Recent Progress in the Mathematical Analysis of the Infinite[1]

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One theme of research that has developed in the recent history of mathematics is the study of the infinite. In mathematics, the notion of the infinite is approached by studying sets having infinitely many members. The evolution of the mathematical investigation of the infinite has uncovered a fundamental question—known as the *Problem of Large Cardinals*—for which the usual tools and techniques of mathematics no longer seem to be adequate. A deeper insight into the structure of the mathematical universe itself seems to be necessary to provide a solution. This article discusses how Maharishi Vedic Science has been used to provide the necessary insight, leading to a solution to the Problem of Large Cardinals—a solution that has appeared recently in the mathematics literature.

The Classical Theory

The classical theory of the infinite began about 150 years ago. Progress in the classical theory is indicated by three significant milestones.

Milestone #1: The Discovery That Infinity Exists*.* Prior to the second half of the nineteenth century, the subject of infinite sets was a forbidden topic. It was believed, for example, that, although we can imagine the natural numbers 1, 2, 3, . . . continuing on indefinitely, any notion of a *single*, *completed set* containing all the natural numbers must be viewed as fanciful, lying outside the domain of rigorous mathematics.

There were many reasons for this taboo (see Hallett, 1988). First, as Aristotle observed nearly two thousand years ago in his *Metaphysics* (1941, book 9, chapter 6), we don't find completed infinities in nature (for instance, seasons return year after year, but at no point can it be said that "infinitely many seasons have passed," even though they could potentially continue forever), so one would not expect such a notion to make sense in mathematics either. A second reason had to do with theological beliefs: One objection (Hallet, 1988, p. 13) asserted that a study of a completed infinity amounts to a study of God; but God cannot be bound by the mathematical conclusions of man. A third reason was that analysis of completed infinities seems to lead to paradoxes: For instance, the infinite sum $1 - 1 + 1 - 1 \ldots$ appears to have two values, depending on how parentheses are inserted:

$$(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0$$
$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 1$$

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The work of Georg Cantor (1845 – 1918) and other pioneers working during this time dispelled these doubts. Cantor addressed not only the mathematical issues, but wrote extensive rebuttals to all philosophical and theological doubts that had been raised. In response to the long-held argument of Aristotle, Cantor observed that the long, "potentially" infinite sequence 1, 2, 3, . . . of natural numbers *presupposes* that all the natural numbers already exist as a completed collection, an "actual" infinite:

> . . . in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence depends on it. (Rucker, 1982, p. 3)

Cantor and other researchers of that period resolved the apparent mathematical paradoxes that had been put forth. For instance, the problem about computing the sum of the terms 1 -1 + 1 – 1 + . . . was resolved some years before Cantor's campaign by observing that, while some infinite summations like this *do* indeed have a value—a *sum*—others do not. We would not expect the summation 1 + 2 + 3 + 4 + . . . to have a particular natural number value because *every natural number* is already a term in the series, so the "sum" would have to be bigger than any number. Such summations—among them, the series 1 – 1 + 1 – 1 + . . . —are said to *diverge*, to have no final sum. The precise notion of *convergence*, originated by A.L. Cauchy (1789 – 1857) in the first half of the nineteenth century, which put to rest this paradox, is now a core element of calculus and higher mathematics.

Cantor also addressed, in several ways, theological and philosophical concerns about studying completed infinities. One of his key arguments, which eventually transformed the way mathematics was understood, was that mathematics is not inherently tied to any of the ways in which it is interpreted or applied. A large bulk of the problems that have been researched in mathematics have arisen from the sciences: The use of mathematical models to understand and predict nature's behavior has suggested hundreds of mathematical problems and has motivated significant advancements in mathematics itself. But the mathematics in such cases, Cantor argued, is a *description* of natural phenomena, not the phenomena themselves. This separation of mathematics from its applications freed mathematics from irrelevant restrictions on the allowed topics of mathematical study, and, in particular, from non-mathematical views about non-mathematical notions of "the infinite."

Cantor's heroic efforts to argue the case for the mathematical infinite did not bear fruit, however, until a very practical need at the foundation of the mathematics of the day loomed large and was recognized as solvable only through the use of completed infinities. In Cantor's time, the basic tenets or "theorems" of calculus—the same calculus that is studied in mathematics curricula today—were well-understood but could not be rigorously proven using the tools available at the time. The difficulty, as Cantor and Dedekind observed, could be traced to an imprecise understanding of the idea of a "real number," a quantified mathematical point on a line. It was discovered that to give a precise definition of the real numbers *required* completed infinities (this point is familiar even in high school mathematics today:

To represent a number like π precisely as a decimal requires infinitely many decimal places).

With the recognition that completed infinities are necessary in mathematics, the first milestone in the classical era was achieved. With this discovery in place, Cantor went on to unveil another surprise about the infinite.

Milestone #2: There is an endless hierarchy of infinite sizes. Allowing the mathematical study of infinite *sets* gave the mathematician freedom to perform all the operations upon infinite sets that are ordinarily performed on finite sets. One such operation is the formation of the *power set* of a given set.

We can illustrate the power set operation with a simple example. Consider the set $S = \{1, 3, 4\}$. The set S has three elements. The *subsets* of S are $\{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}$, and the empty set, denoted Ø. These subsets can be arranged into a new set, denoted $P(S)$, called the *power set of S*:

$$P(S) = \{\, \emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\} \,\}.$$

The set $P(S)$ has *eight* elements, and so is larger than the set S that we started with. In Cantor's time, it was well known that the power set of any finite set is always bigger than the original set: For any *finite* set S, $P(S) > S$.

Cantor's suprising discovery was that the same could be said about *infinite* sets: For any set, finite or infinite, the power set must always be bigger.

The natural question, raised vigorously by Cantor's contemporaries (and perhaps equally vigorously by students even today) is: How can one infinite set be "bigger than" another?

