Vedic Wholeness and the Mathematical Universe: Maharishi's Vedic Science as a Tool For Research in the Foundations of Mathematics

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ABSTRACT

In this paper we make use of Maharishi Vedic Science as a tool to consolidate mathematical intuition about the structure of the mathematical universe and the nature of mathematical infinity. We consider the inability of ZFC to account for the presence of large cardinals in mathematics as a serious failing and suggest that the shortcoming at the root of this failure is the omission of any axiomatic principle describing the nature of the wholeness of the universe V. We then formulate such an axiomatic principle, called the Wholeness Axiom, *which is based on insights into the nature of wholeness derived from Maharishi Vedic Science and from the dynamics suggested by the strongest large cardinal axioms, well-known to set theorists. We illustrate how the universe V exhibits new dynamics in the presence of the Wholeness Axiom, more in accord with the dynamics of wholeness described in Maharishi Vedic Science. We then show that virtually all known large cardinal axioms are naturally accounted for by this new axiom. We conclude that Maharishi Vedic Science, used in conjunction with the frontiers of modern mathematics, can provide the profound intuition needed to build a truly successful foundation for all of mathematics.*

§1. Introduction

> *If the expansion of rishi, devata, and chhandas into the infinite universe does not remain in contact with the source, then the goal of expansion will not be achieved.*
>
> (Maharishi 1991)

> *And do you not also give the name dialectician to the man who is able to exact an account of the essence of each thing? And will you not say that the one who is unable to do this...does not possess full reason and intelligence about the matter?*
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> Plato, *The Republic* (SN 534)

For nearly 100 years, mathematicians interested in the foundations of mathematics have sought a simple set of axioms from which the rest of mathematics could be derived. Georg Cantor, the founder of modern set theory, was among the first to notice that the fundamental concepts used in mathematics—numbers, points, lines, circles, ordered pairs, functions—could be formulated as sets. His insight led to the conclusion that a theory of sets could provide a foundation for mathematics.

Unfortunately, in Cantor's time, the notion of sets was not well understood; the common idea that a set is simply any collection of objects led to logical contradictions. No direct definition of set seemed to avoid basic paradoxes. As an alternative, mathematicians at the turn of the century devised a set of axioms which would describe properties that sets ought to have; these axioms would then provide a basis for proving theorems about sets, and hence about all objects of study in mathematics.

The set of axioms which has become most widely accepted as the foundation for set theory is known as *Zermelo-Fraenkel Set Theory with the Axiom of Choice*, or ZFC for short. In addition to setting forth basic properties of sets, these axioms have, buried within them, "instructions" for building a universe of sets, a universe in which all mathematical objects could, in principle, be located. In order to indicate that the construction of sets begins with the merest point value, the empty set, and expands outward to generate all possible sets, the universe of sets is denoted by the letter *V*.

As a unifying foundation, ZFC, together with its universe *V*, has been highly successful. Yet, in the past few decades, several advances in mathematics have challenged its adequacy as a foundation. One of the most serious concerns has been the discovery of extremely large infinite sets, called *large cardinals*, whose existence cannot be proven from ZFC, yet whose central presence in a significant portion of mainstream mathematics makes it unreasonable to simply deny their existence. It was the hope of many set theorists that an "intuitively evident" principle would emerge that would provide sufficient motivation for including (or excluding) large cardinal axioms among basic axioms of set theory. Efforts to formulate such motivation have been only partially successful; the problem has been that there is no fundamental intuition concerning the nature of enormous mathematical infinities that is generally agreed upon by experts in Foundations—even less so among mathematicians generally.

Traditionally, mathematicians have derived their mathematical intuition on the basis of long years of experience with the objects of study in their respective fields. Certainly the axioms of ZFC arose from an intuitive familiarity with sets; the axioms had to be formulated so as to preserve this familiarity while eliminating undesirable paradoxes. But how does one decide, on an intuitive basis, whether certain types of enormous infinities exist or belong in the universe? An evaluation of the *consequences* of assuming—or not assuming—that various large cardinals exist has not helped to answer the question. [1]

The general feeling in the set theory community concerning the universe of sets is that it is supposed to represent, in an imprecise sense, the "real" world. Sets in the universe should combine the way we expect "real" sets in the "real" world to combine. This "real world" is a combination of the natural world and the world of mathematics as it has developed through its long history.[2] Certainly, observing the physical world tells us how to form the union of two disjoint sets and how to extract a subset from a given set. On the other hand, mathematical experience is required to form and study the collection of all subsets of a given set. Likewise, although most people are not accustomed to locating anything infinite in Nature, still, mathematical experience guides the mathematician to postulate that indeed there is an infinite set.

