbf{G}(\varepsilon_{\mathbf{F}(A)})\big)(x)$.

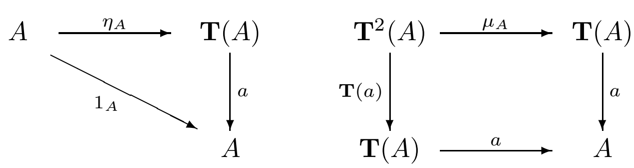
A monad $\mathbf{T}$ can be used to define a new category, entirely within $\mathcal{C}$, called the category of $\mathbf{T}$-algebras, denoted $\mathcal{C}^{\mathbf{T}}$. This category is defined as follows:

$$\mathcal{C}^{\mathbf{T}} = \{(A, a) \mid A \text{ is a } \mathcal{C}\text{-object and } a : \mathbf{T}(A) \to A\},$$

where each (A, a) in $\mathcal{C}^{\mathbf{T}}$ satisfies the equations:

(79) $$a \circ \eta_A = 1_A \qquad a \circ \mathbf{T}(a) = a \circ \mu_A.$$

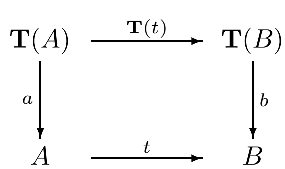
(80)



Morphisms $t : (A, a) \to (B, b)$ in $\mathcal{C}^{\mathbf{T}}$ are $\mathcal{C}$-morphisms $t : A \to B$ making Diagram (82) commutative:

(81) $$b \circ j(t) = t \circ a.$$

(82)



Functors $F^{\mathbf{T}} : \mathcal{C} \to \mathcal{C}^{\mathbf{T}}$ and $G^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$ (the forgetful functor) can be defined as follows:

$$\begin{aligned} G^{\mathbf{T}}((A,a)) &= A \\ F^{\mathbf{T}}(A) &= (\mathbf{T}(A), \mu_A) \\ G^{\mathbf{T}}(t) &= t : A \to B \ \text{ where } t : (A,a) \to (B,b) \\ F^{\mathbf{T}}(f) &= \mathbf{T}(f) \ \text{ where } f : A \to B \text{ and} \\ & \quad \text{the diagram below commutes} \end{aligned}$$

(84)
$$\begin{array}{ccc} \mathbf{T}^2(A) & \xrightarrow{\mathbf{T}^2(f)} & \mathbf{T}^2(B) \\ \Big\downarrow{\scriptstyle \mu_A} & & \Big\downarrow{\scriptstyle \mu_B} \\ \mathbf{T}(A) & \xrightarrow{\mathbf{T}(f)} & \mathbf{T}(B) \end{array}$$

It can be shown that $F^{\mathbf{T}} \dashv G^{\mathbf{T}}$ and $\mathbf{T} = G^{\mathbf{T}} \circ F^{\mathbf{T}}$. The adjunction $F^{\mathbf{T}} \dashv G^{\mathbf{T}}$ is called the *induced* $\mathbf{T}$*-algebra adjunction*.

When a monad $\mathbf{T}$ arises from an adjunction $\mathbf{F} \dashv \mathbf{G}$, where $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathbf{G} : \mathcal{D} \to \mathcal{C}$ (and this is the situation we have in our case of interest), there is a way to measure the degree to which $\mathcal{C}^{\mathbf{T}}$ is an isomorphic copy of $\mathcal{D}$: The *Eilenberg-Moore comparison functor* $\Phi : \mathcal{D} \to \mathcal{C}^{\mathbf{T}}$ is defined as follows for any $\mathcal{D}$-object D and morphism $f : D \to D'$

$$\begin{aligned} \Phi(D) &= (\mathbf{G}(D), \mathbf{G}(\varepsilon_D)) \\ \Phi(f) &= \mathbf{G}(f) : (\mathbf{G}(D), \mathbf{G}(\varepsilon_D)) \to (\mathbf{G}(D'), \mathbf{G}(\varepsilon_{D'})) \end{aligned}$$

where $\varepsilon : \mathbf{F} \circ \mathbf{G} \to 1_{\mathcal{D}}$ is the co-unit of the adjunction $\mathbf{F} \dashv \mathbf{G}$. One can show [43] that Φ is the unique functor satisfying

(85)
$$G^{\mathbf{T}} \circ \Phi = \mathbf{G} \text{ and } \Phi \circ \mathbf{F} = F^{\mathbf{T}}.$$

In many important cases, Φ is an isomorphism. An adjunction $\mathbf{F} \dashv \mathbf{G}$ is said to be *monadic* if Φ is an isomorphism. Moreover, for any functor $\mathbf{G}$, if $\mathbf{G}$ has a left adjoint $\mathbf{F}$ so that the corresponding Eilenberg-Moore comparison functor Φ is an isomorphism, then $\mathbf{G}$ is said to be a *monadic functor*.

Remark 18. We note that whenever $(\mathbf{T}, \mu, \eta)$ is a monad, the induced $\mathbf{T}$-algebra adjunction $F^{\mathbf{T}} \dashv G^{\mathbf{T}}$ is monadic. This follows immediately from the fact that $\mathbf{T} = G^{\mathbf{T}} \circ F^{\mathbf{T}}$. □

The following lemma explains our interest in monadic functors, which we state without proof.[133]

Lemma 55. *Let* $\mathbf{G} : \mathbf{SM} \to V$ *be the forgetful functor. Then for every left adjoint* $\mathbf{F}$ *of* $\mathbf{G}$*, the adjunction* $\mathbf{F} \dashv \mathbf{G}$ *is monadic. In particular, the functor* $\mathbf{G}$ *is monadic.*

[133] For an outline of a proof, see `http://mathoverflow.net/questions/256392/becks-theorem-and-the-category-of-endos`.

Remark 19. (**T**-Algebras) It is known (see for example `http://mathoverflow.net/questions/55182/what-is-known-about-the-category-of-monads-on-set`) that it is almost always the case that, for any monad $(\mathbf{T}, \eta, \mu)$ with $\mathbf{T} : V \to V$, the category $V^{\mathbf{T}}$ contains at least one **T**-algebra that has more than one element. The only two exceptions are the monads induced by the following two functors S, T:[134]

- for all $A \in V$, $S(A) = 1$
- for all nonempty $A \in V$, $T(A) = 1$, but $T(0) = 0$.

Note that neither of these functors is obtainable as $j = \mathbf{G} \circ \mathbf{F}$ where $\mathbf{G} : \mathbf{SM} \to V$ is the forgetful functor and $\mathbf{F} \dashv \mathbf{G}$ since, as we have already shown, for any such functor, we have that $\mathbf{G} \circ \mathbf{F}(1)$ is infinite.

Using the Eilenberg-Moore comparison isomorphism $\Phi : \mathbf{SM} \to V^j$ in this case, we can say more. By Lemma 56, the adjunction $\mathbf{F} \dashv \mathbf{G}$ is monadic. Therefore, Φ satisfies:

$$G^j \circ \Phi = \mathbf{G} \text{ and } \Phi \circ \mathbf{F} = F^j.$$

It follows that $F^j(1) = \Phi(\mathbf{F}(1)) = (X_1, j(X_1) \to X_1)$ yields a j-algebra based on an infinite set. □

The next two lemmas are key elements in the proof that Dedekind monads (to be defined shortly) produce Dedekind self-maps.

Lemma 56. *Suppose* $\mathbf{G} : \mathcal{D} \to \mathcal{C}$ *is a monadic functor. Then*

(1) $\mathbf{G}$ *is faithful; that is,* $\mathbf{G} \restriction \mathcal{D}(D, D')$ *is 1-1, for every pair of* $\mathcal{D}$*-objects* D, D'.

---

[134] A proof of this fact can be found in the blog `http://mathoverflow.net/questions/279321/monads-on-set-with-trivial-algebras`, authored by Tom Leinster. We give an outline of the proof.

The strategy is this: We define two monads with functor parts T_0, T_1, whose algebras always have at most one element. Then we show that if $(\mathbf{T}, \eta, \mu)$ is a monad whose **T**-algebras always contain at most one element, **T** is naturally isomorphic to one of T_0, T_1.

The functor $T_0 : V \to V$ is defined on objects by

$$T_0(A) = \begin{cases} 0 & \text{if } A = 0 \\ 1 & \text{if } A \neq 0 \end{cases}$$

The definition of T_0 on functions $f : A \to B$ is uniquely determined by its definition on objects. Likewise, $\mu^0_A : T_0^2(A) \to T_0(A)$ is uniquely determined. Notice that $\eta^0_A : A \to T_0(A)$ must be the empty function if $A = 0$ or the unique function $! : A \to 1$ otherwise. Verification of the Unit and Associative Laws is easy.

The free T_0 algebras are of the form $(T_0(A), \mu^0_A)$ (namely $F^{T_0}(A)$), so each such algebra has at most one element. All other T_0-algebras are algebra morphism images of these free algebras, so they also must contain at most one element.

The functor $T_1 : V \to V$ is defined on objects by

$$T_1(A) = 1.$$

Definition of the unit η^1 and multiplication μ^1, and verification of the Unit and Associative Laws, is even easier in this case. In this case, all T_1-algebras, being algebra morphism images of the free T_1-algebras $F^{T_1}(A)$, must contain exactly one element.

Now suppose $(\mathbf{T}, \eta, \mu)$ is a monad on V for which all **T**-algebras contain at most one element. For each set A, $\mathbf{T}(A)$ is the underlying set of a free **T**-algebra (namely $(\mathbf{T}(A), \mu_{\mathbf{T}(A)})$), and so $|\mathbf{T}(A)| \leq 1$. If $A \neq \emptyset$, since there is a unit $\eta_A : A \to \mathbf{T}(A)$, it follows that in this case $|\mathbf{T}(A)| = 1$. If $A = \emptyset$, exactly one of the following holds: $\mathbf{T}(A) = 0$ or $|\mathbf{T}(A)| = 1$. In the first case, **T** is naturally isomorphic to T_0 via $\mathbf{T}(A) \mapsto T_0(A)$; in the second case, **T** is naturally isomorphic to T_1 via $\mathbf{T}(A) \mapsto T_1(A)$.

(2) $\mathbf{G}$ *reflects isomorphisms; that is, whenever* $f : A \to B$ *is a* $\mathcal{D}$*-morphism and* $\mathbf{G}(f)$ *is an isomorphism in* $\mathcal{C}$*, then* f *itself is also an isomorphism (in* $\mathcal{D}$*).*

Proof. Let $\mathbf{F}$ be left adjoint to $\mathbf{G}$ so that the adjunction $\mathbf{F} \dashv \mathbf{G}$ is a monadic adjunction and let $\mathbf{T} = \mathbf{G} \circ \mathbf{F} : \mathcal{C} \to \mathcal{C}$. Define, as described above, the functor $G^{\mathbf{T}} : \mathcal{C}^{\mathbf{T}} \to \mathcal{C}$.

For part (1), we first prove that $G^{\mathbf{T}}$ is faithful. Suppose $f, g : (D, d) \to (D', d')$ in $\mathcal{C}^{\mathbf{T}}$, and suppose $G^{\mathbf{T}}(f) = G^{\mathbf{T}}(g)$. This means that $f : D \to D'$ and $g : D \to D'$ are equal as $\mathcal{C}$-morphisms, as required.

To finish the proof of (1), suppose $f, g : D \to D'$ are $\mathcal{D}$-morphisms and $\mathbf{G}(f) = \mathbf{G}(g)$. By Equation (85), $G^{\mathbf{T}}(\Phi(f)) = G^{\mathbf{T}}(\Phi(g))$. By the first part of the proof of (1), $\Phi(f) = \Phi(g)$. Applying Φ^{-1} to both sides yields $f = g$, as required.

For part (2), we first prove that $G^{\mathbf{T}}$ reflects isomorphisms. Suppose $f : (A, a) \to (B, b)$ is a $\mathcal{C}^{\mathbf{T}}$-morphism, so that $f : A \to B$ is a $\mathcal{C}$-morphism for which the following is commutative:

(86)
$$\begin{array}{ccc} \mathbf{T}(A) & \xrightarrow{\mathbf{T}(f)} & \mathbf{T}(B) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

Suppose also that $f : A \to B$ is a $\mathcal{C}$-isomorphism. Let g be the inverse of f in $\mathcal{C}$. We show that $g : B \to A$ is in fact a $\mathcal{C}^{\mathbf{T}}$-morphism $g : (B, b) \to (A, a)$ by showing that the following diagram is commutative:

(87)
$$\begin{array}{ccc} \mathbf{T}(B) & \xrightarrow{\mathbf{T}(g)} & \mathbf{T}(A) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{g} & A \end{array}$$

From Diagram (86) we have the following in $\mathcal{C}$:

$$b \circ \mathbf{T}(f) = f \circ a.$$

Since $\mathbf{T}(f) \circ \mathbf{T}(g) = \mathbf{T}(f \circ g) = 1_B$, we can compose with $\mathbf{T}(g)$ on the right and compose with g on the left to obtain:

$$g \circ b = a \circ \mathbf{T}(g),$$

which demonstrates commutativity of Diagram (87). Now $g : (B, b) \to (A, a)$ is the inverse of $f : (A, a) \to (B, b)$ in $\mathcal{C}^{\mathbf{T}}$ since composition of $\mathcal{C}^{\mathbf{T}}$-morphisms is done by composing the corresponding $\mathcal{C}$-morphisms and corresponding diagrams.

Next, we show that $\mathbf{G} : \mathcal{D} \to \mathcal{C}$ itself reflects isomorphisms. Suppose $f : D_1 \to D_2$ is a $\mathcal{D}$-morphism and $\mathbf{G}(f) : \mathbf{G}(D_1) \to \mathbf{G}(D_2)$ is a $\mathcal{C}$-isomorphism. By Equation (85), $\mathbf{G}(f) = G^{\mathbf{T}}(\Phi(f))$. Now $\Phi(f) : \Phi(D_1) \to \Phi(D_2)$ is a $\mathcal{C}^{\mathbf{T}}$-morphism. Since $G^{\mathbf{T}}(\Phi(f))$ is a $\mathcal{C}$-isomorphism, by the first half of the proof of (2), it follows that $\Phi(f)$ is a $\mathcal{C}^{\mathbf{T}}$-isomorphism. Since Φ^{-1} is also an isomorphism and since functors that are isomorphisms preserve isomorphisms, it follows that $\Phi^{-1}(\Phi(f)) = f$ is an isomorphism in $\mathcal{D}$, and the result follows. □

Lemma 57. (Faithful Functors Reflect Isomorphisms) *Suppose* $F : \mathbf{Set} \to \mathcal{D}$ *is a faithful functor. Then* F *reflects isomorphisms.*

Note. See the previous lemma for definition of terms.

Proof. We introduce two new bits of terminology to facilitate the proof. In any category, a morphism $\alpha : B \to C$ is *monic* if for all morphisms $x, y : A \to B$, $\alpha \circ x = \alpha \circ y$ implies $x = y$. On the other hand, given α, if, for all morphisms $u, v : C \to D$, $u \circ \alpha = v \circ \alpha$ implies $u = v$, then α is said to be epic. In **Set**, a morphism that is both monic and epic is an isomorphism (that is, a bijection). In general categories, while it is true that every isomorphism is both monic and epic, it is not true that a morphism which is both monic and epic must be an isomorphism.

We show that, in general, faithful functors reflect both monics and epics. By the above observations, it will follow that if a functor is defined on **Set**, it must reflect isomorphisms.

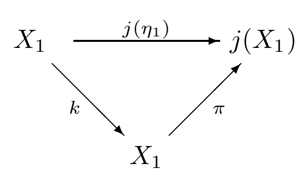
Suppose $F : \mathcal{C} \to \mathcal{D}$ is faithful and suppose $f : B \to C$ is a $\mathcal{C}$-morphism.

To see that F reflects monics, assume $F(f) : F(B) \to F(C)$ is monic and let $x, y : X \to B$ be $\mathcal{C}$-morphisms such that $f \circ x = f \circ y$. It follows that $F(f) \circ F(x) = F(f) \circ F(y)$, and so, since $F(f)$ is monic, $F(x) = F(y)$. Now since F is faithful, it follows $x = y$.

To see that F reflects epics, assume $F(f) : F(B) \to F(C)$ is epic and let $u, v : C \to Y$ be $\mathcal{C}$-morphisms such that $u \circ f = v \circ f$. It follows that $F(u) \circ F(f) = F(v) \circ F(f)$ and, since $F(f)$ is epic, $F(u) = F(v)$. Again, since F is faithful, $u = v$. □

We can now show that, whenever (j, η, μ) is a monad obtained as $j = \mathbf{G} \circ \mathbf{F}$, where $\mathbf{G} : \mathbf{SM} \to V$ is the forgetful functor and $\mathbf{F} \dashv \mathbf{G}$, there is a naturally defined *set* Dedekind self-map $k : X_1 \to X_1$.

Theorem 58. *Let* (j, η, μ) *be the monad defined above. Let* $X_1 = j(1)$. *Then* $j(\eta_1) : X_1 \to j(X_1)$ *is a Dedekind map with factorization* $j(\eta_1) = \pi \circ k$, *as described above.*



In particular, $k : X_1 \to X_1$ *is a Dedekind self-map.*

This theorem will follow as a corollary to a more general result, formulated in Theorem 63 below.

Theorem 59 tells us that if a functor $j : V \to V$ happens to admit the special factorization $j = \mathbf{G} \circ \mathbf{F}$ (where $\mathbf{F}$ and $\mathbf{G}$ are as in the Lawvere construction), then a *set* Dedekind self-map is derivable. But is there some criterion for emergence of a set Dedekind self-map that is "internal" to j, and not dependent upon externally defined categories and functors (namely, $\mathbf{SM}, \mathbf{F}$, and $\mathbf{G}$)? We suggest one such criterion, based on the following definition.

Definition 17. (Dedekind Monad) Suppose (j, η, μ) is a monad, where $j : V \to V$. Then (j, η, μ) is a *Dedekind monad* if the following properties hold:

(1) On objects, j is 1-1 but not onto.
(2) $j(0) = 0$.
(3) For some V-object $c \notin \operatorname{ran} j$,

$$|c| < |j(c)| = |j(j(c))|.$$

The set c mentioned in (3) will be called a *canonical critical point of* j. If $j : V \to V$, (j, η, μ) is a monad, and j is naturally isomorphic to the functor part of a Dedekind monad, then we shall say that (j, η, μ) is an *essentially Dedekind monad*.[135]

We will now show from the theory ZFC−Infinity that, whenever we have a functor $j : V \to V$ that is the functor part of an essentially Dedekind monad (j, η, μ), then there is a set Dedekind self-map k that naturally arises from j. We begin with a lemma that tells us that whenever the components of the unit of a monad are 1-1, the monad reflects isos; this property of a monad is one of the keys to ensure that it gives rise to a Dedekind self-map.

Lemma 59. (ZFC−Infinity) *Suppose $j : V \to V$ and (j, η, μ) is a monad. Suppose also that for each set B, $\eta_B : B \to j(B)$ is 1-1. Then*

(A) *j is a faithful functor, and*
(B) *j reflects isomorphisms.*

Proof of (A). Let $F^j : V \to V^j, G^j : V^j \to V$ be the adjoint functors induced by j, as described above, and let Θ be the natural bijection for the adjunction. By Remark 18, G^j is monadic. It follows from Lemma 57 that G^j is faithful. Recalling that $j = G^j \circ F^j$, it therefore suffices to show that F^j is faithful.

Let $f, g : A \to B$ be functions and assume $F^j(f) = F^j(g)$; we show $f = g$. We let $\overline{F^j(f)} = \Theta_{A,F^j(B)}(F^j(f))$ and $\overline{F^j(g)} = \Theta_{A,F^j(B)}(F^j(g))$. Applying Lemma 91 (see the Appendix), we have

$$\eta_B \circ f = \overline{F^j(f)} = \overline{F^j(g)} = \eta_B \circ g.$$

Since η_B is 1-1 (by assumption), $f = g$, as required.

Proof of (B). We show $j = G^j \circ F^j$ reflects isomorphisms. Since G^j is monadic (by Remark 18), G^j reflects isos (by Lemma 57), so it suffices to show that F^j reflects isomorphisms. But this follows from the fact that F^j is faithful (by Lemma 58). □

[135] Note that an essentially Dedekind monad is not necessarily isomorphic to a Dedekind monad in the sense of *monad isomorphism*; having the functor parts of two monads be naturally isomorphic is a weaker condition than requiring the two monads to admit a monad isomorphism between them. This stronger requirement, though perhaps more natural, is not necessary for our purposes.

Lemma 60. (ZFC – Infinity) *Suppose $j : V \to V$ and (j, η, μ) is an essentially Dedekind monad. Then for each set B, $\eta_B : B \to j(B)$ is 1-1. In particular, j reflects isomorphisms.*

Proof. The last clause follows from Lemma 60. Suppose $j : V \to V$ and (j, η, μ) is an essentially Dedekind monad. Let $j' : V \to V$ be such that (j', η', μ') is a monad and j' is naturally isomorphic to j.

Suppose x, y are distinct elements of B. (Note that if c has fewer than two elements, there is nothing to prove.) We show $\eta_B(x) \neq \eta_B(y)$. Let $G^j : V^j \to V$ and $F^j : V \to V^j$ be the adjoint functors defined from j, as described above.

We wish to obtain a j-algebra (X, a) for which X has two or more elements. Recall from Remark 19 that among monads whose functor part T is defined on V, there are only two for which all T-algebras have at most one element. One of these takes every set to 1; the other takes every nonempty set to 1 and takes 0 to 0.

We first show why j cannot be a functor of the first type. Since j' is a Dedekind monad, $j'(0) = 0$. Since $j \cong j'$, $|j(0)| = |j'(0)| = 0$, and so $j(0) = 0$ also. Therefore, j could not be a functor of the first type.

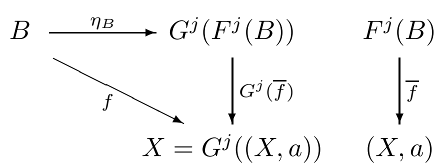
Suppose j is a functor of the second type. Let c be a canonical critical point for j'. Notice $c \neq 0$ since $j'(0) = 0$ but $|c| < |j'(c)|$. Therefore, we have $1 \leq |c| < |j'(c)| = |j(c)|$. It follows that $|j(c)| > 1$, which contradicts the assumption that j is of the second type (which would require that $j(c) = 1$).

We have shown that there is a j-algebra (X, a) for which $|X| > 1$. Pick two distinct elements u, v in X.

Next, we define $f : B \to G^j((X, a)) = X$ by

$$f(z) = \begin{cases} u & \text{if } z = x \\ v & \text{if } z = y \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

By the universal property of η, there is a (unique) $\overline{f} : F^j(B) \to (X, a)$ so that the following diagram is commutative:



Now if $\eta_B(x) = \eta_B(y)$, it follows by commutativity of the diagram that $f(x) = f(y)$, which is impossible. We have shown $\eta_B(x) \neq \eta_B(y)$. Therefore η_B is 1-1. □

Our final lemma shows that, although an essentially Dedekind monad may not be a Dedekind self-map (as a functor), it does have the other two properties of Dedekind monads.

Lemma 61. (ZFC – Infinity) *Suppose (j, η, μ) is an essentially Dedekind monad and (j', η', μ') is a Dedekind monad with $j' : V \to V$ and $j \cong j'$. Let c be a canonical* *critical point of* j'. *Then* $j(0) = 0$ *and*

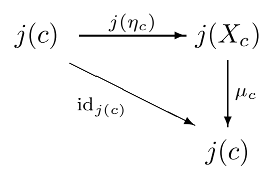
$$|c| < |j(c)| = |j(j(c))|.$$

Proof. Let $(j, \eta, \mu), (j', \eta', \mu'), c$ be as in the hypothesis. The fact that $j(0) = 0$ was shown in the proof of Lemma 60. Because j' is a Dedekind monad and $j \cong j'$, we have $|c| < |j'(c)| = |j(c)|$.

Let $\sigma : j \to j'$ be a natural iso. Since each of $j(\sigma_c) : j(j(c)) \to j(j'(c))$ and $\sigma_{j'(c)} : j(j'(c)) \to j'(j'(c))$ is a bijection, the composition $\sigma_{j'(c)} \circ j(\sigma_c) : j(j(c)) \to j'(j'(c))$ is a bijection as well. It follows that $|j(c)| = |j'(c)| = |j'(j'(c))| = |j(j(c))|$. □

Theorem 62. (ZFC – Infinity) *Suppose* $j : V \to V$ *and* (j, η, μ) *is an essentially Dedekind monad. Let* (j', η', μ') *be a Dedekind monad with* $j \cong j'$ *and such that* c *is a canonical critical point of* j'. *Let* $X_c = j(c)$. *Then there is a Dedekind self-map* $k : X_c \to X_c$.

Proof. By diagram (78), since (j, η, μ) is a monad, we have



Commutativity of the diagram implies that $j(\eta_c)$ is 1-1. Note that if $j(\eta_c)$ were a bijection, then, since j reflects isos (by Lemmas 60(B) and 61), it would follow that $\eta_c : c \to j(c) = X_c$ is also a bijection, which is impossible since, by Lemma 62, $|c| < |j(c)| = |X_c|$. Let $b \in j(X_c) - \operatorname{ran} j(\eta_c)$.

Next we observe that $j(\eta_c)$ is a Dedekind map (Definition 14). We have just shown $j(\eta_c)$ is 1-1 and has a critical point b. The fact that $|X_c| = |j(X_c)|$ follows from Lemma 62. We may therefore apply Proposition 55 to conclude that there is a Dedekind self-map $k : j(c) \to j(c)$. □

A reasonable question is whether there could exist a monad $j : V \to V$ that is *not* a Dedekind monad. In other words, does property (2) in the definition of Dedekind monad hold for every Dedekind self-map monad on V?

We give an example to show that this is not the case; verification of details for this example can be found in [2]. Consider the mapping $\mathcal{P} : V \to V$ defined by sending every set X to its power set $\mathcal{P}(X)$. For any $f : X \to Y$, define $\mathcal{P}(f)$ by

$$\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) : x \mapsto f[x].$$

Therefore, $\mathcal{P}$ is a functor. One may easily verify that $\mathcal{P}$ is 1-1 on objects and has a critical point (also a strong critical point) $\emptyset$.

Define the unit $\eta : 1 \to \mathcal{P}$ by

$$\eta_A : A \to \mathcal{P}(A) : x \mapsto \{x\}.$$

Finally, the multiplication operation $\mu : \mathcal{P}^2 \to \mathcal{P}$ is defined by

$$\mu_A : \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A) : Z \mapsto \bigcup Z.$$

It can be shown [2] that $(\mathcal{P}, \eta, \mu)$ is a monad on V—the *power set monad*—that satisfies part (1) of the definition of a Dedekind monad. However, $(\mathcal{P}, \eta, \mu)$ is not a Dedekind monad since, for any set c, $|\mathcal{P}(c)| < |\mathcal{P}(\mathcal{P}(c))|$, violating part (2) of the definition. Note that existence of the power set monad is provable in ZFC − Infinity; it does not imply the existence of an infinite set.

As promised earlier, we can now show that any monad (j, η, μ) obtained from the Lawvere construction is an essentially Dedekind monad, and so, by Theorem 63, gives rise to a set Dedekind self-map $X_1 \to X_1$ (recalling that in this case, 1 is always a critical point of j and $X_1 = j(1)$). Theorem 59 will then follow as a corollary.

Theorem 63. *Suppose $j : V \to V$ is obtained as $j = \mathbf{G} \circ \mathbf{F}$ as in the Lawvere construction and let (j, η, μ) be the monad induced by j. Then (j, η, μ) is an essentially Dedekind monad.*[136]

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[136] It is an instructive exercise to work out the details of a specific example of the Lawvere construction to see how an essentially Dedekind monad arises. We outline here some of the computations for one such example and leave the verifications to the reader. For this purpose, the reader may find some of the details provided in the Appendix to be useful.

As shown in the Appendix, in the proof of Theorem 93(1), in the presence of ω, a left adjoint $\mathbf{F}$ to the forgetful functor $\mathbf{G} : \mathbf{SM} \to V$ is defined on objects by $\mathbf{F}(A) = 1_A \times s : A \times \omega \to A \times \omega$, where $(1 \times s)(a, n) = (a, s(n)) = (a, n + 1)$. The definition of $\mathbf{F}$ on $\mathbf{SM}$-morphisms is given by the following: Given $f : A \to B$ in $\mathbf{SM}$, $\mathbf{F}(f) : \mathbf{F}(A) \to \mathbf{F}(B)$ is the $\mathbf{SM}$-morphism $\phi = \phi_f$ defined by $\phi(a, n) = (f(a), n)$; one verifies that this definition of ϕ makes the following diagram commutative:

(88)
$$\begin{array}{ccc} A \times \omega & \xrightarrow{1_A \times s} & A \times \omega \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ B \times \omega & \xrightarrow{1_B \times s} & B \times \omega \end{array}$$

Given a set A and an $\mathbf{SM}$ object $\beta : B \to B$, the natural bijection $\Theta_{A,\beta} : \mathbf{SM}(\mathbf{F}(A), \beta) \to \mathbf{Set}(A, \mathbf{G}(\beta))$ is defined, for any $\rho \in \mathbf{SM}(\mathbf{F}(A), \beta)$, by

$$\Theta_{A,\beta}(\rho)(a) = \rho(a, 0).$$

Subclaim (ii) in the proof of Theorem 93(1) shows how $\Theta^{-1}_{A,\beta} : \mathbf{Set}(A, \mathbf{G}(\beta)) \to \mathbf{SM}(\mathbf{F}(A), \beta)$ is defined: For any $f : A \to B = \mathbf{G}(\beta)$,

(89)
$$\Theta^{-1}_{A,\beta}(f) : A \times \omega \to B : (a, n) \mapsto \beta^n(f(a)).$$

The unit $\eta : 1 \to \mathbf{G} \circ \mathbf{F}$ of the adjunction is defined as follows. For each A, $\eta_A : A \to \mathbf{G}(\mathbf{F}(A)) = j(A) = A \times \omega$ is defined by

$$\eta_A = \Theta_{A,\mathbf{F}(A)}(1_{\mathbf{F}(A)}),$$

so that

$$\eta_A(a) = \Theta_{A,\mathbf{F}(A)}(1_{\mathbf{F}(A)})(a) = 1_{\mathbf{F}(A)}(a, 0) = (a, 0).$$

Dually, we define the co-unit $\varepsilon : \mathbf{F} \circ \mathbf{G} \to 1$. For each $\beta : B \to B$, $\varepsilon_\beta : \mathbf{F}(\mathbf{G}(\beta)) \to \beta$ (that is, $\varepsilon_\beta : (1_B \times s : B \times \omega \to B \times \omega) \to (\beta : B \to B)$) is defined by

$$\varepsilon_\beta = \Theta^{-1}_{\mathbf{G}(\beta),\beta}(1_{\mathbf{G}(\beta)}).$$

Applying (89), we have

(90)
$$\varepsilon_\beta = \Theta^{-1}_{A,\beta}(1_{\mathbf{G}(\beta)}) : (b, n) \mapsto \beta^n(b).$$

Proof. We observed earlier that 1 is a critical point of j. In particular, we show in the Appendix (Theorem 93(2)) that $1 < j(1)$. We also show in the Appendix (Theorem 93(1)) that j is naturally isomorphic to a Dedekind self-map $j' : V \to V$ functor, defined on objects by $j'(A) = A \times \omega$. Using any such natural isomorphism,

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Since $(\mathbf{F}, \mathbf{G}, \eta, \varepsilon)$ is an adjunction, it determines a monad (j, η, μ), where $j = \mathbf{G} \circ \mathbf{F}$ and $\mu : j^2 \to j$ is defined by

$$\mu_A(x) = \left(\mathbf{G}(\varepsilon_{\mathbf{F}(A)})\right)(x),$$

for all sets A in V and $x \in j^2(A)$.

We unwind the definition of μ. First, we observe that the domain of $\varepsilon_{\mathbf{F}(A)}$ is

$$(\mathbf{F} \circ \mathbf{G} \circ \mathbf{F})(A) = \mathbf{F}(\mathbf{G}(1_A \times s)) = \mathbf{F}(A \times \omega) = 1_{A\times\omega} \times s : A \times \omega \times \omega \to A \times \omega \times \omega,$$

and the codomain is $1_A \times s$. We have, for every $((a, m), n) = (a, m, n) \in A \times \omega \times \omega$,

$$\varepsilon_{\mathbf{F}(A)}((a, m), n) = (1_A \times s)^n(a, m) = (a, s^n(m)) = (a, m + n)$$

(recalling that this map also makes the appropriate diagram commutative).

Now

$$\mu_A = \mathbf{G}(\varepsilon_{\mathbf{F}(A)}) : (\mathbf{G} \circ \mathbf{F} \circ \mathbf{G} \circ \mathbf{F})(A) \to (\mathbf{G} \circ \mathbf{F})(A),$$

—in other words

$$\mathbf{G}(\varepsilon_{\mathbf{F}(A)}) : A \times \omega \times \omega \to A \times \omega$$

—is computed as follows:

$$\mu_A(a, m, n) = \mathbf{G}(\varepsilon_{\mathbf{F}(A)})(a, m, n) = (a, m + n).$$

This monad (j, η, μ)—called the *Lawvere monad*—generates the category V^j of j-algebras, where $V^j = \{(A, a) \mid a : A \times \omega \to A\}$, where each j-algebra (A, a) satisfies

$$a \circ \eta_A = 1_A \quad \text{and} \quad a \circ j(a) = a \circ \mu_A.$$

Note that $j(a) : A \times \omega \times \omega \to A \times \omega$.

We also obtain an adjunction $F^j \dashv G^j$ where $F^j : V \to V^j$, $G^j : V^j \to V$, are defined on objects by

$$\begin{array}{rcl} F^j(A) & = & (A \times \omega, \mu_A) \\ G^j(A, a) & = & A, \end{array}$$

Next, we look more closely at the Dedekind characteristics of this monad. We first recall that, since $j(1) = 1\times\omega$ and $j(j(1)) = 1\times\omega\times\omega$, we have the Dedekind properties $|\mathrm{crit}(j)| < |j(\mathrm{crit}(j))| = |j(j(\mathrm{crit}(j)))|$. Next, we compute η_1 and $j(\eta_1)$.

(91) $$\eta_1 : 1 \to j(1) = 1 \times \omega : 0 \mapsto (0, 0).$$

Referring to the definition of $\mathbf{F}$ on **SM**-morphisms, $\mathbf{F}(\eta_1) : 1 \times \omega \to 1 \times \omega \times \omega$ is defined to be the map $\phi = \phi_{\eta_1}$—defined by $\phi(0, n) = (\eta_1(0), n) = (0, 0, n)$—making the following commutative:

(92) $$\begin{array}{ccc} 1 \times \omega & \xrightarrow{1_1 \times s} & 1 \times \omega \\ \downarrow \phi & & \downarrow \phi \\ 1 \times \omega \times \omega & \xrightarrow{1_{1\times\omega} \times s} & 1 \times \omega \times \omega \end{array}$$

Therefore,

$$j(\eta_1) = \mathbf{G}(\mathbf{F}(\eta_1)) : 1 \times \omega \to 1 \times \omega \times \omega : (0, n) \mapsto (0, 0, n).$$

We wish to obtain a Dedekind self-map $k : 1 \times \omega \to 1 \times \omega$ following the techniques of the proof of Proposition 55. For this purpose, we offer a fairly natural bijection $\pi : 1 \times \omega \times \omega \to 1 \times \omega$:

$$\pi(0, m, n) = (0, \langle m, n \rangle),$$

and recalling that $X_1 = j(1)$, it follows that

$$|j(X_1)| = |j'(X_1)| = |X_1 \times \omega| = |1 \times \omega \times \omega| = |1 \times \omega| = |j'(1)| = |j(1)| = |X_1|.$$

We have shown therefore that $1 < j(1)$ and $|j(1)| = |j(j(1))|$. It follows that (j, η, μ) is an essentially Dedekind monad. □

The concept of a Dedekind monad leads to yet another equivalent of the Axiom of Infinity:

Theorem 65. (ZFC − Infinity) *The following are equivalent.*

(1) *There is an infinite set.*

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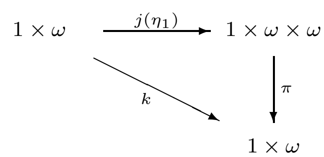
where $\langle - \rangle$ is a definable (bijective) pairing function $(m, n) \mapsto n + \frac{(n+m)(n+m+1)}{2}$. The Dedekind self-map k is defined, as in the proof of Proposition 55, to be $\pi \circ j(\eta_1)$.



A straightforward computation yields

$$k(0, n) = \pi(j(\eta_1)(0, n)) = (0, \langle 0, n \rangle).$$

For instance,

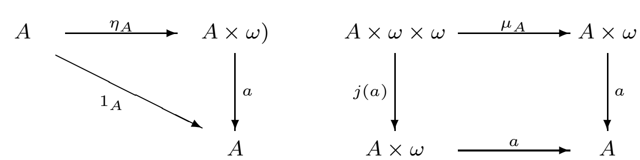
$$k(0, 0) = (0, 0); \quad k(0, 1) = (0, 2); \quad k(0, 2) = (0, 5).$$

Since k is strictly increasing in the second component, the lexicographically least critical point of k is $(0, 1)$.

Finally, let us make an observation about the j-algebras for the Lawvere monad.

Theorem 64. (j-Algebras for the Lawvere Monad) (ZFC − Infinity) *Let (j, η, μ) denote the Lawvere monad and let V^j be the category of j-algebras. Then for each set A, A is the underlying set for a j-algebra (A, a).*

Proof. Given a set A, we obtain $a : j(A) = A \times \omega \to A$ so that the commutative diagrams (80) are commutative. We define a by $a(x, m) = x$. We check the diagrams (80), which assume the following form in the present context:

(93)


In the present context, note that $j(a)$ is defined by $j(a)((x, m), n) = (x, m)$. For the left diagram, we have, for each $x \in A$:

$$a(\eta_A(x)) = a(x, 0) = x = 1_A(x).$$

And for the right diagram, we have

$$a(\mu_A(x, m, n)) = a(x, m + n) = x = a(x, n) = a\Big(j(a)(x, m, n)\Big). \square$$

(2) *There is a Dedekind monad.*[137]

Proof. For (1) $\Rightarrow$ (2), use the fact that existence of an infinite set implies existence of ω. Theorem 64 implies that any left adjoint F' of the forgetful functor $\mathbf{G} : \mathbf{SM} \to V$ yields a Dedekind monad $\mathbf{G} \circ F'$; in the presence of ω, such an F' may be defined on objects by $F'(A) = 1_A \times s : A \times \omega \to A \times \omega$, as discussed in Theorem 93 in the Appendix. Conversely, (2) $\Rightarrow$ (1) follows from Theorem 59 and the fact that existence of a Dedekind self-map on a set implies existence of an infinite set. □

21. Tools for Generalizing to a Context for Large Cardinals

As we discussed at the beginning of this paper, one of our motivations for formulating the New Axiom of Infinity is to provide, in the structure of the axiom itself, the sort of intuition about the "Infinite" that could be useful in addressing the Problem of Large Cardinals. In particular, we are looking for patterns in the statement of the axiom that would be natural choices for generalization; the intended result would be stronger, naturally motivated axioms that could possibly be used to derive known large cardinal properties.

In this section, we collect together patterns of this kind that seem amenable to natural strengthenings.

21.1. The Structure of a Dedekind Self-Map.

Our New Axiom of Infinity states simply that a Dedekind self-map on a set exists. The underlying intuition is that the discrete values that belong to an infinite collection arise as a consequence of more fundamental transformational dynamics, namely, some $j : A \to A$ having a critical point a. We found that such a j has three salient characteristics:

(1) It preserves essential features of its domain: The image $B = j[A]$ is itself a Dedekind infinite set, and $j \upharpoonright B : B \to B$ is a Dedekind self-map with critical point $j(a)$.
(2) It is not simply the identity map—some element of A is moved by j.
(3) Something interesting happens in the interaction between j and its critical point. In this case, a blueprint $W \subseteq A$ of the set ω of natural numbers arises from repeated application of j to a: $W = \{a, j(a), j(j(a)), \ldots\}$.

The intuition that arises from this formulation of the mathematical infinite is that perhaps large cardinal notions also arise from the dynamics of a Dedekind self-map. Since large cardinal properties often involve the entire universe, it is reasonable to look for answers regarding large cardinals in Dedekind self-maps from V to V.

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[137] Some care is needed in the formulation of this theorem since we appear to be quantifying over a class. To re-state (1) $\Rightarrow$ (2) properly, we would specify a formula for the functor $\mathbf{F} : V \to \mathbf{SM}$ (defined above) that takes a set A to $1_A \times s : A \times \omega \to A \times \omega$ and assert that it is left adjoint to the forgetful functor $\mathbf{G} : \mathbf{SM} \to V : (h : A \to A) \mapsto A$, and that the monad induced by $\mathbf{G} \circ \mathbf{F}$ is a Dedekind monad. To re-state (2) $\Rightarrow$ (1) properly requires a schema of statements, one for each formula ψ that defines a Dedekind monad (i, η, μ); let ρ be a subformula of ψ that defines i. For each such ψ, we would have a formula $\sigma(a, k)$ that asserts (using ρ) that a is a critical point of j satisfying the Dedekind conditions and $k : j(a) \to j(a)$ is a Dedekind self-map. Then, for each such ψ, we would include the following statement in the schema:

$$\psi \to \exists a, k\, \sigma(a, k).$$

As we have already observed, existence of a Dedekind self-map $j : V \to V$ is not enough to imply the existence of infinite sets, even though the *set* version of Dedekind self-maps *is* sufficient. We collect together the techniques we have discussed for strengthening Dedekind self-maps $j : V \to V$ so that existence of infinite sets can be derived. We view these techniques as candidates for further development in the direction of deriving large cardinals.

We review the content of Theorems 44, 45, and 47:

Theorem 44. (ZFC – Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map. Suppose also that j preserves disjoint unions, the empty set, and singletons. Then there is an infinite set. Indeed, every critical point of j is contained in an infinite transitive set.*

Theorem 45. (ZFC – Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map with critical point a. Suppose also that j preserves disjoint unions, intersections, and the empty set, and that there is a set A such that $a \in j(A)$. Then there is an ultrafilter D on A. Moreover, if either of the following conditions holds, then D is nonprincipal and A is infinite.*

(1) *j preserves singletons;*

(2) *j preserves terminal objects and $\{a\}$ is also a critical point of j.*

Theorem 47. (ZFC – Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map with a strong critical point. Suppose j preserves finite coproducts and terminal objects. Then there is an infinite set. Indeed, any strong critical point of j is infinite.*

We see in each of these results how requiring a class Dedekind self-map $j : V \to V$ to have certain combinations of preservation properties results in an infinite set, and that these combinations produce an infinite set that can typically be located at a (strong) critical point of j. A natural direction for further development, then, would be to require j to satisfy additional preservation properties and to see whether the properties of a critical point for j are strengthened in the direction of large cardinals.

Corollary 51 shows that whenever j strongly preserves $\in$ and preserves rank and ordinals, and has any type of critical point, we can locate a *least ordinal moved* by j. Recall that for such j (see p. 122), we let $\text{crit}(j)$ denote the least ordinal moved by j; $\text{crit}(j)$ will be a natural critical point to examine as we seek to "generate" large cardinals from preservation properties.

In this section we will provide two strengthenings of these preservation properties that lead to existence of large cardinals. We focus in this section on perhaps the simplest (and certainly one of the weakest) large cardinal notions—inaccessible cardinals—in order to illustrate the techniques. In Section 22, we will discuss other large cardinals and expand the techniques introduced here so that even the strongest types of large cardinals can be derived.

If κ is an infinite cardinal,[138] a subset A of κ is *unbounded in* κ if, for all $\beta < \kappa$, there is $\gamma \in A$ such that $\beta < \gamma$. An infinite cardinal κ is *regular* if unbounded subsets of κ have size κ;[139] κ is a *strong limit* if for every $\lambda < \kappa$, $2^\lambda < \kappa$. An uncountable cardinal κ is inaccessible if it is regular and a strong limit. We begin with a few new preservation properties that Dedekind self-maps $j : V \to V$ may satisfy.

Definition 18. Suppose $j : V \to V$ is a Dedekind self-map.