Cantor was able to answer this question by developing a rigorous theory of infinite sets, which provided a precise definition of what it means for two sets to have the *same size.* Roughly speaking, two sets are said to have the same size if their elements can be matched up one for one. For instance, the set $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$ of natural numbers is shown to have the same size as the set $\mathbf{W} = \{0, 1, 2, 3, 4, \ldots\}$ of whole numbers by matching the elements of **N** with those of **W** in the following way:

N		**W**
1	→	0
2	→	1
3	→	2
.		.
.		.
.		.
n	→	$n - 1$
.		.
.		.
.		.

Cantor then showed, with very clever reasoning, that, for any (infinite) set S, it is *impossible* for there to be a one-to-one correspondence between the elements of S and the elements of its power set. Therefore, in particular, the power set $P(\mathbf{N})$ of the infinite set $\mathbf{N}$ of natural numbers represents a *bigger size* of infinity than that of $\mathbf{N}$ itself. And his reasoning opens the door to yet bigger infinite sizes, since one can then apply the power set operation to $P(\mathbf{N})$ to obtain a still larger infinite set. In fact, there is an infinite hierarchy of ever larger infinite sizes:

$$\mathbf{N} < P(\mathbf{N}) < P(P(\mathbf{N})) < \ldots$$

The discovery that, for any set—even any *infinite* set—its power set is always bigger is known today as *Cantor's Theorem.* Cantor's Theorem marks the second big milestone in the classical theory of the infinite. From this second milestone we learn that not only does the mathematical infinite *exist*, but it also has a nature, a texture, an internal multiplicity, and even its own internal transformational dynamics.

Cantor's discoveries brought a long-sought sense of completion to the business of pure mathematics. Yet, soon after this sense of completion and balance had taken hold, a flaw was discovered—a flaw that would surprise even Cantor. To correct the problem, the foundations of mathematics were led to the third major milestone of this classical period.

Milestone #3: To understand the very idea of a set, it is necessary to understand wholeness, the ultimate infinite. As Cantor developed his theory of infinite sets, he apparently did not think to examine too closely his own definition of the concept of a *set.* Like most students of mathematics (and even many mathematicians) today, he simply assumed that the meaning was obvious: A set is simply a collection of objects. At the turn of the century, Bertrand Russell (1906) and others noticed, however, that this imprecise definition is flawed and leads to paradoxes that undermine the consistency of mathematics itself: Using Cantor's naïve notion of a set, Russell demonstrated that a certain paradoxical set T must exist (T is defined to be the "set" consisting of all sets that are not members of themselves). What makes T paradoxical is that one can prove that it has the following property:

T is a member of itself if and only if T is *not* a member of itself.

If a set with an inconsistent property such as this were allowed into the mathematical universe, it would lead to an inconsistent mathematics, making it possible to prove absolutely anything.

To resolve the paradox, the approach that was taken was to return to Cantor's vision of the infinite. Cantor had not only shown that there is an endless hierarchy of infinite sizes—or infinite *cardinals* as they are called—but also declared (Hallett, p. 42) that this multiplicity of infinite sizes in no way represents the *ultimate* infinite. For Cantor, the *Absolute Infinite*—the totality of all possible mathematics, beyond the possibility of increase or diminution, and beyond all mathematical determination—was the ultimate infinite, and provided the context in which mathematics should be done. Using Cantor's Absolute Infinite and a number of its properties as a guiding intuition, early set theorists developed an intuitive model for the universe of mathematics, the *universe of sets*, denoted *V.* The idea was that the legitimate sets, the sets that are to be used in mathematics, belong to *V*; but paradoxical sets would not appear in *V*.

With this intuitive model *V,* researchers formulated a collection of axioms that express the essential characteristics of *V.* The axioms describe which sets exist and how to obtain new sets from already existing sets. The axioms that were developed in this way, now known as the *Zermelo-Fraenkel axioms with the Axiom of Choice*, or *ZFC*, were sufficiently complete to support the formal construction of the stages of *V*, transforming the stages of *V* from the realm of motivating intuition into formal, rigorously defined mathematical structures.

The construction of *V* is simple and elegant. *V* is built in stages, V_0, V_1, V_2, . . .; *V* itself is obtained by putting together all the stages, that is, by forming the *union* of the stages. Stage V_0 is just the empty set Ø. Then, each successive stage is obtained from the previous stage by taking the power set of the previous stage; it follows that each successive stage *includes* all previous stages. This strategy leads to the following construction:

$$V_0 = \emptyset, \quad V_1 = P(V_0) = \{\emptyset\}, \quad V_2 = P(V_1) = \{\emptyset, \{\emptyset\}\} \quad \ldots$$

Basing their intuition on long years of experience with sets, coupled with the properties of sets that could be seen to hold in the universe *V*, the fathers of modern set theory formulated the axioms of ZFC. Below is a sampling of some of these axioms.

Axiom of Pairing. If *A* and *B* are sets, there is another set *C* whose only elements are *A* and *B*; in other words, there is a set $C = \{A,B\}$.

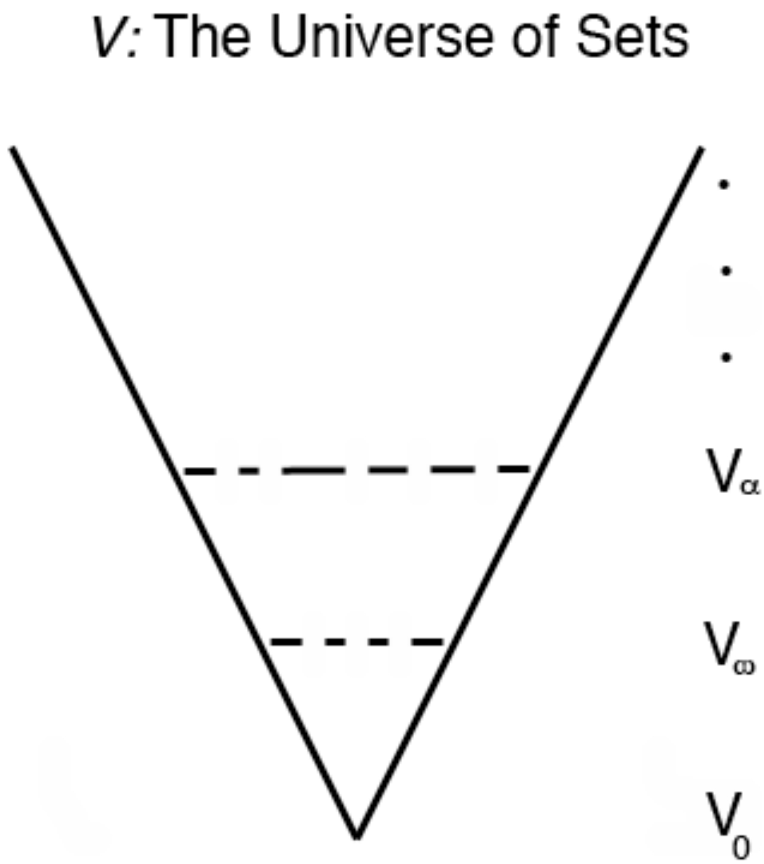
Power Set Axiom. If *A* is a set, then the collection $P(A)$ of all subsets of *A* is also set.