However, when mathematicians try to decide about whether the universe should include large cardinals, they are faced with a unique problem: Nature does not provide well-known examples of enormous infinities, and mathematical experience, although it can provide an intuitive feel for mathematical consequences of large cardinal axioms, does not equip the mathematician to decide whether such cardinals should *exist*. Indeed, P. Maddy [1988a/1988b] carried out a fascinating survey of philosophical justifications for large cardinals; her work detailed virtually all known intuitive principles that have ever been used to justify the better known large cardinal axioms. Each principle has clear intuitive motivation but succeeds in justifying only a very few of these large cardinal axioms. As Maddy herself aptly remarks, "...the axiomatization of set theory has led to the consideration of axiom can-

1. As we shall see, none of the other attempts to find an answer to this question have been successful either.

2. According to P. Maddy's account (Maddy 1988b, p. 758), our basic intuitions concerning mathematical objects like sets *begin* with our first perceptual encounters with objects in the world and then are shaped by the mathematical concepts and training we encounter later.

didates that no one finds obvious, not even their staunchest supporters." (Maddy 1988a, p. 481.)

We can imagine a number of different reasons for this wide variation in the mathematical intuitions that guide set theorists in their attempt to answer the deepest questions about the structure of the universe and the Infinite. One reason could be, as a formalist might argue, that there is no basic underlying reality about which to have clear intuitions in the first place; talk about the "right" foundation for mathematics should not be understood as a commitment to believe that there is some underlying "reality" that is being "described," but rather as a device to motivate new and interesting—but purely formal—mathematical systems. On the other hand, if indeed there is an underlying truth which mathematicians with growing clarity are glimpsing as they formulate ever more fundamental axioms for mathematical foundations—and this was certainly the view of Plato,[3] Cantor,[4] and Gödel[5]—then it may well be that foundational experts are glimpsing this basic reality with quite different levels of lucidity. Certainly Cantor and Gödel are examples of mathematicians whose intuitions went far deeper than those of their contemporaries in foundational matters. Indeed, Plato anticipated such disparities as inevitable (*Republic*, SN 518); according to his

3. Plato held that the objects of mathematical study, however much they may resemble objects in the physical world, properly belong to an independent timeless world beyond the senses, apprehended by a higher faculty of reason (for instance: "... geometry is the knowledge of the eternally existent" (*Republic*, SN 527); also, cf. Plato's *Meno*). Moreover, he held that, whereas ordinary mathematics begins with certain unquestioned assumptions and derives rigorous conclusions from these, the process by which these assumptions are themselves questioned and transcended activates a new kind of knowing in which the highest level of reality begins to be known (cf. Republic SN 510-511).

4. M. Hallett [1988] remarks: "As Cantor himself says ([1883], p. 206, n. 6), what he proposes is a Platonic principle: the `creation' of a consistent coherent concept in the human mind is actually the uncovering or discovering of a permanently and independently existing real abstract idea.

5. According to Gödel, "Evidently the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality." [1947/1983, p. 60]

account, these variations are due to varying degrees of skill in the use of those mental faculties which allow clear perception of these underlying mathematical realities. For those whose intuitive faculties are, in a sense, sleeping, this underlying mathematical reality will be simply a fiction, much as a blue sky must remain a fiction for one deprived of the sense of sight. For those whose faculties are awake, the reality of mathematical objects and those forms which give rise to them will comprise a quite lively reality, which they could hardly consider refuting as a fiction, even as one having the sense of sight would be unable to deny the existence of a lustrous blue sky.

Whether or not this Platonic view of mathematical reality is correct, it certainly has led to fruitful mathematical consequences. Gödel for example claims to have arrived at his famous completeness and incompleteness theorems in logic precisely because of this Platonic world view; he felt that, had he viewed the symbolism of Peano arithmetic as mere formalisms, he would never have made his discoveries (see (Wang 1974)). Indeed, according to Moschovakis,[6] it is the experience of nearly all mathematicians—whether they admit it or not!—that a world view which takes mathematical objects to be "real" and which views theorems as discoveries rather than inventions accords more with their experience of creative research than does a more formalistic view. There is also a certain amount of evidence within foundational studies themselves that suggests that mathematicians are "discovering" a mathematical landscape rather than "inventing" it; we have in mind the remarkable confluence of mathematical methods and insights that occurs in the large cardinal hierarchy. It has been observed by many set theorists that the fact that this hierarchy of principles, demarcating the extensions of ZFC in terms of their consistency strength, form a linear order and yet arise from such diverse mathematical considerations suggests that something profound about the hidden fabric of mathematics has been unearthed.