(1) *j preserves countable disjoint unions* if, whenever $\langle X_n \mid n \in N\rangle$ (where either $N \in \omega$ or $N = \omega$) is a sequence of disjoint sets, $j\left(\bigcup_{n\in N} X_n\right) = \bigcup_{n\in N} j(X_n)$.
(2) *j preserves unboundedness* if, whenever $\alpha \in \mathrm{ON}$ and $A \subseteq \alpha$ is unbounded in α, then, if $j(A) \subseteq j(\alpha)$, then $j(A)$ is unbounded in $j(\alpha)$.
(3) *j preserves power sets* if, for all X, $j(\mathcal{P}(X)) = \mathcal{P}(j(X))$.
(4) Suppose $f : A \to B$ and $g : A \to B$ are functions. The *equalizer $E = E_{f,g}$ of f and g* is the set $\{x \in A \mid f(x) = g(x)\}$. If j is a functor, j is said to *preserve equalizers* if for any f, g as above, $j(E_{f,g}) = E_{j(f),j(g)}$; that is, if E is the equalizer of f and g, then $j(E)$ is the equalizer of $j(f)$ and $j(g)$.

Theorem 66. (Generalization of Theorem 44 to Inaccessibles) (ZFC − Infinity) *Suppose $j : V \to V$ is a Dedekind self-map that strongly preserves $\in$ and preserves ordinals and rank. Let $\kappa = \mathrm{crit}(j)$.*

(A) *Suppose j preserves countable disjoint unions, the empty set, and singletons. Then $\kappa > \omega$.*
(B) *Suppose j is BSP and preserves countable disjoint unions, the empty set, singletons, and unboundedness. Then κ is an uncountable regular cardinal.*
(C) *Suppose j is BSP and preserves countable disjoint unions, the empty set, singletons, unboundedness, and power sets. Then κ is an inaccessible cardinal.*

Proof of (A). By Theorem 44, ω exists; the proof of that theorem shows that, under these hypotheses, $j \upharpoonright \mathbf{HF} = \mathrm{id}_{V_\omega}$; it follows that $\kappa \geq \omega$. We have, by Theorem 50 and the properties that j preserves:

$$\begin{aligned}
\omega &= \{0\} \cup \{1\} \cup \{2\} \cup \cdots \\
&= \{j(0)\} \cup \{j(1)\} \cup \{j(2)\} \cup \cdots \\
&= j(\{0\}) \cup j(\{1\}) \cup j(\{2\}) \cup \cdots \\
&= j\left(\{0\} \cup \{1\} \cup \{2\} \cup \cdots\right) \\
&= j(\omega).
\end{aligned}$$

It follows that $\kappa > \omega$.

Proof of (B). By Theorem 50(5), κ is a cardinal. By (A), κ is an uncountable cardinal. Suppose $f : \alpha \to \kappa$ and $\operatorname{ran} f$ is unbounded in κ. Using the BSP property of j, we can reason as in the proof of Theorem 50(5) to show that $j(f) : j(\alpha) \to j(\kappa)$

[138] Recall that since each cardinal is an ordinal, as a set it consists of all the ordinals that precede it.

[139] A different but equivalent definition of *regular cardinal* was given on page 4.

has domain α and $j(f)(\beta) = f(\beta)$ for all $\beta < \alpha$. Because j preserves images, we have

$$\text{(94)} \qquad \operatorname{ran} j(f) = j(\operatorname{ran} f) \subseteq \kappa,$$

but, because j also preserves unboundedness and $\operatorname{ran} f$ is unbounded in κ,

$$\text{(95)} \qquad j(\operatorname{ran} f) \text{ is unbounded in } j(\kappa).$$

Clearly, since $\kappa < j(\kappa)$, (94) and (95) contradict each other. Therefore, all functions from α to κ have bounded range. It follows, therefore, that κ is regular. We have shown κ is an uncountable regular cardinal.

Proof of (C). We begin by showing that, for every $A \subseteq \alpha$, where $\alpha < \kappa$, we have $j(A) = A$. Because $\alpha < \kappa$ and $\in$ is preserved, $A \subseteq j(A)$. To show $j(A) \subseteq A$, we first observe that $j(A) \subseteq \alpha$. Using the fact that j preserves $\in$ and power sets, we have

$$A \in \mathcal{P}(\alpha) \Rightarrow j(A) \in j(\mathcal{P}(\alpha)) = \mathcal{P}(j(\alpha)) = \mathcal{P}(\alpha).$$

Now, suppose $\gamma \in j(A) - A$. Note that α is the disjoint union of A and $\alpha - A$: $\alpha = A \cup (\alpha - A)$. Since $\gamma \notin A$, it follows that $\gamma \in \alpha - A$. Since j preserves disjoint unions, $\alpha = j(\alpha) = j(A) \cup j(\alpha - A)$. Since $\gamma \in \alpha - A$, $\gamma = j(\gamma) \in j(\alpha - A)$, and so $\gamma \notin j(A)$, which is a contradiction. We have shown $A = j(A)$.

Continuing with the proof that κ is a strong limit, suppose, for a contradiction, that there is a surjective function $g : \mathcal{P}(\alpha) \to \kappa$, where $\alpha < \kappa$. Since j preserves functions and power sets and $j(\alpha) = \alpha$, we have that $j(g) : \mathcal{P}(\alpha) \to j(\kappa)$. Note that, for each $A \in \mathcal{P}(\alpha)$, $g(A) \in \kappa$, whence $j(g(A)) = g(A)$. Therefore, for each $A \in \mathcal{P}(\alpha)$,

$$j(g)(A) = j(g)(j(A)) = j(g(A)) = g(A).$$

Therefore, we have

$$\text{(96)} \qquad \operatorname{ran} j(g) = \operatorname{ran} g = \kappa.$$

We also have, by preservation of images,

$$\text{(97)} \qquad \operatorname{ran} j(g) = j(\operatorname{ran} g) = j(\kappa) > \kappa.$$

Clearly, (96) and (97) contradict each other, and so no such function g exists. We have shown κ is a strong limit, and hence, inaccessible. □

The next result generalizes the approaches in Theorems 45 and 47, where different preservation properties, in some cases of a more category-theoretic flavor, were used. These results also incorporate observations from Example 3, which illustrate the role of ultrafilters in climbing further up the hierarchy of infinities. And finally, the important role of universal elements, which, as the remarks on p. 131 indicate, was the key to the Lawvere construction and to the construction of a Dedekind monad, shows up again in the next theorem as an important element in lifting results leading to infinite sets to results leading to large cardinals.

Theorem 67. (Generalization of Theorems 45 and 47 to Inaccessibles) (ZFC − Infinity) *Suppose $j : V \to V$ is a functor, having a strong critical point, which preserves countable disjoint unions, intersections, equalizers, the empty set, and*

terminal objects, and which has a weakly universal element $a \in j(A)$ for j, for some set A. Then there is a collection $\mathcal{A}$ of subsets of A with the following properties:
(1) *For every $X \in \mathcal{A}$, $|X|$ is inaccessible.*
(2) *If $X, Y \in \mathcal{A}$, then $|X| \neq |Y|$;*
(3) *The size of $\mathcal{A}$ itself is inaccessible.*[140]

Remark 20. We will show that j induces a nonprincipal ω_1-complete ultrafilter on A. The properties (1)–(3) listed above are known consequences[141] of the existence of such an ultrafilter, so we do not provide proofs of these. Note that properties (1)–(3) imply that there are many large cardinals less than $|A|$.

We have not required j to be a Dedekind self-map; we do not have a proof, under the given hypotheses, that it must have this property. We can show, however, that j must be *essentially* Dedekind (see p. 132 for the definition). In the proof given below, Claim 1 shows that j and j_D are naturally isomorphic. As mentioned in Example 5, j_D itself is 1-1.

Also, although the hypotheses require j to have a strong critical point, they do not require the weakly universal element $a \in j(A)$ itself to be a strong critical point. However, under somewhat stronger hypotheses, this does turn out to be the case—see Claim 4 of Example 5, where existence of a measurable cardinal is shown to suffice to obtain this consequence. $\Box$

Proof of Theorem 68. Let $D = \{X \subseteq A \mid a \in j(X)\}$. As in the first part of Theorem 45, D is an ultrafilter (we do not claim yet that it is nonprincipal). Define $j_D : V \to V$ by $j_D(X) = X^A/D$.

Claim 1. j is naturally isomorphic to j_D. That is, there are, for all sets B, bijections $\overline{\phi}_B : j_D(B) \to j(B)$, natural in B.

Proof. We first define an onto map $\phi_B : B^A \to j(B)$, and then show that ϕ_B induces a bijection $\overline{\phi}_B : B^A/D \to j(B)$. Define ϕ_B by

$$\phi_B(f) = j(f)(a).$$

We use the fact that a is a weakly universal element to show that ϕ_B is onto: Suppose $y \in j(B)$. By weak universality, there is $f : A \to B$ so that $y = j(f)(a) = \phi_B(f)$, as required.

Now define $\overline{\phi}_B : B^A/D \to j(B)$ by

$$\overline{\phi}_B([f]) = j(f)(a).$$

Like ϕ_B, $\overline{\phi}_B$ is onto. To see it is well-defined and 1-1, it suffices to show that, for all partial functions $f, g : A \to B$, $f \sim g$ if and only if $j(f)(a) = j(g)(a)$.

$$\begin{aligned} f \sim g \quad &\Leftrightarrow \quad E = E_{f,g} \in D \\ &\Leftrightarrow \quad a \in j(E) \\ &\Leftrightarrow \quad a \in \{z \mid j(f)(z) = j(g)(z)\} = E_{j(f),j(g)} \\ &\Leftrightarrow \quad j(f)(a) = j(g)(a). \end{aligned}$$

[140] These three properties follow from the fact that, under the hypotheses of the theorem, there must exist a measurable cardinal $\kappa \leq |A|$. See Section 22.

[141] For example, see [34, Lemma 27.1].

The proof that the $\overline{\phi}_B$ are components of a natural transformation is straightforward. □

Claim 2. D is nonprincipal.

Proof. Let Z be a strong critical point for j. By Claim 1, Z is also a strong critical point for j_D. Assume D is principal. Then there is $u \in A$ such that $\{u\} \in D$; it follows that $D = \{X \subseteq A \mid u \in X\}$. For a contradiction, it suffices to exhibit a bijection $j_D(Z) \to Z$.

For each $z \in Z$, let $c_z : A \to Z$ be the constant function defined by $c_z(x) = z$ for all $x \in A$. Suppose $g : A \to Z$ is total and let $z = z_g = g(u)$. Let $E = E_{c_z,g} = \{x \in A \mid c_z(x) = g(x)\}$. Since $u \in E$, it follows that $E \in D$, and so $[c_z] = [g]$. Since every $[f] \in Z^A/D$ has a representative g that is total, and observing that for any such g, $[g] = [c_{z_g}]$, we have that $Z^A/D = \{[z] \mid z \in Z\}$. The map $[c_z] \mapsto z$ is therefore a 1-1 correspondence between Z^A/D and Z. □

Claim 3. D is ω_1-complete.

Proof. It suffices to show that if $\{X_n \mid n \in \omega\}$ are disjoint subsets of A with $\bigcup_{n\in\omega} X_n \in D$, then for some $n \in \omega$, $X_n \in D$. Let $X = \bigcup_{n\in\omega} X_n$. Since $X \in D$ and since j preserves countable disjoint unions,

$$a \in j(X) \iff a \in \bigcup_{n\in\omega} j(X_n) \iff \exists n \in \omega\, a \in j(X_n) \iff \exists n \in \omega\, X_n \in D. \square$$

After introducing large cardinals more formally in Section 22, we will give examples of Dedekind self-maps $j : V \to V$ that exhibit the properties mentioned in the two previous theorems, under the assumption of existence of large cardinals.

The most productive direction for generalizing further arises from strengthening the results of Theorem 67. To generalize this kind of preservation, a natural choice is some form of *elementary embedding.* A map $j : V \to V$ is an elementary embedding if, for any formula $\phi(x_1, x_2, \ldots, x_n)$ in the language of sets and for any particular sets $a_1, a_2, \ldots, a_n$,

$$V \models \phi(a_1, a_2, \ldots, a_n) \text{ if and only if } V \models \phi\left(j(a_1), j(a_2), \ldots, j(a_n)\right).$$

(Statements of the form $V \models \phi$ are defined on p. 53.) Intuitively speaking, elementarity of a Dedekind self-map $j : V \to V$ means that *all* properties[142] and relationships of sets that hold in V are preserved by j; preservation of disjoint unions and terminal objects, for example, are special cases of this much stronger form of preservation.

Though elementary embeddings seem natural to consider, without some modification the preservation that they introduce is too strong. A consequence of early work by K. Kunen [38] is that whenever $j : V \to V$ is *definable in* V (as all *class* Dedekind self-maps from V to V must be), if j is elementary, j must be simply the

[142]Technically speaking, all *first-order* properties are preserved by an elementary embedding.

identity map $\mathrm{id}_V : V \to V$; it cannot have a critical point (or even a weak critical point).[143]

One way to pursue this approach in a consistent way is to eliminate the "definability" of j. This can be done by expanding the language of set theory. Ordinarily, the only "extralogical" symbol that is used in set theory is $\in$, which is a formal symbol intended to represent the membership relation. In the expanded set theory we are suggesting, we would introduce one additional extralogical symbol $\mathbf{j}$, intended to represent a map from V to V. One can then introduce axioms that collectively assert that $\mathbf{j}$ is elementary and has a weak critical point. In fact, the statement that there is a weak critical point is usually strengthened to assert "there is a least ordinal moved."[144] This axiom schema is called the *Basic Theory of Elementary Embeddings*, or BTEE.[145] Here is a formal statement:

Axioms of BTEE

(1) (*Elementarity Schema for $\in$-formulas*). Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \dots, x_m)$ is an $\in$-formula:

$$\forall x_1, x_2, \dots, x_m \left(\phi(x_1, x_2, \dots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \dots, \mathbf{j}(x_m))\right);$$

(2) (*Critical Point*). "There is a least ordinal moved by $\mathbf{j}$."

A convention we will adopt for the rest of the paper is to refer to the least ordinal moved by any BTEE-embedding $j : V \to V$ as *the critical point* and denote it by $\mathrm{crit}(j)$; the least ordinal moved does indeed turn out to be a critical point—in fact, it is the *least* critical point in ON—in the usual sense of Dedekind self-maps. This convention accords with the notational convention we adopted earlier (p. 122) since whenever j is a BTEE-embedding, j strongly preserves $\in$, preserves ordinals, and has a weak critical point.

Having removed the "definability" of j by requiring it to be an interpretation of a new function symbol $\mathbf{j}$ in the extended language $\{\in, \mathbf{j}\}$, we are free to consider elementary embeddings from V to V, as formulated in the expanded theory ZFC + BTEE, and to consider elementary embeddings in strengthenings of this theory. In this context, the Kunen inconsistency result arises as an extreme special case in which ZFC + BTEE is strengthened "too much" with additional axioms. We examine consequences of the theory ZFC + BTEE after a brief but more formal introduction to large cardinals in Section 22.

[143] Kunen's result is somewhat stronger than this; a detailed discussion is given in [10, 11]. His work shows that whenever $j : V \to V$ is elementary and *weakly definable*, j must be the identity.

[144] Details of this approach can be found in [11].

[145] In the literature, the Critical Point axiom usually has a weaker formulation, namely, that for some x, $\mathbf{j}(x) \neq x$. However, from the theory ZFC + BTEE alone one cannot prove that an *ordinal* is moved by $\mathbf{j}$, nor that if an ordinal is moved, that there is a *least* ordinal moved [11]. These slightly stronger statements require that we add two very weak instances of Separation for $\mathbf{j}$-formulas to our axiomatic framework. To avoid needless complications, we have wrapped these instances into our formulation of Critical Point for the purposes of this article.

22. Introduction to Large Cardinals

In the previous section we mentioned some principles and strategies for strengthening a proper class Dedekind self-map in such a way that it could give rise to large cardinals. In this section, we give some background information about large cardinals, and then, in the following section, examine two approaches to achieving our goal of accounting for all large cardinals.

In the 19th century, Georg Cantor demonstrated that, assuming there is an infinite set (like the set ω of natural numbers), there must be an endless hierarchy of ever-larger infinite sizes, called *infinite cardinals*. He gave these different infinite sizes names: $\aleph_0, \aleph_1, \aleph_2, \ldots$ (pronounced "aleph-zero, aleph-one, ..."). Later it was recognized that the best way to define cardinal numbers is as special kinds of ordinal numbers. Viewed in this way, the infinite cardinals are $\omega = \omega_0, \omega_1, \omega_2, \ldots$. It is known that every infinite set has size that is equal to exactly one of these infinite cardinals (though it is not always possible to determine, for a given set, *which* of the cardinals represents its size).

In the early 20th century, certain combinations of properties of infinite cardinals were discovered to be quite strong; it was not clear that any infinite cardinal could actually have such combinations of properties. We give an example of one such combination. One of the properties involved is that of being an *aleph fixed point*. A cardinal κ is an aleph fixed point if $\omega_\kappa = \kappa$; in other words, the *index* of the aleph is identical to the aleph itself. Existence of such a cardinal seems at first to be unlikely when one considers that, among the smallest infinite cardinals, the index of an aleph is always much smaller than the aleph itself:

$$0 < \omega_0, \quad 1 < \omega_1, \quad 2 < \omega_2, \ldots.$$

It is possible however to construct an aleph fixed point.

A second property is *regularity*. Recall that a cardinal κ is regular if unbounded subsets of κ always have cardinality κ (see the definition given on p. 152). As we observed before, ω is a good example of a regular cardinal since any unbounded subset of ω must be infinite (and therefore must have size ω).

A question that was asked early in the history of set theory is whether there could exist an infinite cardinal that is both regular and an aleph fixed-point. As was discovered many years later, even if such a cardinal does exist, it cannot in any case be *proven* to exist. Any regular cardinal that is an aleph-fixed point is now known as a *weakly inaccessible* cardinal.[146] Weakly inaccessibles were among the first large cardinals to be discovered. We spend a moment to explain why this type of cardinal, like any other large cardinal, cannot be proved to exist using the ZFC axioms alone.

In the 1930s, Gödel established two important results in mathematical logic that hold the key to understanding the limitative results surrounding large cardinals:

Gödel's Theorems

(1) (Completeness Theorem) A mathematical theory is consistent if and only if it has a model.

[146] A good exercise is to verify that every inaccessible cardinal is weakly inaccessible. See p. 152 for the definition of *inaccessible*.

(2) (Second Incompleteness Theorem) If a mathematical theory is strong enough to prove[147] the axioms of Peano Arithmetic, it cannot prove its own consistency, unless it is inconsistent.

We begin the discussion by defining some terms used in these statements that may be unfamiliar. A *mathematical theory* is a collection of axioms—like the ZFC axioms or axioms for Euclidean geometry—together with all the theorems that can be proven from the axioms. A theory is *consistent* if one cannot prove from the theory a self-contradictory statement such $0 \neq 0$. A *model* for a theory is a set (or for some purposes, we can consider a proper class as well) in which the extralogical symbols are interpreted and in which all the axioms hold true. A model of set theory, for example, interprets the $\in$ symbol as usual membership, and the axioms of ZFC hold true in the model. An example of a model of set theory is V. However, the Second Incompleteness Theorem tells us that if ZFC is consistent, we cannot prove that ZFC is consistent using ZFC alone; in particular, there is no way to formally assert "every axiom of ZFC holds in V" (otherwise we would have defined from ZFC a model of ZFC).[148]

Now we can explain why large cardinals are "large": Suppose κ is a large cardinal—say κ is a weakly inaccessible cardinal. It can be shown [35] that from

[147]More precisely, it must be possible to *interpret* the axioms of Peano Arithmetic into the mathematical theory in question and then derive these interpreted axioms within the theory.

[148]The ZFC axioms are sufficient to establish that each stage V_α in the construction of V exists. However, because ZFC talks only about *sets*, it is not possible to explicitly build V as the union of its stages, since the stages are indexed by the proper class ON and V itself is a proper class.

Nevertheless, one can get around this limitation and *almost* prove that all axioms of ZFC hold in V: First of all, we show how to give a formally acceptable way of asserting "σ holds in V." Any class is understood to be a collection defined by a formula. One may formally write down a formula $\phi_{\mathrm{wf}}(x)$ that asserts $\exists\alpha\,(\text{“}\alpha \text{ is an ordinal”} \wedge x \in V_\alpha)$. The proper class $\mathbf{WF}$ of all wellfounded sets, defined by $\mathbf{WF} = \{x \mid \phi_{\mathrm{wf}}(x)\}$, can then be shown, by the Axiom of Foundation, to include all sets; that is, one shows $\mathbf{WF} = V$ (or, more formally, $\forall x\,(\phi_{\mathrm{wf}}(x))$).

If $\mathcal{M} = (M, E)$ is a model of ZFC—a "universe of sets" with a membership relation denoted E—and σ is a sentence in the language of set theory, then one may write $\sigma^{\mathcal{M}}$ to indicate that σ holds true in $\mathcal{M}$.

Now, for each axiom σ of ZFC, one can prove *within* ZFC the following:

$$\sigma^{\mathbf{WF}}.$$

This shows that we can assert "σ holds in V" (equivalently, "σ holds in $\mathbf{WF}$") in a formally valid way. In fact, it follows that for any *finite* collection $\sigma_1, \dots, \sigma_n$ of ZFC axioms, the conjunction $\sigma_1^{\mathbf{WF}} \wedge \cdots \wedge \sigma_n^{\mathbf{WF}}$ is provable from ZFC. However, first-order logic does not allow us to conjoin the infinitely many axioms $\sigma_1, \sigma_2, \dots$ of ZFC to prove the infinitely long "sentence"

$$\sigma_1^{\mathbf{WF}} \wedge \sigma_2^{\mathbf{WF}} \wedge \sigma_3^{\mathbf{WF}} \wedge \cdots.$$

Moreover, by Gödel's work, assuming ZFC is consistent, we *cannot* prove in ZFC

$$\text{“for all } \sigma \in \text{ZFC}, \sigma^{\mathbf{WF}}.\text{”}$$

κ one can define a transitive model of ZFC.[149] It follows, therefore, that, assuming ZFC is consistent, one cannot prove from ZFC the existence of any large cardinals.[150]

Despite this fact, large cardinals show up as key ingredients in the solutions of many problems in analysis, topology, algebra, and logic.[151] The Problem of Large Cardinals asks for naturally motivated axioms that could be added to the axioms of ZFC so that the known large cardinals could be derived.

Some of the better known types of large cardinals are listed below, in increasing order of strength:

weakly inaccessible
inaccessible
Mahlo
weakly compact
Ramsey
measurable
supercompact
extendible
huge
superhuge
super-n-huge for every $n \in \omega$.

In previous sections, we have already encountered inaccessible cardinals (p. 152), and the essential ingredients of measurable cardinals have also been introduced (Theorem 68 together with Remark 20). Since we will have more to say about measurable cardinals in the next section, we give a definition here.

Definition 19. (Measurable Cardinals) Suppose κ is an infinite cardinal and U is an ultrafilter on κ. Then U is said to be *κ-complete* if, for any collection $\{X_\alpha \mid \alpha < \lambda\}$ of elements of U, with $\lambda < \kappa$, we have that $\bigcap_{\alpha<\lambda} X_\alpha \in U$. If κ is uncountable, we say that κ is a *measurable cardinal* if there is a nonprincipal, κ-complete ultrafilter on κ.

In Theorem 68, we showed how a nonprincipal ω_1-complete ultrafilter on some set A could be obtained from a Dedekind self-map $j : V \to V$ with sufficiently strong preservation properties, and indicated that the presence of such an ultrafilter guaranteed existence of many inaccessible cardinals. Following this thread for a moment, it can be shown [34] that, whenever such an ultrafilter on a set A exists, there is a measurable cardinal κ such that $\kappa \le |A|$, with a corresponding κ-complete

[149] For most large cardinals κ, this model is the κth stage of the universe, V_κ, but in some cases, especially for some of the smaller large cardinals (for instance, the weakly inaccessibles), a modification of V_κ is needed. One way to handle these special cases is to use the *relativized model* $V_\kappa^{\mathbf{L}}$, where $\mathbf{L}$ denotes Gödel's *constructible universe* (another proper class model of set theory) and where the notation $V_\kappa^{\mathbf{L}}$ signifies "the κth stage of the universe, built inside $\mathbf{L}$." The model $\mathbf{L}$ is defined formally on p. 166.

[150] The unprovability of existence of large cardinals may remind the reader of another famous result in set theory: *The Continuum Hypothesis (CH) is independent of ZFC.* The situation with large cardinals is, however, quite different. The result on CH tells us that it is at least *consistent* with ZFC for CH to be true, but this is not the case for large cardinals. Large cardinals cannot be proven even to be consistent with ZFC. A proof is given in the Appendix—Theorem 96.

[151] See [16] for an accessible introduction to some of these problems and their connection to large cardinals.

ultrafilter D over κ. It can then be shown [34] that the set $\{\lambda < \kappa \mid \lambda \text{ is inaccessible}\}$ belongs to D, so that, in a sense, "almost all" cardinals below κ are inaccessible. It is by way of this fact that the conclusion to Theorem 68 can be demonstrated.

23. Some Strengthenings of Dedekind Self-Maps to Account for Large Cardinals

In Section 17, we saw how certain combinations of preservation properties belonging to a class Dedekind self-map $j : V \to V$ resulted in the emergence of an infinite set, and then, with the introduction of a few more properties, the emergence of inaccessible cardinals. Examples 3 and 4 (p. 116 and p. 117) showed that class Dedekind self-maps with the required properties, at least in some cases, could be constructed under sufficiently strong assumptions. In this section, we begin by showing that the preservation properties of the Dedekind self-map described in Theorem 68, which produced many inaccessibles, can be realized under the assumption of a measurable cardinal. We also give a class Dedekind self-map example to show that the properties of Theorem 67 can also be realized, and carry this work further still to give a much fuller account of nearly all large cardinals.

We reproduce Theorem 68 here and then give the related example.

Theorem 68. (ZFC − Infinity) *Suppose $j : V \to V$ is a functor having a strong critical point, which preserves countable disjoint unions, intersections, equalizers, the empty set, and terminal objects, and which has a weakly universal element $a \in j(A)$ for j, for some set A. Then there is a collection $\mathcal{A}$ of subsets of A with the following properties:*

(1) *For every $X \in \mathcal{A}$, $|X|$ is inaccessible.*
(2) *If $X, Y \in \mathcal{A}$, then $|X| \neq |Y|$.*
(3) *The size of $\mathcal{A}$ itself is inaccessible.*

As was mentioned briefly before (p. 160), the hypotheses of Theorem 68 imply the existence of a measurable cardinal $\kappa \leq |A|$. The next example exhibits a functor $V \to V$ having the properties listed in Theorem 68.

Example 5. Suppose κ is a measurable cardinal and D is a nonprincipal, κ-complete ultrafilter on κ. Define $j_D : V \to V$ as was done in the proof of Theorem 68: $j_D(X) = X^\kappa/D$. Defining j_D on functions $f : X \to Y$ by $j_D(f)(g) = [f \circ g]$, as before, turns j_D into a functor. The proof of the fact that j_D is 1-1 and preserves disjoint unions, intersections, and terminal objects is essentially the same (replacing ω with κ) as the corresponding verifications given in Example 3.

Claim 1. $[\mathrm{id}_\kappa] \in j_D(\kappa)$. Moreover,

$$D = \{X \subseteq \kappa \mid [\mathrm{id}_\kappa] \in j_D(X)\}. \tag{98}$$

In addition, $[\mathrm{id}_\kappa] \in j_D(\kappa)$ is a universal element for j_D.

Proof. The proof of (98) is like the proof of Theorem 68. The proof that $[\mathrm{id}_\kappa] \in j_D(\kappa)$ is a universal element for j_D follows the logic given on p. 130. □

Claim 2. j_D preserves countable disjoint unions.

Proof. Suppose $X = \bigcup_{n\in\omega} X_n$ is a countable disjoint union. It is clear that the sets in $\{X_n^\kappa/D \mid n \in \omega\}$ are also disjoint and that $\bigcup_{n\in\omega}(X_n^\kappa/D) \subseteq X^\kappa/D$. To show the converse, let $f : \kappa \to X$. For each $n \in \omega$, let $S_n = \{\alpha < \kappa \mid f(\alpha) \in X_n\}$. By κ-completeness, some S_n belongs to D. It follows that $[f] \in X_n^\kappa/D \subseteq \bigcup_{n\in\omega}(X_n^\kappa/D)$. □

Claim 3. j_D preserves equalizers.

Proof. Let $f, g : X \to Y$ and let $E = E_{f,g} = \{x \in X \mid f(x) = g(x)\}$. We show that $j_D(E) = E^\kappa/D$ is the equalizer $E_{j_D(f),j_D(g)}$ of $j_D(f), j_D(g)$:

$$\begin{aligned} [h] \in j_D(E) &\iff \{\alpha < \kappa \mid h(\alpha) \in E\} \in D \\ &\iff \{\alpha < \kappa \mid f(h(\alpha)) = g(h(\alpha))\} \in D \\ &\iff [h] \in E_{j_D(f),j_D(g)}.\square \end{aligned}$$

Claim 4. κ is a strong critical point for j.

Proof. By κ-completeness and the fact that D is nonprincipal, all members of D have size κ; in particular, all final segments $[\alpha, \kappa)$ belong to D. We show $\kappa < |j_D(\kappa)| = |\kappa^\kappa/D|$. Let $\langle f_\alpha \mid \alpha < \kappa\rangle$ be a sequence of κ functions $\kappa \to \kappa$. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = \sup\{f_\beta(\alpha) \mid \beta < \alpha\} + 1.$$

Then for each β, $\{\alpha \mid f_\beta(\alpha) < g(\alpha)\} \supseteq (\beta, \kappa)$ and $(\beta, \kappa) \in D$. Therefore, $|\kappa^\kappa/D| > \kappa$, as required. □

It can also be shown [14] that κ is a critical point of j_D and the least ordinal moved by j_D. Furthermore, since D is (at least) ω_1-complete, it can be shown that, if we identify κ^κ/D with its transitive collapse, $[\mathrm{id}_\kappa]$ is mapped to κ by the collapsing map; in this sense, $\kappa \in j_D(\kappa)$ itself is a weakly universal element[152] of j_D. □

This example, combined with Theorem 68, provides the following characterization: There is a measurable cardinal if and only if there is a Dedekind self-map $j : V \to V$ that is a functor, preserves unions, intersections, equalizers, the empty set, and terminal objects, and has a weakly universal element. In [3], Trňkova-Blass strengthen this characterization by using the concept of an *exact functor*: A functor is exact if it preserves all finite limits and colimits.[153] The main result of [3] is the following:

Theorem 68. (Trňkova-Blass Theorem) *There exists a measurable cardinal if and only if there is an exact functor from V to V having a strong critical point.* □

[152]We note, however, that $j_D : V \to V$ is not a cofinal functor. The usual construction of an ultrapower embedding, however, provides an example, familiar to set theorists, in which κ is weakly universal element for the embedding, and the embedding *is* cofinal. The construction is outlined in the footnote on p. 171.

[153]Limits and colimits are informally defined on p. 134.

We summarize the results related to Dedekind self-maps and measurable cardinals in the following corollary.

Corollary 69. (Measurable Cardinals and Dedekind Self-Maps) (ZFC − Infinity) *The following statements are equivalent.*

(1) *There is a measurable cardinal.*
(2) *There is a Dedekind self-map* $j : V \to V$ *having the following properties:*
  (i) j *is a functor;*
  (ii) j *has a strong critical point;*
  (iii) j *preserves countable disjoint unions, intersections, equalizers, the empty set, and terminal objects;*
  (iv) *there is a weakly universal element* $a \in j(A)$ *for* j*, for some set* A*.*
(3) *There is an exact functor* $j : V \to V$ *having a strong critical point.*

We consider next an example showing that the properties of a $j : V \to V$ mentioned in Theorem 67 can be realized. We reproduce the theorem here for easy reference.

Theorem 67. (ZFC − Infinity) *Suppose* $j : V \to V$ *is a Dedekind self-map that strongly preserves* $\in$ *and preserves ordinal and rank. Let* $\kappa = \text{crit}(j)$*.*

(A) *Suppose* j *preserves countable disjoint unions, the empty set, and singletons. Then* $\kappa > \omega$*.*
(B) *Suppose* j *is BSP and preserves countable disjoint unions, the empty set, singletons, and unboundedness. Then* κ *is an uncountable regular cardinal.*
(C) *Suppose* j *is BSP and preserves countable disjoint unions, the empty set, singletons, unboundedness, and power sets. Then* κ *is an inaccessible cardinal.*

We mentioned in Section 21 that any elementary embedding $j : V \to V$ for which $(V, \in, j)$ satisfies $\text{ZFC} + \text{BTEE}$ will possess all the preservation properties that we have required of a class Dedekind self-map in the previous few sections. We formulate this observation as a proposition:

Proposition 70. *Suppose* $(V, \in, j)$ *is a model of* $\text{ZFC} + \text{BTEE}$*, and* $\text{crit}(j) = \kappa$*. Then*

(A) $j : V \to V$ *is a Dedekind self-map, with critical point* κ *and strong critical point* κ*.*
(B) j *is BSP, strongly preserves* $\in$*, and preserves ordinals, rank, countable disjoint unions, the empty set, singletons, unboundedness, and power sets; in addition,* j *reflects ordinals.*
(C) j *preserves intersections, equalizers, terminal objects.*
(D) j *is an exact functor.*

Remark 21. Part (B) lists the properties mentioned in Theorem 67, while (C) lists the additional properties mentioned in Theorem 68.

Proof. We prove some of these claims. Assume, as in the hypothesis, that $(V, \in, j)$ is a model of $\text{ZFC} + \text{BTEE}$ and $\text{crit}(j) = \kappa$.

Claim 1. j strongly preserves $\in$.

Proof. Suppose $u \in v$. Consider the formula $\phi(x, y) : x \in y$; certainly $V \models \phi[u, v]$. Then by elementarity, $V \models \phi[j(u), j(v)]$; in other words, $j(u) \in j(v)$. Likewise, if $V \models \neg\phi[u, v]$, then $V \models \neg\phi[j(u), j(v)]$, from which it follows that if $j(u) \in j(v)$, then $u \in v$. □

Claim 2. j preserves and reflects ordinals.

Proof. Let $\phi(x)$ be the formula that asserts "x is an ordinal." Let α be an ordinal. Then $V \models \phi[\alpha]$. By elementarity, $V \models \phi[j(\alpha)]$. Therefore, $j(\alpha)$ is an ordinal. (To be more precise, we would unwind the definition of an ordinal and verify that the elements of the definition are preserved by j. x is an ordinal if and only if $(x, \in)$ is a well-ordered set and x is transitive. x is transitive if and only if, for all u, v, if $u \in x$ and $v \in u$, then $v \in x$. Suppose $\psi(x)$ asserts "x is transitive." Then, formally, $\psi(x)$ is the formula $\forall x \forall u \forall v\, (u \in x \wedge v \in u \rightarrow v \in x)$. So, if w is transitive, $V \models \psi[w]$. By elementarity, $V \models \psi[j(w)]$; that is, $V \models \forall x \forall u \forall v\, (u \in j(w) \wedge v \in u \rightarrow v \in j(w))$. It follows that $j(w)$ is transitive. Similar steps of analysis allow us to conclude that if $(x, \in)$ is a well-ordered set, so is $(j(x), \in)$. From these observations, whenever α is an ordinal, $j(\alpha)$ is also an ordinal. In future arguments, we will be more informal.)

Suppose now that β is not an ordinal; it follows that $V \models \neg\phi[\beta]$. By elementarity, $V \models \neg\phi[j(\beta)]$, and so $j(\beta)$ is not an ordinal. This proves that j reflects ordinals. □

Claim 3. j is a Dedekind self-map, with critical point κ. Moreover, $j(\kappa) > \kappa$.

Proof. The fact that j is 1-1 follows from elementarity: Let $\phi(x, y)$ be the formula "$x \neq y$." Suppose $u \neq v$. Then $V \models \phi[u, v]$, so $V \models \phi[j(u), j(v)]$, and $j(u) \neq j(v)$. So j is 1-1. Since κ is least for which $j(\kappa) \neq \kappa$ (by assumption), if $j(\kappa) < \kappa$, we arrive at a contradiction by noticing that now $j(\kappa)$ is an ordinal $\beta < \kappa$ with $j(\beta) \neq \beta$ (note that $j(\kappa) \in \kappa$ implies $j(j(\kappa)) \in j(\kappa)$). □

Claim 4. j is BSP and j is a functor.

Proof. Let $\rho(x, y)$ be a formula that asserts "there is a 1-1 correspondence from x to y." Now recall that the formula $\phi(x)$ that asserts "x is a cardinal" is formulated as follows:

$$\phi(x) : \text{ "}x \text{ is an ordinal, and for all } \alpha < x,\ \neg\, \rho(\alpha, x).\text{"}$$

Clearly, if γ is a cardinal, then $V \models \phi[\gamma]$; by elementarity, $V \models \phi[j(\gamma)]$. In particular,

$$V \models \text{"}j(\gamma) \text{ is an ordinal, and for all } \alpha < j(\gamma),\ \neg\, \rho(\alpha, \gamma)\text{"}.$$

Thus, $j(\gamma)$ is a cardinal.

We verify that j is a functor. If $f : A \rightarrow B$, since

$$V \models \text{"domain of } f \text{ is } A, \text{ codomain of } f \text{ is } B\text{"},$$

we have

$$V \models \text{“domain of } j(f) \text{ is } j(A), \text{ codomain of } j(f) \text{ is } j(B)\text{”},$$

so $j(f) : j(A) \to j(B)$. We check that j preserves identity morphisms: We represent $\mathrm{id}_A : A \to A$ as the set $\mathrm{id}_A = \{(x, y) \in A \times A \mid x = y\}$. Then

$$j(\mathrm{id}_A) = j\Big(\{(x, y) \in A \times A \mid x = y\}\Big) = \{(x, y) \in j(A) \times j(A) \mid x = y\} = \mathrm{id}_{j(A)}.$$

To see j preserves composition of morphisms, suppose $f : A \to B$ and $g : B \to C$. Representing morphisms as sets of ordered pairs again, we have

$$\begin{aligned} j(g \circ f) &= j\Big(\{(x, z) \in A \times C \mid \exists y \in B\,(y = f(x) \wedge z = g(y))\}\Big) \\ &= \{(x, z) \in j(A) \times j(C) \mid \exists y \in B\,(y = f(x) \wedge z = g(y))\} \\ &= j(g) \circ j(f). \end{aligned}$$

We verify that j preserves functions. Suppose $f : A \to B$. Then for all $a \in A$, by elementarity, $j(a) \in j(A)$. Note that, for each $a \in A$ and each $b \in B$, if $f(a) = b$, then we have:

$$V \models \text{“} b \text{ is the value of } f \text{ at } a.\text{”}$$

and so

$$V \models \text{“} j(b) \text{ is the value of } j(f) \text{ at } j(a).\text{”}$$

But this implies that $j(f)(j(a)) = j(b)$, as required.

We omit the proofs that j preserves images and rank. □

Claim 5. *j preserves intersections and equalizers.*

Proof. Suppose A, B are sets. Then by elementarity of j,

$$j(A \cap B) = j(\{x \in A \mid x \in B\}) = \{x \in j(A) \mid x \in j(B)\} = j(A) \cap j(B).$$

Suppose $f, g : A \to B$ are functions. Then

$$\begin{aligned} j(E_{f,g}) &= j(\{x \in A \mid f(x) = g(x)\}) \\ &= \{x \in j(A) \mid (j(f))(x) = (j(g))(x)\} \\ &= E_{j(f),j(g)}. \square \end{aligned}$$

Claim 6. *j is exact.*

Proof. We show j is left exact; the proof that j is right exact is similar. It suffices to show that j preserves products, equalizers, and terminal objects. We have already established the latter two of these. If A, B are sets, we have

$$\begin{aligned} j(A \times B) &= j(\{(a, b) \mid a \in A \text{ and } b \in B\}) \\ &= \{(a, b) \mid a \in j(A) \text{ and } b \in j(B)\} \\ &= j(A) \times j(B). \square \end{aligned}$$

The next proposition shows that BTEE is a kind of axiom of infinity. Although many of the facts about a BTEE-embedding $j : V \to V$ follow directly from the fact that such embeddings have all the preservation properties listed in Theorem 67, the proof that $j(n) = n$ for all finite ordinals is somewhat different. To see the issue, suppose we wish to show that $j(n) = n$ for each finite ordinal n. We could let $\phi(x)$

be the sentence "x is a finite ordinal and $j(x) = x$" and attempt to prove $\phi(x)$ by induction. But the formula upon which the induction is based is a formula that has an occurrence of j; this means it is a formula of the expanded language $\{\in, \mathbf{j}\}$, and the theory ZFC+BTEE cannot prove that induction holds for this kind of formula; see [11]. Therefore, we give a different proof.

Proposition 71. (ZFC − Infinity) *Suppose* $(V, \in, j)$ *is a model of* ZFC − Infinity + BTEE*, with* $\kappa = \mathrm{crit}(j)$*.*

(A) *For each finite ordinal* n*,* $j(n) = n$*.*
(B) V *contains an infinite set; in particular,* $V \models$ *"*ω *exists".*
(C) $\kappa > \omega$*.*

Proof of (A). If for some finite ordinal m, $j(m) \neq m$, then $\kappa \leq m$ (since κ is the least ordinal with this property). Since $j(0) = 0$, it follows that $\kappa = n + 1$ for some finite ordinal $n \geq 0$. But now since j preserves singletons and disjoint unions, we have

$$\begin{aligned} j(\kappa) &= j(n+1) \\ &= j(n \cup \{n\}) \\ &= j(n) \cup j(\{n\}) \\ &= j(n) \cup \{j(n)\} \\ &= n \cup \{n\} \\ &= n+1 \\ &= \kappa, \end{aligned}$$

contradicting the fact that j moves κ. Therefore $j(m) = m$ for all finite ordinals m.

Proof of (B). Since $j(\kappa)$ is an ordinal and $j(\kappa) > \kappa$, κ is not a finite ordinal and therefore must be infinite. Existence of an infinite set implies existence of ω.

Proof of (C). The set ω may be defined as the least nonzero limit ordinal; it follows that $j(\omega)$ satisfies the same formula, and so $j(\omega) = \omega$. It follows that $\kappa > \omega$. □

Proposition 72 implies that the theory ZFC − Infinity + BTEE is the same as the theory ZFC+BTEE; therefore, from now on, we refer to this theory as ZFC+BTEE.

Corollary 72. *Suppose* $(V, \in, j)$ *is a model of* ZFC + BTEE*, with* $\kappa = \mathrm{crit}(j)$*. Then* κ *is inaccessible.*

Proof. Proposition 72 guarantees that $\kappa > \omega$. The remaining steps of the proof derive from the observation (Proposition 71) that any BTEE-embedding $j : V \to V$ satisfies the preservation properties mentioned in Theorem 67. □

Exhibiting models of ZFC + BTEE of the form $(V, \in, j)$, where V is the universe of sets, presents a peculiar problem. To see the difficulty, recall how we were able to obtain an example of the properties of Theorem 68: We started with a nonprincipal κ-complete ultrafilter D and *defined* a $j_D : V \to V$ with the required properties. However, Kunen's Theorem [38] tells us that it is impossible to *define* a $j : V \to V$

for which $(V, \in, j) \models \mathrm{ZFC} + \mathrm{BTEE}$; such a j, if it exists at all, must be undefinable (in V).

Assuming large cardinals, one can define set models, and even certain class models, of $\mathrm{ZFC} + \mathrm{BTEE}$; we just cannot expect that the embedding j is definable in the model. We describe such a class model next in the spirit of providing examples to show that the conditions described in our earlier results—in this case, Theorem 67—are consistent.