Axiom of Infinity. There is an infinite set.

The ZFC axioms, together with their natural model *V*, have provided a powerful unification of all areas of mathematics. This is because

- Every mathematical notion can be represented as a set.
- Every set that is used in mathematics belongs to *V.*
- Every mathematical theorem in any known field of mathematics can be formulated in the language of sets and derived directly from the ZFC axioms.

In hindsight, the fact that so much emerged from the simple question, "What is a set?" is surprising. One might have expected that a more careful definition of the notion of a set would have solved the problem. Instead, the notion of "set" never was defined at all; in fact, in the solution we have outlined, a set is now to be understood as a *primitive*, an undefined notion, whose meaning emerges from the ZFC axioms. Another way to express this point is to say that a set is any member of the universe *V*. This means that the basic unit of all mathematics, the "point value" from which everything else is built—the notion of a *set*—can only be understood with reference to *wholeness*, with reference to the totality to which it belongs.[2]



The universe *V* represents the wholeness of all of mathematics, at least from the historical perspective that we have been discussing so far. *V* is in fact a concrete realization of Cantor's own vision of wholeness, of his Absolute Infinite: *V* contains all mathematical constructions; it is bigger than any conceivable infinite size; it is not subject to increase or decrease in size (that is, one cannot perform an operation

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[2] The principle behind this phenomenon is expressed by Maharishi (1996, p. 538): ". . . without reference to wholeness, parts will always remain undefined."

on *V* to produce something bigger or smaller). Most of these characteristics follow from the simple fact that *V* itself *is not a set!* It is too big to be a set. One way to see this is to consider what would happen if *V were* a set: If it were, then we could apply the power set operation to it to produce an even bigger set *P*(*V*)—a set that would have to be bigger than the universe itself, which already contains everything! The impossibility of such a consequence leads to the conclusion that *V* is, with respect to sets, *bigger than the biggest*, and therefore not a set.

In this way, the effort to provide a rigorous formulation of the notion of "set" resulted in the discovery of the biggest infinity of them all, the totality *V*, together with the laws that govern the unfoldment of sets within *V*—the ZFC axioms. This achievement marked the accomplishment of the third milestone in the classical theory of the infinite.

The Modern Era

Just as it seemed that the nature of the universe, the extent of mathematics, and the nature of the infinite were all beginning to be well understood, a new kind of infinity appeared. Certain combinations of the properties of the different infinite sizes—called *infinite cardinals*—were found to produce very strong notions of infinity, so strong that the ZFC axioms could not derive the existence of such infinities. Yet, these notions of enormous infinities started turning up as key elements in solutions to well-known research problems in analysis, topology and algebra. These infinities have come to be known as *large cardinals.*

Aleph Fixed Points. To give a sense of the enormity of large cardinals, we spend a moment here considering one property that all large cardinals have in common. We begin with notation for infinite cardinals. Just as the familiar whole numbers 0, 1, 2, 3, . . . are used to denote the sizes of finite sets (for instance, the size of the set {2, 9, 17} is 3), so likewise the sizes of infinite sets are specified using *infinite cardinal numbers*. Some of these infinite cardinals are:

$$\omega_0, \omega_1, \omega_2, \omega_3, \ldots$$

The smallest infinite size, ω_0 (pronounced "omega-zero"), is the size of the set **N** of natural numbers (and is also the size of the sets of whole numbers, of integers, of rationals, and of algebraic numbers). The bigger cardinals, which come after ω_0, represent sizes of much bigger sets. One well-known set that is of the bigger variety is the set **R** of real numbers; though it is impossible to determine exactly which of the cardinals ω_1, ω_2, ω_3, . . . is the exact size of **R**, it can be shown that the size of **R** must lie among these.

An easily observed feature of the list of infinite cardinals displayed above is that, at least at the beginning of the list, we find that the *index* of a cardinal is always smaller than the cardinal itself. For instance the index of ω_0 is 0, and certainly 0 is smaller than ω_0. Likewise, the index of ω_1 is 1, and 1 is smaller than ω_1. This obvious pattern continues far into the endless list of infinite cardinals. However, eventually, something new appears. Eventually, one arrives at a cardinal whose index is *equal to* the cardinal itself. In other words, there must exist a cardinal ω_κ with the property that

$$\kappa = \omega_\kappa.$$

Such a cardinal is called an *aleph fixed point.* Certainly this is an extraordinary property of infinite cardinals, and it is one that belongs to every large cardinal. However, the first aleph fixed point that one encounters on the list is not big enough to be a large cardinal. Nor is the second or third or even the ω_0th. In fact, no conceivable mathematical procedure could ever result in a precise specification of a large cardinal. And this limitation is not simply the result of lack of persistence or skill on the part of mathematicians. Rather, this limitation is a provable theoretical result: *It is impossible to prove from ZFC that any large cardinal exists at all.* (See Jech, 2003.)

Why, then, may we not conclude that such "large cardinals" simply don't exist? Strangely enough, large cardinals do appear as key elements in the solutions of quite a number of significant mathematical problems that have arisen in the past century, so they cannot be dismissed so easily. We provide a list of some of the large cardinals (see Jech, 2003) that have turned up in mathematical research—in increasing order of strength—and then give two examples of well-known mathematical problems whose solutions do depend on large cardinals.