For these reasons, we take the view in this paper that there is indeed an underlying reality that set theorists and experts in foundations are,

6 "The main point in favor of the realistic approach to mathematics is the instinctive certainty of most everybody who has ever tried to solve a problem that he is thinking about `real objects,' whether they are sets, numbers, or whatever; and that these objects have intrinsic properties above and beyond the specific axioms about them on which he is basing his thinking for the moment." [1980, p. 605]

consciously or not, attempting to describe;[7] and we adopt this point of view for the pragmatic reason that this stance has, historically, proven to lead to more interesting and deeper mathematics than the opposite view. We propose to lay the groundwork for a program of research that will (1) clarify the nature of this reality and determine its structure and salient characteristics, and (2) use these insights as the basis for building a new, enriched consistent foundation for mathematics that accomplishes the purpose of foundations.

How are we to study the "underlying reality" of mathematics? From our observations so far about research in Foundations, it should be clear that the depth and clarity of intuition in this sphere of mathematical endeavor tends to vary widely from one mathematician to the next. For most specialized mathematical endeavors, a disparity in facility at an intuitive level balances out as experts keep abreast of the main new theorems and proofs in their fields. In Foundations, however, there is a need for more uniformity of vision; to formulate the right axioms we need to be seeing the same reality with equal clarity. More theorems using the same old axiom system will not in any significant way equalize our collective vision.

We suggest that the reality that the deepest thinkers in Foundations have been glimpsing on an intuitive basis and have been attempting to express through various axiom systems like ZFC has in fact already been systematically investigated by great seers throughout history.[8] We feel that the deep research of these exceptional individuals has tended to be overlooked by those working in the foundations of all the sciences.

In this paper, we propose to make use of the most ancient of these systematic investigations, the *Ved*, revived by Maharishi Mahesh Yogi in the form of *Vedic Science*, as a new, explicit source of intuition for advancing the current research into intuitive principles on which to

7. See (Weinless 1987, pp. 157 ff) for his discussion of *Vedic objectivism* – an approach that uses Maharishi's Vedic Science as a basis for a philosophy of mathematics that takes "mathematical reality" to be as "real" as any "physical reality" since these are all simply expressions of the internal dynamics of the field of pure consciousness.

8. The traditions of knowledge of every culture include insights and information about the fundamental intelligence underlying Nature. Among these, the tradition most often linked with modern mathematics is ancient Greek philosophy, most notably, Plato's philosophy. Plato's philosophy offers a wealth of insight about the ultimate nature of existence as a fundamental wholeness; see his discussions of the *good* and the *one* in the *Republic, Parmenides, Timaeus, Phaedrus,* and *Sophist*.

found set theory and account for large cardinals. This tradition, like many others, asserts that the vast diversity of the universe is the expression of a single wholeness; it offers a detailed account of the structure of this wholeness and its relationship to the manifest universe.

Among the many traditions of knowledge that speak of such a wholeness, we have chosen the Vedic tradition for several reasons. First, this tradition provides the most comprehensive extant treatment of the nature of wholeness, not only from the point of view of detailed insights and information, but also because it provides the procedures and technologies needed to awaken and refresh individual awareness[9] so that these truths of wholeness may be perceived directly; secondly, these procedures, unlike those of many other traditions, are readily available to individuals everywhere, so that truths about wholeness need not be taken merely as articles of faith; and finally, there is at least some evidence that all the most deeply cherished traditions of knowledge of mankind may have had their origin in the Vedic tradition[10]--whether or not this is the case, the many connections and similarities to be found in comparing the Vedic tradition with those that have historically followed it suggest, at the very least, that these traditions are giving expression to the same basic reality.

The central truth unfolded through Maharishi's Vedic Science is that the natural world, from the microcosm to the macrocosm, is the lively expression of a fundamental infinite wholeness; that this wholeness has its own qualities and dynamics; that it can be experienced directly and effortlessly as the most intimate, quiet level of one's own awareness; and that direct experience of this wholeness enlivens the entire range of its life-nourishing qualities and dynamics within the individual and the society in which he lives, resulting in a more successful, powerful, fulfilling, and stress-free life.

Maharishi Vedic Science has been used successfully by physicists,

9. The value of—and, indeed, the need for—such technologies is recognized in most traditions of knowledge in the world. For the reader most familiar with the Western tradition of knowledge, consider these remarks of Plato: ". . .there is in every soul an organ or instrument of knowledge that is purified and kindled afresh by such studies when it has been destroyed and blinded by our ordinary pursuits, a faculty whose preservation outweighs ten thousand eyes, for by it only is reality beheld." (*Republic*, SN 527.)