Example 6. (Gödel's Constructible Universe) Let $\mathbf{L}$ denote Gödel's constructible universe, whose construction we describe now. The universe $\mathbf{L}$, like V, is built in stages, except that at each successor step of the construction, instead of requiring the next stage to contain *all* subsets of the previous stage, we require only the subsets of the previous stage that are *definable* (with parameters) from the previous stage. Suppose X is a set and M is a model of a reasonable fragment of ZFC. Then X is said to be a definable subset of M with parameters if, for some formula $\phi(x, y_1, \ldots, y_n)$ and some elements $u_1, \ldots, u_n \in M$ (the parameters), $X = \{w \in M \mid M \models \phi[w, u_1, \ldots, u_n]\}$. We can define the stages of $\mathbf{L}$:

$$
\begin{aligned}
L_0 &= \emptyset, \\
L_{\alpha+1} &= \{X \subseteq L_\alpha \mid X \text{ is a definable subset of } L_\alpha \text{ with parameters}\}, \\
L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \qquad (\lambda \text{ a limit}), \\
\mathbf{L} &= \bigcup_{\alpha\in \mathrm{ON}} L_\alpha.
\end{aligned}
$$

In the presence of large enough cardinals, such as a measurable cardinal, it is known that $\mathbf{L}$ is much smaller than the full universe V and, in this situation, $\mathbf{L}$ can be used as the underlying class for a model of $\mathrm{ZFC} + \mathrm{BTEE}$. We state the result, which is due to Kunen.[154]

Theorem 73. (Kunen) *Suppose there is a measurable cardinal. Then there is an elementary embedding $j : \mathbf{L} \to \mathbf{L}$ such that $\langle \mathbf{L}, \in, j\rangle \models \mathrm{ZFC} + \mathrm{BTEE}$. Moreover j is definable in V but not in $\mathbf{L}$.* □

Theorem 74 demonstrates that the theory $\mathrm{ZFC}+\mathrm{BTEE}$ is consistent, without requiring us to explicitly define a Dedekind self-map $j : V \to V$ that yields a model of BTEE. As we consider further strengthenings of $\mathrm{ZFC} + \mathrm{BTEE}$, in order to provide evidence of the consistency of these stronger theories, we will follow the example of Theorem 74—building a model of the axioms that is properly contained in V. This step is important since one could introduce "natural"-looking preservation properties or other strengthenings of BTEE that are not in fact consistent; producing models of the axioms we consider avoids this pitfall.

[154] Hereafter, when we talk about *Kunen's Theorem*, we will not be referring to the result mentioned in Theorem 74, but rather the result that says there is no weakly definable nontrivial elementary embedding $j : V \to V$; see page 155.

By Kunen's Theorem, the embedding j of Theorem 74 cannot be definable in $\mathbf{L}$.[155] This fact is important because one might attempt to invoke the Trňkova-Blass Theorem to conclude that a measurable cardinal is derivable from the theory ZFC + BTEE; certainly an elementary embedding $j : \mathbf{L} \to \mathbf{L}$ has all the preservation properties mentioned in Theorem 71—in particular, j is an exact functor with a strong critical point. However, it is not correct to conclude that a measurable cardinal is therefore derivable from ZFC + BTEE. The reason is that one can obtain a transitive model of ZFC + BTEE from a large cardinal that is much weaker than a measurable cardinal; a *Ramsey cardinal*, for example, is sufficient. But if a measurable cardinal could be derived from ZFC + BTEE, this would mean that, from a Ramsey cardinal, one could obtain a model of ZFC + "there exists a measurable cardinal", and this is not possible.

The reason that the Trňkova-Blass Theorem is not applicable here can be found by taking a closer look at the proof of that theorem, or equally well at the proof of Theorem 68. In those proofs, the definition of the ultrafilter D from j makes sense only if j itself is definable in V. In the context of Theorem 68, it is implicitly assumed that the Dedekind self-map $j : V \to V$ in the hypothesis is definable in V; if it were not, defining D by $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ provides no guarantee that D is a *set*. On the other hand, by Kunen's Theorem, we may *not* assume j is definable when $j : V \to V$ is an elementary embedding for which $(V, \in, j)$ is a model of ZFC+BTEE. Therefore, from ZFC+BTEE, the class D cannot be proven to be a *set*, and the argument that D is an ultrafilter, used for the Trňkova-Blass Theorem, cannot be carried out when working in ZFC + BTEE. Therefore, in this case, where j is not definable, exactness of j (and the presence of a strong critical point) is not sufficient to produce a measurable cardinal.

This somewhat subtle limitation suggests a way to strengthen the theory ZFC + BTEE further. It is known that ZFC + BTEE, on its own, implies the existence of weakly compact cardinals, but certainly not the existence of a measurable cardinal. By adding to the axioms of ZFC+BTEE an axiom that asserts that the ultrafilter D naturally defined from j is a set,[156] the theory is strengthened considerably, so that large cardinals quite a bit stronger than a measurable cardinal become derivable.

> **Measurable Ultrafilter Axiom (MUA).** The class $\{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$ is a set.

We remind the reader that the boldface $\mathbf{j}$ signifies that it is a symbol of the expanded language of set theory that we are working in: ZFC + BTEE + MUA. Any model $\langle V, \in, j\rangle$ of ZFC+BTEE+MUA will interpret $\mathbf{j}$ as a nontrivial elementary embedding $j : V \to V$, where V contains, as one of its elements, $\{X \subseteq \kappa \mid \kappa \in j(X)\}$.

Although it is *convenient* to consider this additional axiom MUA, by what criterion can we claim that this is a legitimate axiom to introduce? Our earlier work showed that a measurable cardinal arises from existence of a Dedekind self-map $j : V \to V$ with sufficient preservation properties; moreover, a central theme of the paper, which we have argued to support, is that adding such preservation properties is "natural." We have therefore, at this point in our development, made a case

[155] However, it is quite important for j to be definable (or at least partially definable) in V itself. It is shown in [11, Example 9.2] that Theorem 74 fails if j is not sufficiently definable in V.

[156] Such an assertion can be formulated as an instance of the Separation axiom for $\mathbf{j}$-formulas.

for the existence of measurable cardinals. The assertion therefore that a measurable cardinal could arise as the critical point κ of a Dedekind self-map $j : V \to V$ having strong preservation properties follows naturally from what has already been achieved; in particular, the ultrafilter over κ that is derived from j seems to arise naturally. For this reason, we take MUA to be a legitimate axiom, justified by the criteria of naturalness that we have been using up to this point.

Theorem 74. *The theory* ZFC + BTEE + MUA *proves the existence of many measurable cardinals. Moreover, existence of a supercompact cardinal (even a 2^κ-supercompact, where κ is the critical point of the embedding) is sufficient to produce a transitive model of* ZFC + BTEE + MUA.[157]

In [11], a variety of strengthenings of the theory ZFC+BTEE, like ZFC+BTEE+MUA, are studied. The stronger the models are, the more fully they provide an account of the known large cardinals.

Before examining the ultimate limit of this process of generalization, we take a closer look at the theory ZFC+BTEE+MUA to see the extent to which our original intuitions, based on our analysis of Dedekind self-maps and the bare notion of "infinity," have been successful in leading to an account for the existence of "many" measurable cardinals—entities that exist in the middle-range of the spectrum of large cardinals.

24. The Theory ZFC + BTEE + MUA

We recall significant properties of *set* Dedekind self-maps, which we expected would generalize to the context of Dedekind self-maps $V \to V$ in our search for a natural account of the origin of large cardinals (p. 94). We found that if $j : A \to A$ is a Dedekind self-map on a set A with critical point a, j exhibits the following characteristics:

Properties of Set Dedekind Self-Maps

(A) j preserves essential properties of its domain (A is Dedekind-infinite, and so is the image $j[A]$ of j) and of itself (the property of being a Dedekind self-map propagates to the restriction $j \restriction j[A]$).

(B) The definition of j entails a *critical point* that plays a key role in its dynamics. One aspect of those dynamics is that repeated restrictions of j to successive images give rise to a *critical sequence* $W = \{a, j(a), j(j(a)), \ldots\}$ which is a precursor to the set ω of finite ordinals; the emergence of this critical sequence provides a strong analogy to the ancient and quantum field theoretic perspectives that "particles arise from the dynamics of an unbounded field," where, in this context, particular finite ordinals are viewed

[157] We give a definition of supercompact cardinals here. Suppose $\lambda \geq \kappa$ and λ is a cardinal. Let $P_\kappa\lambda = \{X \subseteq \lambda \mid |X| < \kappa\}$. A nonprincipal, κ-complete ultrafilter D on $P_\kappa\lambda$ is *fine* if for each $X \in P_\kappa\lambda$, $\{Y \in P_\kappa\lambda \mid Y \supseteq X\} \in D$. If D is fine, then D is said to be a *normal* ultrafilter if, whenever $f : P_\kappa\lambda \to \kappa$ has the property that $\{X \in P_\kappa\lambda \mid f(X) \in X\} \in D$, then it follows that, for some fixed $\alpha \in \kappa$, $\{X \in P_\kappa\lambda \mid f(X) = \alpha\} \in D$. Now, given cardinals $\kappa \leq \lambda$, κ is said to be λ-*supercompact* if there exists a fine, normal, κ-complete nonprincipal ultrafilter on $P_\kappa\lambda$. Finally, an infinite cardinal κ is *supercompact* if, for all cardinals $\lambda \geq \kappa$, κ is λ-supercompact.

as "particles." The sequence of restrictions $j_0, j_1, j_2, \ldots$ of j that produce the critical sequence is defined by the following equations:

$$\begin{aligned} A_0 &= A; \\ j_0 &= j : A \to A; \\ \mathrm{crit}(j_0) &= a; \\ A_1 &= j[A_0]; \\ j_1 &= j \upharpoonright A_1; \\ \mathrm{crit}(j_1) &= j(a); \\ A_{n+1} &= j[A_n]; \\ j_{n+1} &= j \upharpoonright A_{n+1}; \\ \mathrm{crit}(j_{n+1}) &= j^{n+1}(a). \end{aligned}$$

Therefore $\mathrm{crit}(j_n) = j^n(a)$ is a critical point of j_n.

(C) Through the interplay between j and its critical point a, a *blueprint* $(j \upharpoonright W, a, \mathcal{E})$ for ω arises, where $W = \{a, j(a), j(j(a)), \ldots\}$. In particular, $j \upharpoonright W$ is a Dedekind self-map, and for every $n \in \omega$, there is $i \in \mathcal{E}$ such that $i(j \upharpoonright W)(a) = n$.

(D) There is also a *strong blueprint* $(j \upharpoonright W, h, a, \mathcal{E})$ for ω. In particular, not only is $(j \upharpoonright W, a, \mathcal{E})$ a blueprint for ω, but also h returns elements to a in the sense that, for every $n \in \omega$, there is $i \in \mathcal{E}$ such that $i^*(h)(n) = a$.

Theorems 44, 45, and 47 illustrate the role of preservation properties, motivated by (A), in strengthening preservation properties of Dedekind self-maps $j : V \to V$ sufficiently to give rise to infinite sets. Moreover, the infinite set generated by each of the self-maps of Theorems 44 and 47 arose as the (strong) critical point of the self-map, as anticipated by (B). Theorem 45 accords with (B) in another way: The critical point of the map is the seed for defining a nonprincipal ultrafilter on a set A; the existence of such an ultrafilter implies that its underlying set is infinite. Elaborating on these results through the introduction of additional preservation properties led to an account of two types of large cardinals—inaccessibles (Theorem 67) and measurables (Theorem 68). In Theorem 67, the critical point turned out to be inaccessible; in Theorem 68, the critical point was the seed that was used to build an ω_1-complete nonprincipal ultrafilter, the existence of which implies the existence of a measurable cardinal. Our approach suggests that these types of large cardinals arise in the same "natural" way as infinite sets do, from Dedekind self-maps $V \to V$ with suitable preservation properties, and with the critical point playing a key part in the emergence of the large cardinal at hand.

Generalizing preservation properties of Dedekind self-maps to the concept of *elementary embedding*, as we do when we consider the theory ZFC+BTEE, provides the ultimate generalization of (A). If $j : V \to V$ is a Dedekind self-map given by a transitive model of ZFC+BTEE, its critical point is necessarily a large cardinal—at least a weakly compact cardinal.

Examining the Dedekind self-maps $j : V \to V$ for any model of ZFC + BTEE + MUA will show how other characteristics of Dedekind self-maps that we have identified (and listed in (B)–(D) above) come into play to give rise to still stronger large cardinals. Since such a map j is a BTEE-embedding, it is already clear that j generalizes properties mentioned in (A) and (B) (though the significance of repeatedly restricting j to subsets of its domain is not yet apparent). In particular, in the theory ZFC + BTEE + MUA, if $\kappa = \mathrm{crit}(j)$, then κ is measurable, and is in fact the κth measurable cardinal.

For ZFC + BTEE + MUA there are also natural analogues to (C)–(D), which we describe next. The fact that the collection $U_j = \{X \subseteq \kappa \mid \kappa \in j(X)\}$, where $j : V \to V$ is the embedding provided by a model of ZFC+BTEE+MUA (with $\kappa = \mathrm{crit}(j)$), is a *set* in ZFC + BTEE + MUA opens the door for a new construction, allowing us to generate (and collapse) sets starting from the critical point κ. It can be shown that U_j is a nonprincipal, κ-complete ultrafilter on κ, so we have immediately that κ is measurable. It has one additional property that should be mentioned here: U_j is *closed under diagonal intersections*. This means that, whenever $\langle X_\alpha : \alpha < \kappa\rangle$ are elements of U_j, the diagonal intersection $\{\xi < \kappa \mid \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$ also belongs to U_j. Any nonprincipal, κ-complete ultrafilter on an uncountable cardinal κ that is closed under diagonal intersections is said to be a *normal measure on κ*. We will therefore refer to U_j as the *normal measure derived from j*.

As we will show now, if $j : V \to V$ is an MUA-embedding with critical point κ, we can obtain from j and κ a blueprint $(\ell, \kappa, \mathcal{E})$ for the stage $V_{\kappa+1}$ of the universe, where $\ell : V_\kappa \to V_\kappa$ is the blueprint map and $\mathcal{E}$ consists of certain elementary embeddings (described below). In other words, the blueprint generates all sets in the universe through stage $V_{\kappa+1}$.

In order to understand the definitions of ℓ and $\mathcal{E}$ it is necessary to introduce a method for constructing new universes of set theory—called the *ultrapower construction*. We will describe the construction and its properties without proofs; proofs may be found in [35].

Suppose κ is a measurable cardinal and U is a κ-complete nonprincipal ultrafilter over κ. Recall that the collection V^κ consists of all functions having domain κ. We declare two such functions f, g *equivalent*, and write $f \sim_U g$, if f and g agree on a set in U, that is, if $\{\alpha \mid f(\alpha) = g(\alpha)\} \in U$. It is easy to check that $\sim_U$ is an equivalence relation. We denote the equivalence class containing f by $[f]$. Since $[f]$ is a proper class, we re-define it in the following way: Let g be a function $\kappa \to V$ of least rank that is equivalent to f; so, $g \sim_U f$ and g is of least possible rank with this property. Then define $[f]$ to be $\{h : \kappa \to V \mid h \sim_u f \text{ and } \mathrm{rank}(h) = \mathrm{rank}(g)\}$. Under this definition, $[f]$ is always a set (being a subset of $V_{\alpha+1}$, where $\alpha = \mathrm{rank}(g)$).

Let $V^\kappa/U = \{[f] \mid f : \kappa \to U\}$. Note that two elements $[f], [g]$ of V^κ/U are equal if $f \sim_U g$. To turn V^κ/U into a model of set theory, we need to define its membership relation. We cannot use the usual membership relation here since members of equivalence classes are not generally other equivalence classes. Instead we define a new relation $\in^*$: We write $[f] \in^* [g]$ if and only if $\{\alpha \mid f(\alpha) \in g(\alpha)\} \in U$. With these definitions of equality and membership, V^κ/U can be shown to be a model of ZFC. Moreover, because U is κ-complete, V^κ/U is a *wellfounded model*; this means that there is no infinite decreasing $\in^*$-chain in V^κ/U—that is, no sequence $f_1, f_2, f_3, \ldots$ so that

$$\ldots \in^* [f_3] \in^* [f_2] \in^* [f_1].$$

Because V^κ/U is wellfounded and has the property that for each $[f] \in V^\kappa/U$, $\{[g] \in V^\kappa/U \mid [g] \in^* [f]\}$ is a *set*, Mostowski's Collapsing Theorem can be applied [34, Chapter 28] to produce an isomorphic transitive image $(N, \in)$ of V^κ/U, which of course is also a model of ZFC. We identify elements of V^κ/U with those of N. We explain this point a bit more. Note that if $\pi : V^\kappa/U \to N$ is the collapsing isomorphism, then $[f] \in^* [g]$ if and only if $\pi([f]) \in \pi([g])$; we will identify $\pi([f])$ with $[f]$.

Finally, we mention that the map $i_U : V \to V^\kappa/U \cong N$ defined by $i_U(x) = [c_x]$, where c_x is the constant function with value x, is an elementary embedding with critical point κ. The map i_U is called the *canonical embedding derived from* U.[158]

We now start moving toward the construction of the blueprint $(\ell, \kappa, \mathcal{E})$ for $V_{\kappa+1}$. The map ℓ will turn out to be a co-Dedekind self-map, and the elements of $\mathcal{E}$ will be restrictions of ultrapower embeddings. We will be able to show that for each $X \subseteq V_\kappa$, there is a normal measure U on κ such that if i_U is the canonical embedding derived from U, $i_U(\ell)(\kappa) = X$. In this way, every set in the stage $V_{\kappa+1}$, the initial part of the universe up to subsets of V_κ, can be seen as arising from or being generated by the interplay of κ, j, and ℓ, where ℓ is encoded and decoded by $\mathcal{E}$.

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[158]This canonical embedding provides an example, familiar to set theorists, of a map that both is cofinal and has a weakly universal element. We outline the ideas here. Suppose we are given $i_U : V \to V^\kappa/U \cong N$, as described in the text, where U is a *normal* measure on κ (note that any measurable cardinal admits a normal measure). We treat V and N as categories, whose objects are the sets they contain and whose morphisms are the functions between sets. It follows that i_U is a functor. It is well-known [36, Exercise 5.11] that, because U is normal, $\kappa = [\mathrm{id}_\kappa]_U$. (More precisely, if $\pi : V^\kappa/U \to N$ is the collapsing isomorphism, then $\pi([\mathrm{id}_\kappa]) = \kappa$.) We first observe that $\kappa \in i_U(\kappa)$ is a weakly universal element for i_U: Suppose $x \in i_U(A)$ for some $A \in V$. Then for some $f : \kappa \to V$, $x = [f]$, and $i_U(A) = [c_A]$. Since $\{\alpha < \kappa \mid f(\alpha) \in c_A(\alpha)\} \in U$, then we can find $g : \kappa \to A$ (in V) so that $f \sim_U g$, and so $x = [g]$.

Now we show $i_U(g)(\kappa) = x$:

$$\begin{aligned} i_U(g)(\kappa) = x \quad &\Leftrightarrow \quad \Big([c_g]([\mathrm{id}_\kappa]) = [g]\Big)^N \\ &\Leftrightarrow \quad \{\alpha < \kappa \mid c_g(\alpha)(\mathrm{id}_\kappa(\alpha)) = g(\alpha)\} \in U \\ &\Leftrightarrow \quad \{\alpha < \kappa \mid g(\alpha) = g(\alpha)\} \in U, \end{aligned}$$

and the last of these statements is true. We have shown $\kappa \in i_U(\kappa)$ is a weakly universal element of i_U. We wish to show that i_U is cofinal (using the definition for two categories given on p. 130). Suppose $x \in N$. We need to find $A \in V$ so that, in N, $x \in i_U(A)$. Let α be such that $x \in V_\alpha^N$. But now $x \in i_U(V_\alpha)$ since

$$i_U(V_\alpha) = V^N_{i_U(\alpha)} = V_{i_U(\alpha)} \cap N \supseteq V_\alpha \cap N = V_\alpha^N.$$

It therefore follows from equation (68) (on p. 130) that

$$N = \{i_U(f)(\kappa) \mid f : \kappa \to V \text{ and } \kappa \in \mathrm{dom}\ i_U(f)\}.$$

This is a well-known result about such ultrapowers, usually proved in a different way [36, Proposition 5.13(a)]. An important related fact is that one cannot carry out a similar argument for any kind of elementary embedding $j : V \to V$; the argument breaks down because V cannot be represented as the transitive collapse of an ultrapower (for example, see [36, Proposition 5.7(e)]). In fact, as is shown in the Appendix, Theorem 95, if j is a WA_0-embedding, then for *no* sets a, A for which $a \in j(A)$ is it the case that a is a universal element for j.

The essence of the construction of ℓ is a $V_{\kappa+1}$*-Laver function*. For any set X, a function $f : \kappa \to V_\kappa$ is an X-Laver function at κ if, for each $x \in X$, there is a normal measure U on κ such that $i_U(f)(\kappa) = x$.

In the present setting, the function f will be defined by a clever method, due to R. Laver, of encoding information about all possible normal measures over κ. We define f and then explain how ℓ is obtained from f. We first define a formula $\psi(g, x, \lambda)$ that is needed both in the definition of f and in the proof that f has the desired properties:

$$\psi(g, x, \lambda): \quad g : \lambda \to V_\lambda \wedge x \subseteq V_\lambda \wedge \text{“for all normal measures } U \text{ on } \lambda,\ i_U(g)(\lambda) \neq x\text{”}.$$

When $\psi(g, x, \lambda)$ holds true, it means that g is *not* a $V_{\lambda+1}$-Laver sequence at λ: Some subset x of V_λ cannot be computed as $i_U(g)(\lambda)$ for any choice of U. We can now define f:

$$f(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } f \restriction \alpha \text{ is } V_{\alpha+1}\text{-Laver at } \alpha, \\ x & \text{otherwise, where } x \text{ satisfies } \psi(f \restriction \alpha, x, \alpha). \end{cases} \tag{99}$$

The definition tells us that $f(\alpha)$ has nonempty value just when the restriction $f \restriction \alpha$ is *not* $V_{\alpha+1}$-Laver at α, and in that case, its value is a witness to non-Laverness.

Theorem 75. ($V_{\kappa+1}$-Laver Functions Under MUA) *The function f defined in (99) is a $V_{\kappa+1}$-Laver function at κ.*

Proof. Let $j : V \to V$ be the Dedekind self-map, with critical point κ, given to us in a model of ZFC + BTEE + MUA. Suppose f is not $V_{\kappa+1}$-Laver at κ, so, in particular, for some y, $\psi(f, y, \kappa)$ holds. We consider $j(f) : j(\kappa) \to V_{j(\kappa)}$.

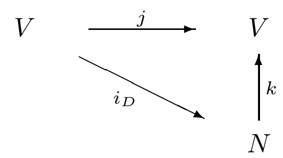
First we show that $j(f) \restriction \kappa = f$. For each $\alpha < \kappa$, we have $j(f)(\alpha) = j(f)(j(\alpha)) = j\,(f(\alpha))) = f(\alpha)$ (since $f(\alpha) \in V_\kappa$). By elementarity, $j(f)$ has the same definition as f. In particular, we have that, for each $\alpha < j(\kappa)$,

$$j(f)(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } j(f) \restriction \alpha \text{ is } V_{\alpha+1}\text{-Laver at } \alpha, \\ x & \text{otherwise, where } x \text{ satisfies } \psi(j(f) \restriction \alpha, x, \alpha). \end{cases}$$

In particular, since $f = j(f) \restriction \kappa$ is not $V_{\kappa+1}$-Laver, computation of $j(f)(\kappa)$ uses the second clause of the definition for $j(f)$ and $\psi(j(f) \restriction \kappa, j(f)(\kappa), \kappa)$ is true. Therefore, $x = j(f)(\kappa)$ is a witness to the fact that $f = j(f) \restriction \kappa$ is not $V_{\kappa+1}$-Laver. Recall from the definition of ψ that any such witness x must be a subset of V_κ, so we have that $j(f)(\kappa) \subseteq V_\kappa$.

Let $D = U_j$ be the normal measure derived from j; that is, $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. By MUA, D is a *set*. Let $i = i_D : V \to V^\kappa/D \cong N$ be the canonical embedding, and define $k : N \to V$ by $k([h]) = j(h)(\kappa)$. One can show [34] that k is an elementary embedding with critical point $> \kappa$ and makes the following diagram commutative:

(100)



By diagram (100), we have

$$\begin{aligned} j(f)(\kappa) &= (k \circ i)(f)(\kappa) \\ &= \big(k(i(f))\big)(k(\kappa)) \\ &= k\big(i(f)(\kappa)\big). \end{aligned}$$

Now since $j(f)(\kappa) \subseteq V_\kappa$ and $\mathrm{crit}(k) > \kappa$, $k\big(j(f)(\kappa)\big) = j(f)(\kappa)$. Since k is 1-1 and

$$k\big(j(f)(\kappa)\big) = j(f)(\kappa) = k\big(\big(i(f)(\kappa)\big),$$

it follows that

$$j(f)(\kappa) = i(f)(\kappa).$$

The import of this last equation is that, while it is claimed that $\psi(j(f) \restriction \kappa, j(f)(\kappa), \kappa)$ holds true, we have just exhibited a normal measure D on κ such that $i_D(f)(\kappa) = j(f)(\kappa)$, and we have a contradiction. We conclude, therefore, that f is $V_{\kappa+1}$-Laver after all. □

We mention briefly the significance of having V itself as the codomain of j in Theorem 76. If the codomain of j were some smaller transitive model M, then, assuming f is not $V_{\kappa+1}$-Laver at κ does imply the triple $j(f) \restriction \kappa, j(f)(\kappa), \kappa$ satisfies the formula $\psi(g, x, \lambda)$, but this formula is now relativized to the model M, and so asserts that for no normal measure U in M is it true that $i_U(f)(\kappa) = j(f)(\kappa)$. So, although it is true that for D as defined in the proof, $i_D(f)(\kappa) = j(f)(\kappa)$, D may not be one of M's normal measures, and so would not give us a contradiction.

We turn to the construction of ℓ:

(101) $$\ell(x) = \begin{cases} f(x) & \text{if } x \text{ is an ordinal} < \kappa, \\ x & \text{otherwise.} \end{cases}$$

We now show that ℓ is the blueprint map for a blueprint $(\ell, \kappa, \mathcal{E})$ for $V_{\kappa+1}$, assuming MUA. We will describe the members of $\mathcal{E}$ in a more precise way in the discussion in Remark 22, below.

Theorem 76. (Existence of Blueprint Self-Maps under MUA) (ZFC + BTEE + MUA). *Suppose $j : V \to V$ is a Dedekind self-map given by a model of* ZFC + BTEE+MUA*, with critical point κ. Then the function $\ell : V_\kappa \to V_\kappa$ defined in (101) has the following property:*

(102) $$\forall X \subseteq V_\kappa\, \exists U \text{ (U is a normal measure on κ and $i_U(\ell)(\kappa) = X$).}$$

Moreover, ℓ is a co-Dedekind self-map.

For the rest of this section, we shall say that a self-map having the property (102) has the *Laver property at* κ.

Proof. Note that for all $\alpha < \kappa$, $f(\alpha) = \ell(\alpha)$. In particular, if $T = \{\alpha < \kappa \mid f(\alpha) = \ell(\alpha)\}$ and U is a normal measure on κ, then $T \in U$. Therefore, if $i = i_U$ is the canonical embedding,

$$\kappa \in i(T) = i\left(\{\alpha < \kappa \mid f(\alpha) = \ell(\alpha)\}\right) = \{\alpha < i(\kappa) \mid i(f)(\alpha) = i(\ell)(\alpha)\}.$$

It follows that, for every normal measure U on κ, $i_U(f)(\kappa) = i_U(\ell)(\kappa)$. It follows that ℓ has the Laver property at κ since f does.

To see that ℓ is a co-Dedekind self-map, it is sufficient to show that, whenever $f : \kappa \to V_\kappa$ is $V_{\kappa+1}$-Laver for subsets of V_κ, for each $x \in V_\kappa$, $|f^{-1}(x)| = \kappa$. Given $x \in V_\kappa$, let $T_x = \{\alpha < \kappa \mid f(\alpha) = x\}$. Let U be a normal measure on κ such that $i_U(f)(\kappa) = x$. Note that $i_U(x) = x$ since $x \in V_\kappa$. Then since $\kappa \in \{\alpha < i(\kappa) \mid i(f)(\alpha) = x\} = i(T_x)$, it follows that $T_x \in U$. Therefore $|f^{-1}(x)| = |T_x| = \kappa$. □

Remark 22. (The Blueprint Coder $\mathcal{E}$) We describe the blueprint coder for the blueprint of $V_{\kappa+1}$ in more detail. Suppose $j : V \to V$ is a Dedekind self-map given by a model of ZFC + BTEE + MUA, with critical point κ. For each normal measure U on κ, let $i_U : V \to V^\kappa/U \cong M_U$ be the canonical embedding and let $\overline{i_U} = i_U \upharpoonright V_{\kappa+1}$ (equivalently, $\overline{i_U} : V_{\kappa+1} \to V^\kappa_{\kappa+1}/U$ defined in the same way as i_U). We define $\mathcal{E}$ by

$$\mathcal{E} = \{\overline{i_U} \mid U \text{ is a normal measure on } \kappa\}.$$

Taking this step allows us to formally define $\mathcal{E}$ as a class; this step is necessary since, formally speaking, we cannot collect all embeddings of the form $i : V \to M$ into a single class. Nothing is lost in restricting the embeddings in this way: Tracing through the proofs above, it is straightforward to verify that a $V_{\kappa+1}$-Laver function can be obtained by making use of $\mathcal{E}$ in place of the full elementary embeddings $i : V \to M$ that were used previously. Moreover, one verifies that ℓ is defined in the same way as before.

We show that $(\ell, \kappa, \mathcal{E})$ is a blueprint for $V_{\kappa+1}$; we use the criteria specified in Definition 7 (p. 83). The map ℓ satisfies (1) because it is a co-Dedekind self-map. For (2), each element of $\mathcal{E}$ is weakly elementary relative to $V_{\kappa+1}$ since, in fact, each is an elementary embedding. Verification of the compatibility requirement makes use of [10], as discussed briefly in Remark 7. Translating this work into the present setting, we observe that there is $i : V_{\kappa+1} \to N \in \mathcal{E}$ and an elementary embedding $k : N \to V_{j(\kappa)+1}$ with the following two properties:

(a) $j \upharpoonright V_{\kappa+1} = k \circ i$;
(b) $k \upharpoonright V_{\kappa+1} = \mathrm{id}_{V_{\kappa+1}}$

In the language of [10, Definition 4.18], these facts tell us that $\mathcal{E}$ is *locally* compatible with j.[159]

[159] We explain more details about this compatibility criterion. In [10], familiar *globally defined* large cardinal notions (like supercompactness and superhugeness) were represented by classes of set embeddings. The reason for this representation was to provide a uniform setting for discussing existence of Laver sequences for arbitrary globally defined large cardinals. The starting point for this study is the notion of a *suitable formula*.

Continuing our verification of (2), let us define $\mathcal{E}_0$ by

$$\mathcal{E}_0 = \{\bar{i} \upharpoonright V_\kappa^{V_\kappa} \mid \bar{i} \in \mathcal{E}\}.$$

It can be shown that every function $V_\kappa \to V_\kappa$ belongs to the range of each element of $\mathcal{E}_0$. For each $i \in \mathcal{E}_0$ and each $f : V_\kappa \to V_\kappa$, since i is in reality the restriction of an elementary embedding $V \to M$ with critical point κ, it can be shown that $i(f) : V^M_{i(\kappa)} \to V^M_{i(\kappa)}$ (where $V^M_{i(\kappa)}$ denotes the set X that satisfies, in the model M,

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Let $\theta(x, y, z, w)$ be a first-order formula (in the language $\{\in\}$) with all free variables displayed. We will call θ a *suitable formula* if the following sentence is provable in ZFC:

$$\forall x, y, z, w\big[\theta(x, y, z, w) \implies \text{"}w \text{ is a transitive set"} \wedge z \in \text{ ON} \wedge \text{"}x : V_z \to w \text{ is an elementary embedding with critical point } y\text{"}\big].$$

For each cardinal κ and each suitable $\theta(x, y, z, w)$, let

$$\mathcal{E}^\theta_\kappa = \{(i, M) : \exists\beta\, \theta(i, \kappa, \beta, M)\}.$$

The codomain of an elementary embedding i needs to be explicitly associated with i in the definition for technical reasons; for practical purposes, we think of $\mathcal{E}^\theta_\kappa$ as a collection of elementary embeddings $i : V_\beta \to M$ with critical point κ.

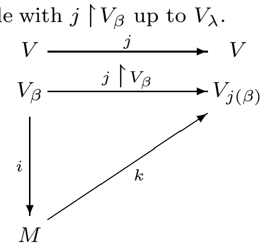
In [10], familiar large cardinals, like supercompact and superhuge, are re-defined in terms of classes $\mathcal{E}^\theta_\kappa$ of embeddings. With this general concept of classes of set embeddings, we defined in [10] a general notion of Laver sequence, which we reproduce here:

Given a class $\mathcal{E}^\theta_\kappa$ of embeddings, where θ is a suitable formula, a function $g : \kappa \to V_\kappa$ is defined to be *$\mathcal{E}^\theta_\kappa$-Laver at κ* if for each set x and for arbitrarily large λ there are $\beta > \lambda$, and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ such that $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$.

A key sufficient condition for Laverness, mentioned in [10], is *compatibility* with an ambient elementary embedding $j : V \to V$ having critical point κ, such as the embedding given by MUA. Suppose $\kappa < \lambda < \beta$, and $i_\beta : V_\beta \to M$ is an elementary embedding with critical point κ. Then i_β is *compatible with j up to V_λ* if there is an elementary embedding $k : M \to V_{j(\beta)}$ with

$$j \upharpoonright V_\beta = k \circ i \text{ and } k \upharpoonright V_\lambda \cap M = \mathrm{id}_{V_\lambda \cap M}.$$

If θ is suitable, then $\mathcal{E}^\theta_\kappa$ is said to be *compatible with j* if for each $\lambda < j(\kappa)$ there is a $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ that is compatible with $j \upharpoonright V_\beta$ up to V_λ.



In the present context of ZFC + BTEE + MUA, the large cardinal under consideration is not globally defined and therefore we do not expect to obtain a Laver sequence for such a cardinal. Our concern is to demonstrate the existence of a $V_{\kappa+1}$-Laver sequence, appropriate for strong kinds of measurable cardinals. We can adapt the definitions given here in the following way. First, we can represent the concept of a measurable cardinal with a suitable formula θ_m:

$$\theta_m(i, \kappa, \beta, M) : \quad \beta = \kappa + 1 \wedge M \text{ is transitive} \wedge i : V_\beta \to M \text{ is elementary.}$$

Now κ is measurable if and only if there exist i, β, M such that $\theta_m(i, \kappa, \beta, M)$: If κ is measurable, let $e : V \to N$ be an elementary embedding with critical point κ, and consider $i = e \upharpoonright V_{\kappa+1} : V_{\kappa+1} \to V^N_{e(\kappa)+1} = M$. Conversely, given $i : V_{\kappa+1} \to M$ with critical point κ, define

$$U = \{X \subseteq \kappa \mid \kappa \in i(X)\}.$$

U is well-defined since $\mathcal{P}(\kappa) \subseteq V_{\kappa+1}$, and is easily seen to be a normal measure on κ; thus κ is measurable.

the statement "X is the stage $V_{i(\kappa)}$"). It can be shown that $V_{\kappa}^{V_{\kappa}} \subseteq {V_{i(\kappa)}^{M}}^{V_{i(\kappa)}^{M}}$, and so (2)(b) is satisfied. Verification of (2)(c) is automatic, and (2)(d) follows since $\kappa \in i(\kappa)$ for any elementary embedding $i : V \to M$ with critical point κ.

For (3), we must argue that ℓ is definable from $\mathcal{E}, j, \kappa$. Certainly, the $V_{\kappa+1}$-Laver function that we defined is derived from $\mathcal{E}, j, \kappa$, and ℓ is definable from this function.

Finally, to prove (4), suppose $X \subseteq V_\kappa$. Since, from the previous theorem, ℓ has the Laver property at κ, we can find U such that $X = i_U(\ell)(\kappa) = \overline{i_U}(\ell)(\kappa)$. Since $\ell \in V_{\kappa}^{V_{\kappa}}$, it follows that $\overline{i_U} \upharpoonright V_{\kappa}^{V_{\kappa}}(\ell)(\kappa) = X$ as well, and $\overline{i_U} \upharpoonright V_{\kappa}^{V_{\kappa}} \in \mathcal{E}_0$. $\square$

Next, we show that there is a strong blueprint for $V_{\kappa+1} - V_\kappa$, the collection of subsets of V_κ of size κ. This strong blueprint shows the generating and collapsing effects of ℓ and its dual, ℓ^{op}, which will be defined to be a certain section of ℓ.

We remark here that we cannot do much better than $V_{\kappa+1} - V_\kappa$. In fact, as we now show, there is no way to define a section s of ℓ with the property that, for each $x \in V_\kappa$, there is $i \in \mathcal{E}_0$ for which $i(s)(x) = \kappa$. Let $s : V_\kappa \to V_\kappa$ be a section of ℓ and let $x \in V_\kappa$. Let $i = i_U$ be a canonical embedding, as usual. Then

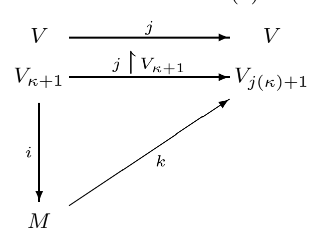
$$i(s)(x) = i(s)\,(i(x)) = i(s(x)) = s(x) \neq \kappa.$$

The fact that elements of V_κ are "left out" of the dynamics of the strong blueprint accords with our expectation: For any MUA-embedding $j : V \to V$ with critical point κ, we are *given* κ, and thereby V_κ as well, so the only sets that need to "return" to κ are those that lie outside of V_κ.

Example 7. (Strong Blueprint for $V_{\kappa+1} - V_\kappa$) Let $j : V \to V$ be a Dedekind self-map given by a model of $\text{ZFC} + \text{BTEE} + \text{MUA}$, with critical point κ. Let $(\ell, \kappa, \mathcal{E})$ be a blueprint for $V_{\kappa+1}$. We define the dual map ℓ^{op} so that $(\ell, \ell^{\text{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V_{\kappa+1} - V_\kappa$.

Next, we can define the concept of an X-$\mathcal{E}_\kappa^{\theta_m}$-Laver sequence, adapting the definition given in the main text: For any set X, a function $f : \kappa \to V_\kappa$ is an X-$\mathcal{E}_\kappa^{\theta_m}$-Laver function at κ if, for each $x \in X$, there is $i \in \mathcal{E}_\kappa^{\theta_m}$ such that $i(f)(\kappa) = x$. It is straightforward to show that for any $f : \kappa \to V_\kappa$ and any set X, f is X-Laver at κ if and only if f is X-$\mathcal{E}_\kappa^{\theta_m}$-Laver at κ.

These definitions naturally lead to a *local* form of compatibility of $\mathcal{E}_\kappa^{\theta_m}$ with $j : V \to V$, which is discussed in the main text: Starting with an MUA-embedding $j : V \to V$ with critical point κ, we declare that $\mathcal{E}_\kappa^{\theta_m}$ is *locally compatible with* j if there is $i : V_{\kappa+1} \to M \in \mathcal{E}_\kappa^{\theta_m}$ that is compatible with $j \upharpoonright V_{\kappa+1}$ up to $V_{\kappa+1}$. (And $\mathcal{E}_\kappa^{\theta_m}$ is indeed *locally compatible with* j since i can be obtained as the canonical embedding i_D derived from the ultrafilter $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$.) This local compatibility definition is equivalent to the statements (a) and (b) given in the main text.



We define $\ell^{\mathrm{op}} : V_\kappa \to V_\kappa$ as follows:

$$\ell^{\mathrm{op}}(x) = \begin{cases} \alpha & \text{if } \alpha \text{ is the least ordinal in } \ell^{-1}(x)\text{, if there is one,} \\ y & \text{otherwise, where } y \text{ is an arbitrary element of } \ell^{-1}(x). \end{cases}$$

As $\ell^{\mathrm{op}}(x) \in \ell^{-1}(x)$ for each $x \in V_\kappa$, it is obvious that ℓ^{op} is a Dedekind self-map and a section of ℓ. We point out here that, while the definition of $\ell^{\mathrm{op}}(x)$ requires finding the least ordinal α satisfying a certain formula—namely that α belongs to $\ell^{-1}(x)$—this formula is *not* a **j**-formula. This is important because, from the theory $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{MUA}$, it is not in general possible to compute the least ordinal for which a **j**-formula holds; see [11].

Claim. Suppose $X \in V_{\kappa+1} - V_\kappa$, U is a normal measure on κ, and $i = i_U$ is the canonical embedding with critical point κ for which $i(\ell)(\gamma) = X$. Then $\gamma \geq \kappa$.

Note that the Claim (once proven) continues to hold true if i_U is replaced with $\overline{i_U}$.

Proof of Claim. Suppose $\alpha < \kappa$. We compute $i(\ell)(\alpha)$, using the fact that $i(\alpha) = \alpha$:

$$i(\ell)(\alpha) = i(\ell)\left(i(\alpha)\right) = i(\ell(\alpha)) = \ell(\alpha) \in V_\kappa.$$

The fact that $i(\ell(\alpha)) = \ell(\alpha)$ follows because $\ell(\alpha) \in V_\kappa$ and i is the identity on V_κ. We have shown that if $X \in V_{\kappa+1} - V_\kappa$ and $i(\ell)(\gamma) = X$, then $\gamma \geq \kappa$. $\square$

We verify the main property of ℓ^{op}: Let $X \in V_{\kappa+1} - V_\kappa$. Let U be a normal measure on κ and $i = i_U$ the canonical embedding so that $i(\ell)(\kappa) = X$. Note that, by elementarity, $i\left(\ell^{\mathrm{op}}\right)$ is defined, for each $x \in V_{i(\kappa)}$, by

$$i\left(\ell^{\mathrm{op}}\right)(x) = \begin{cases} \alpha & \text{if } \alpha \text{ is the least ordinal in } i(\ell)^{-1}(x)\text{, if there is one,} \\ y & \text{otherwise, where } y \text{ is an arbitrary element of } i(\ell)^{-1}(x). \end{cases}$$

By the claim, κ is the least ordinal in $i(\ell)^{-1}(X)$. By definition of $i(\ell)^{\mathrm{op}}$, it follows that $i(\ell)^{\mathrm{op}}(X) = \kappa$. Again, note that the same argument goes through if i_U is replaced by $\overline{i_U}$.

We can now formally establish that $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V_{\kappa+1} - V_\kappa$ by verifying the properties in Definition 8. We have already shown that $(\ell, \kappa, \mathcal{E})$ is a *blueprint* for X, where $X = V_{\kappa+1} - V_\kappa$ and where $\mathcal{E}_0$ is the set of restrictions of elements of $\mathcal{E}$ to $V_\kappa^{V_\kappa}$. What remains is to establish the following points, and these were demonstrated in the paragraphs above:

(A) $\mathrm{dom}\ \ell^{\mathrm{op}} = \mathrm{dom}\ \ell$.
(B) ℓ^{op} is a section of ℓ.
(C) For each $i \in \mathcal{E}_0$, $\mathrm{dom}\ i(\ell^{\mathrm{op}}) = \mathrm{dom}\ i(\ell)$
(D) For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(\ell^{\mathrm{op}})(x) = \kappa$. $\square$

We arrived at the theory $\mathrm{ZFC} + \mathrm{BTEE}$ by noticing (1) the needed strengthening of a Dedekind self-map $j : V \to V$ to produce an infinite set was obtained by requiring j to have fairly natural preservation properties; (2) by strengthening these preservation properties further, certain large cardinals could be derived; (3) the strongest kind of preservation possible is obtained when j is an elementary embedding, and the theory ZFC + BTEE is the formal assertion of the existence of such an embedding from V to V.

In our initial study of Dedekind self-maps, we found they exihibited not only interesting preservation properties, but led naturally to the concept of a nonprincipal ultrafilter. Generalizing these ideas led to an example and corresponding theoretical results in which a Dedekind self-map $j : V \to V$ exhibits strong preservation properties and a nonprincipal ultrafilter plays a key role. The results in this case provided motivation for the existence of a measurable cardinal. The construction of the ultrafilter in this case could not be carried out directly in the theory ZFC + BTEE because of definability restrictions, and so we were led to postulate a supplementary axiom to ZFC + BTEE, namely, MUA, which asserts that the ultrafilter naturally derived from a BTEE-embedding exists as a set. The strengthened theory ZFC + BTEE + MUA implies existence of many measurable cardinals and also exhibits many more of the characteristics that we specified in the case of a set Dedekind self-map (described in Properties of Set Dedekind Self Maps, p. 168)—in particular, a blueprint for $V_{\kappa+1}$ and a strong blueprint for $V_{\kappa+1} - V_\kappa$.

However, one characteristic of Dedekind self-maps that we did not discuss—mentioned in part (B) of Properties of Set Dedekind Self-Maps (p. 168)—is the generation of a critical sequence and the role of restrictions of j. Although it is true that the "generating" effect of j, in the theory ZFC + BTEE + MUA, was captured nicely by the blueprint and strong blueprint that are derived from j, it is still natural to ask about the properties of the sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ and the role of restrictions of j.