Some Common Large Cardinals

- *Inaccessible*
- *Mahlo*
- *Weakly Compact*
- *Ramsey*
- *Measurable*
- *Strong*
- *Woodin*
- *Supercompact*
- *Extendible*
- *Huge*
- *Superhuge*
- *Super-n-huge for every natural number n*

Two Examples of Theorems That Depend on Large Cardinals. The many examples of mathematical theorems that are tied to large cardinals have the unfortunate characteristic of being difficult to understand for the non-expert. For the sake of the mathematically experienced reader, we take a short detour to examine two such examples. The reader who does not wish to follow this detour may safely skip to the next section.

The Normal Moore Space Conjecture. A *metric space* is a set that admits a *metric* or *distance function.* A familiar example is the real line **R** whose distance function *d* is defined using absolute value: The distance between reals x and y is the absolute value of their difference: $d(x, y) = |x - y|$. The Pythagorean Theorem is used to obtain the usual distance function for the plane **R x R**: The distance between points (x, y) and (u, v) in the plane is the square root of $(x - u)^2 + (y - v)^2$. And there are many other more advanced examples.

Metric spaces have a number of very nice properties that make them easier to work with than more general topological spaces. One such property is *normality*: In any metric space, disjoint closed sets can be separated by disjoint open sets.

A generalization of metric spaces, called *Moore spaces*, named after their inventor R. L. Moore (1882 – 1974), replaced the use of the metric in a topological space with a countable sequence of covers of the space (called a *development*) having special separation properties. A Moore space is defined to be a developable, regular, Hausdorff space.

Many of the nice properties of metric spaces also hold in Moore spaces, but Moore spaces are more general: Many examples of nonmetrizable Moore spaces are known (a topological space X is *metrizable* if a metric can defined on X that is compatible with the topology on X). Moreover, the known examples as of the mid-twentieth century of nonmetrizable Moore spaces were also non-normal. In this context, the question arose: Is every normal Moore space metrizable? The conjecture that this is indeed the case is called the *Normal Moore Space Conjecture (NMSC).*

The conjecture was settled through the use of advanced techniques in set theory. There are many aspects of the solution, but the result we wish to mention here is that the truth of NMSC is intimately tied to the existence (or at least the consistency of) large cardinals. More precisely:

Theorem. [Nyikos, Fleissner]

1. Assuming there is a strongly compact cardinal, there is a model of set theory (a universe of mathematics) in which NMSC holds.
2. If NMSC is true, then there is a model of set theory in which there is a measurable cardinal.

The conclusion is that if NMSC is true, then large cardinals must be lurking in at least some of the universes of set theory, and, conversely, if there exists a sufficiently strong large cardinal, then NMSC must hold in at least some of the universes of set theory. For a detailed discussion on NMSC, see Kunen & Vaughan (1984).

Determinacy of Analytic Games. An *infinite game* based on a subset A of the unit interval [0,1] involves two players, Player I and Player II, who take turns picking elements of the two-element set {0,1}. Their successive plays result in an infinite sequence of 0s and 1s, which we denote $s = s_1s_2s_3 \dots$. Player I wins the game if the sum $s_1/2 + s_2/4 + s_3/8 + \dots$ lies in A; otherwise Player II wins. The set A is said to be *determined* if one of the two players has a *winning strategy.*

The natural question that arises in this context is, Is *every* subset of [0,1] determined? The Axiom of Choice shows that the answer is "no," but there are other related questions for which a positive answer does not conflict with AC.

To arrive at these related questions, we consider "nicely defined" subsets of [0,1]. These nice sets are called the *Borel sets*; they are obtained by beginning with the closed subsets of [0,1], then considering all possible countable unions of closed sets, then all countable intersections of these, all countable unions of these, and so forth. After the process is repeated ω_1 times, no more new sets can be obtained by this procedure; the sets that are obtained in this way are the Borel sets. One can obtain a somewhat larger class than the Borel sets—a class that also has many nice properties—by considering *continuous images* of Borel sets. The class of all continuous images of Borel sets is called the class of *analytic sets.*

Two questions that relate the notion of determinacy with these classes of sets of reals are:

1. Is every Borel set determined?
2. Is every analytic set determined?

The work of set theorists D. A. Martin and L. A. Harrington established the following:

Theorem. [Martin, Harrington]

1. Every Borel set is determined.
2. If there is a measurable cardinal, every analytic set is determined.
3. If every analytic set is determined, there is a model of set theory in which there is a proper class of weakly compact cardinals.

This theorem demonstrates, perhaps even more dramatically than the results on NMSC, how large cardinals can be inextricably tied to the solution of a research problem in mathematics. For a fuller discussion of determinacy, see Jech (2003).

Where Do Large Cardinals Come From?

In the 1960s, many new kinds of large cardinals began to emerge from various quarters. At this time, a more pressing need was felt in the set-theoretic community to come to terms with this phenomenon. Reactions to the challenge varied among researchers. Among those who participated in moving toward a solution, some hoped to "debunk" large cardinals, while others sought to provide a foundational basis for them. Some in the former category—including some of the brightest set theorists of the time—dedicated many years in an attempt to prove that some or all large cardinals are inconsistent with ZFC. And, although many deep results in set theory did get established in these large projects, none of them resulted in a proof that any large cardinal, big or small, is inconsistent.

The view of the other group of researchers was that some or all large cardinals should indeed be thought of as an authentic part of the mathematical universe. To travel this course required answers to the following questions:

1. Which large cardinals are *legitimate?* It is possible that some may have arisen in such an arbitrary and ill-motivated way that there is no justification for them.
2. How can those large cardinals that are legitimate be derived from the foundational axioms? Certainly ZFC is not strong enough to derive any large cardinal, but can we find an axiom (or possibly several) that expresses some intuitively clear truth about the universe and, at the same time, is strong enough to provide a proof of the existence of these large cardinals?

These are the questions of the *Problem of Large Cardinals.* To address these questions, many researchers in the field turned to Cantor's original vision of the universe V as a guide to intuition—what is it about the structure of V that would suggest that large cardinals should really exist?

One approach was to recognize that V itself represents the "ultimate infinite" and an intuition that emerged was that large cardinals are "reflections" of that totality into the realm of ordinary sets. Large cardinal properties that seemed to hold true of V itself were thereby legitimized, and the result was that some of the smaller large cardinals, such as inaccessible and Mahlo cardinals, found a high degree of acceptance.

Another approach was the observation that many large cardinal properties happen to belong to the smallest infinite cardinal, ω_0, the size of the set of natural numbers. Since ω_0 is so small—being the least among the infinite cardinals ω_0, ω_1, ω_2, ω_3, . . . —it can't actually be a large cardinal. Nevertheless, it does possess many large cardinal characteristics, which, were they to belong to any cardinal λ bigger than ω_0, would cause λ to be a *bona fide* large cardinal. It is accurate to say that ω_0 is to the world of the finite what large cardinals are to the world of the infinite; indeed, to the world of the finite, ω_0 appears to be a "large cardinal."[3] It is for this reason, one can argue, that ω_0 exhibits so many large cardinal characteristics. In any case, the fact that these properties belong to one infinite cardinal was used as a justification for the belief that *other* infinite cardinals should have the same properties.

The logic for this justification again comes from Cantor; in Cantor's vision, the landscape of infinite cardinals exhibits a certain *uniformity*: Properties found to hold for one infinite cardinal should be found in other cardinals throughout the universe.

[3] The "world of the finite" is described by the axioms of ZFC with the Axiom of Infinity removed, denoted ZFC – Infinity—a natural model for this theory is V_ω (though there are other models of this theory in which infinite sets do exist, like V itself). Then, just as in the case of large cardinals relative to ZFC, it is *impossible to prove* the existence of any infinite set from ZFC – Infinity, but no one has ever proven, from ZFC – Infinity, that infinite sets do *not* exist. Carrying the analogy further, from our vantage point of ZFC, it is "obvious" that, the truth of the matter is that infinite sets *do* exist, but the theory ZFC – Infinity was unable to demonstrate this because its axiom system is not rich enough. The same can be argued regarding ZFC relative to large cardinals: From a certain vantage point (a vantage point we attempt to articulate in this article), it is "obvious" that large cardinals exist, but ZFC has not been able to demonstrate this point because the collection of ZFC axioms is not rich enough.

Justifying large cardinals on the basis of properties found to hold for ω_0 is known as *generalization.* Generalization was used to legitimize several large cardinals, such as weakly compact and measurable cardinals.

Efforts to justify large cardinals using such heuristics have met with limited success. The really big and often complex large cardinals, such as supercompact and superhuge cardinals, could not be justified using these approaches. Cantor's vision was able to carry us only so far in our understanding of the structure of the universe *V*—a fact that should not be surprising since Cantor himself was entirely unaware of the phenomenon of large cardinals.

To make further progress toward a solution to the Problem of Large Cardinals, the following questions naturally present themselves:

1. Beyond Cantor's vision of the wholeness of the universe *V*, what source of intuition can we draw upon to decide which large cardinals really do belong in the universe?
2. Can we draw upon this new source of intuition to help in the formulation of a new axiom for set theory that would provide an axiomatic basis for these large cardinals?

Insights from Maharishi Vedic Science

A natural approach to consider in addressing these questions is Maharishi Vedic Science. A Vedic mathematician's hunch, using this approach, might be something like this:

Everything to do with the infinite arises from
the self-interacting dynamics of wholeness.

In attempting to use this apparently non-mathematical principle as an intuitive guideline that could provide insight into the structure of the universe and even possibly a new axiom of set theory, we need to identify the mathematical analogues to the notions of "wholeness" and "self-interacting dynamics."

We have already seen that, from the mathematical point of view, *V* already naturally represents a kind of wholeness for mathematics. Examining its properties further, we can see even more clearly that it is a natural analogue to the notion of wholeness in Maharishi Vedic Science. The universe *V* is the source and container of all sets, and yet is not itself a set; since it is not a set, and therefore cannot be directly referred to in the formal theory, it exhibits the property of being *unmanifest.* Also, being bigger than any possible set, it also exhibits the properties of being *unbounded* and *bigger than the biggest*; and, being the container of all possible mathematical structures from any area of mathematics, it exhibits the quality of *ominipresence.* In addition, *V* is, in a natural sense, the *home of all the laws of nature* in the sense that the laws that govern the unfoldment of sets—the ZFC axioms—occur in *V* (coded as sets); moreover, *V* can regenerate its own stages using its own internally coded ZFC axioms, thereby expressing its *self-sufficient* quality.

Next, to represent transformational dynamics in a mathematical way, it is natural to consider the mathematical concept of a *function.* A function from one collection A to another collection B is a rule for uniquely associating elements of A with elements of B. As a simple example, if one were to take a straight piece of wire and bend it so that it forms a circle, one could represent this transformation with a function that assigns to the position of each point on the wire in its starting position the corresponding position of that same point after the wire has been bent; such a function gives a meaningful and precise description of the physical change applied to the wire. In a similar way, all types of transformation in the sciences are represented by functions.

With these analogies in mind, we can now ask whether the universe V, as it is presently understood in set theory, comes equipped, in some natural way, with some kind of function that transforms V to V, and that mirrors essential features of the dynamics of wholeness as described in Maharishi Vedic Science. If we can locate such a function, we can examine it closely and see if it provides hints about the origin of large cardinals.

To narrow the search somewhat and to aim for the fullest use of Maharishi Vedic Science, we will attempt to find a function j transforming V to V that has the some additional characteristics. The dynamics represented by j should

a. transform wholeness and yet leave wholeness *unchanged by the transformation*;[4]
b. be *unmanifest*;[5]
c. be *present at each point in the universe.*[6]

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[4] In the *Science of Being,* Maharishi (1966, p. 39) remarks, "This absolute state of pure consciousness is of unmanifested nature, which is ever maintained by cosmic law. Pure consciousness, pure Being, is maintained always as pure consciousness and pure Being, and yet It is transformed into all the different forms and phenomena."