This topic reveals interesting limitations in the theory ZFC + BTEE + MUA. We mention some known results [11] and introduce some new refinements. To state the results, let us recall that the universe V satisfies all the axioms of ZFC; we say that V is a *model* of ZFC. This means that, for each axiom σ in ZFC, σ *is true in* V, and we denote this fact by writing $V \models \sigma$.[160]

In fact, once we know ZFC has any model at all, it is guaranteed to have *many models*. One can show that if λ is inaccessible (or any of the other stronger large cardinals), V_λ is also a model of ZFC. Now suppose $\lambda < \rho$ are both inaccessible cardinals. We write $V_\lambda \prec V_\rho$ to indicate the following: For any formula $\phi(x_1, x_2, \ldots, x_n)$ and any sets $a_1, a_2, \ldots, a_n \in V_\lambda$,

$$V_\lambda \models \phi[a_1, a_2, \ldots, a_n] \text{ if and only if } V_\rho \models \phi[a_1, a_2, \ldots, a_n].$$

We say that V_λ is an *elementary submodel of* V_ρ. One consequence of this property is that V_λ and V_ρ satisfy the same sentences; they have "identical views" of the world.

We may now list some known limitations of the theory ZFC + BTEE + MUA:

(1) *Critical sequence may not exist.* In the theory ZFC + BTEE + MUA, the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ cannot be shown to "exist," even as a j-class, without supplementing the theory with additional axioms. (One cannot even guarantee that whenever n is a nonstandard integer in the theory, $j^n(\kappa)$ exists as a set.) It can be shown that, for each particular

[160] Models were introduced on p. 22 and the satisfaction relation $\models$ was introduced on p. 53.

(metatheoretic) natural number n, the theory proves that the sequence $\langle\kappa, j(\kappa), \ldots, j^n(\kappa)\rangle$ exists as a set. On the other hand, whenever we work in a *transitive model* of ZFC + BTEE + MUA, this problem is corrected, and the critical sequence can indeed be shown to be a j-class in the model.

(2) *Stages $V_{j^n(\kappa)}$ may not form an elementary chain.* The theory ZFC+BTEE+MUA shows, for each particular (metatheoretic) natural number n, that $V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)}$, but the sequence of models $V_\kappa, V_{j(\kappa)}, V_{j^2(\kappa)}, \ldots$ cannot be shown to be a j-class. Once again, this sequence *can* be shown to be a j-class inside any transitive model of the theory. However, even inside such a transitive model, without additional axioms, the reasonable conjecture $V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)} \prec \ldots \prec V$ cannot be proven.

(3) *Boundedness of the critical sequence is undecidable.* A natural question, which the theory ZFC + BTEE + MUA cannot answer, is whether the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is bounded. Assuming existence of a 2^κ-supercompact cardinal,[161] there is a transitive model of ZFC+BTEE+MUA in which the critical sequence is bounded, but, on the other hand, letting σ denote the sentence "the critical sequence is unbounded," any I_3 embedding $i : V_\lambda \to V_\lambda$ (with critical point κ and with λ a limit greater than κ) gives rise to a transitive model $(V_\lambda, \in, i)$ of ZFC + BTEE + MUA + σ.[162]

(4) *Restrictions $j \restriction V_{j^n(\kappa)}$ $(n \geq 1)$ may not exist.* The restriction $j \restriction V_\kappa$ can be shown to exist as a set in ZFC + BTEE + MUA (it is equal to id_{V_κ}); using elementarity of j, one can show that $V_\kappa \prec V_{j(\kappa)}$, and hence that $j \restriction V_\kappa : V_\kappa \to V_{j(\kappa)}$ is an elementary embedding.[163] However, it is not possible to show that restrictions of j to $V_{j^n(\kappa)}$, for $n \geq 1$, exist as sets in ZFC+BTEE+MUA. In fact, as we show in the Appendix, Theorem 97, the theory ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction \mathbf{j}(\kappa))$ is sufficient to prove the consistency of the ZFC + BTEE + MUA. Therefore, by Gödel's Second Incompleteness

[161]Supercompactness is defined in a footnote on p. 168.

[162]An I_3 embedding $i : V_\lambda \to V_\lambda$ is an elementary embedding for which λ is a limit ordinal and whose critical point lies below λ.

[163]Here is a short proof: We first show that $V_\kappa \prec j(V_\kappa)$. Arguing in the metatheory: Suppose $\phi(x_0, x_1, x_2, \ldots, x_n)$ is a formula, $a_1, a_2, \ldots, a_n$ are sets, and there is $b \in j(V_\kappa)$ such that $j(V_\kappa) \models \phi[b, j(a_1), \ldots, j(a_n)]$. Then $j(V_\kappa) \models \exists x\, \phi[x, j(a_1), \ldots, j(a_n)]$. By elementarity of j, $V_\kappa \models \exists x\, \phi[x, a_1, \ldots, a_n]$. Therefore, for some $a_0 \in V_\kappa$, $V_\kappa \models \phi[a_0, a_1, \ldots, a_n]$. By elementarity again, $j(V_\kappa) \models \phi[j(a_0), j(a_1), \ldots, j(a_n)]$. By the Tarski-Vaught criterion for elementary submodels, it follows that $V_\kappa \prec j(V_\kappa)$.

Since $j(V_\kappa) = V_{j(\kappa)}$, and since the statement that $V_\kappa \prec V_{j(\kappa)}$ is equivalent to the assertion that $j \restriction V_\kappa : V_\kappa \to V_{j(\kappa)}$ is elementary, we have therefore established the latter assertion mentioned in the text.

To make this metatheoretic argument formal, since we cannot quantify over formulas, we use *codes* for formulas; codes are sets belonging to V_ω whose structure reflects the build-up of formulas. Every formula in the language $\{\in\}$ of ZFC can be coded as a set in V_ω. Given a formula $\phi(x_0, x_1, \ldots, x_n)$ as before, let p be a code for ϕ. Let $f : \mathrm{rank}(p) \to V_\kappa$; f represents a possible finite sequence of parameters for the formula coded by p; in the above example, the parameters of f were $a_1, \ldots, a_n$. Then $j(p) = p$ and $j(f) : \mathrm{rank}(p) \to V_{j(\kappa)}$. By elementarity of j, we have

$$\mathrm{Sat}(p, V_\kappa, f) \iff \mathrm{Sat}(p, V_{j(\kappa)}, j(f)),$$

where, in general $\mathrm{Sat}(x, M, g)$ is the Δ_1 statement that asserts formally that M satisfies the formula coded by x with parameter values given by g. This argument shows formally $V_\kappa \prec V_{j(\kappa)}$.

Theorem, assuming ZFC + BTEE + MUA is consistent,

$$\text{ZFC} + \text{BTEE} + \text{MUA} \nvdash \exists z\, \exists n \geq 1\, (z = \mathbf{j} \restriction \mathbf{j}^n(\kappa)).$$

The limitation described in (4) holds the key to pushing beyond ZFC + BTEE + MUA toward a theory in which all the characteristics (A)–(D) that we initially identified for set Dedekind self-maps (listed in Properties of Set Dedekind Self-Maps on p. 168) are realized and all the limitations described in (1)–(4) above can be removed. It can be shown that, as we consider extensions of ZFC + BTEE in which axioms asserting existence of restrictions of j to ever larger sets, we arrive at theories having ever stronger large cardinal consequences. The limit of this direction of generalization is the assertion that the restriction of j to *every* set exists as a set. We consider this very strong extension of ZFC + BTEE in the next section.

To close this section, we review what is known about various strengthenings of ZFC + BTEE, obtained by adding axioms that assert existence of a restriction of $\mathbf{j}$ to some set. Each of the theories mentioned below consists of ZFC + BTEE plus some statement of the form "$\mathbf{j} \restriction A$ is a set," for some set A. Most of these theories are stronger than ZFC + BTEE + MUA, as indicated by their strong large cardinal consequences (see [11]).

(A) The theory ZFC + BTEE + $\exists z\, (z = \mathbf{j} \restriction \kappa^+)$ has consistency strength[164] at least that of a strong cardinal.

(B) The theory ZFC + BTEE + $\exists z\, (z = \mathbf{j} \restriction \mathcal{P}(\kappa))$ has consistency strength at least that of a Woodin cardinal. Moreover, from this theory, it is possible to derive the axiom MUA.

(C) The theory ZFC + BTEE + $\exists z\, (z = \mathbf{j} \restriction \mathbf{j}(\kappa))$ is strong enough to prove the consistency of the theory ZFC + BTEE + MUA, as mentioned above. The proof is given in the Appendix, Theorem 97.

(D) The theory ZFC+BTEE+$\exists z\, \big(z = \mathbf{j} \restriction \mathcal{P}(\mathcal{P}(\mathbf{j}(\kappa)))\big)$ directly proves κ is huge with κ huge cardinals below it.[165]

(E) The theory ZFC + BTEE + $\exists \lambda \exists z\, (\lambda$ is an upper bound for the critical sequence and $z = \mathbf{j} \restriction \lambda)$ is inconsistent.

The "moreover" clause in (B) can be shown as follows: Working in the theory ZFC + BTEE + $\exists z\, (z = j \restriction P(\kappa))$, where κ is the critical point of the embedding $j : V \to V$, let $g = j \restriction P(\kappa)$. Now the ultrafilter derived from j has the following simple ZFC definition: $U = \{X \subseteq \kappa \mid \kappa \in g(X)\}$.

It can be shown that, for any set X for which $|X| \leq \kappa$, the restriction $j \restriction X$ does exist as a set in ZFC+BTEE+MUA (however, note that restrictions of j to sets of larger cardinality require stronger axioms, as (A) demonstrates). This follows from two observations, which we prove below.

[164] Suppose $A(x)$ is a large cardinal property. For instance, $A(x)$ could be the statement, "x is a measurable cardinal." To say that the *consistency strength* of an extension T of ZFC is at least that of a cardinal κ for which $A(\kappa)$ means that one can prove in ZFC that if T is consistent, then so is the theory ZFC+ "there exists a cardinal κ such that $A(\kappa)$". For example, ZFC+BTEE has consistency strength at least that of a weakly compact cardinal, and ZFC + BTEE + MUA has consistency strength at least that of a measurable cardinal.

[165] We note here that assuming $\exists z\, \big(z = \mathbf{j} \restriction \mathbf{j}(\kappa)\big)$ is already enough to obtain a blueprint for $V_{\kappa+2}$. The proof is given in the Appendix, Theorem 98.

Lemma 77. *Suppose T is an extension of the theory* ZFC + BTEE *and $j : V \to V$ is the embedding with critical point κ. Suppose $j \upharpoonright X$ can be proven to exist (as a set) in the theory T. Then:*

(i) *For any $Y \subseteq X$, $j \upharpoonright Y$ is also a set*
(ii) *For any Y for which there is a bijection $X \to Y$, $j \upharpoonright Y$ is also a set.*

Proof of (i). Suppose $Y \subseteq X$ and let $i = j \upharpoonright X$. Note that

$$j \upharpoonright Y = \{(u, v) \mid u \in Y \text{ and } i(u) = v\},$$

which is a set by ordinary Replacement.

Proof of (ii). Suppose $f : X \to Y$ is a bijection. By elementarity, $j(f) : j(X) \to j(Y)$ is also a bijection. Let $i = j(f)$ and $k = j \upharpoonright X$, both of which are sets in T. We have

$$\begin{aligned} j \upharpoonright Y &= \{(y, z) \mid z = j(y)\} \\ &= \{(y, z) \mid \exists x \in X\, y = f(x) \text{ and } z = i(k(x))\}. \end{aligned}$$

and the last expression is a set by ordinary Replacement. □

A consequence of (ii) is that the following theories are equivalent: ZFC + BTEE + $\exists z\,(z = \mathbf{j} \upharpoonright V_{\kappa+1})$ and ZFC + BTEE + $\exists z\,(z = \mathbf{j} \upharpoonright P(\kappa))$. For similar reasons, the following theories are also equivalent: ZFC + BTEE + $\exists z\,(z = \mathbf{j} \upharpoonright V_{\mathbf{j}(\kappa)})$ and ZFC + BTEE + $\exists z\,(z = \mathbf{j} \upharpoonright \mathbf{j}(\kappa))$. To see this, recall that κ is inaccessible (Corollary 73), so, in V, $j(\kappa)$ is also inaccessible, and so it follows that $|V_{j(\kappa)}| = j(\kappa)$.

As our sampling of results suggests, the large cardinal consequences of asserting the existence of restrictions of j to sets X increase in strength as those sets X increase in size. Part (E) shows one limitative result in this direction. When the critical sequence is unbounded, however, we are free to require restrictions of j to sets of any size without introducing inconsistency. The next section explores this possibility.

25. The Theory ZFC + WA

Our analysis of the embedding $j : V \to V$ that we get from a model of ZFC + BTEE + MUA shows that our strategy for strengthening the notion of a Dedekind self-map $V \to V$ based on observations we have made about set Dedekind self-maps, in Properties of Set Dedekind Self-Maps (p. 168), has been successful so far: We have, using techniques of generalization that are suggested to us by our New Axiom of Infinity, arrived at a formulation of a Dedekind self-map of the universe whose properties ensure the existence of many measurable cardinals. However, we have yet to tap the full potential of these properties. For example, our blueprint for generating sets (discussed in part (C) of Properties) has taken us only up to $V_{\kappa+1}$. Also, property (B) in Properties suggests that the critical sequence derived from j plays a special role (for set Dedekind self-maps, it forms a blueprint for ω) and that the critical sequence "emerges" from consideration of a sequence of restrictions of the Dedekind self-map under consideration. However, under MUA, the critical sequence for j is not even formally defined. Worse, it is not possible to define restrictions $j \upharpoonright V_{j^n(\kappa)}$ for $n > 0$; doing so entails much stronger large cardinal consequences than those available in the theory ZFC + BTEE + MUA. One of our original observations about a Dedekind self-map $j : A \to A$ was that restrictions of j to subsets of A play an important role in unfolding the dynamics of j. The theory ZFC + BTEE + MUA has the effect of masking dynamics of this kind because restrictions of j to sets of size $> \kappa$ cannot be proven to exist in the universe.

The last paragraph of the previous section suggested a way to proceed further and to address the limitations we have just outlined. Working in the theory ZFC+BTEE provides us with a nontrivial elementary embedding $j : V \to V$ and maximizes the preservation properties a Dedekind self-map from V to V could have. Insisting also that restrictions of j to various sets are also sets in the universe not only agrees with our original intuition about properties j *should* have (based on our analysis of Dedekind self-maps), but also leads to significant strengthenings of the theory, in the direction of stronger large cardinals. For instance, as we mentioned in the last section, the theory $\text{ZFC} + \text{BTEE} + \exists z\,(z = \mathbf{j} \restriction \mathbf{j}(\kappa))$ is already strong enough to produce a model of ZFC + BTEE + MUA, while the theory

(103)

$$\text{ZFC} + \text{BTEE} + \exists z\,(z = \mathbf{j} \restriction \lambda) + \text{“}\lambda \text{ is} \geq \text{the supremum of the critical sequence”}$$

is already so “strong” that it is inconsistent.

These considerations suggest the following stronger axiom as a natural strengthening of MUA:

> **Axiom of Amenability**. For every set X, $\mathbf{j} \restriction X : X \to \mathbf{j}(X)$ is a set.

Recall that the boldface $\mathbf{j}$ signifies that it is a symbol of the expanded language of set theory that we are working in, as we saw in our discussion of ZFC+BTEE+MUA. Any model $\langle V, \in, j\rangle$ of ZFC + BTEE + Amenability will interpret $\mathbf{j}$ as a self-map $j : V \to V$.

Intuitively, the Axiom of Amenability is a way of ensuring that the “dynamics” of an elementary embedding $j : V \to V$ are present “everywhere” within the universe by requiring that the restriction of j to any set is also a set in the universe.[166] The inconsistency result (103) tells us that the only way the theory ZFC + BTEE + Amenability could be consistent is if the critical sequence has no upper bound in the universe,[167] and, in particular, the embedding $j : V \to V$ must be *cofinal*.[168]

[166]The intuition that suggests that the dynamics of j should not be divorced from the sets in the universe, but should somehow be an integral part of that world, closely parallels the viewpoint expressed in ancient texts. For instance, Maharishi remarks [45], “The deepest level of every grain of creation is the self-referral field, the transcendental level of pure intelligence, the self-referral state of Unity—pure wakefulness, pure intelligence—*Chiti Shaktiriti*—as expressed by the last *Yog-Sutra*—that self-referral intelligence which is the common basis of all expressions of Natural Law” (p. 425).

[167]Moreover, ZFC + BTEE + Amenability *is consistent*, relative to large cardinals that are even stronger than those implied by this theory; for instance if there is an I_3-cardinal κ, with corresponding I_3-embedding $i : V_\lambda \to V_\lambda$ with critical point κ (these are defined in the footnote on p. 179), $(V_\lambda, \in, i)$ is a model of ZFC + BTEE + Amenability.

[168]See the definition on p. 105.

In the literature, the set of axioms BTEE + Amenability is given the name the *(Weak) Wholeness Axiom* or WA_0.[169] We have the following result:

Theorem 78. (Consequences of the Wholeness Axiom) [10] *Working in* ZFC + WA_0*, let* $j : V \to V$ *denote the* WA_0*-embedding and let* κ *denote the least ordinal moved by* j*. Then* j *is a Dedekind self-map (neither a set nor a proper class)*[170] *that is BSP and has critical point* κ*. Moreover,* κ *has "virtually all" the known large cardinal properties; in particular,* κ *is super-n-huge for every* n *(and is in fact the* κ*th such cardinal).* □

Using WA_0, we are in a position to see even more clearly the extent to which the concept of a Dedekind self-map points the way to a deeper understanding of the origin of large cardinals. We once again refer to the Properties of Set Dedekind Self-Maps (A)–(D) mentioned earlier (p. 168). As we discussed for MUA embeddings, any WA_0-embedding, being an elementary embedding, will be a good example of a generalization of (A) and most of (B). In particular, relative to (B), the critical point κ in this case has extremely strong large cardinal properties; indeed, as Theorem 79 shows, κ has virtually all known large cardinal properties.[171]

There are also natural analogues to (C)–(D), which we discuss next. These are very similar to those described in our analysis of ZFC + BTEE + MUA, except that we are now able to obtain a blueprint for *all sets in the universe*, rather than just for all subsets of V_κ. The steps of reasoning are very similar to the MUA case, so we just outline the results.

In place of elementary embeddings derived from a normal measure, which we used in our analysis of ZFC + BTEE + MUA, we use *extendible embeddings.* To begin, then, we discuss the notion of an *extendible cardinal* a bit further. A perusal of our list of large cardinals given earlier (p. 159) shows that extendible cardinals are among the very strongest large cardinals in the list. A cardinal κ is extendible if,

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[169] It is called "weak" because a slightly stronger version of the Wholeness Axiom is also known. The (full) Wholeness Axiom (WA) is BTEE + $\mathrm{Separation}_{\mathbf{j}}$, where $\mathrm{Separation}_{\mathbf{j}}$ is the usual Separation axiom, applied to **j**-formulas. Amenability is a consequence [11] of $\mathrm{Separation}_{\mathbf{j}}$. In the literature, BTEE + Amenability is denoted WA_0 to indicate it is slightly weaker than WA. Nevertheless, it has been shown that all the known large cardinal consequences of WA can also be shown to be consequences of WA_0. Therefore, in this paper, as we introduce the Wholeness Axiom, we have emphasized the more intuitively appealing Amenability axiom as a starting point.

[170] The theory ZFC + WA_0 provides examples of phenomena that were alluded to earlier in the paper. First of all, any WA_0-embedding $j : V \to V$, viewed as a collection of ordered pairs, is an example of a subcollection of V that is neither a set nor a proper class. (In fact, the range of j is also an example of this kind.) The question of whether such entities could meaningfully exist was raised on p. 99. A second interesting phenomenon arises because of the fact (which we do not prove here) that the sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots, j^n(\kappa), \ldots$ has no upper bound in ON. It follows that the function f defined on ω by $f(n) = j^n(\kappa)$ is a member of $V^{<V}$ but not a member of $V^{<V} \cap V$. This is because the range of f is unbounded in V, so f cannot be a set. We asked (p. 59) whether a function with these properties could exist. Note that even though the domain of f is a set, and the collection of ordered pairs that determine f is only countably infinite, it is not possible to view f as a set since its range is unbounded in the universe; and f is not a proper class since, if it were, Replacement would imply that $\mathrm{ran}\, f$ has an upper bound.

[171] There are a few exceptions. Neither WA_0 nor the slightly stronger WA is strong enough to prove the consistency of the very strongest large cardinals; there are axioms, notably I_1, I_2, I_3, that have greater consistency strength, and so the large cardinals they produce are, consistencywise, stronger than those arising from WA_0 and WA. See [10, 36].

for every $\eta > \kappa$, there is an elementary embedding $i : V_\eta \to V_\xi$—called an *extendible embedding*—having critical point κ. Intuitively, extendible cardinals arise from a WA_0-embedding (and its iterates) by restriction: Suppose $j : V \to V$ is a WA_0-embedding and $\kappa = \mathrm{crit}(j)$. Suppose η is an ordinal $> \kappa$. If $\kappa < \eta < j(\kappa)$, it is easy to see that $j \restriction V_\eta : V_\eta \to V_{j(\eta)}$ is an extendible embedding with critical point κ. Likewise, if $j(\kappa) \leq \eta < j(j(\kappa))$, $(j \circ j) \restriction V_\eta : V_\eta \to V_{j(j(\eta))}$ is another extendible embedding with critical point κ. Proceeding in this way demonstrates that κ itself is an extendible cardinal and that this fact is witnessed by the j-class of restrictions of iterates of j to various V_η, $\eta > \kappa$.

Associated with any extendible embedding $i : V_\eta \to V_\xi$, with critical point κ and $\eta \geq \kappa + 1$, is a particular normal measure U_i defined, exactly as for the theory ZFC + BTEE + MUA, by

$$U_i = \{X \subseteq \kappa \mid \kappa \in i(X)\}.$$

Using the notion of extendible elementary embeddings, which, as we have seen, are naturally related to any ambient WA_0-embedding $j : V \to V$, one can define from j and its critical point κ a blueprint $\ell : V_\kappa \to V_\kappa$ for *all sets in the universe*; the function ℓ will be, as in our analysis of ZFC + BTEE + MUA, a co-Dedekind self-map. We will once again call ℓ a *blueprint self-map*. We will show that, for every set $x \in V$, there is an extendible embedding i such that $i(\ell)(\kappa) = x$. In this way, every set in the universe can be seen as arising from or being generated by the interplay of κ, j and ℓ.

The essence of the construction of ℓ is a *Laver function* $f : \kappa \to V_\kappa$. A Laver function[172] is a function $f : \kappa \to V_\kappa$ with the property that for any set x, there is an extendible embedding $i : V_\eta \to V_\xi$ such that:

(1) $\kappa = \mathrm{crit}(i)$
(2) $\mathrm{rank}(x) < \eta < i(\kappa) < \xi$
(3) $i(f)(\kappa) = x$.

This definition is in contrast with that for an X-Laver function (in particular, a $V_{\kappa+1}$-Laver function, as discussed in connection with ZFC + BTEE + MUA) which restricts the possible values of x to the set X.

The weak Wholeness Axiom guarantees the existence of a Laver function:

Theorem 79. [10] (WA_0) *Suppose $j : V \to V$ is a WA_0-embedding with critical point κ. Then there is a Laver function $f : \kappa \to V_\kappa$.*

Proof. We build a formula $\phi(g, x)$ that asserts that g is *not* Laver, with witness x, as follows: Let $\psi(\eta, \zeta, i, \alpha)$ be a formula that states formally "$i : V_\eta \to V_\zeta$ is an elementary embedding with critical point α." We let $\phi(g, x)$ be the following formula:

$$\exists\alpha \Big[g : \alpha \to V_\alpha \wedge \forall\eta\forall\zeta\forall i\, [(\psi(\eta,\zeta,i,\alpha) \wedge \mathrm{rank}(x) < \eta < i(\alpha) < \zeta) \\ \to i(g)(\alpha) \neq x]\Big].$$

[172] In the literature, such a function is called a *weakly extendible* Laver function. Extendible Laver functions were introduced in [10] where the existence of such a function was proved to follow from (and to be equivalent to) the existence of an extendible cardinal. Weakly extendible Laver functions were introduced in [14] to simplify some of the proofs in the context of ZFC + WA.

Define $f : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} \emptyset & \text{if } f \restriction \alpha \text{ is Laver at } \alpha \text{ or } \alpha \text{ is not a cardinal,} \\ x & \text{otherwise, where } x \text{ satisfies } \phi(f \restriction \alpha, x). \end{cases}$$

Let $D = U_j$ be the normal ultrafilter over κ that is derived from j; that is:

$$D = \{X \subseteq \kappa \,|\, \kappa \in j(X)\}.$$

By Amenability, D is a set, since $D = \{X \subseteq \kappa \mid \kappa \in g(X)\}$, where $g = j \restriction \mathcal{P}(\kappa)$.

Define sets S_1 and S_2 by

$$\begin{array}{rcl} S_1 & = & \{\alpha < \kappa \,|\, f \restriction \alpha \text{ is Laver at } \alpha\} \\ S_2 & = & \{\alpha < \kappa \,|\, \phi(f \restriction \alpha, f(\alpha))\}. \end{array}$$

Clearly, $S_1 \cup S_2 \in D$. To complete the proof, it suffices to prove that $S_1 \in D$, and for this, it suffices to show $S_2 \notin D$.

Toward a contradiction, suppose $S_2 \in D$. Reasoning as in the ZFC + BTEE + MUA case (p. 172), we have $f = j(f) \restriction \kappa$ and so $\phi(f, j(f)(\kappa))$ holds in V. Let $x = j(f)(\kappa)$. Since $j(f) : j(\kappa) \to V_{j(\kappa)}$, $\text{rank}(x) < j(\kappa)$, so we can pick $\eta > \kappa$ so that $\text{rank}(x) < \eta < j(\kappa)$. Let $i = j \restriction V_\eta : V_\eta \to V_\zeta$, where $\zeta = j(\eta)$. By Amenability, i is a set, and is an elementary embedding with critical point κ. Clearly, in V, $\text{rank}(x) < \eta < i(\kappa) < \zeta$ and $i(f)(\kappa) = x$, contradicting the fact that $\phi(f, j(f)(\kappa))$ holds in V. Therefore $S_2 \notin D$, as required. □

We can now state the main fact about ℓ:

Theorem 80. (Existence of Blueprint Self-Maps) (WA_0) *Suppose $j : V \to V$ is a WA_0-embedding with critical point κ. Then there is a blueprint self-map $V_\kappa \to V_\kappa$; that is, there is an $\ell : V_\kappa \to V_\kappa$ such that for any set x, there is an extendible elementary embedding $i : V_\eta \to V_\xi$ with critical point κ and $\eta \geq \kappa + 1$ such that $i(\ell)(\kappa) = x$. Moreover, ℓ is a co-Dedekind self-map.*

The proof is exactly the same as the one given for the MUA case in Theorem 77, replacing canonical embeddings i_U with extendible embeddings i.

For (D) in the Properties list, we show that there is, just as in the MUA case, a natural dual to ℓ, which we will once again denote ℓ^{op}, which sends every sufficiently large set back to the "point" κ. The precise statement is given in the following theorem:

Theorem 81. (WA_0) *Let $j : V \to V$ be a WA_0-embedding with critical point κ. For each blueprint self-map $\ell : V_\kappa \to V_\kappa$, there is a Dedekind self-map $\ell^{\text{op}} : V_\kappa \to V_\kappa$ with the following properties:*

(1) *For every $x \notin V_\kappa$, there is an extendible elementary embedding $i : V_\eta \to V_\xi$ with critical point κ and $\eta \geq \kappa + 1$ such that*

$$i(\ell^{\text{op}})(x) = \kappa$$

(2) *ℓ^{op} is a section of ℓ; in particular, $\ell \circ \ell^{\text{op}} = \text{id}_{V_\kappa}$.*

Again, the proofs are essentially identical to those given in the MUA case (see p. 177), replacing embeddings i_U obtained from a normal measure with extendible embeddings. In this case, we are not restricted to subsets of V_κ, as we were in the

MUA case, but the reasoning is the same since now we have a (full) Laver function f that gives access to sets of arbitrarily large rank.

As before, we do not claim that there is an i for which $i(\ell^{\mathrm{op}})(x) = \kappa$ when $x \in V_\kappa$. Indeed, as before, there is no way to define a section s of ℓ so that this is true, and the proof of this is identical to the one given in the previous section.

We turn now to a more detailed examination of our blueprint for the universe under WA.

Remark 23. (The Blueprint Coder $\mathcal{E}$ for V) We describe now in more detail the blueprint coder for the blueprint of V, given to us by WA_0. Suppose $j : V \to V$ is a WA_0-embedding, given to us by a model of $\mathrm{ZFC} + \mathrm{WA}_0$, with critical point κ. The collection $\mathcal{E}$ is the class of extendible embeddings $V_\beta \to V_\eta$ with critical point κ. The set $\mathcal{E}_0$ is defined to be the restriction of $\mathcal{E}$ to $V_\kappa^{V_\kappa}$, that is, $\mathcal{E}_0 = \{i \restriction V_\kappa^{V_\kappa} \mid i \in \mathcal{E}\}$. Note that each element of $\mathcal{E}_0$ is of the form $i : V_\kappa^{V_\kappa} \to V_{i(\kappa)}^{V_{i(\kappa)}}$.

We indicate why the triple $(\ell, \kappa, \mathcal{E})$ is a blueprint for V, and also why $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V - V_\kappa$. We refer to Definitions 7 and 8.

We start by verifying the properties mentioned in Definition 7. For (1), we have already seen that ℓ is a co-Dedekind self-map with co-critical point κ. For (2), note that, because each $i \in \mathcal{E}$ is an elementary embedding, each is weakly elementary as well. Verification of properties (2)(a)–(d) is straightforward, and is like the MUA case. To establish the compatibility requirement for j, mentioned in (2), we again, as with MUA, defer to the notion of compatibility developed in [10]. In the present context, this notion of compatibilty can be described as follows: Given a WA_0-embedding $j : V \to V$ with critical point κ, for each $\beta > \kappa$, there is an extendible embedding $i : V_\beta \to V_\eta \in \mathcal{E}$ with critical point κ such that $j \restriction V_\beta = i$.[173]

For (3), we must argue that ℓ is definable from $\mathcal{E}, j, \kappa$. A review of the definition of ℓ and the extendible Laver function on which it is based makes this point clear. Finally, for (4), the fact that, for each $x \in V$, there is $i \in \mathcal{E}$ such that $i(\ell)(\kappa) = x$ guarantees that $\left(i \restriction V_\kappa^{V_\kappa}\right)(\ell)(\kappa) = x$, and $i \restriction V_\kappa^{V_\kappa} \in \mathcal{E}_0$.

Verification of the remaining points in Definition 8 to show that $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V - V_\kappa$ is now straightforward in light of Theorem 82. □

25.1. **Restrictions of a $\mathbf{WA}_0$-Embedding and Its Critical Sequence.** In our analysis of the theory $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{MUA}$, we showed that most of the interesting properties of Dedekind self-maps, which we listed in the section Properties of Set Dedekind Self-Maps (p. 168), had natural generalizations to MUA-embeddings j :

[173] To clarify the details here, we continue the discussion that we began in the footnote on p. 174, concerning compatibility, as it was developed in [10], and as it pertains to the class of extendible elementary embeddings. In [10], the class of extendible embeddings was shown to be captured by the suitable formula:

$$\theta_{\mathrm{ext}}(i, \kappa, \beta, M) : \quad \exists \delta > 0\, \exists \zeta\, [\beta = \kappa + \delta \wedge M = V_\zeta \wedge i : V_\beta \to M \text{ is elementary with critical point } \kappa \wedge \beta < i(\kappa) < \zeta].$$

Applying the definition of compatibility to the class $\mathcal{E}_\kappa^{\theta_{\mathrm{ext}}}$ leads to the following: Suppose $\kappa < \lambda < \beta$, and $i_\beta : V_\beta \to V_\eta \in \mathcal{E}_\kappa^{\theta_{\mathrm{ext}}}$. Then compatibility of i_β with j up to V_λ simply means that $j \restriction V_\lambda = i \restriction V_\lambda$. Then, to say that $\mathcal{E}_\kappa^{\theta_{\mathrm{ext}}}$ is *compatible with* j means that for each $\lambda < j(\kappa)$ there is a $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}_\kappa^{\theta_{\mathrm{ext}}}$ that is compatible with $j \restriction V_\beta$ up to V_λ. To show this requirement is met, the fact that for each $\beta > \kappa$, there is an extendible embedding $i : V_\beta \to V_\eta \in \mathcal{E}_\kappa^{\theta_{\mathrm{ext}}}$ such that $j \restriction V_\beta = i$ suffices. (See [10, Theorem 4.33].)

$V \to V$. However, one of the properties in the original Properties list had to do with the critical sequence of the self-map and its emergence on the basis of successive restrictions of the original self-map. We listed several points (p. 178) that show that these particular characteristics of Dedekind self-maps do not generalize well to the context of MUA embeddings.

The situation with the Wholeness Axiom is quite different. However, for this discussion, some of our proofs will require the stronger version of the Wholeness Axiom. Recall that WA_0 is BTEE+Amenability, whereas WA is the stronger theory BTEE+Separation$_{\mathbf{j}}$.[174] (It is known that our discussion below about indiscernibility of the terms of the critical sequence $\langle \kappa, j(\kappa), j(j(\kappa)), \ldots \rangle$ and regarding the self-application operator $\cdot$ on j cannot be carried out in $\mathrm{ZFC} + \mathrm{WA}_0$.) Therefore, for the remainder of this section, our base theory will be the *full* ZFC + WA.

We recall that, given a Dedekind self-map $j : A \to A$ with critical point a, the terms of j's critical sequence $a, j(a), j(j(a)), \ldots$ arose as critical points of successive restrictions of j:

$$\begin{aligned}
A_0 &= A; \\
j_0 &= j : A \to A; \\
\mathrm{crit}(j_0) &= a; \\
A_1 &= j[A_0]; \\
j_1 &= j \upharpoonright A_1; \\
\mathrm{crit}(j_1) &= j(a); \\
A_{n+1} &= j[A_n]; \\
j_{n+1} &= j \upharpoonright A_{n+1}; \\
\mathrm{crit}(j_{n+1}) &= j^{n+1}(a).
\end{aligned}$$

Something similar occurs when we consider a certain sequence of restrictions of a given WA-embedding $j : V \to V$ with critical point κ. We observed in our study of ZFC + BTEE + MUA that $j \upharpoonright V_\kappa : V_\kappa \to V_{j(\kappa)}$ is an elementary embedding, but we were unable to consider restrictions like $j \upharpoonright V_{j(\kappa)}, j \upharpoonright V_{j(j(\kappa))}, \ldots$ because the theory was not strong enough to admit such restrictions as sets in the universe. In the theory ZFC + WA, we no longer have this limitation and we are now able to observe that each of these restricted embeddings serves to bring to light the next term in the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$, as we observed in the case of set Dedekind self-maps. In particular, the observation that $j \upharpoonright V_\kappa : V_\kappa \to j(V_\kappa) = V_{j(\kappa)}$ is an elementary embedding brings to light the image $j(\kappa)$ of κ. This is analogous to the discovery of the critical point $j(a)$ for $j \upharpoonright B$ obtained by restricting $j : A \to A$ to its range: $j \upharpoonright B : B \to B$, with $B = j[A]$.

If we now restrict j to the codomain of $j \upharpoonright V_\kappa$, we obtain (the set) $j \upharpoonright V_{j(\kappa)}$. Elementarity tells us that, for any $x \in V_{j(\kappa)}$, $j(x) \in j(V_{j(\kappa)}) = V_{j(j(\kappa))}$. In this way, the next term of j's critical sequence appears, namely, $j(j(\kappa))$. In general, the critical sequence for j can be seen to arise as the sequence of successive ranks of the codomains obtained by considering restrictions $j \upharpoonright V_\kappa, j \upharpoonright V_{j(\kappa)}, j \upharpoonright V_{j(j(\kappa))}$, and so forth.

[174]See the footnote on p. 183 for more details about BTEE + Separation$_{\mathbf{j}}$.

The critical sequence of j that we obtain in the theory ZFC + WA must be unbounded in the universe: As we mentioned at the end of Section 24, any theory which includes both Amenability and the statement that the critical sequence is bounded must be inconsistent.

A surprising fact about the terms of the critical sequence of a WA-embedding j is that they are *indistinguishable from each other* on the basis of properties formulated in the language $\{\in\}$ (that is, formulas that do not include the symbol $\mathbf{j}$). More formally, the critical sequence $\langle\kappa, j(\kappa), j(j(\kappa)), \ldots\rangle$ *is a j-class of $\in$-indiscernibles* [10, Theorem 3.18]. This means that, for any $\in$-formula $\phi(x_1, \ldots, x_m)$ and any two finite increasing sequences of natural numbers $i_1 < i_2 < \ldots < i_m$, $k_1 < k_2 < \ldots < k_m$,

$$V \models \phi[j^{i_1}(\kappa), \ldots, j^{i_m}(\kappa)] \text{ if and only if } V \models \phi[j^{k_1}(\kappa), \ldots, j^{k_m}(\kappa)].$$

One consequence is that each $j^n(\kappa)$ has all the same large cardinal properties as κ itself (that is, all the same large cardinal properties that can be stated in the language $\{\in\}$). Using this observation, one can show that "almost all" cardinals in the universe are super-n-huge for every $n \in \omega$: Given $m \in \omega$, let X_m denote the set of all cardinals $\alpha < j^m(\kappa)$ that are super-n-huge cardinals for every n. By indiscernibility, $j^m(\kappa)$ is also super-n-huge for every n, and by indiscernibility again, $j^m(\kappa)$ admits a normal measure D_m that contains X_m (since κ has this property relative to X_0).[175] Finally, let us say that for any class $\mathbf{C}$ of cardinals, *almost all cardinals belong to* $\mathbf{C}$ if, for all but finitely many $m \in \omega$, $\mathbf{C} \cap j^m(\kappa) \in D_m$. It follows that if $\mathbf{C} = \{\alpha \mid \alpha \text{ is super-}n\text{-huge for every } n\}$, then almost all cardinals in the universe belong to $\mathbf{C}$.

We consider next another sequence of restrictions of j that also leads to the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$. First, we can apply j to the set function $j \restriction V_\kappa : V_\kappa \hookrightarrow V_{j(\kappa)}$ to obtain $j(j \restriction V_\kappa) : V_{j(\kappa)} \to V_{j(j(\kappa))}$.

By elementarity of j, we expect $j(j \restriction V_\kappa)$ to be an inclusion map and a restriction of some elementary embedding, but which one? Formally applying j suggests that the new map is $j(j) \restriction j(V_\kappa)$ but the meaning of $j(j)$ is not clear (certainly j itself is not in the domain of j).

We can address the problem in the following way. Let $\alpha > \kappa + 1$. Let $f = j \restriction V_\alpha$. Certainly

$$j \restriction V_\kappa = f \restriction V_\kappa.$$

Now we can apply j:

$$j(j \restriction V_\kappa) = j(f \restriction V_\kappa) = j(f) \restriction V_{j(\kappa)}.$$

This example motivates the following definition:

Definition 20. Suppose $i, k : V \to V$ are WA_0-embeddings. Then

$$i \cdot k = \bigcup_{\alpha \in \mathrm{ON}} i(k \restriction V_\alpha).$$

The operator $\cdot$ is called *application.* It can be shown that, if i, k are WA_0 embeddings $V \to V$, then $i \cdot k$ is also a WA_0-embedding; likewise for WA-embeddings.[176]

[175] An explicit definition of D_m is given in [10, pp. 192-93].

[176] We prove these observations here. As we proceed with the discussion, it will be useful to record the fact that most of the results that we seek to obtain for WA (or WA_0)-embeddings will

also hold, *mutatis mutandis*, for I_3-embeddings $i : V_\lambda \to V_\lambda$ (recall that i is an I_3 embedding if it is an elementary embedding for which $\mathrm{crit}(i) < \lambda$ and λ is a limit ordinal; see the footnote on p. 179). The definition of the application operator $\cdot$ for such embeddings $i, k : V_\lambda \to V_\lambda$ is as follows:

$$i \cdot k = \bigcup_{\alpha \in \lambda} i(k \restriction V_\alpha).$$

In fact, typically, proofs of these results are more transparent in the context of I_3 embeddings; we will therefore prove our main results for these embeddings and ask the reader to make the small necessary adjustments to provide proofs for the WA_0 and WA cases.

Our target theorem is the following.

Theorem 82.

(A) *Suppose $i, k : V_\lambda \to V_\lambda$ are nontrivial I_3 embeddings. Then $i \cdot k : V_\lambda \to V_\lambda$ is also a nontrivial I_3 embedding.*

(B) *Suppose $i, k : V \to V$ are WA_0-embeddings (WA-embeddings). Then $i \cdot k : V \to V$ is also a WA_0-embedding (WA-embedding).*

The theorem will follow as a corollary to a somewhat more general result:

Theorem 83.

(A) *Suppose λ is an infinite limit ordinal, $f : V_\lambda \to V_\lambda$ is any function, and $i : V_\lambda \to V_\lambda$ is an I_3 embedding. Then for all formulas $\phi(x_1, \ldots, x_m)$ in the language $\{\in, \mathbf{f}\}$ (where $\mathbf{f}$ is a unary function symbol) and all $a_1, \ldots, a_m \in V_\lambda$,*

$$(V_\lambda, \in, f) \models \phi[a_1, \ldots, a_m] \iff (V_\lambda, \in, i \cdot f) \models \phi[i(a_1), \ldots, i(a_m)].$$

That is, i lifts to an elementary embedding $(V_\lambda, \in, f) \to (V_\lambda, \in, i \cdot f)$.

(B) *Suppose $f : V \to V$ is a function (not necessarily definable in V) for which $f \restriction X$ is a set for every set X. Suppose $(V, \in, j) \models \mathrm{ZFC} + \mathrm{BTEE}$. Then for all formulas $\phi(x_1, \ldots, x_m)$ in the language $\{\in, \mathbf{f}\}$ (where $\mathbf{f}$ is a unary function symbol) and all $a_1, \ldots, a_m \in V_\lambda$,*

$$(V, \in, f) \models \phi[a_1, \ldots, a_m] \iff (V, \in, j \cdot f) \models \phi[j(a_1), \ldots, j(a_m)].$$

We outline a proof of part (A); the proof of part (B) is essentially the same. We begin with a few special cases that illuminate the proof. First, we note that the structure of the language $\{\in, \mathbf{f}\}$ is more complex than that of $\{\in\}$ because, in order to form formulas in this language, we must first define *terms*. The terms in the language $\{\in\}$ are just the variables of the language, but in $\{\in, \mathbf{f}\}$, the terms consist of all variables together with all expressions of the form $\mathbf{f}^n(x)$, where x is a variable and $\mathbf{f}^n$ is $\mathbf{f} \circ \cdots \circ \mathbf{f}$ (n factors). The key to the proof of (A) is to establish the result in the special cases in which the formula ϕ is of the form

$$\mathbf{f}^m(x) \in \mathbf{f}^n(y) \qquad \text{or} \qquad \mathbf{f}^m(x) = \mathbf{f}^n(y).$$

Beginning with a simple example, suppose $a, b \in V_\lambda$ are such that $(V_\lambda, \in, f) \models \mathbf{f}[a] \in \mathbf{f}[b]$. Notice there is a limit ordinal $\beta < \lambda$

$$\{a, b\} \in V_\beta.$$

It is easy to see that (both in V and in V_λ)

$$f(a) \in f(b) \quad \text{if and only if} \quad (f \restriction V_\beta)(a) \in (f \restriction V_\beta)(b).$$

Applying i to the second expression, it follows from elementarity of i that (both in V and in V_λ)

$$i(f \restriction V_\beta)(i(a)) \in i(f \restriction V_\beta)(i(b)).$$

Since, by definition of $\cdot$, $i(f \restriction V_\beta)(i(a)) = (i \cdot f)(i(a))$ and $i(f \restriction V_\beta)(i(b)) = (i \cdot f)(i(b))$, it follows that

$$(V_\lambda, \in, i \cdot f) \models \mathbf{f}[i(a)] \in \mathbf{f}[i(b)],$$

where we understand that in the latter formula, $\mathbf{f}$ is interpreted by $i \cdot f$.

In this example, we were able to choose β large enough so that, in our formula ϕ, $f \restriction V_\beta$ is an element of V_λ that faithfully represents f (which is *not* an element of V_λ); since $f \restriction V_\beta$ is an element of V_λ, we can apply i to it to obtain the necessary conclusion. This is the main trick for handling all the atomic formula cases.

Notice that the definition of $i \cdot k$ makes sense only if restrictions of k to arbitrarily

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We consider one more simple example that highlights the need for a bit more care. Consider the atomic formula

$$\mathbf{f}(x) \in \mathbf{f}^2(y).$$

Suppose $a, b \in V_\lambda$ satisfy this formula. If we attempt to mechanically apply the technique in the previous example, where $\{a, b\} \in V_\beta$ for some limit β, we encounter the difficulty that $\mathrm{ran}(f \upharpoonright V_\beta)$ may not be a subset of V_β, and so it may not be possible to perform the iteration $(f \upharpoonright V_\beta) \circ (f \upharpoonright V_\beta)$. To handle this, we pick β so that $\{a, b, f(b)\} \in V_\beta$. Now, although $(f \upharpoonright V_\beta)$ may still not be composable with itself, we may still argue (in V or in V_λ) that

$$f(a) \in (f \circ f)(b) \quad \text{if and only if} \quad f_\beta(a) \in f_\beta(f_\beta(b)),$$

where $f_\beta = f \upharpoonright V_\beta$, since we have guaranteed that $f(b) \in \mathrm{dom}\, f_\beta$.

In general, suppose $\phi(x_1, \ldots, x_m)$ is a formula in the language $\{\in, \mathbf{f}\}$. Obtain another formula $\overline{\phi}(x_1, \ldots, x_m)$ as follows: Replace each occurrence of an atomic formula of the form $\mathbf{f}^m(x) \in \mathbf{f}^n(y)$ by the formula

$$\exists\beta\, [\text{“}\beta \text{ is a limit”} \wedge \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(x)) \cdots) \in \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(y)) \cdots)],$$

where in the first subformula, the composition has m components and in the second, n components, and where $\mathbf{f}_\beta$ is a defined constant, defined by $\mathbf{f}_\beta = \mathbf{f} \upharpoonright V_\beta$. And make a similar substitution for each atomic formula of the form $\mathbf{f}^m(x) = \mathbf{f}^n(y)$.

Claim. For all $a_1, \ldots, a_m \in V_\lambda$,

$$(V_\lambda, \in, f) \models \phi[a_1, \ldots, a_m] \Longleftrightarrow V_\lambda \models \overline{\phi}[a_1, \ldots, a_m].$$

Proof. Consider a subformula θ of $\phi[a_1, \ldots, a_m]$ of the form $f^m(a) \in f^n(b)$; the corresponding subformula $\overline{\theta}$ of $\overline{\phi}$ is

$$\exists\beta\, [\text{“}\beta \text{ is a limit”} \wedge \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(a) \cdots)) \in \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(b) \cdots))].$$

If $(V_\lambda, \in, f) \models \theta$, we can pick a limit β large enough so that $\{a, f(a), \ldots, f^m(a)\} \cup \{b, f(b), \ldots, f^n(b)\} \in V_\beta$. Then clearly

$$V_\lambda \models \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(a) \cdots)) \in \mathbf{f}_\beta(\mathbf{f}_\beta(\ldots \mathbf{f}_\beta(b) \cdots)). \tag{104}$$

Therefore, $\overline{\theta}$ holds in V_λ.

Conversely, if the formula (104) holds, then β must be large enough for the compositions to be defined, and so f_β agrees with f. Therefore, θ holds in $(V_\lambda, \in, f)$.

A similar argument can be carried out when $\in$ is replaced by $=$ in the formulas θ and $\overline{\theta}$. □

Proof of Theorem 84(A). Proof is by induction on the length of formulas. In the usual way, one begins by establishing the result for atomic formulas, and then for formulas built up by connectives and quantifiers. We will prove the atomic formula case only; the other steps of the proof are straightforward.

Suppose ϕ is the formula $\mathbf{f}^m(x) \in \mathbf{f}^n(y)$. Let $a, b \in V_\lambda$. Then

$$\begin{aligned}
(V_\lambda, \in, f) \models \phi[a, b] \quad &\Longleftrightarrow \quad V_\lambda \models \overline{\phi}[a, b] && \text{(by the claim)} \\
&\Longleftrightarrow \quad V_\lambda \models \overline{\phi}[i(a), i(b)] && \text{(by elementarity of } i\text{)} \\
&\Longleftrightarrow \quad (V_\lambda, \in, i \cdot f) \models \phi[i(a), i(b)] && \text{(by the proof of the claim).}
\end{aligned}$$

The same logic also establishes the case in which the atomic formula is an $=$-formula. □

Proof of Theorem 83. Suppose $i, k : V_\lambda \to V_\lambda$ are elementary embeddings. We show $i \cdot k$ is elementary. Let $\phi(x_1, \ldots, x_m)$ be a formula. Let ϕ_k be the $\{\in, \mathbf{k}\}$-formula obtained from ϕ by replacing each free variable x_ℓ in ϕ by the term $\mathbf{k}(x_\ell)$. We observe that, for all $a_1, \ldots, a_m \in V_\lambda$,

$$(V_\lambda, \in, k) \models \phi_k[a_1, \ldots, a_m] \Longleftrightarrow V_\lambda \models \phi(k(a_1), \ldots, k(a_m)). \tag{105}$$

Replacing k by $i \cdot k$ (so that $\mathbf{k}$ is interpreted in $(V_\lambda, \in, i \cdot k)$ as $i \cdot k$) yields

$$(V_\lambda, \in, i \cdot k) \models \phi_{i \cdot k}[a_1, \ldots, a_m] \Longleftrightarrow V_\lambda \models \phi((i \cdot k)(a_1), \ldots, (i \cdot k)(a_m)). \tag{106}$$

large stages of the universe are *sets*; this shows that the assumption that k is a WA_0 embedding cannot be weakened.

We shall often write ik for $i \cdot k$. Note that $j \cdot j$ gives precise expression to the intuitive notation $j(j)$.[177] In particular, we may now write:

$$j(j \upharpoonright V_\kappa) = jj \upharpoonright V_{j(\kappa)}.$$

Moreover, by elementarity of j, we have:

$$jj \upharpoonright V_{j(\kappa)} : V_{j(\kappa)} \hookrightarrow V_{j(j(\kappa))}.$$

One may apply j in this way repeatedly (for instance, the next iteration yields the fact that $j(j(j \upharpoonright V_\kappa))$ is the inclusion map, describing the fact that $V_{j(j(\kappa))} \prec V_{j(j(j(\kappa)))}$). This sequence of steps leads to the conclusion that there is an elementary chain of elementary submodels of V [10, Proposition 3.12]:

$$V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)} \prec \ldots \prec V.$$

This statement provides us with one other powerful consequence: Because $V_\kappa \prec V$, we may conclude that every first-order sentence that holds in V must also hold in V_κ. Moreover, since $\kappa = |V_\kappa|$, this tells us that the "point" κ is a set representative of the wholeness V—intuitively, κ can declare,

"I am the totality."

We spend a moment to clarify the details here. We will show that κ itself may be taken to be the universe of mathematics. In order for κ to be a universe in actual fact, it needs to have a membership relation. The usual membership relation for the universe V is $\in$, but κ is going to be a "miniature" universe, which still is supposed to "contain everything," so the usual membership relation won't work (recall that, with the usual membership relation, the only elements of κ are other

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Therefore, for all $a_1, \ldots, a_m \in V_\lambda$,

$$\begin{aligned} V_\lambda \models \phi(a_1,\ldots,a_m) &\iff V_\lambda \models \phi(k(a_1),\ldots,k(a_m)) && \text{(elementarity of } k) \\ &\iff (V_\lambda, \in, k) \models \phi_k(a_1,\ldots,a_m) && \text{(by (105))} \\ &\iff (V_\lambda, \in, i\cdot k) \models \phi_k(i(a_1),\ldots,i(a_m)) && \text{(Theorem 84(A))} \\ &\iff V_\lambda \models \phi((i\cdot k)(a_1),\ldots,(i\cdot k)(a_m)) && \text{(by (106))} \end{aligned}$$

Restricting to the case of I_3 embeddings, we observe next that if k is nontrivial, $i \cdot k$ must also be nontrivial. This follows immediately from Theorem 84(A): If $(V_\lambda, \in, k) \models \exists x\,(k(x) \neq x)$, then $(V_\lambda, \in, i \cdot k) \models \exists x\,((i\cdot k)(x) \neq x)$. In the case of I_3-embeddings, nontriviality is enough to guarantee that there is a least ordinal moved.

If we begin instead with WA_0-embeddings i, k, we can use Theorem 84 in the following way. Suppose $\mathrm{crit}(k) = \kappa$. Using the theorem, it follows that $\mathrm{crit}(i \cdot k) = i(\kappa)$.

Assuming i, k are WA_0-embeddings, we also need to show that $i \cdot k$ is not only elementary with a critical point, but also satisfies Amenability; in the case of WA-embeddings, we must show that $i \cdot k$ satisfies $\mathrm{Separation}_{i\cdot k}$. However, these are immediate consequences of Theorem 84(B). For instance, if k is a WA_0-embedding, then

$$(V, \in, k) \models \forall X \exists Y\,(Y = k \upharpoonright X).$$

By the theorem,

$$(V, \in, i\cdot k) \models \forall X \exists Y\,(Y = (i\cdot k) \upharpoonright X).$$

[177] The intuitively natural notation "$j(j)$" can be justified even more than this; see the footnote about this starting on p. 220.

ordinals, so many other sets are not included). We define a new binary relation R on κ as follows: First, let $g : \kappa \to V_\kappa$ be any bijection. Define R on κ by:

$$\alpha\, R\, \beta \quad \Leftrightarrow \quad g(\alpha) \in g(\beta).$$

This clever definition of R makes it so that g is now an *isomorphism* from (κ, R) to $(V_\kappa, \in)$; that is, for any formula $\phi(x, y)$ and objects $a, b \in \kappa$, $\phi[a, b]$ holds in the model (κ, R) if and only if $\phi[g(a), g(b)]$ holds in $(V_\kappa, \in)$. But since $V_\kappa \prec V$, $\phi[a, b]$ holds in (κ, R) if and only if $\phi[g(a), g(b)]$ holds in $(V, \in)$. Therefore, $(V, \in)$ and (κ, R) satisfy exactly the same *sentences*.[178]

Another way to say it is that the map $i : (\kappa, R) \to (V, \in)$, defined by

$$i = \mathrm{inc} \circ g : \kappa \to V_\kappa \hookrightarrow V,$$

where $\mathrm{inc} : V_\kappa \hookrightarrow V$ is the inclusion map, is an elementary embedding.

We summarize our results on the theory ZFC + WA:

Theorem 84. *Suppose $j : V \to V$ is a* WA*-embedding with critical point κ.*

(1) *The critical sequence of j, $\langle \kappa, j(\kappa), j(j(\kappa)), \ldots \rangle$, is unbounded in ON and is a j-class of $\in$-indiscernibles.*
(2) $V_\kappa \prec V_{j(\kappa)} \prec \cdots \prec V_{j^n(\kappa)} \prec \cdots \prec V$.
(3) $V_\kappa \prec V$. *Moreover, there exist a binary relation R on κ and an elementary embedding $i : (\kappa, R) \to (V, \in)$. In particular, for every sentence σ in the language $\{\in\}$, $(V, \in) \models \sigma$ if and only if $(\kappa, R) \models \sigma$.*

26. Conclusion

We began our investigation of infinite sets in this article with the intention of improving the usual formulation of the Axiom of Infinity so that it could provide a richer intuition about the concept of "infinite sets." The hope was that, with a clearer idea about "the infinite," we could address the long-standing Problem of Large Cardinals. With a clearer idea about the intuition that underlies the concept of "infinite set," we could perhaps see clearly what large cardinals actually are and why they naturally belong in the universe.

We observed that the "philosophy of the infinite" that underlies the usual formulation of the Axiom of Infinity—in particular, the view concerning the emergence of the set of natural numbers—actually differs quite a bit from perspectives that we find in ancient traditions of knowledge. In those traditions, the natural numbers are understood to have a transcendental source, and that, as the diversity of the natural numbers unfolds, parts remain connected to their source, and transformational dynamics that occur within the source do not alter the fundamental nature of the source. In the usual mathematical treatment however, the idea that there could be a "source" of natural numbers, or that the natural numbers could be anything other than a sequence of discrete, disconnected quantities, is unfamiliar.

To help formulate an alternative Axiom of Infinity, we considered, in addition to the viewpoints from ancient traditions, the approach taken by modern physics in its quest to locate the ultimate consituents of the material universe. With the emergence of quantum field theory, there was a shift in the world view about the nature of discrete particles, which at an earlier time were presumed to be what

[178] A *sentence* is a formula that has no parameters.

all things are ultimately made of. In quantum field theory, however, all particles are precipitations of the dynamics of unbounded quantum fields. The "reality" of particles is the dynamics of the field that underlies them. This viewpoint accords well with the ancient view since a quantum field plays the role of the "source" of particles quite naturally.

Adopting this approach from physics as an intuition for understanding the mathematical infinite, we sought a mathematical way of saying that the discrete values of ω are expressions or precipitations of some sort of unbounded field. A formulation that seems to embody this idea was the concept of a *Dedekind self-map* $j : A \to A$ with critical point a: The "unbounded field" is A (since A must in fact be infinite); the dynamics of the field are represented by j; and the precipitated values that emerge are $a, j(a), j(j(a)), \ldots$.

We went on to show that the similarity between these precipitated values and the actual natural numbers $0, 1, 2, \ldots$ is not coincidental; indeed, a careful analysis showed that we can form the set $W = \{a, j(a), j(j(a)), \ldots\}$ and, defining concepts of induction in that context, we can define the Mostowski collapsing map π, which, when applied to W and $j \upharpoonright W$, neatly outputs ω and the successor function $s : \omega \to \omega$. With a bit more work in this direction, we were able to formalize the general notion of a *blueprint*, which allowed us to conclude in an even more rigorous way that $j \upharpoonright W$, together with its critical point and a collection $\mathcal{E}$ of iterator maps, form a blueprint for generating ω; also, a section h of $j \upharpoonright W$ was shown to be a blueprint for *returning* elements of ω to a. This rather involved mathematical definition of blueprints turned out to exhibit the essential characteristics of the notion of "blueprint" that occurs in many ancient traditions of knowledge in their account of the origin of manifest existence from the source.

In studying the dynamics of a Dedekind self-map and the concept of a blueprint, we hypothesized that large cardinals, by analogy with the natural numbers, should also arise as precipitations of a Dedekind self-map, in this case having domain the universe V. We anticipated that the right sort of $j : V \to V$ for this purpose should have strong preservation properties, by analogy with the version defined on sets; that sets it generates should arise from dynamics between j and its critical point; that j would give rise to a blueprint for a significant collection of sets in the universe; and that the sequence of critical points generated should emerge from considering restrictions of j to smaller domains.

When we began our study of Dedekind self-maps $j : V \to V$, we first had the task of strengthening j's properties sufficiently to ensure even the existence of ω, since a bare $j : V \to V$ could be defined even in the theory ZFC $-$ Infinity. We developed two ways to accomplish this. One way involved strengthening j with preservation properties; one result in this direction was the following: If j preserves disjoint unions, the empty set, and singletons, the universe must contain an infinite set. Another approach was to obtain an infinite set directly from the action by a suitably defined Dedekind self-map $j : V \to V$ on its least critical point. An example of this approach was the Lawvere Construction (Theorem 54) wherein $j = \mathbf{G} \circ \mathbf{F}$, $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ is the forgetful functor, $\mathbf{F}$ is a left adjoint of $\mathbf{G}$, 1 is the least critical point of j, and $j(1)$ is infinite. As discussed earlier in the paper (page 103), we consider both of these ways of arriving at an infinite set to be in keeping with our theme for finding the right generalization of a Dedekind self-map on a *set*, based on insights culled from ancient texts.

Generalizing the first of these approaches, we added other naturally motivated preservation properties. The resulting stronger versions of Dedekind self-maps of the universe led to the emergence of inaccessible cardinals and measurable cardinals.

We observed that the fullest possible way of requiring $j : V \to V$ to exhibit preservation properties is to require j to be an *elementary embedding*, which, by definition, preserves *all* first-order properties of its domain. In particular, we considered the requirement that $(V, \in, j)$ should be a model of ZFC + BTEE. Taking this step allowed us to explore further possibilities afforded by such strong mappings without the danger of falling into the inconsistency indicated by Kunen's famous theorem.[179] We were able to avoid this pitfall because the embedding j is obtained as a realization of an additional extralogical symbol $\mathbf{j}$, whose properties as a nontrivial elementary embedding are expressed in the BTEE axiom schema; in particular, j is never definable in V.

Working in a transitive model $M = (V, \in, j)$ of the theory ZFC + BTEE obtainable from a *Ramsey cardinal* (which is much weaker as a large cardinal notion than a measurable cardinal), a natural question arises: This model satisfies the criteria given in the Trňkova-Blass Theorem for existence of a measurable cardinal in that j itself is an exact functor with a strong critical point. Yet, as we argued, there is no measurable cardinal in this model (since a Ramsey cardinal cannot be used to create a model of a measurable cardinal). The reason for the apparent paradox is that, in order for the Trňkova-Blass criteria to hold, the functor (j in this case) must be definable in the ambient universe. But, by Kunen's inconsistency result, no elementary emdedding $j : V \to V$ having a critical point could possibly be definable in V. Thus, whereas for the class Dedekind self-maps j that are discussed in the Trňkova-Blass Theorem, the collection $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ (where $\kappa = \mathrm{crit}(j)$) is provably a *set*, this is not generally the case when j is a BTEE-embedding.

Having already established the naturalness of measurable cardinals based on the techniques developed so far, we postulated, by way of a new axiom MUA, the existence (as a set) of this ultrafilter U, derived from j. The resulting theory ZFC + BTEE + MUA is strong enough to imply not only that the critical point κ is measurable, but that "almost all" cardinals below κ are measurable as well. In addition, through its interaction with its critical point κ, j gives rise to a blueprint for the set $V_{\kappa+1}$.

The theory ZFC+BTEE+MUA is limited, however, by the fact that it provides limited information about its critical sequence $\kappa, j(\kappa), \ldots$, and most of the natural restrictions of j to sets of the form $V_{j^n(\kappa)}$ that we would like to study, by analogy with the Dedekind self-maps defined on a set, cannot be proved to exist in an MUA universe.

Replacing MUA by the stronger Axiom of Amenability gave us a much stronger theory: WA_0 is the theory BTEE+Amenability. We showed that from this stronger theory, the strong properties of all the major large cardinals can be accounted for as properties of the critical point of the embedding. And, moreover, the embedding gives rise to a blueprint for the *entire universe of sets*.

[179]This result was discussed on page 155.

The theory $\mathrm{ZFC}+\mathrm{WA}_0$ provides a nearly complete[180] solution to the Problem of Large Cardinals, and was motivated, as this article shows, by natural generalizations of a new Axiom of Infinity (which in turn was motivated by reflecting on ancient perspectives on the origin of the natural numbers).

This step in our work marks the completion of the goal we set for ourselves at the beginning, to reformulate the Axiom of Infinity so that a deeper intuition about the nature of the mathematical infinite would become apparent and point a direction for generalization that could ultimately account for large cardinals.

As we began to discuss in the last section, beyond $\mathrm{ZFC}+\mathrm{WA}_0$, the stronger theory $\mathrm{ZFC}+\mathrm{WA}$ provides full support for restrictions of j to stages of the form $V_{j^n(\kappa)}$, and this fact leads to a number of attractive results, even beyond a solution to the Problem of Large Cardinals. We conclude the paper by discussing one final result in this direction.

We work now in the theory $\mathrm{ZFC}+\mathrm{WA}$, where the embedding is j and critical point is κ. We observed before that, if $\mathbf{S} = \{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$ is the critical sequence for j, then $\mathbf{S}$ is a j-class of indiscernibles (relative to $\in$-formulas). From this observation, we were able to conclude that "almost all" cardinals in the universe are super-n-huge for every $n \in \omega$ (see p. 188).

We can say somewhat more. Recall that we also have the result (Theorem 85(2))

$$V_\kappa \prec V_{j(\kappa)} \prec V_{j(j(\kappa))} \prec \cdots \prec V. \tag{107}$$

This fact tells us that, not only is it the case that κ can declare "I am totality," but in fact each $j^n(\kappa)$ can make the same declaration, since $V_{j^n(\kappa)} \prec V$ for each n.

This observation allows us to form a rather special blueprint for ω. Adapting the proof of the Mostowski Collapsing Theorem (Theorem 10), one shows that the Mostowski collapsing map $\pi : \mathbf{S} \to \omega$ shown in diagram (108) is a bijection, takes κ to 0, and makes the diagram commutative:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\;j\restriction\mathbf{S}\;} & \mathbf{S} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \omega & \xrightarrow{\;s\;} & \omega \end{array} \tag{108}$$

In particular, $j \restriction \mathbf{S} : \mathbf{S} \to \mathbf{S}$ is an initial Dedekind self-map with critical point κ. Moreover, in the language of blueprints, we have that $(j \restriction \mathbf{S}, \kappa, \mathcal{E})$ is a *bona fide* blueprint for ω, where, as described earlier, $\mathcal{E} = \{i_n \mid n \in \omega\}$ and, for each $n \in \omega$ and each $g : \mathbf{S} \to \mathbf{S}$, $i_n(g) = \pi \circ g^n$.[181]

It follows that $\mathbf{S}$ is a blueprint for ω of a rather special kind. Unlike the ordinary natural numbers, each element of $\mathbf{S}$, as we have been saying, "embodies the totality." Though members of $\mathbf{S}$ are distinct and correspond to distinct natural numbers, each stands for totality and, at that level, each is indistinguishable from the others.

[180] Since there are a few extremely strong large cardinal notions that are not derivable from $\mathrm{ZFC}+\mathrm{WA}_0$, it is necessary to qualify in this way.

[181] Actually, we cannot quite fit the triple $(j \restriction \mathbf{S}, \kappa, \mathcal{E})$ into the universe (since even $\mathbf{S}$ itself is too big to be a set, even a $\mathbf{j}$-set), but the statement can be reworded to avoid this technical difficulty.

One of the motivating themes of this paper has been to consider the set of natural numbers in a different light—as precipitations of transformational dynamics of an unbounded field, embodied in the concept of a Dedekind self-map. One philosophical motivation for this change of viewpoint is the desire to recognize, in accord with points made by Maharishi and numerous early philosophers (Section 2), the natural numbers as *different on the surface but fundamentally the same*, being in each case an individual expression of wholeness. In Maharishi's approach, this viewpoint is expressed by declaring that each natural number is an expression of the Absolute Number. Viewing the natural numbers as arising from the Dedekind self-map $(\mathbf{S}, j \restriction \mathbf{S}, \kappa)$ expresses this point of view, in two ways. First, *any* initial Dedekind self-map gives expression to the idea that individual natural numbers are precipitations of an underlying field (represented by the self-map). Second, and more significantly, each natural number arising from this *particular* self-map $(\mathbf{S}, j \restriction \mathbf{S}, \kappa)$ arises as the (Mostowski) collapse of "wholeness" (represented by the elements of $\mathbf{S}$) to a particular value.[182]

We have in $\mathbf{S}$ therefore a representation of the natural numbers in which each number has been raised to the dignity of the "Absolute Number" (as far as this can be done in the realm of Western mathematics). The set $\mathbf{S}$ is by no means an ordinary set: It is not a member of V and it is in fact completely undefinable in V. In this sense, it represents the *unmanifest* unfoldment of the natural numbers, entirely beyond the reach of ordinary mathematics.

In [45], Maharishi introduced his Absolute Number and discussed the need to restore the connection of each natural number to its source, to the Absolute Number. Restoring this connection can be envisioned by putting a circle around each natural number. The effect is to cause the boundaries of each number to melt away, allowing each to thereby reconnect to its true, ultimate nature:

> This means that any number zeroed transforms itself into the Absolute Number—any number zeroed becomes unmanifest. Thus, different exprssions of the Absolute Number are: ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑧, ⑨, ⑩.
>
> By circling any number, the number begins to indicate that it is part and parcel of the Absolute Number—that its boundaries are unmanifest or, in spite of its boundaries, it is a continuum—it plays its part in explaining the eternal order that sustains the evolution of the universe. Its individual status has become Cosmic—as an individual it has been elected to be a ruler—the full potential of its creativity has blossomed (p. 614).

Elaborating this point a bit further, he says:

> We conceptualize the Absolute Number by circling any number, making it self-referral, making it infinite, making it an Absolute Number (p. 625).

[182] Maharishi remarks, "At the door of the Transcendent, the finite numbers begin to tap the unlimited reservoir of the Absolute Number, rendering every finite value on the ground of the infinite—the Absolute" [45, p. 633].

Referring to diagram (108) above, we see an analogy between the process of "circling" each natural number to "make it an Absolute Number," and the map $\pi^{-1} : \omega \to \mathbf{S}$, which transforms each natural number to its "absolute" counterpart—one of the $j^n(\kappa)$. For instance, the computation $\pi^{-1}(1) = j(\kappa)$[183] illustrates how the concrete natural number 1 is elevated by π^{-1} to its counterpart $j(\kappa)$ in $\mathbf{S}$.

Diagram (108) suggests the hidden dynamics by which the natural numbers emerge from the Absolute Number. For instance, we can ask, how does 1 arise from 0? In other words, how does the successor function s transform 0 to 1? Following the diagram, we see:

$$1 = s(0) = \pi(j(\pi^{-1}(0))).$$

The first step of computation is $0 \to \pi^{-1}(0)$. This step lifts 0 to its "absolute" value κ. The next step is $\kappa \to j(\kappa)$. This step represents the *unmanifest dynamics of computation*, since j itself is (a strong analogy for) the unmanifest dynamics of wholeness moving within itself. In the third and final step of computation, $j(\kappa) \to 1$, we have the *collapse* (recall π is the Mostowski collapsing map) of a kind of "absolute number" to a concrete manifest number.[184] This process gives expression to the following description [45]:

> It [the Absolute Number] is its own reality which functions within itself and gives a structure to knowledge and its infinite organizing power, and therefore it is the basis of all numbers and mathematical structures—just as the Unified Field of Natural Law is the basis of all force and matter fields (Physics)—the common source of all the Laws of Nature (pp. 625–626).

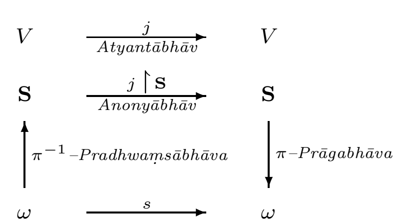
[183] Formally, following the diagram, we have the computation:

$$\pi(j(\kappa)) = s(\pi(\kappa)) = s(0) = 1.$$

This yields $\pi(j(\kappa)) = 1$. Applying π^{-1} to both sides yields the desired result.

[184] These steps of unfoldment correspond to the process, described by Maharishi, by which one sound or expression in the Veda gives rise to the next, by means of a collapse into the gap between the two. P. Oates summarizes the process [58]:

> First the previous sound (word, etc.) dissolves or collapses into the gap; this is the stage of *Pradhwaṃsābhāva*. Next is the step of absolute abstraction, complete silence, or *Atyantābhāva*. Within *Atyantābhāva*, pure consciousness, however, exists the self-interacting dynamics of consciousness, the seed of all dynamism, the structuring dynamics of the gap, called *Anonyābhāva*. ... From *Anonyābhāva*, the final stage occurs, *Prāgabhāva*, which is the mechanics by which a syllable, or sound, of Veda and Vedic literature emerges from the gap (p. 129).



The diagram, in which the expressed value of the natural number is seen integrated with its underlying, "unmanifest" dynamics, presents a holistic view of the natural numbers in a *circular form*,[185] rather than simply as a sequence $0, 1, 2, \ldots$. The fact that this more complete view of the natural numbers naturally assumes this shape provides an analogy for what Maharishi calls an *eternal structure* [47]:

> For any structure to be immortal, it must be inexhaustible; for any structure to be inexhaustible, it must be self-referral, which means it must refer to its source, it must refer to itself, it must be in a circular form (p. 75).

Just as the ultimate nature of the natural numbers and their sequential unfoldment is the Absolute Number and its unmanifest dynamics, so likewise is the creation itself, according to Maharishi, entirely a matter of unmanifest dynamics [45]:

> This means that the creative process does not put the creation out of the self-referral ocean of consciousness for the simple reason that there is nothing that can be out of the unbounded ocean of silence of the Unified Field (p. 539).

Elsewhere, he explains [45]:

> So in the last example we see that the deepest level of Nature's functioning, the Unified Field of Natural Law, glimpsed today by modern physics, is completely self-referral and infinite; it is of unmanifest nature—the source of creativity—changeless, and yet the source of all change (p. 249).

In the present example, the transformation $\pi^{-1} : 0 \mapsto \pi^{-1}(0) = \kappa$ is the breaking apart of the boundaries of the concrete number 0, and so corresponds to *Pradhwaṃsābhāva*. After this step has occurred, the process is governed by $j : V \to V$, which represents not only the dynamism underlying wholeness, but also complete silence, since j is an *elementary embedding* and, as such, preserves the integrity of every object and relationship in V. This silent aspect of j is not seen in the computation explicitly, but provides the context for the computation $j \restriction \mathbf{S} : \kappa \mapsto j(\kappa)$. This silent unseen aspect of the computation corresponds to *Atyantābhāva*.

Recall that κ itself is the seed of dynamism for the universe. (This follows from the role of κ as the seed in the blueprint $(\ell, \kappa, \mathcal{E})$, as in Theorem 81, and also from the fact that κ can be seen as a model of set theory that is elementarily equivalent to V itself, via the elementary embedding $(\kappa, R) \to (V, \in)$, as in Theorem 85(3).) Therefore, the computation $j \restriction \mathbf{S} : \kappa \mapsto j(\kappa)$ is the first impulse of that dynamism and so corresponds to *Anonyābhāva*.

Finally, the "unmanifest, cosmic" value $j(\kappa)$ collapses to the concrete value 1 in the computational step $\pi : j(\kappa) \mapsto 1$, and this restructuring into the concrete level corresponds to *Prāgabhāva*. This application of these four qualities of the gap, as described in Maharishi Vedic Science, closely parallels a similar application to the study of the dynamics of a chemical reaction [45, p. 542–544].

[185] Following the diagram in the way described above in order to display the unmanifest dynamics of computation leads us around the "circular" shape of the diagram. The diagram is "circular" in two senses. First, structurally, it is a *cycle*, as an undirected graph. Second, computationally, applying maps consecutively, starting from the lower left corner, returns one to the starting point. For instance: $s^{-1} \circ \pi \circ (j \restriction \mathbf{S}) \circ \pi^{-1} = \mathrm{id}_\omega$. This is a consequence of the fact that the diagram is commutative.

Following the path along the diagram for the computation $s(2)$, displayed in a linear format here, illustrates this idea. First note that $s(2) = \pi(j(\pi^{-1}(\pi(j(\pi^{-1}(0))))))$. In other words, we have:

$$0 \to \kappa \to j(\kappa) \to 1 \to j(\kappa) \to j(j(\kappa)) \to 2.$$

This point, he explains, expresses the reality declared in the *Upanishads* [45]:

Brahm-satyaṁ-jaganmitihyā.
Brahm is real and the world only appears to be real (p. 250).

These passages suggest a need not only for an "unmanifest mathematics" of the natural numbers but even an "unmanifest mathematics" of the physical world. We can take some steps in this direction by looking more closely at the "unmanifest" j-class $\mathbf{S}$. Just as the mathematical continuum of real numbers is derivable from the ordinary natural numbers through the process of expanding to the set $\mathbb{Q}$ of rational numbers and then forming the *completion* of $\mathbb{Q}$, so can $\mathbf{S}$ be expanded to an *absolute* version of "rational numbers," whose completion can represent a field of "absolute numbers" on the basis of which physical theories can be constructed.

We elaborate these points a bit further here; this work is the basis of research that will appear at a later time.[186] Recall from previous work[187] that we can view the blueprint $\mathbf{S}$ equally well as a sequence of compositions of j with itself:

$$\mathbf{S}_j = \{j, j \circ j, j \circ j \circ j, \ldots\}.$$

The analogue to the rational numbers, in which the "natural numbers" $\mathbf{S}_j$ are naturally embedded, is obtained by considering an alternative sequence of iterates, obtained by forming all possible *applications* of j to itself; we denote this collection $\mathcal{A}_j$:

$$\mathcal{A}_j = \{j,\ j \cdot j,\ j \cdot (j \cdot j),\ (j \cdot j) \cdot j \ldots\}.$$ [188]

All the elements of $\mathcal{A}_j$ are WA embeddings of V to itself, and the collection of all critical points of elements of $\mathcal{A}_j$, denoted $\mathrm{crit}(\mathcal{A}_j)$, is a countably infinite collection that properly includes $\mathbf{S}$. Although it is *not* true that $\mathbf{S}_j \subseteq \mathcal{A}_j$, it is nevertheless the case that $\mathbf{S}_j$ is naturally embedded in $\mathcal{A}_j$ by the map $j^n \mapsto j^{[n]}$, where, for any WA-embedding $k : V \to V$, we define the sequence of applicative iterates $k^{[1]}, k^{[2]}, \ldots, k^{[n]}, \ldots$ inductively by

$$\begin{array}{rcl} k^{[1]} & = & k, \\ k^{[n+1]} & = & k \cdot k^{[n]}. \end{array}$$

We define $\mathbf{S}'_j = \{j^{[n]} \mid n \geq 1\}$. Like $\mathbf{S}$ and $\mathbf{S}_j$, $\mathbf{S}'_j$ is another way to represent ω.

Returning now to $\mathcal{A}_j$, since each of its elements is also a WA-embedding, each also represents the dynamics of wholeness. To understand how $\mathcal{A}_j$ plays a role, relative to $\mathbf{S}'_j$, which is analogous to the rationals, one defines a *metric* d on $\mathcal{A}_j$,[189] which

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[186] Many of the technical details that underlie the brief treatment given here can be found in the Appendix, starting on p. 219.

[187] See for instance Theorem 22.

[188] This collection seems to be too big to fit inside V, but coding tricks can be used to solve this problem.

[189] We give the technical definition of the metric here: We first define $e : \omega \to \mathrm{crit}(\mathcal{A}_j)$ to be the unique increasing enumeration of $\mathrm{crit}(\mathcal{A}_j)$. Then, for every $i, k \in \mathcal{A}_j$ for which $i \neq k$, let $m_{i,k}$ be the least $n \in \omega$ such that $i(e(n)) \neq k(e(n))$. It can be shown [41] that $m_{i,k}$ always exists. Define $d : \mathcal{A}_j \times \mathcal{A}_j \to \mathbb{R}$ by

$$d(i,k) = \begin{cases} 0 & \text{if } i = k, \\ 1/(m_{i,k}+1) & \text{otherwise.} \end{cases}$$

One can show that d is a metric.

turns $\mathcal{A}_j$ into a *dense-in-itself* space.[190] The following analogy is an immediate consequence:

$$\omega : \mathbb{Q} :: \mathbf{S}'_j : \mathcal{A}_j.$$

Finally, a metric space *completion* of $\mathcal{A}_j$ can be constructed; this new space can be shown [9] to be a *perfect, complete, separable metric space*—a context in which much of modern mathematical analysis can be carried out. From here, the traditional construction of a Hilbert space can be performed, providing a context for formalizing the quantum mechanical aspects of the material universe. In this way, the unmanifest dynamics of pure consciousness are linked to the concrete dynamics of the physical universe. We have established a framework for doing "absolute mathematics" for studying manifest existence.

27. Appendix: Additional Proofs

In this Appendix, we record proofs of results mentioned in the main part of the article that were omitted for the sake of expository flow.

27.1. Hereditarily Finite Sets.

In this subsection, we show that **HF** is equal to V_ω whenever the universe contains an infinite set, but is equal to V if ω is not in the universe. We begin with a lemma.

Lemma 85. (ZFC – Infinity)

(A) *For all* $n \in \overline{\omega}$, $V_n \in \mathbf{HF}$.

(B) *Suppose* $Y \subseteq \bigcup_{n\in\overline{\omega}} V_n$ *and* Y *is finite. Then for some* $n \in \overline{\omega}$, $Y \in V_n$.

Proof of (A). Proceed by induction on $n \in \overline{\omega}$. The Base Case is obvious. For the Induction Step, assume $V_n \in \mathbf{HF}$. Then $V_{n+1} = \mathcal{P}(V_n)$ is finite and transitive, hence $V_{n+1} \in \mathbf{HF}$.

Proof of (B). Let $Y = \{y_1, \ldots, y_k\}$. For $1 \le i \le k$, let n_i be such that $y_i \in V_{n_i}$. Let $n = \max\{n_i \mid 1 \le i \le k\}$. Then $Y \subseteq V_n$ and so $Y \in V_{n+1}$.

Theorem 86. (ZFC – Infinity) $\mathbf{HF} = \bigcup_{n\in\overline{\omega}} V_n$.

Proof. For this proof, denote $\bigcup_{n\in\overline{\omega}} V_n$ by $V_{\overline{\omega}}$. By part (A) of the lemma, $V_{\overline{\omega}} \subseteq \mathbf{HF}$. We show $\mathbf{HF} \subseteq V_{\overline{\omega}}$. Define a class C by

$$C = \{z \in \mathbf{HF} \mid \forall n \in \overline{\omega}\,(z \notin V_n)\}.$$

We prove $C = \emptyset$; assume not.

Claim. C has an $\in$-minimal element.

Proof. Let $X \in C$. If $X \cap C = \emptyset$, we are done. Assume $X \cap C \neq \emptyset$. Let T be a finite transitive set with $X \in T$; then $X \subseteq T$. Let $Y = T \cap C$. Note $Y \neq \emptyset$; let $y \in Y$ be $\in$-minimal in Y, so $y \cap Y = \emptyset$. We show y is $\in$-minimal in C. Certainly $y \in C$; we show $y \cap C = \emptyset$.

[190] A metric space (X, ρ) is *dense-in-itself* if for every $x \in X$, there is a sequence $\langle y_n \rangle_{n\in\omega}$ that converges to x, where each y_n is different from x (equivalently, the sequence $\langle \rho(x, y_n) \rangle_n$ converges to 0). The set $\mathbb{Q}$ of rationals is the classic example of a dense-in-itself space.

Suppose, for a contradiction, that $z \in y \cap C$. Since $y \in T$ and T is transitive, $z \in T$. We have $z \in y \cap T \cap C = y \cap Y$—a contradiction.

Continuation of the Proof of Theorem 87. Let $z \in C$ be $\in$-minimal. No element of z belongs to C (by $\in$-minimality), so $z \subseteq V_{\overline{\omega}}$, and z is finite. By part (B) of the lemma, $z \in V_{\overline{\omega}}$, and we have a contradiction. Therefore $C = \emptyset$, and $\mathbf{HF} = V_{\overline{\omega}}$. □

Corollary 87.

(1) ZFC *proves* $\mathbf{HF} = V_\omega$, *where* $\mathbf{HF}$ *is the class of hereditarily finite sets and* $V_\omega = \bigcup_n V_n$.

(2) ZFC $-$ Infinity $+ \neg$Infinity *proves* $\mathbf{HF} = V$. □

27.2. **Results in Category Theory.** In this subsection, we give precise definitions and prove background results in category theory, which were referenced in the main text. In the main text (p. 126), the concept of adjoint functors was defined in the context of functors $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ and $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$. The definition states that $\mathbf{F}$ is a left adjoint of $\mathbf{G}$ if, for any A in $\mathbf{Set}$ and $g : B \to B$ in $\mathbf{SM}$, there is a bijection $\Theta_{A,g} : \mathbf{SM}(\mathbf{F}(A), g) \to \mathbf{Set}(A, \mathbf{G}(g))$, and, moreover, the maps are "natural" in A and g. This latter requirement was not defined explicitly; we explain those details here, and then prove several results concerning adjunctions.

To begin, we need the concept of *opposite category.* For any category $\mathcal{C}$, $\mathcal{C}^{\text{op}}$ is another category, the *opposite category of* $\mathcal{C}$, whose objects are the same as those of $\mathcal{C}$ and whose morphisms are those of $\mathcal{C}$, but *reversed* (so if $f : A \to B$ is a morphism of $\mathcal{C}$, then $f^{\text{op}} : B \to A$ is a morphism of $\mathcal{C}^{\text{op}}$).

Returning now to the functors $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ and $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$, we specify functors $\Gamma_{\mathbf{Set}} : \mathbf{Set}^{\text{op}} \times \mathbf{SM} \to \mathbf{Set}$ and $\Gamma_{\mathbf{SM}} : \mathbf{Set}^{\text{op}} \times \mathbf{SM} \to \mathbf{Set}$ defined on objects by $\Gamma_{\mathbf{Set}}(A, \beta) = \mathbf{Set}(A, \mathbf{G}(\beta)))$, and $\Gamma_{\mathbf{SM}}(A, \beta) = \mathbf{SM}(\mathbf{F}(A), \beta)$, where $\beta : B \to B$ is any $\mathbf{SM}$-object. These functors are defined on respective morphisms as follows: Given any $(A, \beta), (C, \delta) \in \mathbf{Set}^{\text{op}} \times \mathbf{SM}$ and any $\mathbf{Set}^{\text{op}} \times \mathbf{SM}$-morphism $\langle f^{\text{op}}, g\rangle : (A, \beta) \to (C, \delta)$, and any $h : A \to G(\beta)$ in $\mathbf{Set}$, we have

$$\Gamma_{\mathbf{Set}}(\langle f^{\text{op}}, g\rangle)(h) = \mathbf{G}(g) \circ h \circ f,$$

and given $h : F(A) \to \beta$ in $\mathbf{SM}$, we have

$$\Gamma_{\mathbf{SM}}(\langle f^{\text{op}}, g\rangle)(h) = g \circ h \circ \mathbf{F}(f).$$

Then to say that $\Theta_{A,\beta} : \mathbf{SM}(A, \mathbf{G}(\beta)) \to \mathbf{Set}(\mathbf{F}(A), \beta)$ is *natural in* A *and* β means Θ is a natural transformation from $\Gamma_{\mathbf{SM}}$ to $\Gamma_{\mathbf{Set}}$, which in turn means that, for each $\mathbf{Set}^{\text{op}} \times \mathbf{SM}$-morphism $\langle f^{\text{op}}, g\rangle : (A, \beta) \to (C, \delta)$, with $\beta : B \to B$ and $\delta : D \to D$, the following diagram is commutative (here, natural maps are drawn horizontally rather than vertically, as was originally done in the definition of "natural"):

$$\begin{array}{ccc}
\Gamma_{\mathbf{SM}}(A, \beta) & \xrightarrow{\Theta_{A,\beta}} & \Gamma_{\mathbf{Set}}(A, \beta) \\
\Big\downarrow{\scriptstyle \Gamma_{\mathbf{SM}}(f^{\text{op}}, g)} & & \Big\downarrow{\scriptstyle \Gamma_{\mathbf{Set}}(f^{\text{op}}, g)} \\
\Gamma_{\mathbf{SM}}(C, \delta) & \xrightarrow{\Theta_{C,\delta}} & \Gamma_{\mathbf{Set}}(C, \delta)
\end{array}$$

It is convenient to draw the diagram in the following way:

$$\begin{array}{ccc}
\mathbf{SM}(\mathbf{F}(A),\beta) & \xrightarrow{\Theta_{A,\beta}} & \mathbf{Set}(A,\mathbf{G}(\beta)) \\
\Big\downarrow{\scriptstyle h\mapsto g\circ h\circ \mathbf{F}(f)} & & \Big\downarrow{\scriptstyle h\mapsto \mathbf{G}(g)\circ h\circ f} \\
\mathbf{SM}(\mathbf{F}(C),\delta) & \xrightarrow{\Theta_{C,\delta}} & \mathbf{Set}(C,\mathbf{G}(\delta))
\end{array}$$

As a matter of terminology, notice that in the above discussion, we begin with a morphism $\langle f^{\mathrm{op}},\beta\rangle : (A,\beta) \to (C,\delta)$ in $\mathbf{Set}^{\mathrm{op}} \times \mathbf{SM}$, which, by definition, is precisely the morphism $\langle f,\beta\rangle : (C,\beta) \to (A,\delta)$ in $\mathbf{Set} \times \mathbf{SM}$. Applying $\Gamma_{\mathbf{SM}}$ produces a morphism $\Gamma_{\mathbf{SM}}(A,\beta) \to \Gamma_{\mathbf{SM}}(C,\delta)$ and applying $\Gamma_{\mathbf{Set}}$ produces a morphism $\Gamma_{\mathbf{Set}}(A,\beta) \to \Gamma_{\mathbf{Set}}(C,\delta)$. In each case, the domain and codomain of f, but not of β, have been reversed by the functors. We say that $\Gamma_{\mathbf{SM}}$ and $\Gamma_{\mathbf{Set}}$ are *contravariant in the first argument, covariant in the second argument.*

Theorem 88. (The Unit of an Adjunction) *Let* $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ *and* $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ *be adjoint functors, as defined on p. 126. For each* $\mathbf{Set}$ *object* A*, define a function* $\eta_A : A \to \mathbf{GF}(A)$ *by* $\eta_A = \Theta_{A,\mathbf{F}(A)}(1_{\mathbf{F}(A)})$ *(see Definition 27.2).* η *is called the* unit of the adjunction $F \dashv G$. *Then* η_A *has the following universal property: Given any* $k : A \to \mathbf{G}(\beta)$*, there is a unique* $\overline{k} : \mathbf{F}(A) \to \beta$ *such that the following diagram (on the left) is commutative:*

$$\begin{array}{ccccc}
A & \xrightarrow{\eta_A} & \mathbf{GF}(A) & & \mathbf{F}(A) \\
& \searrow{\scriptstyle k} & \Big\downarrow{\scriptstyle \mathbf{G}(\overline{k})} & & \Big\downarrow{\scriptstyle \overline{k}} \\
& & \mathbf{G}(\beta) & & \beta
\end{array}$$

Proof. We first observe that, for any $g : \mathbf{F}(A) \to \beta$, we have

$$\Theta_{A,\beta}(g) = \mathbf{G}(g) \circ \eta_A.$$

We show this by studying a special case of the main diagram from Definition 27.2, where f^{op} is taken to be the identity map $1_{\mathbf{F}(A)}$:

$$\begin{array}{ccc}
\mathbf{SM}(\mathbf{F}(A),\mathbf{F}(A)) & \xrightarrow{\Theta_{A,\mathbf{F}(A)}} & \mathbf{Set}(A,\mathbf{GF}(A)) \\
\Big\downarrow{\scriptstyle h\mapsto g\circ h} & & \Big\downarrow{\scriptstyle h\mapsto \mathbf{G}(g)\circ h} \\
\mathbf{SM}(\mathbf{F}(A),\beta) & \xrightarrow{\Theta_{A,\beta}} & \mathbf{Set}(A,\mathbf{G}(\beta))
\end{array}$$

By commutativity of the diagram, we obtain, for any $g : \mathbf{F}(A) \to \beta$ in $\mathbf{SM}$,

$$\mathbf{G}(g) \circ \eta_A = \Gamma_{\mathbf{Set}}\left(\Theta_{A,\mathbf{F}(A)}(1_{\mathbf{F}(A)})\right) = \Theta_{A,\beta}\left(\Gamma_{\mathbf{SM}}(1_{\mathbf{F}(A)})\right) = \Theta_{A,\beta}(g). \tag{109}$$

We now prove that η_A has the desired universal property: As described earlier, we are given $k : A \to \mathbf{G}(\beta)$. Since $\Theta_{A,\beta}$ is a bijection, we can obtain $\overline{k}$ such that

$\Theta_{A,\beta}(\overline{k}) = k$. Now by (109) we have

$$k = \Theta_{A,\beta}(\overline{k}) = \mathbf{G}(\overline{k}) \circ \eta_A,$$

as required.