[5] On this point, Maharishi (1990) remarks, "In the state of one-being-three we have the state of complete unified wakefulness. In this is the first value of transformation in the unmanifest value. When we say 'transformation,' we still mean this level is unmanifest."

[6] Maharishi remarks (1966, p. 29), "It has been said that Being is the ultimate reality of creation and that It is present in all strata of creation. It is present in all forms, words, smells, tastes and objects of touch; in everything experienced; in the senses of perception and organs of action; in all phenomena; in the doer and the work done; in all directions-north, south, east and west; in all times past, present and future; It is uniformly present. It is present in front of man, behind him, to left and right of him, above hime, below him and in him. Everywhere and in all circumstances Being, the essential constituent of creation, permeates everything."

Elsewhere, he states (Maharishi, 1994, p. 315), "The three-in-one structure of Samhita of Rishi, Devata, and Chhandas is the basis of all the Laws of Nature at every point in creation; it is the availability of all the Laws of Nature at any one point in the whole span of infinity of space and time."

To meet the first requirement, the function j must, as far as possible, preserve the integrity of the structure of V. Structure-preserving functions are a key notion in nearly every field of mathematics: Continuous functions preserve limits of sequences; homomorphisms preserve the operations of an algebraic structure; order-morphisms preserve the relation of an ordered structure. Likewise, whatever relationships exist within the structure of V should be preserved by j. At the same time, j must do *something*—one could mistakenly let j be simply the identity function that has no transforming effect at all. The identity function id is the function that assigns to each set x the value x itself: $\mathrm{id}(x) = x$. Certainly the identity function preserves all relationships in V, but no transformation occurs either. So, we require j to be a *non-trivial*, structure-preserving function: There must be some x for which $j(x) \neq x$.

For the second point, implementing the idea that j should be *unmanifest* is very natural in the mathematical context. A function with the enormous scope that a function from V to V must have prevents it from being a function in the ordinary sense. The usual functions in mathematics are actually members of V (represented in a standard way as sets). But a function defined on all of V cannot be represented as a set. A usual maneuver to get a handle on such enormous transformations in set theory is to consider whether such a transformation is *definable*. Definability of such a function allows one to say things about the function almost as if it were an actual set. Requiring our function to be *unmanifest* can naturally be done by insisting that it *not* be definable. The Vedic Science perspective suggests that the transformational dynamics represented by j should be hidden from ordinary view and therefore, mathematically speaking, undefinable.

Finally, we wish to ensure that the behavior of our function j, being undefinable, is not divorced from the reality of sets in V; j needs to be somehow "present" everywhere within V. This requirement is realized by declaring that the *restriction of j to any set in V also belongs to V.* The function j itself does not belong to V; it's not even definable in V. But we require every restriction of j to a set to belong to V.

Summarizing these requirements, we can say that we are looking for some evidence of a naturally occurring function j: $V \to V$ with these characteristics:

a. j preserves the internal structure of V
b. j is undefinable in V
c. the restriction $j \mid X$ of j to any set X in V must itself belong to V.

Locating the Seed for a Solution to the Problem of Large Cardinals

In the 1960s, William Lawvere (1969) observed that Cantor's Axiom of Infinity is actually *equivalent* to the existence of a certain very interesting function j: $V \to V$. This j is obtained as the composition[7] of two *functors F* and *G* which have a highly coherent relationship with each other (they are *adjoint functors*):

[7] The *composition* $g \circ f$ of two functions f: $A \to B$ and g: $B \to C$ is a new function h: $A \to C$ defined by $h(a) = g(f(a))$ for any a in A.

$$j = G \circ F.$$

A functor is a special kind of function that exhibits special characteristics when it is applied to other *functions*; it is perfectly legitimate here to think of them as just another kind of function. Because of the adjoint relationship between G and F, both exhibit strong preservation properties: Relationships in the domains of each of the functors are preserved by these functors. In the language of category theory, F preserves all *limits* and G preserves all *colimits*.[8] This gives a hint that even the existence of an infinite set implies that certain truth-preserving dynamics are at work within the wholeness of V.

It is important to keep in mind that this j has a special status among functions that one could define from V to V. The function j is special because its existence is *equivalent to* the Axiom of Infinity. The Axiom of Infinity simply states that there is an infinite set—basically, asserting the existence of **N**. This is a very localized phenomenon: A single set is declared to exist someplace in the universe. On the other hand, this particular j provides transformational dynamics of the entire universe, exhibiting important structure-preserving characteristics. Its equivalence with the Axiom of Infinity tells us that an essential property of V is the existence of a companion transformation j, with structure-preserving characteristics. The seed of the vision of wholeness from the cognition of the ancient seers seems therefore to be already present in the design of the universe V.

When we look at the properties exhibited by Lawvere's functor j through the eyes of a Vedic mathematician, however, we notice that something about j seems amiss. To be a full expression of the transformational dynamics that naturally belong to wholeness, as understood in Maharishi Vedic Science, we expect j to have more fully developed properties. In particular, it would be natural to expect that j itself, rather than just its factors F and G, should exhibit strong preservation properties. This gap between what we expect to find based on our guiding intuition and what we actually find suggests a direction for improvement.

We are expecting that, by implementing principles of Vedic Science in a mathematical context, motivation for large cardinals will naturally appear. What we have now discovered is that our candidate for giving mathematical expression to the dynamics of wholeness is missing some desirable characteristics. A natural hope is that if we attempt to strengthen the properties of j, we will strengthen the ZFC Axiom of Infinity in such a way that deeper properties of wholeness can be brought to light and naturally illuminate the issue of large cardinals.

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[8] In the category of sets, some examples of limits are Cartesian products and pullback diagrams; examples of colimits are disjoint unions and natural maps from a set to its set of equivalence classes mod an equivalence relation (*coequalizers*). To illustrate with two simple examples from the category of sets, when F preserves products, it satisfies $F(A \times B) = F(A) \times F(B)$ for any sets A and B. When G preserves disjoint unions, then, whenever A and B are disjoint, we have $G(A \cup B) = G(A) \cup G(B)$.