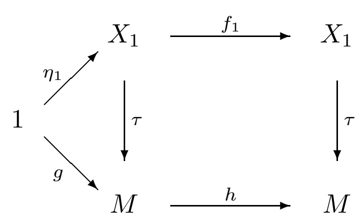
To show that $\overline{k}$ is unique, assume $k = \mathbf{G}(\overline{\ell}) \circ \eta_A$, for some $\overline{\ell} : \mathbf{F}(A) \to \beta$. Let $\ell = \Theta_{A,\beta}(\overline{\ell})$. Then by (109),

$$\ell = \Theta_{A,\beta}(\overline{\ell}) = \mathbf{G}(\overline{\ell}) \circ \eta_A = k.$$

Since $\Theta_{A,\beta}$ is a bijection, $\overline{\ell} = \overline{k}$. $\square$

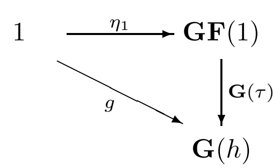
Finally, we show how these observations demonstrate one of the claims in the text. We re-state the claim here (the context for this is p. 127):

Claim. Suppose $\mathbf{F}, \mathbf{G}$ are defined as above and $\mathbf{F} \dashv \mathbf{G}$, and Θ is a witness to the adjunction. Let $f_1 = \mathbf{F}(1) : X_1 \to X_1$. Let $\eta_1 = \Theta_{1,f_1}(1_{f_1}) : 1 \to X_1$. Let $g : 1 \to M = \mathbf{G}(h : M \to M)$. Then there is a unique **SM**-map $\tau : f_1 \to h$ that makes the following diagram commutative:



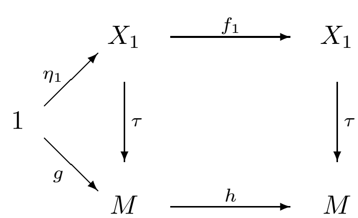
Proof. Suppose we are given $g : 1 \to \mathbf{G}(h)$, where $h : M \to M$ is an object in **SM**. By the universal property of η_1, we can find a unique $\tau : \mathbf{F}(1) \to h$ so that the following is commutative:

(110)



Since $\mathbf{F}(1) = f_1 : X_1 \to X_1$, $h : M \to M$, and $\mathbf{G}(\tau)$ is simply the **Set** morphism $\tau : X_1 \to M$, the commutative triangle implies that $g(0) = \tau(\eta_1(0))$ (in the text, $\eta_1(0)$ was denoted η_0). Also, since $\tau : \mathbf{F}(1) \to h$ is an **SM**-morphism, we obtain the following commutative diagram:

(111)



For uniqueness, if τ' also makes diagram (111) commute (replacing τ with τ'), then $\tau' : \mathbf{F}(1) \to h$ is also an **SM**-morphism making diagram (110) commute. It follows that $\tau' = \tau$. $\square$

In this paper, our primary interest in developing some of the theory of adjoints is in the adjunction $\mathbf{F} \dashv \mathbf{G}$, where $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ is the forgetful functor. However, the preceding definitions and theorems are applicable in a broader context and the proofs go through without much change. We state these more general versions here.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. Generalizing a notational convention, for any objects X, Y in $\mathcal{C}$, we have $\mathcal{C}(X, Y) = \{f \mid f : X \to Y \text{ is a } \mathcal{C}\text{-morphism}\}$, and similarly for $\mathcal{D}$.

Definition (Adjoint Functors) We shall say that F is a *left adjoint of* G, and we write $F \dashv G$, if there is, for every object A in $\mathcal{C}$ and object B in $\mathcal{D}$, a bijection $\Theta_{A,B} : \mathcal{D}(F(A), B) \to \mathcal{C}(A, G(B))$ that is natural in A and B. We say that the triple (F, G, Θ) is an *adjunction*. Naturality in this case means that, for every $f : C \to A$ belonging to $\mathcal{C}$ and $g : B \to D$ belonging to $\mathcal{D}$, the following is commutative.

(112)
$$\begin{array}{ccc} \mathbf{D}(F(A), B) & \xrightarrow{\Theta_{A,B}} & \mathbf{C}(A, G(B)) \\ \Big\downarrow{\scriptstyle \Gamma_F(\rho) = g \circ \rho \circ F(f)} & & \Big\downarrow{\scriptstyle \Gamma_G(h) = G(g) \circ h \circ f} \\ \mathbf{D}(F(C), D) & \xrightarrow{\Theta_{C,D}} & \mathbf{D}(C, G(D)) \end{array}$$

Definition (Unit of an Adjunction) Given an adjunction (F, G, Θ) as above, the *unit* η for the adjunction is defined, for each $A \in \mathcal{C}$, as follows (exactly as before):

$$\eta_A = \Theta_{A,F(A)}(1_{F(A)}).$$

The unit of an adjunction in this general context has the same properties as before:

Theorem 89. (The Unit of an Adjunction) *Suppose we are given categories $\mathcal{C}, \mathcal{D}$ and an adjunction (F, G, Θ), as described above. The unit $\eta : 1_{\mathcal{C}} \to G \circ F$ of the adjunction (F, G, Θ) has the following properties.*

(0) *The unit η satisfies, for each object A in $\mathcal{C}$ and B in $\mathcal{D}$:*

$$\Theta_{A,B}(g) = G(g) \circ \eta_A.$$

(1) *For each set A, η_A is a* universal arrow*; that is, for all $\mathcal{C}$-objects A and $\mathcal{D}$-objects B, given any function $k : A \to G(B)$, there is a unique $\mathcal{D}$-morphism $\overline{k} : F(A) \to B$ such that the following diagram commutes:*

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & G(F((A))) & \quad & F(A) \\ & {\scriptstyle k}\searrow & \Big\downarrow{\scriptstyle G(\overline{k})} & & \Big\downarrow{\scriptstyle \overline{k}} \\ & & G(B) & & B \end{array}$$

(2) *η is a natural transformation.*

We now apply this machinery to prove a lemma that is used in the proof of Lemma 60 in the main text, concerning Dedekind monads.

Lemma 90. *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors with $F \dashv G$. Let Θ be the natural bijection for the adjunction and let η be the unit of the adjunction. Suppose $f : A \to B$ is a $\mathcal{C}$-morphism. Let $\overline{F(f)} : A \to G(F(B))$ be the transpose of $F(f)$; that is, $\overline{F(f)} = \Theta_{A,F(B)}(F(f))$. Then the following holds:*

$$\overline{F(f)} = \eta_\beta \circ f.$$

Proof. We use the following special case of Diagram (112):

$$\begin{array}{ccc}
\mathcal{D}(F(B), F(B)) & \xrightarrow{\Theta_{B,F(B)}} & \mathcal{C}(B, G(F(B))) \\
\Big\downarrow{\scriptstyle \Gamma_F(h) = h \mapsto 1_{F(B)} \circ h \circ F(f)} & & \Big\downarrow{\scriptstyle \Gamma_G(h) = h \mapsto G(1_{G(B)}) \circ h \circ f} \\
\mathcal{D}(F(A), F(B)) & \xrightarrow{\Theta_{A,F(B)}} & \mathcal{C}(A, G(F(B)))
\end{array}$$

We trace through the diagram starting with the value $1_{F(B)}$ in $\mathcal{D}(F(B), F(B))$. Note that, with reference to Diagram (112), the value of g is, in the present context, $1_{F(B)}$.

Tracing right and then down, we have

$$\Gamma_G(\Theta_{B,F(B)}(1_{F(B)})) = \Gamma_G(\eta_B) = \eta_B \circ f.$$

Tracing down and then left, we have

$$\Theta(A, F(B))(\Gamma_F(1_{F(B)})) = \Theta(A, F(B))(F(f)) = \overline{F(f)}.$$

We have shown $\overline{F(f)} = \eta_B \circ f$, as required. $\square$

Lemma 91. (Adjoints Are Isomorphic) *Suppose $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are functors and suppose that each is a left adjoint of a functor $G : \mathcal{D} \to \mathcal{C}$. Then F_1 and F_2 are naturally isomorphic.*

Proof. We exhibit an isomorphism and then show that it is natural. The isomorphism comes from the universal property of the units of the respective adjunctions. Let η^1 and η^2 be the units for the adjunctions (F_1, G, Θ_1) and (F_2, G, Θ_2), respectively. For each set A, we have functions $\eta^1_A : A \to GF_1(A)$ and $\eta^2_A : A \to GF_2(A)$. Using the universal property of η^1, there is a unique function $\overline{\eta^2_A} : F_1(A) \to F_2(A)$ that makes the upper square below commutative. By the universal property of η^2, there is a unique function $\overline{\eta^1_A} : F_2(A) \to F_1(A)$ that makes the second square below commutative. The top square is repeated below the first two, and for the same reason, it is also commutative.

$$
\begin{array}{ccccc}
A & \xrightarrow{\eta^1_A} & G(F_1(A)) & \qquad & F_1(A) \\
\downarrow{\scriptstyle \mathrm{id_A}} & & \downarrow{\scriptstyle G(\overline{\eta^2_A})} & & \downarrow{\scriptstyle \overline{\eta^2_A}} \\
A & \xrightarrow{\eta^2_A} & G(F_2(A)) & & F_2(A) \\
\downarrow{\scriptstyle \mathrm{id_A}} & & \downarrow{\scriptstyle G(\overline{\eta^1_A})} & & \downarrow{\scriptstyle \overline{\eta^1_A}} \\
A & \xrightarrow{\eta^1_A} & G(F_1(A)) & & F_1(A) \\
\downarrow{\scriptstyle \mathrm{id_A}} & & \downarrow{\scriptstyle G(\overline{\eta^2_A})} & & \downarrow{\scriptstyle \overline{\eta^2_A}} \\
A & \xrightarrow{\eta^2_A} & G(F_2(A)) & & F_2(A)
\end{array}
$$

Notice also that $G(\overline{\eta^1_A} \circ \overline{\eta^2_A})$ makes the outer rectangle that encloses the upper two squares commutative; yet, by the universal property of the unit, $1_{F_1(A)}$ is the unique map for which the following square is commutative.

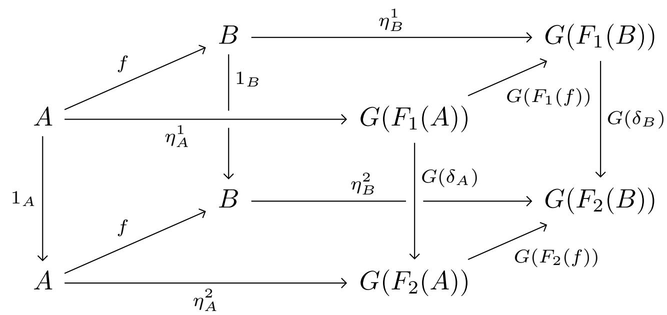
$$
\begin{array}{ccccc}
A & \xrightarrow{\eta^1_A} & G(F_1(A)) & \qquad & F_1(A) \\
\downarrow{\scriptstyle \mathrm{id_A}} & & \downarrow{\scriptstyle G(1_{F_1(A)})} & & \downarrow{\scriptstyle 1_{F_1(A)}} \\
A & \xrightarrow{\eta^1_A} & G(F_1(A)) & & F_1(A)
\end{array}
$$

It follows that $\overline{\eta^1_A} \circ \overline{\eta^2_A} = 1_{F_1(A)}$. A similar argument focused on the bottom two squares shows that $\overline{\eta^2_A} \circ \overline{\eta^1_A} = 1_{F_2(A)}$. These together show that $\overline{\eta^1_A}$ and $\overline{\eta^2_A}$ are inverses, hence isomorphisms.

We now show that $\overline{\eta^2_A}$ are components of a natural transformation. For each A, let $\delta_A = \overline{\eta^2_A} : F_1(A) \to F_2(A)$. Suppose $f : A \to B$ is a $\mathcal{C}$-morphism. We wish to show that $\delta_B \circ F_1(f) = F_2(f) \circ \delta_A$; that is, the following diagram is commutative:

$$
\begin{array}{ccc}
F_1(A) & \xrightarrow{F_1(f)} & F_1(B) \\
\downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_B} \\
F_2(A) & \xrightarrow{F_2(f)} & F_2(B)
\end{array} \tag{113}
$$

(114)

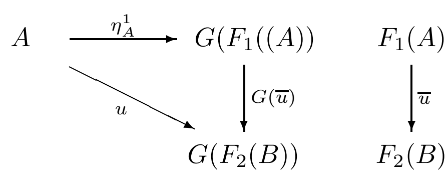


Claim. $G(\delta_B) \circ G(F_1(f)) \circ \eta_1^A = G(F_2(f)) \circ G(\delta_A) \circ \eta_1^A$.

Proof of Claim.

$$\begin{array}{rcll} G(\delta_B) \circ G(F_1(f)) \circ \eta_1^A & = & G(\delta_B) \circ \eta_B^1 \circ f & \text{(because } \eta^1 \text{ is natural)} \\ & = & \eta_B^2 \circ f & \text{(univ property of } \eta_B^1 \text{ wrt } \eta_B^2) \\ & = & G(F_2(f)) \circ \eta_A^2 & \text{(because } \eta^2 \text{ is natural)} \\ & = & G(F_2(f)) \circ G(\delta_A) \circ \eta_1^A & \text{(univ property of } \eta_A^1 \text{ wrt } \eta_A^2). \ \square \end{array}$$

The claim gives us commutativity of Diagram (113) when precomposed with η_A^1. To complete the proof, we use the universal property of η_A^1 in another way. This universal property tells us that if $u = G(F_2(f) \circ G(\delta_A) \circ \eta_A^1 : A \to G(F_2(B))$, then there is a unique $\overline{u} : F_1(A) \to F_2(B)$ for which the triangle below is commutative:



We observe however that, by the claim, there are two values of $\overline{u}$ that make the diagram commutative (recalling that, by definition, $u = G(F_2(f) \circ G(\delta_A) \circ \eta_A^1)$:

$$\begin{array}{rcl} \overline{u} & = & F_2(f) \circ \delta_A \\ \overline{u} & = & \delta_B \circ F_1(f) \end{array}$$

By uniqueness guaranteed by the universal property of η_A^1, we conclude that

$$\delta_B \circ F_1(f) = F_2(f) \circ \delta_A,$$

as required. $\square$

Theorem 92. (Adjoints and Universal Elements) *Let* $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$ *denote the forgetful functor and suppose* $\mathbf{F} : \mathbf{Set} \to \mathbf{SM}$ *is a functor.*

(1) *If* $\mathbf{F}$ *is a left adjoint of* $\mathbf{G}$, $\mathbf{G} \circ \mathbf{F}$ *is essentially Dedekind.*
(2) *Whenever* $\mathbf{F} \dashv \mathbf{G}$ *and* $|A| > 0$, $j(A) = \mathbf{G}(\mathbf{F}(A))$ *is infinite.*

(3) *Whenever* **G** *has a universal element, there is a naturally defined initial Dedekind self-map. Moreover,* **G** *has a left adjoint.*

Proof. For (1), since we have already shown that existence of a left adjoint of **G** produces a Dedekind self-map, existence of the left adjoint guarantees that ω exists. We can then define a functor $\mathbf{F}'$ by $\mathbf{F}'(A) = 1_A \times s : A \times \omega \to A \times \omega$, where the map $1_A \times s$ is defined by $(1_A \times s)(a, n) = (a, n + 1)$. Also, if $h : A \to B$ is a **Set** morphism, then $\mathbf{F}'(h) : \mathbf{F}'(A) \to \mathbf{F}'(B)$ is defined by $\mathbf{F}'(h) = h \times 1_\omega$.

$$\begin{array}{ccc} A \times \omega & \xrightarrow{1_A \times s} & A \times \omega \\ \downarrow{\scriptstyle h \times 1_\omega} & & \downarrow{\scriptstyle h \times 1_\omega} \\ B \times \omega & \xrightarrow{1_B \times s} & B \times \omega \end{array}$$

One can show that $\mathbf{F}' \dashv \mathbf{G}$. In the following claim, we prove that the necessary bijections $\Theta_{A,\beta}$ exist; we omit the proof that these bijections are the components of a natural transformation. (See [25, pp. 445–6] for more details.)

Claim. For any set A and **SM**-morphism β, there is a bijection $\Theta_{A,\beta}$ from $\mathbf{SM}(\mathbf{F}'(A), \beta)$ to $\mathbf{Set}(A, \mathbf{G}(\beta))$.

Proof. We prove two facts that will make it easier to define (and verify the properties of) Θ.

Subclaim.

(i) Suppose $\rho : 1_A \times s \to \beta \in \mathbf{SM}(\mathbf{F}'(A), \beta)$. Then for each $a \in A$, the values of $\rho(a, n)$ are completely determined by the value $\rho(a, 0)$. In particular, for each $n \in \omega$, $\rho(a, n) = \beta^n \rho(a, 0)$ (where $\beta^0 = 1_B$).

(ii) Suppose $f : A \to B$. Then there is a unique element ρ of $\mathbf{SM}(\mathbf{F}'(A), \beta)$ such that $\rho(a, 0) = f(a)$.

$$\begin{array}{ccc} A \times \omega & \xrightarrow{1_A \times s} & A \times \omega \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ B & \xrightarrow{\beta} & B \end{array} \tag{115}$$

Proof of Subclaim (i). Proceed by induction on $n \in \omega$. The result is clear for $n = 0$. Assuming $\rho(a, n) = \beta^n(\rho(a, 0))$, we have by commutativity of the diagram above,

$$\rho(a, n + 1) = \rho(a, (1_A \times s)(a, n)) = \beta(\rho(a, n)) = \beta^n(\rho(a, 0)).$$

Proof of Subclaim (ii). Given $f : A \to B$, we wish to define $\rho_f : A \times \omega \to B$. We begin by letting $\rho_f(a, 0) = f(a)$. To complete the proof, it will be enough to show that the requirement that the diagram above be commutative completely determines the definition of $\rho_f(a, n)$ for $n > 0$. Note that, in order for commutativity to hold, $\rho_f(a, 1) = \rho_f(a, s(0)) = \beta(\rho(a, 0)) = \beta(f(a))$. We can therefore recursively build the values $\rho_f(a, n)$ so that commutativity is preserved; in particular, we obtain $\rho_f(a, n) = \beta^n(f(a))$. Note that this is the only definition of ρ_f for which $\rho_f(a, 0) = f(a)$ that could possibly make the diagram commutative; and indeed, it does make the diagram commutative (this follows by the way ρ_f was constructed, but can be verified in a separate step by induction). □

Let $\rho \in \mathbf{SM}(\mathbf{F}'(A), \beta)$. Note that ρ makes the diagram (115) commutative. We define $\Theta_{A,\beta}(\rho) : A \to B$ by

$$\Theta_{A,\beta}(\rho)(a) = \rho(a, 0).$$

If $\Theta_{A,\beta}(\rho_1) = \Theta_{A,\beta}(\rho_2)$, then $\rho_1(a, 0) = \rho_2(a, 0)$, and so by Subclaim (i), $\rho_1 = \rho_2$. We have shown $\Theta_{A,\beta}$ is 1-1.

Suppose $f : A \to B \in \mathbf{Set}(A, B)$. By Subclaim (ii), there is a unique element ρ of $\mathbf{SM}(\mathbf{F}'(A), \beta)$ such that $\rho(a, 0) = f(a)$. This shows that $\Theta_{A,\beta}$ is onto. □

To see that $\mathbf{F}'$ is 1-1 on objects, suppose $A \neq B$, say $a \in A - B$. Then $(a, 0) \in A \times \omega - B \times \omega$, and it follows that $\mathbf{GF}'(A) \neq \mathbf{GF}'(B)$. Also, $j' = \mathbf{G} \circ \mathbf{F}'$ has a critical point: As before, $j'(0) = 0$, and, for all A for which $|A| \geq 1$, $j'(A)$ is infinite (since $j'(A) = A \times \omega$).

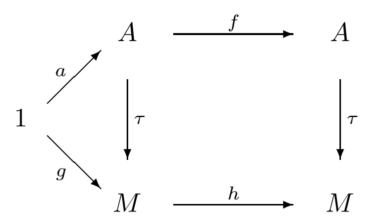
A fact from category theory ([2] or [43]) is that left adjoints of the same functor must be naturally isomorphic. In this case, this means that there is, for each set A, an $\mathbf{SM}$-isomorphism $\sigma_A : \mathbf{F}(A) \to \mathbf{F}'(A)$ that is natural in A. Applying $\mathbf{G}$, we obtain another isomorphism $\mathbf{G}(\sigma_A) : \mathbf{GF}(A) \to \mathbf{GF}'(A)$ (note that every functor preserves isomorphisms). Moreover $\mathbf{G}(\sigma_A)$ can be shown to be natural in A. Therefore, j and j' are naturally isomorphic and j' is a Dedekind self-map. It follows that j is an essentially Dedekind self-map.

For (2), using these ideas, we can now verify that, for any set A having one or more elements, $j(A)$ is infinite. Since $j \cong j'$, it follows that

$$|j(A)| = |j'(A)| = |A \times \omega| \geq \omega.$$

For (3), suppose $a \in \mathbf{G}(f)$ is a universal element for $\mathbf{G}$, where $f : A \to A$. We show (A, f, a) is an initial Dedekind self-map. Identifying $a \in A$ with the map $1 \to A : 0 \mapsto a$, we first show that $f : A \to A$ has the NNO property. Let M be a set and $g : 1 \to M$ and $h : M \to M$ be functions; we must show there is a unique $\mathbf{SM}$-morphism $\tau : f \to h$ making the following diagram commutative:

(116)



Because $a \in \mathbf{G}(f)$ is a universal element, there is a unique **SM**-morphism $\tau : f \to h$ making the following triangle commute:

$$(117) \qquad \begin{array}{ccccc} 1 & \xrightarrow{\;a\;} & A = \mathbf{G}(f) & \qquad & f \\ & \overset{g}{\searrow} & \Big\downarrow {\scriptstyle \mathbf{G}(\tau)} & & \Big\downarrow {\scriptstyle \tau} \\ & & M = \mathbf{G}(h) & & h \end{array}$$

Existence of the **SM**-morphism τ guarantees commutativity of the square in diagram (116); commutativity of the triangle in diagram (117) guarantees commutativity of the triangle in diagram (116); uniqueness of τ in diagram (117) ensures uniqueness of τ in diagram (116).

Now, as indicated by Remark 17, using the NNO property of $f : A \to A$, we can prove the analogues to Claims (A)–(C) on pp. 127–129 to establish the following:

(i) The sequence $s = \langle a, f(a), f(f(a)), \ldots \rangle$ has no repeated terms.
(ii) The function $f \upharpoonright B : B \to B$ is an initial Dedekind self-map, where $B = \operatorname{ran} s$
(iii) $B = A$.

These points establish that $f : A \to A$ is an initial Dedekind self-map with critical point a.

For the "moreover" clause, the proof given in part (1) shows how to define a left adjoint for **G** once existence of ω has been established. □

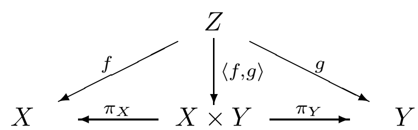
27.3. **Adjoints in the Fabric of Mathematics.** In the main text (see the footnote on p. 134), we raised a philosophical point. We showed in Theorem 54 that an infinite set is derivable from the existence of a left adjoint **F** to the forgetful functor $\mathbf{G} : \mathbf{SM} \to \mathbf{Set}$, and that indeed, an infinite set emerges as $j(\text{crit}(j))$, where $j = \mathbf{G} \circ \mathbf{F}$. Although this conclusion accords well with our expectation, based on points from ancient wisdom (p. 103), the mathematical assumption from which the conclusion is derived—namely, that **G** has a left adjoint—was not explicitly justified on the basis of that wisdom. Nor likewise was there any such justification for postulating the existence of a Dedekind monad, which directly gives rise to a set Dedekind self-map from its critical point, though again, the conclusion accords well with those principles.

We then outlined a proposal for providing such justification; the principle in this case is that diversity emerges from the integration of opposing forces, on the ground of unity. This principle seems to be at work in the structure of adjoints and, according to many category theorists, the dynamics of adjoints can be seen to structure virtually all of mathematics. In this section, we provide evidence for this point of view.

For our first point, we observe that the category of sets is an example of a *cartesian closed* category, whose defining characteristics all arise from adjunctions. Stated simply, a cartesian closed category is one that has a terminal object, is closed under cartesian products (for any objects X, Y, there is another object $X \times Y$ in the usual sense), and is closed under taking exponents (for any objects X, Y, there is an *exponential object* X^Y consisting of all morphisms from Y to X).[191] These defining axioms can be understood as expressions of adjoint situations, as follows:[192] A cartesian closed category can be defined to be a category $\mathcal{C}$ equipped with adjoint situations of the following three sorts:

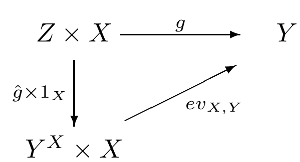
(1) *Existence of Terminal Object*. $\mathcal{C}$ has a terminal object if there is an object 1 with the property that, for every X in $\mathcal{C}$, there is a unique morphism $X \to 1$. Existence of a terminal object for $\mathcal{C}$ is equivalent to existence of a right adjoint $\mathbf{1} \to \mathcal{C}$ to the unique morphism (which, in the category of all categories, is a functor) from $\mathcal{C}$ to the category $\mathbf{1}$.

(2) *Existence of Products*. For any objects X, Y in $\mathcal{C}$, the product of X and Y is an object $X \times Y$ together with projection morphisms $\pi_X : X \times Y \to X, \pi_Y :$ $X \times Y$ having the universal property: For any object Z in $\mathcal{C}$ and morphisms $f : Z \to X, g : Z \to Y$, there is a unique morphism $\langle f, g\rangle : Z \to X \times Y$ making the following diagram commutative:



Existence of products in $\mathcal{C}$ is equivalent to existence of a right adjoint $\Pi : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, where Δ is defined by $\Delta(X) = (X, X)$. For each $X \in \mathcal{C}$, we let $\Pi_X : \mathcal{C} \to \mathcal{C}$ denote the X-product functor defined by $\Pi_X(Y) = X \times Y$, with the obvious definition on morphisms.

(3) *Existence of Exponentials*. For any objects X, Y in $\mathcal{C}$, an exponential object Y^X in $\mathcal{C}$, together with its evaluation map $ev_{X,Y} : Y^X \times X \to Y$, is an object-morphism pair that satisfies the following universal property: Given any object $Z \in \mathcal{C}$ and any $g : Z \times X \to Y$, there is a unique morphism $\hat{g} : Z \to Y^X$ such that the following diagram commutes:



For **Set**, $ev_{X,Y}$ is the *evaluation map* given by $ev_{X,Y}(f, y) = f(y)$. Exponentiation by X can be seen as a functor $E_X : \mathcal{C} \to \mathcal{C}$ that is right adjoint to the X-product functor Π_X, that is, $\Pi_X \dashv E_X$.

[191]The definitions of "product" and "exponential" in this context need to be expressed in the language of categories; each of these is defined entirely in terms or morphisms with universal properties, in the spirit of the definition of coproducts, discussed earlier.

[192]See [42].

The fact that cartesian closed categories are defined in terms of adjoints indicates that the structure and dynamics of the category **Set** are intimately tied to adjunctions.

Another way in which adjunctions make their presence known in the category of sets was discovered relatively recently in work by Rosebrugh and Wood [66]. They show that the category **Set** can be completely characterized in terms of *maximal adjoint strings*. We give an overview of this characterization here.

After stating some preliminaries, we will give the main result, and then explain the terminology somewhat more and list the background theorems that the result depends on. First, we consider a new way to construct categories. Suppose $\mathcal{C}$ is any category with the property that, for any objects C, D of $\mathcal{C}$, the collection $\mathcal{C}(C, D)$ of morphisms from C to D in $\mathcal{C}$ is a *set*; such categories are said to be *locally small*. We denote by $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ the category of contravariant functors from $\mathcal{C}$ to **Set** whose morphisms are natural transformations between functors.[193] $\mathcal{C}$ can always be "embedded" in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ via what is known as the *Yoneda embedding*, which is a special functor $\mathbf{Y}$ defined as follows: For each object C in $\mathcal{C}$, $\mathbf{Y}(C)$ is the functor $yC : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ defined by:

$$yC(D) = \mathcal{C}(D, C).$$

In other words, yC takes each object D of $\mathcal{C}$ to the *set* of $\mathcal{C}$-morphisms from D to C; in order for $\mathcal{C}(D, C)$ to be an object of **Set**, the condition that $\mathcal{C}$ be locally small is necessary. Moreover, for any morphism $f : C \to E$ in $\mathcal{C}$, $\mathbf{Y}(f) : yC \to yE$ is the natural transformation yf defined, for each object D in $\mathcal{C}$, by

$$yf_D(h) = D \xrightarrow{h} C \xrightarrow{f} E = f \circ h.$$

Rosebrugh-Wood Adjoint String Theorem (RAST) [66]. *Suppose $\mathcal{C}$ is a locally small category.*

(A) *Suppose that the Yoneda embedding $\mathbf{Y} : C \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ has a left adjoint $\mathbf{X}$, which in turn has left adjoints $\mathbf{W}, \mathbf{V}, \mathbf{U}$; in other words, suppose $\mathcal{C}$ admits the following adjoint string:*

$$\mathbf{U} \dashv \mathbf{V} \dashv \mathbf{W} \dashv \mathbf{X} \dashv \mathbf{Y}.$$

Then $\mathcal{C}$ is equivalent to **Set**. *Moreover, the maximum possible length of an adjoint string of this type (beginning at the right with a Yoneda embedding) is 5.*

(B) *An example of an adjoint string of length 5, of the type mentioned in* (A)*, is given by*

$$\exists\exists\mathbf{Y}_0 \dashv \mathcal{K}(\exists\mathbf{Y}_0) \dashv \exists\mathbf{Y}_1 \dashv \mathcal{K}(\mathbf{Y}_1) \dashv \mathbf{Y}_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}^{\mathbf{Set}^{\mathrm{op}}}.$$

We explain the meaning of the notation in (B) and derive the adjoint string mentioned there. We will make use of the following standard results from the literature:[194]

[193] The definition of natural transformation is given on p. 124. The opposite category of a category $\mathcal{C}$ was introduced on p. 201, and the notion of a contravariant functor was introduced on p. 202.

[194] These are stated, with references, in [66].

We let $\mathbf{0}$ denote the empty category (no objects or morphisms); it is the initial object in the category **Cat** of categories. We let $\mathbf{1}$ denote the category having just one object and one morphism; it is the terminal object in **Cat**. If $\mathcal{C}$ is an object in **Cat**, the unique functor $\mathbf{F} : \mathcal{C} \to \mathbf{1}$ is denoted $!_{\mathcal{C}}$, or simply ! if the context makes the meaning clear. Note that $\mathbf{0}^{\mathrm{op}} = \mathbf{0}$ and $\mathbf{1}^{\mathrm{op}} = \mathbf{1}$.

For each category $\mathcal{C}$, we let $\mathcal{K}(\mathcal{C})$ denote $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.[195] $\mathcal{K}$ is a contravariant functor from **Cat** to **Cat** (equivalently, a functor $\mathbf{Cat} \to \mathbf{Cat}^{\mathrm{op}}$); it is defined on morphisms of **Cat** in the following way: Suppose $\mathcal{C}, \mathcal{D}$ are categories and $\mathbf{F} : \mathcal{C} \to \mathcal{D}$ is a **Cat**-morphism. Then $\mathcal{K}(\mathbf{F}) : \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is defined by $\mathcal{K}(\mathbf{F})(\mathbf{H}) = \mathbf{H} \circ \mathbf{F}$.

Lemma R_1. *If $\mathbf{L}_1, \mathbf{L}_2$ are left adjoint to a functor $\mathbf{F}$, then $\mathbf{L}_1$ and $\mathbf{L}_2$ are naturally isomorphic. Likewise, if $\mathbf{R}_1, \mathbf{R}_2$ are both right adjoints of $\mathbf{F}$, then $\mathbf{R}_1$ and $\mathbf{R}_2$ are naturally isomorphic.* $\square$

Lemma R_2. *Whenever $\mathbf{L} \dashv \mathbf{F}$, we have $\mathcal{K}(\mathbf{L}) \dashv \mathcal{K}(\mathbf{R})$.* $\square$

Lemma R_3. *If the morphisms of $\mathcal{C}$ form a set and if $\mathcal{D}$ is locally small, and $\mathbf{F} : \mathcal{C} \to \mathcal{D}$, then $\mathcal{K}(\mathbf{F})$ has a left adjoint, denoted $\exists\mathbf{F}$ and a right adjoint, denoted $\forall\mathbf{F}$. Therefore,*

$$\exists\mathbf{F} \dashv \mathcal{K}(\mathbf{F}) \dashv \forall\mathbf{F}.$$

In particular, if $\mathbf{F}$ is the Yoneda embedding $\mathbf{Y}_{\mathcal{C}} : \mathcal{C} \to \mathcal{K}(\mathcal{C}) = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, then $\forall\mathbf{Y}_{\mathcal{C}} \cong \mathbf{Y}_{\mathcal{K}(\mathcal{C})}$. Therefore:

$$\exists\mathbf{Y}_{\mathcal{C}} \dashv \mathcal{K}(\mathbf{Y}_{\mathcal{C}}) \dashv \mathbf{Y}_{\mathcal{K}(\mathcal{C})}. \ \square$$

We build the adjoint string of the RAST Theorem: We begin by considering the Yoneda embeddings $\mathbf{Y_0} : \mathbf{0} \to \mathbf{Set}^{\mathbf{0}} \cong \mathbf{1}$ and $\mathbf{Y_1} : \mathbf{1} \to \mathbf{Set}^{\mathbf{1}} \cong \mathbf{Set}$. Applying R_3 to $\mathbf{Y_0}$, we have the following adjoint string:

$$\exists\mathbf{Y_0} \dashv \mathcal{K}(\mathbf{Y_0}) \dashv \forall\mathbf{Y_0} \cong \mathbf{Y}_{\mathcal{K}(\mathbf{0})} \cong \mathbf{Y_1}. \tag{118}$$

Notice that $\mathcal{K}(\mathbf{Y_0}) : \mathcal{K}(\mathbf{1}) \to \mathcal{K}(\mathbf{0})$, that is, $\mathcal{K}(\mathbf{Y_0})$ is the unique morphism $! : \mathbf{Set} \to \mathbf{1}$. This implies that the other two functors in the adjoint string (118) have the signature $\mathbf{1} \to \mathbf{Set}$.

Next, we apply R_2 to (118):

$$\mathcal{K}(\exists\mathbf{Y_0}) \dashv \mathcal{K}(!) \dashv \mathcal{K}(\mathbf{Y_1}). \tag{119}$$

Since $! : \mathbf{Set} \to \mathbf{1}$, $\mathcal{K}(!) : \mathbf{Set} \to \mathbf{Set}^{\mathbf{Set}^{\mathrm{op}}}$. It follows by adjointness that the other two functors in (119) have signature $\mathbf{Set}^{\mathbf{Set}^{\mathrm{op}}} \to \mathbf{Set}$.

Next, we apply R_3 to the functor $\exists\mathbf{Y_0} : \mathbf{1} \to \mathbf{Set}$ to obtain the following adjoint string:

$$\exists\exists\mathbf{Y_0} \dashv \mathcal{K}(\exists\mathbf{Y_0}) \dashv \forall\exists\mathbf{Y_0}. \tag{120}$$

[195] The notation $\mathcal{K}$ is in honor of *Kan* who discovered this functor and established the first mathematical results about it.

Examining (119) and (120), we see that both $\mathcal{K}(!)$ and $\forall\exists\mathbf{Y_0}$ are right adjoints to $\mathcal{K}(\exists\mathbf{Y_0})$, so, by R_1, $\mathcal{K}(!) \cong \forall\exists\mathbf{Y_0}$. Using this observation, we can combine (118), (119), and (120) to produce a length-4 adjoint string:

$$\exists\exists\mathbf{Y_0} \dashv \mathcal{K}(\exists\mathbf{Y_0}) \dashv \forall\exists\mathbf{Y_0} \cong \mathcal{K}(!) \dashv \mathcal{K}(\mathbf{Y_1}). \tag{121}$$

Next, we apply R_3 to the functor $\mathbf{Y_1} : \mathbf{1} \to \mathbf{Set}$, producing the following adjoint string:

$$\exists\mathbf{Y_1} \dashv \mathcal{K}(\mathbf{Y_1}) \dashv \forall\mathbf{Y_1} \cong \mathbf{Y}_{\mathcal{K}(\mathbf{1})}. \tag{122}$$

Examining (121) and (122), we see that $\mathcal{K}(\mathbf{Y_1})$ has two left adjoints: $\mathcal{K}(!)$ and $\exists\mathbf{Y_1}$, which, by R_1, must be isomorphic. Combining (121) and (122), and noting that $\mathcal{K}(\mathbf{1}) \cong \mathbf{Set}$, yields the final result:

$$\exists\exists\mathbf{Y}_0 \dashv \mathcal{K}(\exists\mathbf{Y}_0) \dashv \exists\mathbf{Y}_1 \dashv \mathcal{K}(\mathbf{Y}_1) \dashv \mathbf{Y}_{\mathbf{Set}}. \ \square \tag{123}$$

These results suggest that adjoint situations are fundamental to the structure of mathematics, particularly the category of sets. As adjunctions seem to naturally embody the principle that diversity of creation emerges from the integrated dynamics of opposing forces, the following "slogan" therefore follows directly from our intention to make use of ancient wisdom in establishing first principles:

Adjoints exist whenever possible.

The wording of the slogan is important. There are many examples of functors $G : \mathcal{D} \to \mathcal{C}$ that do not have a left adjoint; we do not propose to somehow override such mathematical facts using our slogan. On the other hand, in a case in which it cannot be decided (on the basis of axioms of a foundational theory) whether a functor has a left (or right) adjoint, the slogan says that it is reasonable to assume the adjoints exist.

Of course, the slogan then suggests to us that, in the context of ZFC − Infinity, the forgetful functor $\mathbf{G} : \mathbf{SM} \to V$ *should* have a left adjoint.

With this slogan, we have accomplished our aim in this subsection. We were aiming to provide justification, based on ancient principles, for category-theoretic statements that lead to the conclusion that an infinite set (or set Dedekind self-map) emerges from the critical point of a functor $j = \mathbf{G} \circ \mathbf{F} : V \to V$ (either obtained from the Lawvere constructor, or, more abstractly, defined directly as a Dedekind monad); although the conclusion accords with our persective that infinite sets should emerge from the interaction between a class Dedekind self-map and its critical point, the category-theoretic statements that lead to this result were not clearly derivable from ancient wisdom. We reviewed wisdom from the ancients that tells us that all diversity arises from the integrated dynamics of opposing forces; and we have observed that the idea of emergence from such dynamics is naturally embodied in adjunctions. We offered evidence in this subsection that, indeed, "adjoints are everywhere," and that, in a very real sense, the mathematical landscape is constructed from adjunctions. The ancient perspective then motivates our new slogan, which asserts that if an adjoint could possibly exist, it does exist. And finally, we are led to the conclusion, now based on a profound insight into the dynamics of structuring manifest life, that, even in a context in which infinite sets are not postulated, the functor $j = \mathbf{G} \circ \mathbf{F} : V \to V$—and indeed a Dedekind monad—*should* exist.

27.4. **WA$_0$-Embeddings and Universal Elements.** In the footnote on p. 171, we showed that if $i_U : V \to V^\kappa/U \cong N$ is an ultrapower embedding with critical point κ (where U is a normal measure on a measurable cardinal κ), then $\kappa \in i_U(\kappa)$ is a weakly universal element for i_U. It was remarked that something similar cannot be done for WA$_0$-embeddings $j : V \to V$; indeed, whenever $j : V \to V$ is a WA$_0$-embedding, there is no weakly universal element for j. We prove this result here. We need the following lemma:

Lemma 93. [11] *Working in the theory* ZFC + WA$_0$*, if* $j : V \to V$ *is a* WA$_0$*-embedding with critical point* κ*, then, whenever* $\lambda \geq \kappa$ *is a cardinal,* $j(\lambda) > \lambda$*.*

Theorem 94. (No Weakly Universal Element for WA$_0$-Embeddings) *Work in the theory* ZFC + WA$_0$*. Let* $j : V \to V$ *be a* WA$_0$*-embedding with critical point* κ*. Then there is no weakly universal element for* j*; that is, for all sets* a, A *with* $a \in j(A)$*,* a *is not a weakly universal element for* j*.*

Proof. We show that

$$\forall a\, \exists z\, \forall f\, (a \in \mathrm{dom}\ j(f) \Rightarrow j(f)(a) \neq z). \tag{124}$$

We argue that this is sufficient to prove the theorem: Observe first that $j : V \to V$ is cofinal: Suppose $x \in V$. Then $x \in V_\alpha$ for some $\alpha \geq \kappa$; since, by the lemma, $j(\alpha) > \alpha$, we have

$$x \in V_\alpha \subseteq V_{j(\alpha)}.$$

Now, if $a \in j(A)$ were a weakly universal element for j, it would follow (by the remarks on p. 130) that $V = \{j(f)(a) \mid a \in \mathrm{dom}\ (j(f))\}$. Thus, to prove the theorem, it is enough to establish (124). Let a be a set.

Case I: $a \notin \mathrm{ran}\, j$.

By cofinality of j, we can find an infinite set A for which $a \in j(A)$. Let $D = \{X \subseteq A \mid a \in j(X)\}$. By Theorem 45, D is a nonprincipal ultrafilter on A. Let $\lambda \geq \kappa$ be a beth fixed point[196] for which $A \in V_\lambda$. Pick some $y \in A$.

Suppose $g : B \to V$ is such that $a \in \mathrm{dom}\ j(g)$. We show that, for some $f : A \to V$, $j(f)(a) = j(g)(a)$. Since j preserves intersections, we have $a \in j(A) \cap j(B) = j(A \cap B)$, and so $A \cap B \in D$. Define $f : A \to V$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in A \cap B \\ y & \text{otherwise} \end{cases}$$

[196]The *beth numbers* form a class sequence of cardinal numbers $\beth_0, \beth_1, \ldots, \beth_\alpha, \ldots$ ($\alpha \in \mathrm{ON}$) defined by

$$\beth_0 = \omega; \quad \beth_{\alpha+1} = 2^{\beth_\alpha}; \quad \beth_\beta = \bigcup_{\alpha<\beta} \beth_\alpha \ (\beta \text{ a limt}).$$

A *beth fixed point* is a beth number $\beth_\alpha$ for which $\alpha = \beth_\alpha$. The beth fixed points $\beth_\alpha$ are cofinal in ON and have the following properties:

(i) $|V_\alpha| = \beth_\alpha$.
(ii) (*Strong Limit*) Whenever $\mu, \nu < \beth_\alpha$, $|{}^\nu\mu| = |\{f \mid f : \nu \to \mu\}| < \beth_\alpha$.

Since $A \cap B \subseteq \{x \in A \mid f(x) = g(x)\}$, $\{x \in A \mid f(x) = g(x)\} \in D$. It follows that $j(f)(a) = j(g)(a)$.

Let $R = \{j(f)(a) \mid f : A \to V\}$ and let $T = \{j(h)(a) \mid h : A \to V_\lambda\}$. We show that whenever $f : A \to V$ is such that $j(f)(a) \in V_{j(\lambda)}$, there is $h : A \to V_\lambda$ such that $j(f)(a) = j(h)(a)$. Since $j(f)(a) \in V_{j(\lambda)}$, we have $S \in D$, where $S = \{x \in A \mid f(x) \in V_\lambda\}$ (note that, by elementarity of j, $j(V_\lambda) = V_{j(\lambda)}$). Define $h : A \to V_\lambda$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in S \\ y & \text{otherwise} \end{cases}$$

Since $S \subseteq \{x \in A \mid f(x) = h(x)\}$, it follows that $\{x \in A \mid f(x) = h(x)\} \in D$, whence $j(f)(a) = h(f)(a)$.

We have shown that whenever $j(f)(a) \in R$ and $j(f)(a) \in V_{j(\lambda)}$, then $j(f)(a) \in T$; that is, $R \cap V_{j(\lambda)} \subseteq T$. We now show that there must exist $z \in V_{j(\lambda)}$ such that, for all $g : B \to V$ for which $z \in \text{dom } j(g)$, $j(g)(a) \neq z$. Notice by elementarity that $j(\lambda)$ is a beth fixed point, and that, by Lemma 94, $j(\lambda) > \lambda$. Then

$$|T| \leq |V_\lambda|^{|A|} = \lambda^{|A|} < j(\lambda) = |V_{j(\lambda)}|.$$

It follows that for some $z \in V_{j(\lambda)}$, there is no $f : A \to V_\lambda$ such that $j(f)(a) = z$, hence no $g : B \to V$ with $j(g)(a) = z$.

Case II: $a \in \operatorname{ran} j$.

For this case, we show that if $a \in \operatorname{ran} j$ and $f : A \to V$ with $a \in \text{dom } j(f)$, then $j(f)(a) \in \operatorname{ran} j$; it will then follow that there must exist z (for instance, $z = \kappa$) for which $j(f)(a) \neq z$ for any choice of f.

Given a, f as above, let b be such that $a = j(b)$. Then

$$j(f)(a) = j(f)(j(b)) = j(f(b)) \in \operatorname{ran} j,$$

as required. □

27.5. **Large Cardinals and Gödel's Theorems.** In the footnote on p. 159, it was pointed out that the question that asks whether large cardinals exist is quite different from the question that asks if the Continuum Hypothesis (CH) is true: While it is possible to prove that if ZFC is consistent, so is ZFC + CH, it is not possible to prove consistency of large cardinals with ZFC. We prove this fact here.

Theorem 95. (Large Cardinals Not Provably Consistent) *Large cardinals cannot be proven to be consistent with* ZFC. *More precisely,*[197] *let I be the statement "an inaccessible cardinal exists." Then, unless* ZFC *is inconsistent,*

$$\text{ZFC} \not\vdash \text{Con(ZFC)} \to \text{Con}(\text{ZFC} + I).$$

[197] The expression "Con(ZFC)" is the formal sentence in the language of set theory that asserts "ZFC is consistent." It is shorthand notation for the sentence $\neg\text{Prov}_{\text{ZFC}}(0 = 1)$—the encoded assertion that there does not exist a proof from ZFC of the statement "$0 = 1$."

Proof. Assume

(125) $$\text{ZFC} \vdash \text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + I).$$

Then certainly

(126) $$\text{ZFC} + I \vdash \text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + I).$$

Since an inaccessible produces a model of ZFC, we have:

(127) $$\text{ZFC} + I \vdash \text{Con}(\text{ZFC}).$$

Combining (126) and (127) (and using the fact that if ZFC $\vdash \sigma$ and ZFC $\vdash \sigma \rightarrow \psi$, then ZFC $\vdash \psi$), we have

(128) $$\text{ZFC} + I \vdash \text{Con}(\text{ZFC} + I).$$

By Gödel's Second Incompleteness Theorem, (128) is possible only if ZFC $+ I$ is itself inconsistent. By (125), this would give us a proof from ZFC of the inconsistency of ZFC itself and therefore, in truth, ZFC is inconsistent. □

The statement I in the theorem can be modified so that it asserts the existence of any of the large cardinals that are known today—at least any that are not known to be inconsistent. The only property of a large cardinal that is needed for the proof is that from ZFC it can be shown that its existence implies Con(ZFC). Moreover, the same proof can be used to prove the following, assuming ZFC − Infinity is consistent:

$$\text{ZFC} - \text{Infinity} \nvdash \text{Con}(\text{ZFC} - \text{Infinity}) \rightarrow \text{Con}(\text{ZFC}).$$

27.6. **Extensions of ZFC + BTEE Obtained by Restricting j.** In the main text (p. 179), it was mentioned that consistency of ZFC + BTEE + MUA can be proven from the theory ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction V_{\mathbf{j}(\kappa)})$. We prove this here. We also prove, from the same theory, that the embedding $j : V \rightarrow V$ from that theory gives rise to a blueprint for $V_{\kappa+2}$ (see the footnote on p. 180).

Theorem 96. (Strong Forms of Restriction) *The theory* ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction V_{\mathbf{j}(\kappa)})$ *proves "there is a transitive model of* ZFC + BTEE + MUA*."*

Proof. Let $j : V \rightarrow V$ be the embedding given in the hypothesis, having critical point κ and having the property that $\mathbf{j} \restriction V_{\mathbf{j}(\kappa)}$ is a set. Let $g = j \restriction V_{j(\kappa)}$. Since $\mathcal{P}(\mathcal{P}(\mathcal{P}(\kappa))) \in V_{\mathbf{j}(\kappa)}$, one can define a 2^κ-supercompactness measure U_g by

$$U_g = \{X \subseteq P_\kappa 2^\kappa \mid g[2^\kappa] \in g(X)\},$$

which ensures that κ is 2^κ-supercompact. But this degree of supercompactness has been shown [11, Proposition 9.10] to be sufficient to build a transitive model of the theory ZFC + BTEE + MUA. □

Theorem 97. (A Blueprint for $V_{\kappa+2}$) *Work in the theory* ZFC + BTEE+ $\exists z\,\big(z = \mathbf{j} \restriction \mathbf{j}(\kappa)\big)$. *Let* $\lambda = 2^\kappa$ *and let* $h = j \restriction \lambda$. *Let* $\mathcal{E}^+$ *consist of all* λ*-supercompact embeddings (restricted to* $P_\kappa\lambda^+$*), defined from a normal measure* U *over* $P_\kappa\lambda$ *(where* $P_\kappa\lambda = \{X \subseteq \mathcal{P}(\lambda) \mid |X| < \kappa\}$*) as the canonical ultrapower embedding* $i = i_U : V \rightarrow$

$V^{P_\kappa\lambda} \cong M_U$. *Then there is a co-Dedekind self-map* $\ell^+ : V_\kappa \to V_\kappa$ *so that* $(\ell^+, \kappa, \mathcal{E}^+)$ *is a blueprint for* $V_{\kappa+2}$.

Proof. Our extra axiom, $\exists z \big(z = \mathbf{j} \restriction \mathbf{j}(\kappa)\big)$, together with Lemma 78, implies that h exists as a set (since $\lambda < j(\kappa)$).

We describe our plan for building a blueprint $(\ell^+, \kappa, \mathcal{E}^+)$ for $V_{\kappa+2}$: We will define a $V_{\kappa+2}$-Laver function $f^+ : \kappa \to V_\kappa$, from which ℓ^+ can be defined, as was done earlier. We let $\mathcal{E}^+$ consist of all λ-supercompact embeddings (restricted to V_ν for some regular cardinal $\nu > P_\kappa\lambda$), defined from a normal measure U over $P_\kappa\lambda$ (where $P_\kappa\lambda = \{X \subseteq \mathcal{P}(\lambda) \mid |X| < \kappa\}$) as the canonical ultrapower embedding $i = i_U : V \to V^{P_\kappa\lambda} \cong M_U$. As usual, we let $\mathcal{E}_0^+ \sqsubseteq_r \mathcal{E}^+$ be the subset consisting of all restrictions $i \restriction V_\kappa^{V_\kappa}$, for $i \in \mathcal{E}^+$. The Laver property that we will show f^+ has is that for every $X \in V_{\kappa+2}$, there is a λ-supercompact ultrafilter U such that $i_U(f^+)(\kappa) = X$.

We follow the steps of our earlier construction of a $V_{\kappa+1}$-Laver sequence under MUA. As before, we define a formula ψ. Let $\psi(u, x, \gamma)$ be the following formula:

$$u : \gamma \to V_\gamma \;\wedge\; x \subseteq V_{\gamma+1} \;\wedge\; \text{“for all normal measures } U \text{ on } P_\gamma 2^\gamma,\ i_U(u)(\gamma) \neq x\text{”}.$$

When $\psi(u, x, \lambda)$ holds true, it means that u is *not* a $V_{\gamma+2}$-Laver sequence at γ: Some subset x of $V_{\gamma+1}$ cannot be computed as $i_U(u)(\gamma)$ for any choice of U. We can now define $f^+ : \kappa \to V_\kappa$:

$$f^+(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } f^+ \restriction \alpha \text{ is } V_{\alpha+2}\text{-Laver at } \alpha, \\ x & \text{where } x \text{ satisfies } \psi(f^+ \restriction \alpha, x, \alpha). \end{cases} \tag{129}$$

The definition tells us that $f^+(\alpha)$ has nonempty value just when the restriction $f^+ \restriction \alpha$ is *not* $V_{\alpha+2}$-Laver at α, and in that case, its value is a witness to non-Laverness.

Claim. The function f^+ defined in (129) is a $V_{\kappa+2}$-Laver function at κ.

Proof. Let $j : V \to V$ be the embedding, with critical point κ, given to us in a model of our theory $\mathrm{ZFC} + \mathrm{BTEE} + \exists z\,(z = \mathbf{j} \restriction \mathbf{j}(\kappa))$.

Suppose f^+ is not $V_{\kappa+2}$-Laver at κ, so, in particular, for some y, $\psi(f^+, y, \kappa)$ holds. We consider $j(f^+) : j(\kappa) \to V_{j(\kappa)}$. Notice, as in the earlier proof, that $j(f^+) \restriction \kappa = f^+$. Also, by elementarity, $j(f^+)$ has the same definition as f^+:

$$j(f^+)(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } j(f^+) \restriction \alpha \text{ is } V_{\alpha+2}\text{-Laver at } \alpha, \\ x & \text{where } x \text{ satisfies } \psi(j(f^+) \restriction \alpha, x, \alpha). \end{cases}$$

In particular, since $f^+ = j(f^+) \restriction \kappa$ is not $V_{\kappa+2}$-Laver, $j(f^+)(\kappa)$ is itself a witness to non-Laverness of f^+, and $\psi\big(j(f^+) \restriction \kappa, j(f^+)(\kappa), \kappa\big)$ is true.

Let $D = U_j$ be the normal measure derived from j; that is, $D = \{X \subseteq P_\kappa\lambda \mid (j \restriction \lambda)[\lambda] \in j(X)\}$. We claim that, by our extra axiom, D is a set: Arguing in the usual way [35], $j(\kappa) > \lambda$. By inaccessibility of $j(\kappa)$, we have

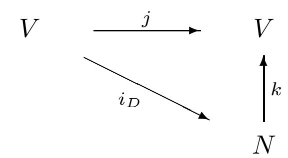
$$|\mathcal{P}(P_\kappa\lambda)| = 2^{\lambda^{<\kappa}} < j(\kappa).$$

Therefore, by Lemma 78, there are set functions r_1, r_2 with $r_1 = j \restriction \mathcal{P}(P_\kappa\lambda)$ and $r_2 = j \restriction \lambda$. It follows that

$$D = \{X \subseteq P_\kappa\lambda \mid r_2[\lambda] \in r_1(X)\},$$

and so D is a set.

Let $i = i_D : V \to V^{P_\kappa\lambda}/D \cong N$ be the canonical λ-supercompact embedding induced by D and define $k : N \to V$ by $k([t]) = j(t)(((j \upharpoonright \lambda)[\lambda])$. One can show that k is an elementary embedding with critical point $> 2^\kappa$ and makes the following diagram commutative:



By the diagram, we compute:

$$\begin{aligned} j(f^+)(\kappa) &= (k \circ i_D)(f^+)(\kappa) \\ &= \big(k(i_D(f^+))\big)(k(\kappa)) \\ &= k\big(i_D(f^+)(\kappa)\big). \end{aligned}$$

Since $i_D : V \to N$ is λ-supercompact, ${}^\lambda N \subseteq N$. It follows that N contains all sets that are hereditarily of size $\le 2^\kappa$—in particular, $V_{\kappa+1}$ and each of its subsets belongs to N. Therefore $V_{\kappa+2} = \mathcal{P}(V_{\kappa+1}) \subseteq N$.

Since $\operatorname{crit}(k) > 2^\kappa$, $k \upharpoonright \mathcal{P}(V_{\kappa+1}) = \operatorname{id}_{V_{\kappa+1}}$. Then since $j(f^+)(\kappa)$ is a subset of $V_{\kappa+1}$, it follows that $k\big(j(f^+)(\kappa)\big) = j(f^+)(\kappa)$. Since k is 1-1 and

$$k\big(j(f^+)(\kappa)\big) = j(f^+)(\kappa) = k\big(\big(i_D(f^+)(\kappa)\big),$$

it follows that

$$j(f^+)(\kappa) = i_D(f^+)(\kappa).$$

Therefore, as in earlier arguments of this kind, though we have claimed that $\psi(j(f^+) \upharpoonright \kappa, j(f^+)(\kappa), \kappa)$ holds true, we have just exhibited a normal measure D on $P_\kappa\lambda$ such that $i_D(f^+)(\kappa) = j(f^+)(\kappa)$. We have a contradiction. Therefore f^+ is $V_{\kappa+2}$-Laver after all. $\square$

Remark 24. Examining the proof shows that the blueprint $(\ell^+, \kappa, \mathcal{E}^+)$ derived from f^+ is actually a blueprint for $H\Big((2^\kappa)^+\Big) \supsetneq V_{\kappa+2}$ (for any infinite cardinal γ, $H(\gamma)$ is the set of sets having hereditary cardinality $< \gamma$). Moreover, the proof goes through in the weaker theory $\mathrm{ZFC} + \mathrm{BTEE} + \exists z\,(z = \mathbf{j} \upharpoonright \mathcal{P}(\mathcal{P}(\mathcal{P}(\kappa))))$. $\square$

27.7. **The Algebra $\mathcal{A}_j$ of Elementary Embeddings $V \to V$.** In the main text, working in the theory $\mathrm{ZFC} + \mathrm{WA}$ with WA-embedding $j : V \to V$ having critical point κ, we introduced a collection $\mathcal{A}_j$ of elementary embeddings $V \to V$, defined from j, as follows:

$$\mathcal{A}_j = \{j,\ j \cdot j,\ j \cdot (j \cdot j),\ (j \cdot j) \cdot j \ldots\}.$$

Elements of $\mathcal{A}_j$ are obtained from j by repeated *application* of j to itself. We recall the definition of application on WA_0-embeddings $V \to V$: Given WA_0-embeddings $i, k : V \to V$,

$$i \cdot k = \bigcup_{\alpha \in \mathrm{ON}} i(k \restriction V_\alpha).$$

Recall also our convention that we shall often write ik for $i \cdot k$. It should be noted here that, although WA_0-embeddings are strong enough for us to properly define the application operation, they are not strong enough to form the collection $\mathcal{A}_j$; for such j, even the term $j \cdot (j \cdot j)$ is not well-defined. Therefore, we shall henceforth restrict our attention to WA-embeddings.

A precise formulation of the elements of $\mathcal{A}_j$ can be given as follows: The elements of $\mathcal{A}_j$ are those obtained by finitely many applications of the following rules:

(i) j is an element

(ii) if k, ℓ are elements, $(k \cdot \ell)$ is an element.

In the main text, we defined a certain kind of exponentiation based on application, providing us, for any WA-embedding $k : V \to V$, with a sequence of applicative iterates $k^{[1]}, k^{[2]}, \ldots, k^{[n]}, \ldots$, defined inductively by

$$\begin{aligned} k^{[1]} &= k, \\ k^{[n+1]} &= k \cdot k^{[n]}. \end{aligned}$$

We defined $\mathbf{S}'_j = \{j^{[n]} \mid n \geq 1\}$. Like $\mathbf{S}$ and $\mathbf{S}_j$, $\mathbf{S}'_j$ was seen to be another way to represent ω.

We mentioned in the main text that $\mathcal{A}_j$ plays a role, relative to $\mathbf{S}'_j$, that is analogous to the role of the rationals relative to ω, and we provided evidence for this claim by defining a metric d on $\mathcal{A}_j$ and pointing out that, in the resulting topology, $\mathcal{A}_j$ is dense in itself.

In this section of the Appendix, we provide a deeper study of $\mathcal{A}_j$ as a metric space and examine in some detail the properties of the completion $\overline{\mathcal{A}}_j$ of $\mathcal{A}_j$ and its suitability as the support for a separable Hilbert space, which could be used, in principle, as a tool in quantum mechanics. To understand $\mathcal{A}_j$ requires a fuller understanding of the set $\mathrm{crit}(\mathcal{A}_j)$ of critical points of the elements of $\mathcal{A}_j$. We develop the ideas far enough to be able to state two key theorems (due to Laver) in this area: that $\mathrm{crit}(\mathcal{A}_j)$ has order-type ω, and that embeddings in $\mathcal{A}_j$ are completely determined by their behavior on $\mathrm{crit}(\mathcal{A}_j)$. These results will allow us to define the metric d on $\mathcal{A}_j$ described in the text and prove a number of its properties. We then give a careful treatment of our approach to completing $\mathcal{A}_j$ and show that the resulting space is a perfect separable Polish space, suitable for extension as a separable Hilbert space.

We begin with a computation of the critical point of $j \cdot j$, where $j : V \to V$ is a WA-embedding with critical point κ. This computation will lead to several generalizations and to a deeper understanding of $\mathrm{crit}(\mathcal{A}_j)$. We give a heuristic proof that the critical point of $j \cdot j$ turns out to be $j(\kappa)$: Apply j to the true sentence "the critical point of j is κ" to obtain "the critical point of $j \cdot j$ is $j(\kappa)$." (A more careful proof is given below, in Proposition 102).[198]

[198] Here, when applying j to "the critical point of j is κ," we resisted the temptation of writing "the critical point of $j(j)$ is $j(\kappa)$," since, as discussed earlier, '$j(j)$' makes no sense; in place of

$j(j)$, we wrote $j \cdot j$ since $j \cdot j$ can be understood as playing the same role as $j(j)$ in elementary embedding arguments.

In this footnote, we clarify the connection between $j \cdot j$ and the intuitively appealing notation $j(j)$. The technique we discuss is valid in the context of elementary embeddings $i : V_\lambda \to V_\lambda$, with critical point $< \lambda$, and for which λ is a limit ordinal. Such embeddings are reflections of elementary embeddings $V \to V$ in the realm of sets, and have essentially the same characteristics as the embeddings $V \to V$. However, embeddings $V_\lambda \to V_\lambda$ provide a special convenience because of the fact that we can explore layers in the universe above V_λ—in particular, the sets belonging to $V_{\lambda+1}$. The technique we describe therefore may rigorously be applied only to embeddings $V_\lambda \to V_\lambda$, but serves as a useful heuristic for our study of elementary embeddings $V \to V$.

We begin with the observation that every elementary embedding $i : V_\lambda \to V_\lambda$ can be lifted to a "partially" elementary embedding $\hat{\imath} : V_{\lambda+1} \to V_{\lambda+1}$. As a function, $\hat{\imath}$ is defined on the new sets in $V_{\lambda+1}$ (namely, the subsets of V_λ) by

$$\hat{\imath}(X) = \bigcup_{\alpha<\lambda} i(X \cap V_\alpha).$$

Now $\hat{\imath}$ is not exactly an elementary embedding on $V_{\lambda+1}$, but it *is* elementary relative to all *bounded formulas.*

We take a moment to discuss bounded formulas. Let α be a nonzero ordinal and suppose ϕ is a formula of set theory. Then ϕ is *bounded relative to* V_α if every quantifier in ϕ is either of the form $\forall x \in X$ or $\exists x \in X$ for some set X belonging to V_α.

The following sentence σ is an example of a bounded formula:

$$\sigma : \ \forall x \in V_\lambda \, \exists y \in V_\lambda \, (x \in y). \tag{130}$$

Written more formally, σ may be expressed in the following way:

$$\sigma : \ \forall x \, \exists y \, (x \in V_\lambda \to (y \in V_\lambda \wedge x \in y)). \tag{131}$$

In σ (either version), V_λ plays the role of X in our general definition. Notice that all quantified variables ($\forall x$, $\exists y$) are bound to V_λ. The formula σ says that for every element of V_λ, there is another element of V_λ which contains the first as an element. This is a true statement. Moreover, it is clear that σ is bounded relative to $V_{\lambda+1}$ because every quantified variable in σ is bound to a set (in this case, V_λ) that is a member of $V_{\lambda+1}$.

It turns out that, because all variables in σ are bound to V_λ, the formula is *true in* $V_{\lambda+1}$. We indicate why σ holds not only in V, but also in the model $V_{\lambda+1}$.

In general, if ψ is a sentence and V_α is some stage in the universe, we say ψ *holds in* V_α if the *relativization* of ψ to V_α, denoted ψ^{V_α}, is true. The formula ψ^{V_α} is obtained from ψ by replacing each universal quantifier $\forall x$ that occurs in ψ by $\forall x \in V_\alpha$, and each existential quantifier $\exists y$ by $\exists y \in V_\alpha$. Using this procedure, and using the version (131) of σ, the relativization $\sigma^{V_{\lambda+1}}$ of σ to $V_{\lambda+1}$ is the following:

$$\sigma^{V_{\lambda+1}} : \ \forall x \in V_{\lambda+1} \, \exists y \in V_{\lambda+1} \, (x \in V_\lambda \to (y \in V_\lambda \wedge x \in y)). \tag{132}$$

Since $V_\lambda \subseteq V_{\lambda+1}$ (and since expressions like $z \in V_\lambda$ mean the same thing in $V_{\lambda+1}$ as in V), $\sigma^{V_{\lambda+1}}$ holds true if and only if σ is true.

In this example, σ and its relativization to $V_{\lambda+1}$ are equivalent. By contrast, consider the following formula θ, which is the same as σ but without bounds for the quantifiers.

$$\theta : \forall x \, \exists y \, (x \in y). \tag{133}$$

The sentence θ says that any set is contained in another set. While θ is certainly true in V, it is not true in $V_{\lambda+1}$. For instance, if $x = V_\lambda$, it is not true that there is another set y in $V_{\lambda+1}$ for which $x \in y$.

Remark 25. We observe that $\sigma = \theta^{V_\lambda}$, where σ, θ are defined as above. The reasoning that showed that θ^{V_λ} holds in $V_{\lambda+1}$ also proves the more general fact that, for any sentence τ and any nonzero ordinals α, β, τ^{V_α} holds in V_β if and only if $\tau^{V_\alpha \cap V_\beta}$ holds.

These considerations lead to the following.

Fact 1. Suppose $i : V_\lambda \to V_\lambda$ is elementary. Let $\hat{\imath}$ be the extension of i to $V_{\lambda+1}$ described above. Then for any formula $\phi(x_1, x_2, \ldots, x_n)$ and any $a_1, a_2, \ldots, a_n \in V_{\lambda+1}$,

(a) $\phi^{V_\lambda}[a_1, a_2, \ldots, a_n]$ is true if and only if it is true in $V_{\lambda+1}$, and
(b) $\phi^{V_\lambda}[a_1, a_2, \ldots, a_n]$ iff $\phi^{V_\lambda}[\hat{\imath}(a_1), \hat{\imath}(a_2), \ldots, \hat{\imath}(a_n)]$.

Fact 1(a) follows from Remark 25. Fact 1(b) is a precise way of saying that $\hat{\imath}$ is elementary relative to bounded formulas (technically speaking, one says that $\hat{\imath} : V_{\lambda+1} \to V_{\lambda+1}$ is a Σ_0-elementary embedding). This latter point follows because, as our earlier example suggests, for any bounded formula ψ, ψ holds in $V_{\lambda+1}$ if and only if ψ^{V_λ} holds in $V_{\lambda+1}$. This follows because, when we are working in $V_{\lambda+1}$, bounded quantifiers $\forall x \in X$ can be replaced by $\forall x \in V_\lambda\,(x \in X \to \cdots)$ and $\exists x \in X$ can be replaced by $\exists x \in V_\lambda\,(x \in X \wedge \cdots)$. We therefore have the following chain of equivalences for any bounded formula $\psi(x_1, \ldots, x_n)$ and sets $a_1, \ldots, a_n \in V_{\lambda+1}$; this shows that (b) is equivalent to the statement that $\hat{\imath} : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_0-elementary.

$$\begin{aligned}
V_{\lambda+1} \models \psi[a_1, \ldots, a_n] \quad &\Longleftrightarrow \quad V_{\lambda+1} \models \psi^{V_\lambda}[a_1, \ldots, a_n] \\
&\Longleftrightarrow \quad \psi^{V_\lambda}[a_1, \ldots, a_n] \\
&\Longleftrightarrow \quad \psi^{V_\lambda}[\hat{\imath}(a_1), \ldots, \imath(a_n)] \\
&\Longleftrightarrow \quad V_{\lambda+1} \models \psi^{V_\lambda}[\hat{\imath}(a_1), \ldots, \imath(a_n)] \\
&\Longleftrightarrow \quad V_{\lambda+1} \models \psi[\hat{\imath}(a_1), \ldots, \imath(a_n)].
\end{aligned}$$

The following lemma establishes several properties of $\hat{\imath}$ which lie at the heart of a proof of Fact 1(b). However, our formal proof of Fact 1(b) will rely on Theorem 84(A), discussed earlier in the text.

Lemma 98. *Suppose $i : V_\lambda \to V_\lambda$ is a nontrivial elementary embedding with critical point κ. Define $\hat{\imath} : V_{\lambda+1} \to V_{\lambda+1}$ as above.*

(1) *$\hat{\imath}$ preserves $\in$.*
(2) *For all $x \in V_{\lambda+1}$, $\mathrm{rank}(x) \le \mathrm{rank}(\hat{\imath}(x))$.*
(3) *$\hat{\imath}$ reflects $\in$; that is, whenever $\hat{\imath}(x) \in \hat{\imath}(y)$, it follows that $x \in y$.*
(4) *$\hat{\imath}$ is a Dedekind self-map.*

Proof of (1). Suppose $x, y \in V_{\lambda+1}$ and $x \in y$. There are, potentially, three cases:

(a) $x, y \in V_\lambda$.
(b) $x, y \in V_{\lambda+1} - V_\lambda$.
(c) $x \in V_\lambda, y \in V_{\lambda+1} - V_\lambda$.

In case (a), the result follows because of elementarity of i. Case (b) is impossible since, by rank considerations, no element of $V_{\lambda+1} - V_\lambda$ is a member of any element in $V_{\lambda+1}$. We prove the result for the remaining case (c). Since $x \in y$ and $y \subseteq V_\lambda$, there is $\alpha < \lambda$ such that $x \in y \cap V_\alpha$. By elementarity of i, $i(x) \in i(y \cap V_\alpha) \subseteq \hat{\imath}(y)$.

Proof of (2). Since i is elementary, it suffices to prove the result for $x \in V_{\lambda+1} - V_\lambda$. We show that, for every $\alpha < \lambda$, there is $z_\alpha \in \hat{\imath}(x)$ of rank $\ge \alpha$. Let $y \in x$ have rank $\ge \alpha$. By elementarity of i, $\mathrm{rank}(i(y)) \ge i(\alpha) \ge \alpha$. By (1), $i(y) = \hat{\imath}(y) \in \hat{\imath}(x)$. We can therefore complete the proof of (2) by letting $z_\alpha = i(y)$.

Proof of (3). Suppose $x, y \in V_{\lambda+1}$ and $x \notin y$. There are two cases to consider:

(a) $x \in V_\lambda$.
(b) $x \in V_{\lambda+1} - V_\lambda$.

For (a), assume, for a contradiction, that $\hat{\imath}(x) \in \hat{\imath}(y)$. Then for some $\alpha < \lambda$, $i(x) \in i(y \cap V_\alpha)$. By elementarity, $x \in y \cap V_\alpha$, which contradicts our assumption that $x \notin y$.

For (b), the hypothesis implies $\mathrm{rank}(x) = \lambda$. By (2), $\mathrm{rank}(\hat{\imath}(x)) = \lambda$. But sets of rank λ do not belong to any set in $V_{\lambda+1}$; in particular, $\hat{\imath}(x) \notin \hat{\imath}(y)$.

Proof of (4). First observe that κ, as the least ordinal moved by $\hat{\imath}$, does not belong to the range of $\hat{\imath}$. Therefore, it remains to show that $\hat{\imath}$ is 1-1. Suppose $x \neq y$ in $V_{\lambda+1}$, and, without loss of generality, let $z \in x - y$. Since $z \in x$, we conclude by (1) that $\hat{\imath}(z) \in \hat{\imath}(x)$. Since $z \notin y$, we conclude by (3) that $\hat{\imath}(z) \notin \hat{\imath}(y)$. Therefore, $\hat{\imath}(z) \in \hat{\imath}(x) - \hat{\imath}(y)$. In particular, $\hat{\imath}(x) \neq \hat{\imath}(y)$, as required. □

We turn to a formal proof of Fact 1(b). One approach that could be taken is to proceed by induction on the length of formulas $\phi(x_1, \ldots, x_n)$. Atomic formulas would then be handled by Lemma 99, parts (1) and (3). However, we shall take a somewhat slicker route by invoking Theorem 84(A).

Given a nontrivial elementary embedding $i : V_\lambda \to V_\lambda$ and any subset $A \subseteq V_\lambda$, let us define application $i \cdot A$ in the following way:

$$i \cdot A = \bigcup_{\alpha \in \lambda} i(A \cap V_\alpha).$$

Lemma 99. *Suppose $i : V_\lambda \to V_\lambda$ is a nontrivial elementary embedding.*

(A) *Suppose $A \subseteq V_\lambda$. Then $i : (V_\lambda, \in, A) \to (V_\lambda, \in, i \cdot A)$ is a nontrivial elementary embedding.*

(B) *Suppose $A_1, \ldots, A_n$ are subsets of V_λ. Then $i : (V_\lambda, \in, A_1, \ldots, A_n) \to (V_\lambda, \in, i \cdot A_1, \ldots, i \cdot A_n)$ is an elementary embedding.*

Proof of (A). By Theorem 84(A), $i : (V_\lambda, \in, \chi_A) \to (V_\lambda, \in, i \cdot \chi_A)$ is an elementary embedding, where $\chi_A : V_\lambda \to \{0, 1\}$ is the characteristic function of A. The conclusion of the lemma follows easily.

Proof of (B). Let $A = A_1 \times \cdots \times A_n \subseteq V_\lambda$ and let $\pi : A \to A_i$ be the usual projection maps, which are definable from A. By (A), $i : (V_\lambda, \in, A) \to (V_\lambda, \in, i \cdot A)$ is a nontrivial elementary embedding. Then, given a formula $\phi(x_1, \ldots, x_m)$ in the language $\{\in, \mathbf{R}_1, \ldots, \mathbf{R}_n\}$, obtain a formula $\overline{\phi}(x_1, \ldots, x_m)$ in the language $\{\in, \mathbf{R}\}$ by replacing occurrences of $\mathbf{R}_i$ by $\pi_i[\mathbf{R}]$, where the π_i are the definable projection maps. Then, given $a_1, \ldots, a_m$, it is easy to see that

$$\begin{aligned}
(V_\lambda, \in, A_1, \ldots, A_n) \models \phi[a_1, \ldots, a_m] &\iff (V_\lambda, \in, \pi_1[A], \ldots, \pi_n[A]) \models \overline{\phi}[a_1, \ldots, a_m] \\
&\iff (V_\lambda, \in, A) \models \overline{\phi}[a_1, \ldots, a_m] \\
&\iff (V_\lambda, \in, A) \models \overline{\phi}[i(a_1), \ldots, i(a_m)] \\
&\iff (V_\lambda, \in, A_1, \ldots, A_n) \models \phi[i(a_1), \ldots, i(a_m)]. \square
\end{aligned}$$

Proof of Fact 1(b). Suppose

(134) $\phi(x_1, \ldots, x_n)$ is a formula in which all quantified variables are bound to V_λ.

Let $A_1, \ldots, A_n$ be elements of $V_{\lambda+1}$. Let σ be the $\{\in, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$-sentence obtained from ϕ by thinking of each A_i as a unary relation symbol. (Without loss of generality, we may assume each $A_i \in V_{\lambda+1} - V_\lambda$; therefore, no subformula will be of the form $A_i \in A_j$. Replace formulas $x \in A_i$ by $A_i(x)$. Replace formulas $A_i = A_j$ with $\forall x \in V_\lambda\, (x \in A_i \leftrightarrow x \in A_j)$.) Clearly, using Fact 1(a) and part (B) of the lemma, we have:

$$\begin{aligned}
V_{\lambda+1} \models \phi[A_1, \ldots, A_n] &\iff V_{\lambda+1} \models \phi^{V_\lambda}[A_1, \ldots, A_n] && \text{(by (134))} \\
&\iff V \models \phi^{V_\lambda}[A_1, \ldots, A_n] && \text{(Fact 1(a))} \\
&\iff (V_\lambda, \in A_1, \ldots, A_n) \models \sigma && \text{(construction of } \sigma) \\
&\iff (V_\lambda, \in \hat{\imath}(A_1), \ldots, \hat{\imath}(A_n)) \models \sigma && \text{(Lemma 100(B))} \\
&\iff V \models \phi^{V_\lambda}[\hat{\imath}(A_1), \ldots, \hat{\imath}(A_n)] && \text{(construction of } \sigma) \\
&\iff V_{\lambda+1} \models \phi^{V_\lambda}[\hat{\imath}(A_1), \ldots, \hat{\imath}(A_n)] && \text{(Fact 1a)} \\
&\iff V_{\lambda+1} \models \phi[\hat{\imath}(A_1), \ldots, \hat{\imath}(A_n)] && \text{(by (134))}
\end{aligned}$$

All this effort justifies the abuse of notation $i(k)$ in place of $i \cdot k$ whenever i, k are WA_0-embeddings since, by definition, $i \cdot k = \hat{\imath}(k)$ and $\hat{\imath} : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_0 elementary.

We illustrate with an example.

Proposition 100. *For any WA_0-embeddings $i, k : V_\lambda \to V_\lambda$,*

$$\mathrm{crit}(i \cdot k) = i(\mathrm{crit}(k)).$$

Remark 26. This proposition was given a quick proof already on p. 191.

The next proposition collects together a few well-known properties of this application operation, applied to elements of $\mathcal{A}_j$, where j is a WA-embedding with critical point κ. For any $n \geq 1$, let $\kappa_n = j^n(\kappa)$, and let $\kappa_0 = \kappa$.

Proposition 101.

(1) $\mathrm{crit}(j \cdot j) = j(\kappa)$.
(2) *For all* $n \geq 1$, $j \cdot j(\kappa_n) = \kappa_{n+1}$.
(3) $(j \cdot j) \circ j = j \circ j$.

Proof of (1). Let $\alpha > \kappa$. Let $i = j \restriction V_\alpha$. Since $\mathrm{crit}(i) = \kappa$, the following is true in V:

$$\forall \beta < \kappa \, (i(\beta) = \beta) \text{ and } i(\kappa) > \kappa.$$

By elementarity of j, the following also holds in V:

$$\forall \beta < j(\kappa) \, (j(i)(\beta) = \beta), \tag{135}$$

and

$$j(i)(j(\kappa)) > j(\kappa). \tag{136}$$

Since (135) and (136) hold for all large enough α, we conclude that both hold true if $j(i)$ is replaced by $j \cdot j$. Therefore: $\mathrm{crit}(j \cdot j) = j(\kappa)$. $\square$

Proof of (2). Let $\alpha > \kappa_{n-1}$. Let $i = j \restriction V_\alpha$. Applying j to the formula

$$i(\kappa_{n-1}) = \kappa_n$$

and noting that $j(\kappa_{n-1}) = \kappa_n$ yields

$$j(i)(\kappa_n) = j(i)(j(\kappa_{n-1})) = j(\kappa_n). \tag{137}$$

Since $\kappa_{n-1} \in \mathrm{dom}\ i$, $\kappa_n \in \mathrm{dom}\ j(i)$, and so $j(i)$ and $j \cdot j$ agree on κ_n. We have from (137):

$$j \cdot j(\kappa_n) = j(i)(\kappa_n) = j(\kappa_n) = \kappa_{n+1}. \square$$

Proof of (3). Let x be a set and α an ordinal such that $x \in V_\alpha$. Let $i = j \restriction V_\alpha$ and let $y = j(x)$. Then, applying j to the formula

$$i(x) = y$$

Proof. Let $\alpha = \mathrm{crit}(k)$. The following formula states that α is the critical point of k:

$$k(\alpha) \neq \alpha \text{ and for all } \beta < \alpha, \, k(\beta) = \beta$$

Denote the underlying formula $\phi(x, y)$; the displayed formula is obtained by the substitutions $x = k$ and $y = \alpha$. There is just one quantified variable: $\forall \beta < \alpha$. This could be written equivalently as $\forall \beta \in V_\lambda \, (\beta < \alpha \longrightarrow \dots)$ (since $\alpha \in V_\lambda$), so the quantifier is bound to V_λ.

So, $\phi(k, \alpha)$ is a true statement in V. Since it is bounded in $V_{\lambda+1}$ (in the sense above), it is also true in $V_{\lambda+1}$. Now we apply $\hat{\imath}$ to $\phi(k, \alpha)$, and we may therefore conclude that $\phi(\hat{\imath}(k), \hat{\imath}(\alpha))$ holds in $V_{\lambda+1}$, since $\hat{\imath}$ is elementary relative to such a formula. Again, by Fact 1(a), $\phi(\hat{\imath}(k), \hat{\imath}(\alpha))$ must also hold in V.

Finally, let's consider what the formula $\phi(\hat{\imath}(k), \hat{\imath}(\alpha))$ actually says. Returning to the informal form of the formula, we have

$$\hat{\imath}(k)(\hat{\imath}(\alpha)) \neq \hat{\imath}(\alpha) \text{ and for all } \beta < \hat{\imath}(\alpha), \, \hat{\imath}(k)(\beta) = \beta.$$

In other words, the critical point of $\hat{\imath}(k)$ is $\hat{\imath}(\alpha)$. Now $\hat{\imath}(\alpha) = i(\alpha) = i(\mathrm{crit}(k))$, so we have shown that

$$\mathrm{crit}(i \cdot k) = \mathrm{crit}(\hat{\imath}(k)) = i(\mathrm{crit}(k)). \square$$

yields

$$(j \cdot j)(j(x)) = j(i)(j(x)) = j(y) = (j \circ j)(x),$$

as required. (j was applied in the middle step.) □

Suppose $k : V \to V$ is a WA_0-embedding (whose critical point may or may not be κ). We defined before (p. 220) the applicative iterates of k, namely $k^{[n]}$, $n \in \omega$. We review this definition and introduce a second kind of iterate, $k_{[n]}$, defined once again inductively:

$$\begin{aligned} k^{[1]} &= k_{[1]} = k \\ k^{[2]} &= k_{[2]} = k \cdot k \\ k^{[n+1]} &= k \cdot k^{[n]} \\ k_{[n+1]} &= k^{[n]} \cdot k. \end{aligned}$$

Proposition 102. $\mathrm{crit}(j^{[n+1]}) = \kappa_n$.

Proof. By induction on n. □

Proposition 103. *Let* id *be the identity function (on V). Then* id $\notin A_j$.

Proof. We proceed by using the inductive definition of A_j. Certainly $j \neq \mathrm{id}$. Assuming neither i, k equals id, we show $i \cdot k \neq \mathrm{id}$. For this we may invoke Proposition 101: Since $k \neq \mathrm{id}$, k has a critical point, and so by the Proposition, $i(\mathrm{crit}(k)) = \mathrm{crit}(i \cdot k)$. Since $\mathrm{crit}(i \cdot k)$ exists, $i \cdot k \neq \mathrm{id}$. □

We record one well-known additional fact:

Proposition 104. $(\mathcal{A}_j, \cdot)$ *is a left-distributive algebra; that is, for all $i, k, \ell \in \mathcal{A}_j$, $i \cdot (k \cdot \ell) = (i \cdot k) \cdot (i \cdot \ell)$.*[199] □

Proposition 103 tells us that the familiar elements of the critical sequence of j all turn out to be critical points of embeddings belong to $\mathcal{A}_j$. Therefore, $\mathrm{crit}(\mathcal{A}_j)$ must be infinite. A natural question is whether there is an embedding in $\mathcal{A}_j$ whose critical point is *not* a term in the critical sequence for j. It turns out that there are many such embeddings; the least such critical point lies between κ_2 and κ_3.[200]

[199] An easy (heuristic) proof can be given that relies on Fact 1b (p. 222) and remarks following, in an earlier footnote: Think of i, k, ℓ as nontrivial elementary embeddings $V_\lambda \to V_\lambda$. One shows easily that the formula $z = k \cdot \ell$ is a bounded formula in $V_{\lambda+1}$. Thus $\hat{\imath}$ is elementary relative to this formula. We have

$$i \cdot (k \cdot \ell) = \hat{\imath}(k \cdot \ell) = \hat{\imath}(k) \cdot \hat{\imath}(\ell) = (i \cdot k) \cdot (i \cdot \ell).$$

[200] In this footnote, we exhibit an embedding in $\mathcal{A}_j$ whose critical point γ is not a term in the critical sequence for j. It can be shown that $\mathrm{crit}(\mathcal{A}_j) \cap \kappa_3 = \{\kappa_0 < \kappa_1 < \kappa_2 < \gamma\}$, so that γ is the least in $\mathrm{crit}(\mathcal{A}_j)$ that does not belong to the critical sequence.

We prove that $((j \cdot j) \cdot j) \cdot (j \cdot j)$ has a critical point γ strictly between κ_2 and κ_3. As we have already shown, $\mathrm{crit}(j \cdot j) = \kappa_1 < \kappa_2$. We write:

(138) $$\mathrm{crit}(j \cdot j) < \kappa_2$$

Now we apply the map $(j \cdot j) \cdot j$ to the formula (138) (suppressing the $\cdot$ for readability):

(139) $$((jj)j)(\mathrm{crit}(jj)) < ((jj)j)(\kappa_2).$$

Theorem 105. [41] $\mathrm{crit}(\mathcal{A}_j)$ *has order-type* ω. *In fact, for each* $n \in \omega$, $\mathrm{crit}(\mathcal{A}_j) \cap [\kappa_n, \kappa_{n+1})$ *is finite. Also, for each* $n > 2$, $\mathrm{crit}(\mathcal{A}_j) \cap (\kappa_n, \kappa_{n+1})$ *is nonempty.* □

Theorem 106 tells us that the critical sequence $\mathbf{S}$ of j is a proper subcollection of $\mathrm{crit}(\mathcal{A}_j)$. The fact that $\mathrm{crit}(\mathcal{A}_j)$ is well-ordered of type ω implies there is a successor function $s_j : \mathrm{crit}(\mathcal{A}_j) \to \mathrm{crit}(\mathcal{A}_j)$, defined by

$$s_j(\alpha) = \text{least } \beta \in \mathrm{crit}(\mathcal{A}_j) \text{ such that } \beta > \alpha.$$

It is easy to see that $(\mathcal{A}_j, s_j, \kappa)$ is an initial Dedekind self-map.