Enhancing the Preservation Properties of *j*

In the 1970s, Blass and Trnkova (see Blass, 1976) took the step we have just been discussing. They asked, What happens if a j: $V \rightarrow V$ is required to have essentially the same preservation properties as those of the factors F and G of Lawvere's functor j? Such a function, in precise mathematical terms, is called an *exact functor*; an exact functor preserves all *finite* limits and colimits. Blass and Trnkova were able to prove the following interesting theorem:

Theorem. [Blass, Trnkova] The following are equivalent:

1. There is a nontrivial exact functor j: $V \rightarrow V$.
2. There is a measurable cardinal.

The theorem shows that our program of enhancing Lawvere's functor so that it exhibits more preservation properties is on the right track. The Blass-Trnkova functor is already a much fuller expression of the functional dynamics we are seeking. If we replace the Lawvere version of the Axiom of Infinity that we stated before with this new Blass-Trnkova version: "There is a nontrivial exact functor j: $V \rightarrow V$," we have as an immediate and perfectly natural consequence that the universe must contain a measurable cardinal.

The Wholeness Axiom

Taking the next step, we can ask: Can the preservation properties of the Blass-Trnkova functor be enhanced even further so that j preserves *all* properties of V? Can we even require that j have the added characteristics that it is undefinable but its restrictions to sets belong to V?

In the language of set theory, the natural way to require a function to preserve *all* properties is to make it an *elementary embedding.* Therefore, the requirements on a function from V to V mentioned above will be met if we require j to be an undefinable elementary embedding whose restrictions lie in V. We state the existence of such a function as a new, enhanced Axiom of Infinity, which we call the *Wholeness Axiom*, and then examine what new features of wholeness are brought to light by adding it to the existing foundational ZFC axioms.

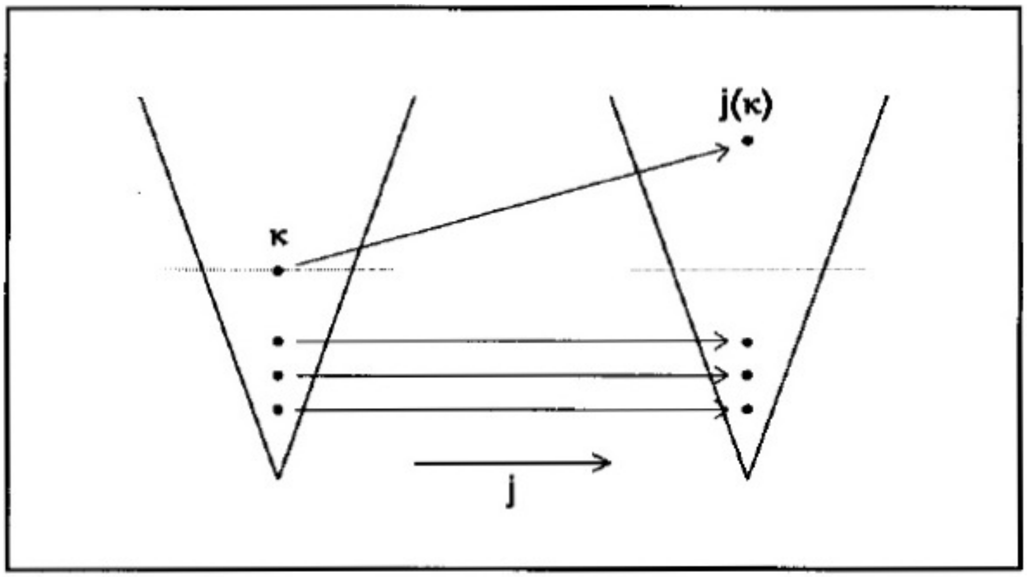
Wholeness Axiom (WA). There is a nontrivial elementary embedding j: $V \rightarrow V$ with the property that for every set X, the restriction $j \mid X$ is also a set.

Notice that the requirement that j should be undefinable has not been mentioned in the definition of the Wholeness Axiom. The reason is that undefinability of j actually can be *proven*: It follows from a theorem by K. Kunen (1971) that if such an embedding exists at all, it cannot be definable.[9]

Also notice that we have required j to be *nontrivial.* This means that for some set x, $j(x) \neq x$; we say that *j moves x.* In fact, it can be shown that some *infinite cardinal* is

[9] Technically, it cannot even be *weakly definable.* See Corazza (2006).

moved by *j.* The *least* cardinal moved by *j* is called the *critical point of j* and is usually denoted by the Greek letter κ (pronounced "kappa"). It can be shown that *j* moves κ to another infinite cardinal $j(\kappa)$ that is bigger than κ; that is, $j(\kappa) > \kappa$.



We can now state the main result of this article, which shows that our efforts to provide a solution to the Problem of Large Cardinals have been successful:

Theorem. [Corazza] Assume WA and let $j: V \rightarrow V$ denote the WA-embedding. Let κ be the first cardinal moved by *j*. Then κ is the κth cardinal that is super-*n*-huge for every *n* in **N**. In particular, the critical point κ has virtually all large cardinal properties.

The theorem tells us where large cardinals come from: In the transformational dynamics of the wholeness *V*, large cardinal properties arise as the characteristics of the first point that is moved under the transformation. At the precise moment when complete silence, represented by the behavior of *j as simply the identity function below* κ, changes to dynamism—in that first impulse of activity, represented by and concentrated at κ—we find that κ is filled with a powerful dynamism, evidenced by the fact that it has essentially all known large cardinal properties.

In fact, it is accurate to say that κ acquires the status of a point in the universe which stands as a *representative* of the totality of *V.* This is seen by the fact that the stage V_κ of the universe indicated (and coded) by κ is in fact an *elementary submodel of V.* This means that everything that is true about *V* is also true about V_κ. It also means that V_κ *knows* all there is to know about the wholeness *V*. Truly, κ can declare "I am wholeness."

These dynamics closely parallel the way in which the totality, as described in Maharishi Vedic Science, represented by the Sanskrit letter A, collapses to its own point, K, in the emergence of manifest existence.

The dynamism that arises in κ in the unfoldment of the embedding j has further parallels to the dynamics of AK in Maharishi Vedic Science. It turns out that further interactions between j and κ lead to a plethora of other *derived* embeddings, also defined on V, which collectively code up a sequence of values that live in the stage V_κ. This sequence of values is known as a *Laver sequence*. And, it can be shown that, from a Laver sequence, every set in the universe can be located. Indeed, if

$$S = \langle s_0, s_1, s_2, \ldots, s_\alpha, \ldots \rangle_{\alpha<\kappa}$$

is the Laver sequence that is obtained from j and its derived embeddings, then, for any set X in V, there is a derived embedding i with the property that the κth term of the expanded sequence $i(S)$ must be X itself. These dynamics parallel the sequential unfoldment of the Veda from AK, giving rise, in turn, to all of creation. We can summarize these results as follows:

Theorem. [Corazza] Assume WA. Let j be the WA-embedding and let κ denote the first cardinal moved by j. Then there is a κ-sequence $S = \langle s_0, s_1, s_2, \ldots, s_\alpha, \ldots \rangle_{\alpha<\kappa}$ with the following property: For every set X, there is a derived embedding i such that if $i(S)$ denotes the sequence obtained by elementarily expanding S by i, then for any set X, X occurs as the κth term of $i(S)$; that is,

$$X = i(S)_\kappa$$

As a final point of interest, once the existence of a WA-embedding is known, the structure of the universe V is seen in a new way. Whereas before, even the existence of a single large cardinal—even a puny inaccessible cardinal—was cause for doubt, now in the presence of a WA-embedding, *almost all cardinals in the universe are large cardinals!*

Theorem. [Corazza] Assume WA and let j denote the WA-embedding. The sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is unbounded in V and each term is a WA-cardinal. Morever, each of these cardinals λ admits a normal measure with the property that the set of cardinals below λ that are super-n-huge for every n has normal measure 1. More succinctly, *almost all* cardinals in the universe are super-n-huge for every n.

Conclusion

In this article, we have reviewed the evolution of the mathematical analysis of the infinite. The classical period in this history achieved important milestones, including the initial recognition that infinitely many objects could be collected together into a single set; that there are many different sizes of infinite sets; and that all of mathematics can be viewed as taking place within—indeed, *originating* within—a vast wholeness V, beyond the limits of any particular set or infinite size. The modern era of this analysis began with the discovery of large cardinals and their underivability from the axioms of set theory. A persistent theme in this period has been the quest to provide an axiomatic account for the presence of large cardinals in the universe.

We found that Cantor's vision of the universe of mathematics as an embodiment of the Absolute Infinite was able to guide the mathematical formulation of the axioms of ZFC and even provided techniques for justifying many of the smaller large cardinals. However, the need for a deeper insight into the structure of the wholeness V led us to seek a deepening of the intuition offered by Cantor.

Our proposed approach to address this need has been to make use of the principles of Maharishi Vedic Science. Maharishi Vedic Science identifies qualities and dynamics of wholeness itself—the wholeness of life and of consciousness. We have applied these to formulate a strategy for locating in the mathematical wholeness V heretofore unrecognized characteristics that could provide natural justification for large cardinals.

In this effort, we discovered in Lawvere's mid-1900s equivalent formulation of the Axiom of Infinity the beginnings of a natural parallel to the self-interacting dynamics of wholeness. Refining Lawvere's results to their logical conclusion, aiming toward the fullest possible representation of Maharishi Vedic Science principles within this context, we were led to the formulation of the Wholeness Axiom. The Wholeness Axiom asserts, in a precise mathematical way, that the wholeness V has at its unmanifest basis transformational dynamics (represented by j) which preserve the internal structure of V and which are present at every point within the universe.

From the Wholeness Axiom, we were able to derive a solution to the Problem of Large Cardinals. The solution shows that large cardinal properties arise as special properties that appear in the first impulse of change arising in the transformational dynamics embodied in j; in particular, that all large cardinal properties arise as properties of the first cardinal κ moved by j. Further examination of the interactions that occur between j and κ led to the observation that a certain sequence S of sets—known as a *Laver sequence*—naturally arises within the κth stage of the universe. This sequence has the special property that it encodes all sets in the universe. In particular, all sets in the universe can be seen to emerge through the interaction of j, κ, and S. These dynamics provide a strong analogy to the dynamics of wholeness described in Maharishi Vedic Science according to which wholeness, represented by the first letter A of Rk Veda, collapses to its point, K (the second letter), in the sequential unfoldment of the entire Veda, which in turn, through its own self-referral dynamics, gives rise to all manifest existence.

The evolution of mathematical insight about the infinite suggests another parallel—a parallel between the quest for the Infinite in mathematics and in the life of the individual. When the quest begins, the "infinite" seems to be an unrealistic fairy tale. In the mathematical world, actual infinities were barred from the mainstream for centuries; and later, in the modern era, large cardinals were viewed with great skepticism for many decades after their initial discovery. So likewise in the life of the individual there is often an initial skepticism at the prospect that something as grandiose as the "Infinite" could really exist, really be experienced.

Then, after a taste of the Infinite, a change occurs. In mathematics, once the infinite was recognized as a reality, the *nature* of the infinite was found to be vast and textured, and its unfolding dynamics were found to be contained in a wholeness vaster than even the biggest notion of "infinity." And then in the modern era, as more attention was paid to the phenomenon of large cardinals and certainty of their validity grew, they became a central tool in contemporary foundational research. In a similar way, once an individual has tasted the infinite and its influence in life, the doorway to a clear perception of the nature and hidden dynamics of the infinite gradually starts to open.

Finally, there is a deeper realization. In the world of mathematics, a notion of infinity that seemed hardly possible or imaginable is finally seen to be nearly omnipresent: Under the Wholeness Axiom, nearly all cardinals are disovered to have essentially all large cardinal properties. And in the life of the individual, the tall tale of the "Infinite," once ignored and pushed aside, at last is seen to be the truest of all realities, awake and present in every aspect of experience.

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