We state next (without proof) a remarkable result due to Laver. It tells us that any element of A_j is completely determined by its behavior on $\mathrm{crit}(\mathcal{A}_j)$.

Theorem 106. [41] *If* $i, k \in \mathcal{A}_j$ *and* $i \restriction \mathrm{crit}(\mathcal{A}_j) = k \restriction \mathrm{crit}(\mathcal{A}_j)$, *then* $i = k$. □

Lemma 107. *Suppose* $k \in \mathcal{A}_j$ *and* $\gamma \in \mathrm{crit}(\mathcal{A}_j)$. *Then* $k(\gamma) \in \mathrm{crit}(\mathcal{A}_j)$.

Proof. We need to find $i \in \mathcal{A}_j$ such that $\mathrm{crit}(i) = k(\gamma)$. Let $\ell \in \mathcal{A}_j$ be such that $\gamma = \mathrm{crit}(\ell)$. Let $i = k \cdot \ell$. Then

$$\mathrm{crit}(i) = \mathrm{crit}(k \cdot l) = k(\mathrm{crit}(\ell)) = k(\gamma).\square$$

Using Theorem 106, we define $e : \omega \to \mathrm{crit}(\mathcal{A}_j)$ to be the unique increasing enumeration of $\mathrm{crit}(\mathcal{A}_j)$. For every pair (i, k) from $\mathcal{A}_j$ for which $i \neq k$, let $m_{i,k}$ be the least $n \in \omega$ such that $i(e(n)) \neq k(e(n))$. By Theorem 107, $m_{i,k}$ exists. Define

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By Proposition 101:

$$\mathrm{crit}\big(((jj)j)(jj)\big) < ((jj)j)(\kappa_2). \tag{140}$$

We compute $((jj)j)(\kappa_2)$: Notice that $jj(\kappa_1) = ((jj) \circ j)(\kappa_0) = (j \circ j)(\kappa_0) = \kappa_2$. Likewise, $jj(\kappa_2) = (j \circ j)(\kappa_1) = \kappa_3$. So we have

$$((jj)j)(\kappa_2) = ((jj)j)(jj(\kappa_1)) = jj(j(\kappa_1)) = jj(\kappa_2) = \kappa_3$$

Combining (139) and (140), we have

$$\mathrm{crit}\big(((jj)j)(jj)\big) < \kappa_3.$$

Let $\gamma = \mathrm{crit}\big(((jj)j)(jj)\big)$; we have shown so far that$\gamma < \kappa_3$. Next we show that $\gamma > \kappa_2$. Again, since $\mathrm{crit}(jj) = \kappa_1$, it follows that

$$\mathrm{crit}(jj) > \kappa_0. \tag{141}$$

We apply $(jj)j$ to (141); applying Proposition 101 as well, we have:

$$\gamma = \mathrm{crit}\big(((jj)j)(jj)\big) > ((jj)j)(\kappa_0). \tag{142}$$

Next, we compute $((jj)j)(\kappa_0)$: Let $y = j(\kappa_0)$. Applying jj, we get,

$$(jj)(y) = ((jj)j)\big((jj)(\kappa_0)\big) = ((jj)j)(\kappa_0).$$

Since $y = j(\kappa_0)$, $(jj)(y) = (jj \circ j)(\kappa_0) = (j \circ j)(\kappa_0) = \kappa_2$, and this leads to:

$$\kappa_2 = ((jj)j)(\kappa_0).$$

Plugging into 142 yields:

$$\gamma = \mathrm{crit}\big(((jj)j)(jj)\big) > \kappa_2. \tag{143}$$

The conclusion is that

$$\kappa_2 < \gamma = \mathrm{crit}\big(((jj)j)(jj)\big) < \kappa_3, \tag{144}$$

and so we have exhibited a member γ of $\mathrm{crit}(A_j)$ that does not lie in the critical sequence of j. □

$d : \mathcal{A}_j \times \mathcal{A}_j \to \mathbb{R}$ by

$$d(i,k) = \begin{cases} 0 & \text{if } i = k \\ 1/(m_{i,k}+1) & \text{otherwise} \end{cases}$$

Proposition 108. *d is a metric on $\mathcal{A}_j$.*

Proof. We'll prove that the triangle inequality holds; the other properties of a metric are immediate. Let $x, y, z \in A_j$. We assume that x, y, z are all different (the other cases are easy). Let x', y', z' denote their corresponding restrictions to $\text{crit}(\mathcal{A}_j)$. Let m be least such that $x'(e(m)) \neq y'(e(m))$ and let n be least such that $y'(e(n)) \neq z'(e(n))$. Let r be least for which $x'(e(r)) \neq z'(e(r))$. We show that $r \geq \min\{m, n\}$. Suppose $s < \min\{m, n\}$. Then

$$x'(e(s)) = y'(e(s)) \text{ and } y'(e(s)) = z'(e(s)),$$

so $x'(e(s)) = z'(e(s))$. Therefore, x' and z' do not differ on any $s < \min\{m, n\}$, whence $r \geq \min\{m, n\}$.

To complete the proof, we must show that

$$\frac{1}{m+1} + \frac{1}{n+1} \geq \frac{1}{r+1}$$

Without loss of generality, assume $m \leq n$. Then, since $r \geq m$, then

$$\frac{1}{r+1} \leq \frac{1}{m+1} \leq \frac{1}{m+1} + \frac{1}{n+1},$$

as required. $\square$

Definition. A metric space (X, ρ) is *dense-in-itself* if for every $x \in X$, there is a sequence $\langle y_n \rangle_{n \in \omega}$, where each y_n is different from x, which converges to x (i.e., the sequence $\langle \rho(x, y_n) \rangle_n$ converges to 0).

Proposition 109. *$(\mathcal{A}_j, d)$ is dense-in-itself.*

Proof. Let $k \in \mathcal{A}_j$. We define a sequence $\langle i_n \rangle_{n \in \omega}$ of elements of $\mathcal{A}_j$, all different from k, such that $i_n \to k$. To define i_n, let $\alpha = e(n) \in \text{crit}(A_j)$. Let m_n be large enough so that $\kappa_{m_n} > k(\alpha)$; it follows that $\kappa_{m_n} > \alpha$. Define

$$i_n = j^{[m_n+3]} \cdot k.$$

Claim 1. For each n, $i_n \upharpoonright \langle e(0), e(1), \ldots, e(n) \rangle = k \upharpoonright \langle e(0), e(1), \ldots, e(n) \rangle$. Therefore, $i_n \to k$.

Proof of Claim 1. Let $\beta \in \text{crit}(\mathcal{A}_j), \beta \leq \alpha$. Let $y = k(\beta)$. Applying $j^{[m_n+3]}$ to the statement

$$k \text{ applied to } \beta = y$$

yields

$$j^{[m_n+3]} \cdot k \text{ applied to } j^{[m_n+3]}(\beta) = j^{[m_n+3]}(y).$$

But $j^{[m_n+3]}(\beta) = \beta$ and $j^{[m_n+3]}(y) = j^{[m_n+3]}(k(\beta)) = k(\beta)$ since $\kappa_{m_n} > k(\beta)$. We therefore have:

$$j^{[m_n+3]} \cdot k \text{ applied to } \beta = k(\beta).$$

The result follows. □

Claim 2. For each n, $i_n \neq k$.

Proof of Claim 2.

Case I. $\mathrm{crit}(k) \geq \kappa_{m_n+2}$. We show i_n and k have different critical points in this case:

$$\mathrm{crit}(i_n) = \mathrm{crit}(j^{[m_n+3]} \cdot k) = j^{[m_n+3]}(\mathrm{crit}(k)) > \mathrm{crit}(k),$$

as required.

Case II. $\mathrm{crit}(k) < \kappa_{m_n+2}$. By Theorem 106, $\mathrm{crit}(A_j) \cap [\mathrm{crit}(k), \kappa_{m_n+2})$ is finite. Let β be the largest element in this finite set. Since $k(\beta) \in \mathrm{crit}(\mathcal{A}_j)$ (by Lemma 108), $k(\beta) > \beta$ (since $\beta \geq \mathrm{crit}(k)$), and β is the largest element of $\mathrm{crit}(\mathcal{A}_j) \cap [\mathrm{crit}(k), \kappa_{m_n+2})$, it follows that $k(\beta) \geq \kappa_{m_n+2}$. Applying $j^{[m_n+3]}$ to the statement

$$k \text{ applied to } \beta \text{ is } k(\beta)$$

yields

$$j^{[m_n+3]} \cdot k \text{ applied to } j^{[m_n+3]}(\beta) \text{ is } j^{[m_n+3]}(k(\beta)).$$

But since $j^{[m_n+3]}(\beta) = \beta$ and $k(\beta) \geq \kappa_{m_n+2}$,

$$(j^{[m_n+3]} \cdot k)(\beta) > k(\beta).$$

Therefore, $i_n(\beta) \neq k(\beta)$. □

Proposition 110 will be useful when we look at the completion of $(\mathcal{A}_j, d)$; it will tell us that the completion must be a perfect Polish space [39, Theorem 1] and must therefore have cardinality $2^{\aleph_0}$ (see for example [55, p. 66]). Before discussing this point, we will describe the completion of $(\mathcal{A}_j, d)$ in some detail. We will consider three ways of performing the construction. The least interesting way is to form equivalence classes of Cauchy sequences of members of $\mathcal{A}_j$. This approach is certainly sufficient to provide a complete metric space in which $\mathcal{A}_j$ is densely embedded. But the elements of the space are not especially useful.

A second approach to forming the completion—and we will use this approach to produce the completion in two other ways—is to embed $(\mathcal{A}_j, d)$ into a complete metric space and take the closure of the image. The resulting set, being a closed subspace of a complete metric space, must itself be a complete metric space.

The all-purpose complete metric space that is often used in this strategy is obtained by defining a certain metric on the set of all functions from the given space into $\mathbb{R}$ that are bounded and continuous. In the present setting, let $Y = \{f : \mathcal{A}_j \to \mathbb{R} \mid f \text{ is bounded and continuous}\}$. Define a metric σ on Y by

$$\sigma(f, g) = \sup\{|f(i) - g(i)| : i \in A_j\}.$$

Now we embed (A_j, d) into (Y, σ) using the following mapping H: For each $k \in \mathcal{A}_j$ obtain an $f_k \in Y$ defined as follows:

$$f_k(i) = d(i, k) - d(i, j).$$

One shows that each f_k is bounded and continuous (relative to the metric d). One also shows that H is an isometry with respect to d and σ; that is, H is one-one and for all $\ell, k \in \mathcal{A}_j$,

$$\sigma(f_\ell, f_k) = d(\ell, k).$$

We outline the proofs:

Claim A. For each $k \in \mathcal{A}_j$, f_k is bounded and continuous.

Proof. It is easy to verify that range(f_k) is a subset of $[-1, 1]$, whence f_k is bounded. To prove continuity we show that for each $m \in \omega$, if $d(i, i') < \frac{1}{2m+2}$, then $|f_k(i) - f_k(i')| < \frac{1}{m+1}$.

$$\begin{array}{rcl} |f_k(i) - f_k(i')| & = & |d(i,k) - d(i,j) - d(i',k) + d(i',j)| \\ & = & |(d(i,k) - d(i',k)) + (d(i',j) - d(i,j))| \\ & \leq & |d(i,k) - d(i',k)| + |d(i',j) - d(i,j)| \\ & \leq & d(i,i') + d(i,i') \\ & \leq & 2d(i,i') \\ & < & \dfrac{1}{m+1}.\Box \end{array}$$

Claim B. H is one-one.

Proof. Suppose $f_k = f_\ell$. We show that $k = \ell$ by showing that $d(k, \ell) = 0$. Since $f_k = f_\ell$, the two functions agree at $\ell \in \mathcal{A}_j$. We have

$$0 = f_k(\ell) - f_\ell(\ell) = (d(\ell,k) - d(\ell,j)) - (d(\ell,\ell) - d(\ell,j)) = d(\ell,k) - d(\ell,\ell) = d(\ell,k).\Box$$

Claim C. For every $k, \ell \in \mathcal{A}_j$, $\sigma(f_\ell, f_k) = d(\ell, k)$.

Proof. First notice that by the triangle inequality, for any $i \in \mathcal{A}_j$,

$$d(k,\ell) \geq |d(i,k) - d(i,\ell)|,$$

and so the supremum of the set $\{|d(i,k) - d(i,\ell)| : i \in \mathcal{A}_j\}$ is achieved when $i = \ell$. Therefore, we have:

$$\begin{array}{rcl} \sigma(f_k, f_\ell) & = & \sup\big(\{|f_k(i) - f_\ell(i)| : i \in A_j\}\big) \\ & = & \sup\big(\{|d(i,k) - d(i,j) + d(i,j) - d(i,\ell)| : i \in A_j\}\big) \\ & = & \sup\big(\{|d(i,k) - d(i,\ell)| : i \in A_j\}\big) \\ & = & |d(\ell,k) - d(\ell,\ell)| \\ & = & d(\ell,k).\Box \end{array}$$

These facts tell us that for each $k \in \mathcal{A}_j$, f_k is the representative of k in the space Y. In this way, we have identified each $k \in \mathcal{A}_j$ with a bounded continuous real-valued function. Unfortunately, in representing k by f_k, one loses much information about the critical point and the values of k at crit$(\mathcal{A}_j)$. In particular, given j and e, one cannot typically compute the critical point of k from f_k without testing f_k against arbitrarily many elements of $\mathcal{A}_j$. For this reason, we will make use of an alternative approach.

Another space in which to embed $(\mathcal{A}_j, d)$ is the space ω^ω of all functions from ω to ω. This space is very useful because of its close relationship to $\mathbb{R}$ and several of its subspaces. A subspace of ω^ω that will be important for our work is the set $\omega^{\uparrow\omega}$ of strictly increasing functions from ω to ω. We record some useful facts about ω^ω and $\omega^{\uparrow\omega}$, and then describe the embedding of $(\mathcal{A}_j, d)$.

Theorem 110. (ω^ω)

(1) *Define* $\rho : \omega^\omega \times \omega^\omega \to \mathbb{R}$ *by*

$$\rho(f,g) = \begin{cases} 0 & \textit{if } f = g \\ 1/(m+1) & \textit{if } f \neq g \textit{ and } m \textit{ is least for which } f(m) \neq g(m) \end{cases}$$

Then (ω^ω, d) *is a complete metric space. The induced topology has a base consisting of sets of the form* $N_p = \{f : \omega \to \omega \mid f \supset p\}$*, where* p *is a finite sequence of elements of* ω*.*

(2) ω^ω*, with the topology induced by the metric* ρ*, is homeomorphic to both* $\mathbb{P}$*, the set of irrationals (with the topology induced by* $\mathbb{R}$*), and* $\mathbb{P}_{[0,1]} = \mathbb{P} \cap [0,1]$*.*

(3) *There is a continuous bijection* $\tau : \omega^\omega \to [0,1)$ *that is "measure-preserving" in the following sense: Let* μ *be the product measure on* ω^ω*, obtained from the measure on* ω *that assigns the value* 2^{-n-1} *to each* $n \in \omega$*, and let* λ *denote Lebesgue measure on* $[0,1)$*. Then for any* μ*-measurable set* X *in* ω^ω*,* $\tau(X)$ *is* λ*-measurable in* $[0,1)$ *and* $\mu(X) = \lambda(\tau(X))$*.*

(4) *The subspace* $\omega^{\uparrow\omega}$ *of strictly increasing functions is closed in* ω^ω*.*

Remark 27. The results mentioned in Theorem 111 are well-known [4, 54] so we omit the proofs, but include a few remarks about them. The proof of (1) is similar to Proposition 109. For (2), we give the construction that underlies the proof; we describe the constructions that demonstrate ω^ω is homeomorphic to $\mathbb{P}$ and also to $\mathbb{P}_{[0,1]}$, simultaneously. We begin by observing that any bijection $v : \omega \to \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers) induces a homeomorphism $\bar{v} : \omega^\omega \to \mathbb{Z}^\omega$, defined by

$$\bar{v}(f)(n) = v(f(n)).$$

Now build a collection $\{I_s \mid s \in \mathbb{Z}^{<\omega}\}$ of open intervals in $\mathbb{R}$ (or $[0,1]$) as follows: For each $n \in \mathbb{Z}$, let $I_{\langle n\rangle} = (n, n+1)$ (or, in the case of $[0,1]$, let $I_{\langle n\rangle}$ be the following collection of subintervals of $(0,1)$:

$$\left\{\dots, \left(\frac{1}{16}, \frac{1}{8}\right), \left(\frac{1}{8}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{7}{8}\right), \left(\frac{7}{8}, \frac{15}{16}\right), \dots\right\}.)$$

For any $s \in \mathbb{Z}^{<\omega}$, assuming I_s has been defined (for either construction), let $\{I_{s^\frown n} \mid n \in \mathbb{Z}\}$ be a disjoint family of open subintervals of I_s, ordered like $\mathbb{Z}$ and lying next to each other: The right endpoint of $I_{s^\frown n}$ is the left endpoint of $I_{s^\frown n+1}$, with union dense in I_s and each having diameter $\leq$ half the diameter of I_s. Now define $\tau : \mathbb{Z}^\omega \to \mathbb{R}$ (or $\tau : \mathbb{Z}^\omega \to (0,1)$) by

$$\{\tau(g)\} = \bigcap \{I_{g \restriction n} \mid n \in \omega\}.$$

Then τ is a homeomorphism $\mathbb{Z}^\omega \to \mathbb{R} - H$ (or $\mathbb{Z}^\omega \to (0,1) - H$), where H is a countable dense set (consisting of the endpoints of the I_s's). If $\mathbb{Z}^\omega$ is given the lexicographic ordering, then τ is order-preserving. Since H is order-isomorphic to $\mathbb{Q}$ (and to $\mathbb{Q} \cap (0,1)$), there is an order isomorphism $\mathbb{R} \to \mathbb{R}$ (or $(0,1) \to (0,1)$) taking H to $\mathbb{Q}$ (or H to $\mathbb{Q} \cap (0,1)$). We have exhibited homeomorphisms $\mathbb{Z}^\omega \to \mathbb{R} - \mathbb{Q} = \mathbb{P}$ and $\mathbb{Z}^\omega \to \mathbb{P}_{[0,1]}$.

For (3), we first describe $\tau : \omega^\omega \to [0,1)$: Given $f : \omega \to \omega$, $\tau(f)$ is the real number $x \in [0,1)$ obtained as follows: We first specify a sequence s of 0s and 1s, coded by f: start with $f(0)$ ones, then append a 0, then $f(1)$ ones, then a 0, and so forth. Now define x by

$$x = \sum_{i=1}^{\infty} \frac{s(i)}{2^i}.$$

It is not hard to see that τ is a continuous bijection; however, τ^{-1} fails to be continuous at precisely the dyadic rationals. Let μ be the product measure on ω^ω, where we first form the measure space based on ω^ω: For each $j \in \omega$, let E_j be the measure space $(\omega, \mathcal{P}(\omega), \nu)$, with $\nu(n) = 2^{-n-1}$. The product measure space is $(\omega^\omega, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the smallest σ-algebra containing subsets $A \subseteq \omega \times \omega \times \ldots = \omega^\omega$ of the form $A_0 \times A_1 \times \cdots$ where all but finitely many of the A_i are equal to ω, and $\mu(A) = \prod_j \nu(\mathcal{A}_j)$.

A couple of simple examples will clarify the relationship between μ and Lebesgue measure λ on $[0,1)$. We consider how μ acts on a couple of basic open sets of ω^ω. Let $p = \langle 0 \rangle$; we compute $\mu(N_p)$: Since, as a product of components, N_p has only one component different from ω, namely $\{0\}$, then $\mu(N_p) = 2^{0-1} \cdot 1 \cdot 1 \cdot \ldots = \frac{1}{2}$. Now $\tau(0) = 0$, and it is easy to see that $\tau(N_p) = [0, \frac{1}{2})$. Thus, $\lambda(\tau(N_p)) = \frac{1}{2} = \mu(N_p)$. Likewise,

$$\mu(N_{\langle 1 \rangle}) = 2^{-2} = \lambda\left(\left[\frac{1}{2}, \frac{3}{4}\right)\right) = \lambda(\tau(N_{\langle 1 \rangle})).$$

We give a proof of (4): Suppose $f \in \omega^\omega - \omega^{\uparrow\omega}$, and let n be least such that $f(n) \geq f(n+1)$. Let $p = f \upharpoonright n+2$. Then $f \in N_p$ and $N_p \cap \omega^{\uparrow\omega} = \emptyset$. Therefore, whenever an element f of ω^ω does not belong to $\omega^{\uparrow\omega}$, there is an open set containing f and missing $\omega^{\uparrow\omega}$; hence, $\omega^{\uparrow\omega}$ is closed. $\Box$

Next, we embed (A_j, d) into (ω^ω, ρ) by mapping each $k \in \mathcal{A}_j$ to the function $f_k : \omega \to \omega$ defined by:

(145) $$f_k(n) = e^{-1}(k(e(n))).$$

One shows easily that the map $F : k \mapsto f_k$ is an isometry: Let $i \neq k$ in $\mathcal{A}_j$. Notice that for each m, $i(e(m)) \neq k(e(m))$ if and only if $f_i(m) \neq f_k(m)$. So

$$\begin{aligned} \rho(f_i, f_k) &= 1/(m+1) \text{ where } m \text{ is least for which } f_i(m) \neq f_k(m) \\ &= 1/(m+1) \text{ where } m \text{ is least for which } i(e(m)) \neq k(e(m)) \\ &= d(i,k) \end{aligned}$$

We also observe here that each f_k is strictly increasing: Certainly any $k \in \mathcal{A}_j$ is strictly increasing. Suppose $m < n \in \omega$. Since both k and e are strictly increasing,

$$\begin{aligned} f_k(m) < f_k(n) &\Leftrightarrow e^{-1}(k(e(m))) < e^{-1}(k(e(n))) \\ &\Leftrightarrow k(e(m)) < k(e(n)) \\ &\Leftrightarrow e(m) < e(n) \\ &\Leftrightarrow m < n. \end{aligned}$$

Therefore, the image B of $(\mathcal{A}_j, d)$ under F is a representation of $\mathcal{A}_j$ by increasing functions $\omega \to \omega$.[201] Let $\bar{\mathcal{A}}_j$ be the closure of B in ω^ω. Let $\bar{d}$ be the restriction of ρ to $\bar{\mathcal{A}}_j \times \bar{\mathcal{A}}_j$. Then $(\bar{\mathcal{A}}_j, \bar{d})$ is (a third form of) the completion of $(\mathcal{A}_j, d)$.

A complete metric space is *separable* if it has a countable dense set. The completion $\bar{X}$ of any countable metric space X is always separable since X is dense in its completion. A point x in a metric space X is said to be an *isolated point* if $\{x\}$ is an open set in X. A *Polish space* is a complete separable metric space. A metric space is *perfect* if it has no isolated points. (Both Euclidean spaces and Hilbert spaces are perfect.) The first part of the following proposition is well-known:

Proposition 111. [39] *The completion of a countable dense-in-itself metric space is a perfect Polish space. Therefore $(\bar{\mathcal{A}}_j, \bar{d})$ is a perfect Polish space.*□

Below, we list some additional properties of $(\bar{\mathcal{A}}_j, \bar{d})$. These require a few preliminary definitions. Let Y be a Polish space. A subset X of Y is *nowhere dense* if, for every open subset U of Y, there is an open set $V \subseteq U$ such that $V \cap X = \emptyset$. A set $X \subseteq Y$ *has universal measure zero* if, for every diffused (takes singletons to 0) Borel measure μ on Y, there is a μ-measure zero Borel set in Y that includes X. X *has Marczewski measure zero* if, for every perfect subset $P \subseteq Y$, there is a perfect subset $Q \subseteq P$ such that $Q \cap X = \emptyset$.

Theorem 112.

(1) *All elements of $\bar{\mathcal{A}}_j$ are strictly increasing; that is, $\bar{\mathcal{A}}_j \subseteq \omega^{\uparrow\omega}$.*
(2) *The set $\omega^{\uparrow\omega}$ is nowhere dense in ω^ω; thus, in particular, $\bar{\mathcal{A}}_j$ is nowhere dense in ω^ω.*
(3) *There exists a finite diffused Borel measure μ on ω^ω such that $\mu(\bar{\mathcal{A}}_j) > 0$ (in other words, $\bar{\mathcal{A}}_j$ does not have universal measure zero).*
(4) *$(\bar{\mathcal{A}}_j, \bar{d})$ is closed and bounded in (ω^ω, ρ) but not compact. Indeed, we can find an infinite subset $X = \{i_1, i_2, \ldots\}$ of A_j such that $F[X]$ has no limit point in $\bar{\mathcal{A}}_j$ (indeed, not even in ω^ω).*
(5) *There is a real number $c > 0$ and an infinite decreasing chain $F_0 \supseteq F_1 \supseteq \ldots$ of nonempty closed subsets of $\bar{\mathcal{A}}_j$ for which $\langle \delta(F_n) \rangle \to c$ and yet $\bigcap_{n \geq 0} F_n = \emptyset$.*

Proof of (1). Suppose $\langle i_n \rangle_n \to k$, where $k \in \bar{\mathcal{A}}_j$ and each i_n belongs to $\bar{\mathcal{A}}_j$. As we observed earlier, each i_n is strictly increasing. Let $m \in \omega$. It suffices to show $k(m) < k(m+1)$. Let $n_0 \in \omega$ be large enough so that

$$\text{for all } n \geq n_0,\ i_n \restriction m+2 = k \restriction m+2. \tag{146}$$

Since i_n is increasing, $i_n(m) < i_n(m+1)$. It follows by (146) that $k(m) < k(m+1)$, as required. □

[201] Dougherty-Jech [19] discuss B starting from first principles. A subset A of ω^ω of strictly increasing functions is an *embedding algebra* if A is equipped with a binary operation $\cdot$ so that, for all $f, g, h \in A$,

(1) $f \cdot (g \cdot h) = (f \cdot h) \cdot (f \cdot h)$ (left-distributivity)
(2) if $\mathrm{cr}(f)$ denotes the least integer moved by f (whenever there is one), then $\mathrm{cr}(f \cdot g) = f(\mathrm{cr}(g))$.

The set B is naturally an embedding algebra: Define $\cdot$ on B by $f_i \cdot f_k = F(i \cdot k)$. Note that for any $f_k \in B$, $\mathrm{cr}(f_k) = e^{-1}(\mathrm{crit}(k))$. With these definitions, $(B, \cdot)$ can be shown to be an embedding algebra.

Proof of (2). The basic open sets of ω^ω are of the form $N_p = \{f : \omega \to \omega \mid f \supset p\}$, where p is a function $p : n \to \omega$, for some $n \in \omega$. Given such a basic open set N_p, we exihibit q for which $N_q \subseteq N_p$ and N_q misses $\bar{\mathcal{A}}_j$. If p is not strictly increasing, then we let $q = p$; this works because all elements of $\bar{\mathcal{A}}_j$ are strictly increasing. If p is strictly increasing, let $q = p^\frown\langle 1, 0\rangle$. Clearly q is not strictly increasing, so N_q misses $\bar{\mathcal{A}}_j$, yet is a subset of N_p. $\square$

Proof of (3). It suffices to show that $\bar{\mathcal{A}}_j$ is not universal measure zero. But this follows from the fact (see [54]) that every universal measure zero set has Marczewski measure zero (but no perfect set could have Marczewski measure zero). $\square$

Proof of (4). Recall that for each $k \in \mathcal{A}_j$, $f_k : \omega \to \omega$ is defined above in (145) by $f_k(m) = e^{-1}(k(e(m)))$. First we note that the ρ-diameter of ω^ω is 1:

$$\sup\{\rho(f, g) \mid f, g \in \omega^\omega\} = 1.$$

Thus, in particular, the space $\bar{\mathcal{A}}_j$ is bounded.

To show $\bar{\mathcal{A}}_j$ is not compact, we will make use of some known results about $\mathcal{A}_j$. Let us say that, for any limit ordinal γ and any $i, k \in \mathcal{A}_j$, $i \overset{\gamma}{=} k$ if for all $x \in V_\gamma$, $i(x) \cap V_\gamma = k(x) \cap V_\gamma$; it can be shown [18] that $\overset{\gamma}{=}$ is an equivalence relation on $\mathcal{A}_j$. We refer the reader to [18] for proofs of the following two facts:

Fact A. [18, Section XII, Lemma 4.4]. Suppose $i \overset{\gamma}{=} k$ and $\alpha < \gamma$. Then if $i(\alpha) < \gamma$, we have that $i(\alpha) = k(\alpha)$. $\square$

Fact B. [18, Section XII, Lemma 4.13]. Let $j : V \to V$ be a WA-embedding with critical point κ. Let $n \in \omega$. Then there is $i \in \mathcal{A}_j$ such that

$$i \overset{\kappa_{n+1}}{=} j^n.\square$$

Claim. For each $n \in \omega$ there is $i_n \in \mathcal{A}_j$ such that $\mathrm{crit}(i_n) = \kappa$ and $i_n(\kappa) = \kappa_n$.

Proof. Given $n \in \omega$, using Fact B, we obtain $i_n \in \mathcal{A}_j$ such that $i_n \overset{\kappa_{n+1}}{=} j^n$. Since $\kappa < \kappa_{n+1}$ and $j^n(\kappa) < \kappa_{n+1}$, by Fact A, we conclude that $i_n(\kappa) = j^n(\kappa)$. $\square$

Using the notation from the Claim, let $X = \{f_{i_n} \mid n \in \omega\}$. Notice that for each n, $f_{i_n} \in N_{\langle n\rangle}$, and for $m \neq n$, $N_{\langle n\rangle} \cap N_{\langle m\rangle} = \emptyset$. It follows that X is infinite. To see X has no limit point (even in ω^ω), let $g \in \omega^\omega$, and let $n = g(0)$. Then $g \in N_{\langle n\rangle}$ and $X \cap N_{\langle n\rangle} = \{f_{i_n}\}$. This shows g is not a limit point of X, since in a metric space, a limit point of a set Y has the property that every open set containing the limit point contains *infinitely many* points of Y. $\square$

Proof of (5). Let $X = \{f_{i_n} \mid n = 1, 2, \ldots\}$, from part (4). Let $F_0 = X$ and for each $n \geq 1$, let $F_n = X - \{f_{i_1}, \ldots, f_{i_n}\}$. Each F_n is closed and $F_0 \supseteq F_1 \supseteq \ldots$. Since for each $n, f_n \notin F_n$, it follows that $\bigcap_{n\geq 0} F_n = \emptyset$. Also, since $\bar{d}(f_{i_m}, f_{i_n}) = 1$ whenever $m \neq n$, it follows that $\delta(F_n) = 1$ for each n and so $\langle\delta(F_n)\rangle_n \to 1$. This completes the proof that the sets $F_0, F_1, F_2, \ldots$ have the required properties. $\square$

Remark 28. Part (4) shows that even though $\bar{A}_j$ is closed and nowhere dense in ω^ω, $\bar{A}_j$ is *not* homeomorphic to the Cantor set (since the Cantor set is compact). Part (5) shows $\bar{A}_j$ is somewhat pathological; the result of (5) contrasts with the fact that, since $\bar{A}_j$ is a complete metric space, it has the property that whenever $F_0 \supseteq F_1 \supseteq \dots$ are closed subsets with $\langle \delta(F_n) \rangle_n \to 0$, then $\bigcap_{n \geq 0} F_n \neq \emptyset$. □

This way of representing the completion of $\mathcal{A}_j$, as in (145), allows us to preserve information about the behavior of elements of $\mathcal{A}_j$ on $\mathrm{crit}(\mathcal{A}_j)$. For instance, if $\mathrm{crit}(k) = e(m)$, we can discover this fact by examining f_k: Look for the least r for which $f_k(r) \neq r$; this r must equal m. In addition, if $k(e(m)) = e(n)$, we can discover this fact by applying f_k to m; it must be the case that $f_k(m) = n$. This feature of strong preservation of structure is in sharp contrast to the characteristic of the space of bounded continuous real-valued functions on $\mathcal{A}_j$ in which, as we have seen, very little of this structure is preserved.

This representation of elements of $\mathcal{A}_j$ also allows us to understand the new elements of $\bar{A}_j$ as other functions from ω to ω, that is, as sequences of natural numbers.

Example 8. We show how a limit of a sequence of elements in $\mathcal{A}_j$ is realized as a function $\omega \to \omega$ in $\bar{\mathcal{A}}_j - \mathcal{A}_j$. Consider the sequence $\langle j^{[m]} : m \geq 1 \rangle$. For each $m \in \omega$, let r_m be such that $e(r_m) = \kappa_m$, which is the critical point of $j^{[m+1]}$. Under F, $j^{[m+1]}$ is mapped to a function f_{m+1} where $f_{m+1} \restriction r_m = \mathrm{id} \restriction r_m$, and $f_{m+1}(r_m) > r_m$. So $\rho(f_{m+1}, \mathrm{id}) = 1/(r_m + 1)$, and it follows that $\langle f_m \rangle_m \to \mathrm{id}$. As above, if we let $B = F''A_j$, I claim that $\mathrm{id} \in \bar{\mathcal{A}}_j - B$: Suppose instead that it lies in B. Then there is an $\ell \in \mathcal{A}_j$ whose restriction to $\mathrm{crit}(\mathcal{A}_j)$ is $\mathrm{id} = \mathrm{id}_C$, where $C = \mathrm{crit}(\mathcal{A}_j)$. But this is impossible because, as we showed earlier, $\mathrm{id} \notin \mathcal{A}_j$, as required. □

Finally, we outline the steps one might typically take to make use of our perfect Polish space $\bar{\mathcal{A}}_j$ in applications to quantum physics. We begin with a complex-valued Borel measure μ on $\bar{\mathcal{A}}_j$; it would suffice to use the measure guaranteed by Theorem 113(3).

As a first step toward defining the Hilbert space $\;^2(\bar{\mathcal{A}}_j, \mu)$, we define:

$$V = \{f : \bar{\mathcal{A}}_j \to \mathbb{C} \mid |f|^2 \text{ is } \mu\text{-integrable over } \bar{\mathcal{A}}_j\},$$

where $\mathbb{C}$ is the set of complex numbers. Defining addition and scalar multiplication pointwise turns V into a vector space. A quasi-inner product can be defined by

$$\langle f_1, f_2 \rangle = \int_{\bar{\mathcal{A}}_j} f_1 f_2^* \, d\mu. \tag{147}$$

As usual in this construction, a property that is needed, but that we do not have, is

$$\langle f, f \rangle = 0 \Leftrightarrow f = 0.$$

This difficulty is corrected by introducing an equivalence relation $\approx$ on V by

$$f_1 \approx f_2 \quad \text{iff} \quad |f_1 - f_2| = 0 \ (\mu\text{-a.e.}) \text{ on } \bar{\mathcal{A}}_j.$$

For each $f \in V$, let $[f]$ denote the $\approx$-equivalence class containing f. Define $\bar{V} = \{[f] \mid f \in V\}$. One shows that the inner product computation (147) does not depend on the choice of representatives, and that addition and scalar multiplication are well-defined. Now, we have properly defined $\;^2(\bar{\mathcal{A}}_j, \mu)$ by the triple $\bar{V}, \mu$ and this inner product defined on equivalence classes. Because $\bar{\mathcal{A}}_j$ is a Polish space, it follows that $^2(\bar{\mathcal{A}}_j, \mu)$ is a separable Hilbert space. We have now the mathematical setting for performing a quantum-mechanical analysis of some system behaving within $\bar{\mathcal{A}}_j$.

We began this section by elaborating on the dynamics of a WA-embedding $j : V \to V$. By studying the self-application operation $\cdot$, which allows us to combine WA-embeddings $V \to V$ to produce others, we discovered an infinity of such embeddings—forming the collection $\mathcal{A}_j$—and a countable j-class $\mathrm{crit}(\mathcal{A}_j)$ of critical points that determine the behavior of the embeddings belonging to $\mathcal{A}_j$. Having observed that $\mathcal{A}_j$ is a dense-in-itself metric space, we took the next logical mathematical step and obtained the completion $\bar{\mathcal{A}}_j$ of $\mathcal{A}_j$, which turned out to be a nowhere dense perfect Polish space—a space suitable for building a canonical Hilbert space $X = \ ^2(\bar{\mathcal{A}}_j, \mu)$. Although we do not have a particular physical application in mind for this formalism, we have taken a step in the direction of forging a theoretical link between, on the one hand, our highly abstract mathematical model of wholeness and its dynamics, and, on the other hand, the mathematical formalism (that of separable Hilbert spaces) for studying the physical universe through the eyes of quantum mechanics. Such a link parallels, we feel, the real-life link that joins the unmanifest dynamics of pure consciousness to the phenomena of manifest existence.

We close this section of the Appendix with a technical question that must not be overlooked, but which we have neglected so far, namely: How can we provide a proper formal treatment of the collection $\mathcal{A}_j$ as a j-class when each $i \in \mathcal{A}_j$ is itself already a j-class? Indeed, the collection $\mathcal{A}_j$ itself is too big to be a j-class; it doesn't "fit" in the universe V. There are several ways to handle this situation. An elegant partial solution would be to make use of the following theorem due to Laver (see [19]):

Theorem 113. $(\mathcal{A}_j, \cdot)$ *is, up to isomorphism, the unique, free, monogenic, left-distributive algebra.*

Left-distributivity asserts that $i\cdot(k\cdot\ell) = (i\cdot k)\cdot(i\cdot\ell)$; $\mathcal{A}_j$ satisfies this property [18]. It is monogenic because the algebra is generated by a single element, namely, j. Because of freeness, this algebra is unique up to isomorphism. But there are examples of such algebras in the realm of sets. One example that has been studied extensively is found in the *braid group* B_∞ *with infinitely many generators* $b_1, b_2, \ldots$[18]. One can define an operation $\cdot$ on B_∞ that makes B_∞ a left-distributive algebra. It follows that if we form the left-distributive subalgebra $B_1 = \langle b_1 \rangle$ of B_∞, then B_1 is an example of a *free* (monogenic) left-distributive algebra. Therefore, many of the interesting applications of $\mathcal{A}_j$ that we have in mind could be carried out using B_1 in its place. There are, however, properties of $\mathcal{A}_j$ that are not known to hold true of all free monogenic left-distributive algebras, and that may well be present only in $\mathcal{A}_j$. For instance, B_1 is not known to yield an embedding algebra (discussed on p. 232) as can be done using $\mathcal{A}_j$.

To provide a more complete solution to the problem, we describe here a direct approach to coding elements of $\mathcal{A}_j$ so that $\mathcal{A}_j$ may be treated as if it were a j-class. We begin by recalling that the elements of $\mathcal{A}_j$ can be obtained by forming the closure of the following statements:

(i) (j) is an element

(ii) if k, ℓ are elements, $(k \cdot \ell)$ is an element.

This finitary definition can be abstracted as a notion of a *set of expressions*, defined independently of the notion of elementary embeddings, and then any particular $i \in \mathcal{A}_j$ can be viewed as a realization of an expression, obtained by substituting suitable restrictions of j for each element in the expression. We elaborate here most of the details of this approach.

We begin with an abstract definition of the set Expr of *expressions*: Start with an alphabet $A = \{a\}$, with a single element a. Expr is the smallest set closed under the following rules:

(i) (a) is an expression
(ii) if b, c are expressions, (bc) is an expression

The *length* of an expression ex, denoted $|ex|$, is the number of occurrences of a in ex. An *indexed expression* is an expression in which each of the integers $1, 2, \ldots, |ex|$ is assigned to an occurrence of a in ex, in descending order, scanning ex from left to right. An *indexing* of an expression ex is the result of assigning integers to occurrences of a so that ex becomes an indexed expression. We shall denote the assignment of an integer i to an occurrence a in an expression by a^i.

For example, an indexing of the expression $(((a)(a))(((a)(a))(a)))(a)$ is

$$(((a^6)(a^5))(((a^4)(a^3))(a^2)))(a^1).$$

One can show by induction that every expression has a unique indexing: Certainly (a) has the unique indexing (a^1). Assuming expressions b and c have unique indexings, obtain an indexing for (bc) by increasing each index in b by the amount $|c|$. Since $|bc|$ is a fixed size and indices must decrease from left to right, the indexing is unique.

To represent a computation of $i(x)$ for some $i \in \mathcal{A}_j$, our plan is to substitute into each indexed expression a term of the form $j \restriction V_\alpha$, for α large enough. The following lemma tells us what "large enough" needs to be.

Lemma 114. *For each $n \in \omega$, $j \restriction V_{\kappa_n+\omega\cdot n} \subseteq V_{\kappa_{n+1}+\omega\cdot n}$.*

Proof. For each $x \in V_{\kappa_n+\omega\cdot n}$,

$$j(x) \in j\left(V_{\kappa_n+\omega\cdot n}\right) = V_{\kappa_{n+1}+\omega\cdot n}.$$

Clearly, then, all ordered pairs of the form $(x, j(x))$ for $x \in V_{\kappa_n+\omega\cdot n}$, belong to $V_{\kappa_{n+1}+\omega\cdot n}$. □

Next, we define a function that will facilitate our intended substitutions, in light of the Lemma.

Definition 21. Define $t : V \to \omega$ by

$$t(z) = \textit{ least } m \in \omega \textit{ such that } \operatorname{rank} z < \kappa_m$$

We define a substitution function **Sub** : (indexed expressions) $\to V$, so that for each ex, z, $\mathbf{Sub}(ex, z)$ is obtained in the following way: For each a^i in ex, substitute $j \restriction V_{\kappa_{t(z)+i}+\omega\cdot i}$ for a^i, and consider concatenations of these restrictions of j to be functional application.

For example, let $ex = (x^3)((x^2)(x^1))$ be an indexed expression, and let $z = \kappa+\kappa$. Then $t(z) = 1$ since $\kappa_1 = j(\kappa)$ is the least cardinal in the critical sequence above z. We have:

$$\begin{aligned}\mathbf{Sub}(ex, z) &= j \restriction V_{\kappa_{t(z)+3}+\omega\cdot 3}\left(j \restriction V_{\kappa_{t(z)+2}+\omega\cdot 2}\left(j \restriction V_{\kappa_{t(z)+1}+\omega}\right)\right)\\ &= j \restriction V_{\kappa_4+\omega\cdot 3}\left(j \restriction V_{\kappa_3+\omega\cdot 2}\left(j \restriction V_{\kappa_2+\omega}\right)\right).\end{aligned}$$

Now let $A^0_j = \{\mathbf{Sub}(ex, z) \mid ex$ is an indexed expression and $z \in V\}$. Note that the range of each $\mathbf{Sub}(ex, z)$ is bounded in V. Therefore, A^0_j is a j-class, and provides us with (a j-class of) codes for $\mathcal{A}_j$ in the following way: For each $i \in \mathcal{A}_j$, let ex be an expression realized by i. Then for any $z \in V$, $i(z)$ can be computed by $\mathbf{Sub}(ex, z)(z)$. (It can be shown that this computation does not depend on the representing expression ex—see [18].)

The class A^0_j of codes is good enough to provide us with j-class versions of the collections/maps:

(a) $\mathrm{crit}(\mathcal{A}_j)$
(b) the increasing enumeration $e : \omega \to \mathrm{crit}(\mathcal{A}_j)$
(c) the operator $m_{i,k}$ defined to be the least n for which $i(e(n)) \neq k(e(n))$.
(d) the metric $d : \mathcal{A}_j \times \mathcal{A}_j \to \mathbb{R}$.

For (a), letting $\mathrm{ord}(z)$ be the formula "z is an ordinal,"

$$\mathrm{crit}(\mathcal{A}_j) = \{\alpha \mid \exists ex, z\ (\mathrm{ord}(z) \wedge \mathbf{Sub}(ex, z)(z) \neq z \wedge \forall \delta < z\ (\mathbf{Sub}(ex, z)(\delta) = \delta))\},$$

Parts (b)–(d) follow easily from (a). ☐

Let us also remark that the seemingly larger completion $\bar{\mathcal{A}}_j$ of $\mathcal{A}_j$ is not difficult to formalize since it has been formally defined as a subspace of ω^ω.

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Department of Mathematics and Computer Science
Maharishi University of Management
Fairfield, IA 52556, USA *Email:* pcorazza@mum.edu