10. Cf. (Mead, 1965, pp. 9–25).

most notably John Hagelin [1987/1989], to understand and motivate research into the functioning of nature at its deepest levels. Indeed, recent research in quantum field theory has led to the discovery that all the fundamental force and matter fields of nature are expressions of a single, infinite, self-interacting, highly energetic, self-created, completely unified superstring field. This field, in any of its formulations, has been shown by Hagelin, in collaboration with Maharishi, to exhibit the very qualities and dynamics that characterize the field of pure consciousness as it has been portrayed in the Vedic tradition of knowledge. Hagelin [1987] argues strongly in favor of the contention that indeed, the unified field discovered by modern physics is nothing other than the field of pure consciousness described by the ancient texts of the Ved.

Our plan is to use this Vedic vision of wholeness as an intuition to guide the construction of the universe *V* and to account for large cardinal axioms. We will begin by examining the structure of the universe *V*, as it is currently understood in modern mathematics, in the light of Maharishi's Vedic Science. We will see that in some respects, the dynamics of unfoldment of *V* directly parallel those of pure consciousness in its expression into manifest existence, and in certain other respects the model falls short. Then, using principles from Maharishi's Vedic Science, we will formulate a new axiom to be added[11] to the present ZFC axiom system with the intention of capturing within the resulting universe more of the qualities and dyanmics of pure consciousness. This axiom, which we will call the *Wholeness Axiom*, asserts (in nontechnical terms) that the nature of the universe of sets as a whole is to move within itself and know itself through its own self-interaction. We will see that this new axiom brings the qualities and dynamics of the universe *V* in much closer alignment with those of the wholeness of pure consciousness. As a satisfying consequence of this new theory, we will be able to give a complete account of the origin of virtually all known large cardinals.

In this paper, we shall not attempt to address the natural question, "What should a foundation of mathematics provide?" We believe that any answer to this question must at least include an account of large cardinal axioms. For the present, we take our success in this latter

11.Our axiom should properly be called a *metatheoretic* axiom since it cannot be directly formulated in the language of ZFC. Nevertheless, it can be formulated in an expanded language as an *axiom schema*. See below and (Corazza 2000).

regard as sufficient evidence that our program is on the right track, and permit this more universal question, concerning the nature of a proper foundation, to motivate further research.

It is important to mention that our solution to the problem of the origin of large cardinals could easily have been discovered without the use of Maharishi Vedic Science; in fact, set theorists are quite familiar with the fact that axioms like the one we propose are strong enough to imply the existence of all known large cardinals.[12] What has been missing up till now has been a compelling reason to adopt such an axiom; without basic insights into the nature of the very wholeness that set theorists have been attempting to give expression to all these years, the large cardinal axioms all appear rather arbitrary. Once we have gained a glimpse of the dynamics that *ought* to underlie *any* foundational wholeness (since these are the dynamics which underlie nature itself), an axiom such as our Wholeness Axiom begins to stand out as extremely natural.

Our audience for this paper is intended to be wide in scope. For nonmathematical readers who are interested in applications of Maharishi Vedic Science, we have attempted to make the threads of reasoning leading to the Wholeness Axiom and its ramifications direct and relatively free from unnecessary technicalities, while elucidating the connections to Maharishi Vedic Science as accurately as possible. For mathematicians who may be unfamiliar with axiomatic set theory, we quickly review the basic ideas of the subject from ground level and include several highly readable references. And we hope that the experienced set theorist will find the strong connections between the familiar world of large cardinals and elementary embeddings on the one hand, and our intuitive model from Maharishi Vedic Science on the other, a pleasant surprise. Readers who are new to Maharishi Vedic Science may find the article (Corazza 1993) to be a good starting point.

The paper is organized as follows. We begin with a brief review of modern set theory and the structure of the universe V. As a first suggestion that this structure differs in important ways from the structure of wholeness described by Maharishi's Vedic Science, we observe that certain central properties of wholeness do not appear to be present in the structure of V, at least not in the way that we would expect to find

12. See for example (Maddy 1988a/1988b).

them. This divergence in structures becomes more evident when we next consider the dynamics of wholeness described in Maharishi Vedic Science—the lack of any real analogue to the self-referral dynamics of wholeness suggests that some new principle of dynamism ought to be introduced. We then give a brief introduction to the theory of large cardinals and the model theory of ZFC, leading to a natural candidate for an explicit representation of the hidden dynamics of the universe of V as a whole: a nontrivial elementary embedding from V to itself. We then discuss K. Kunen's surprising result that, under certain natural assumptions, such embeddings *don't exist*! Because such an embedding has seemed particularly natural to large cardinal experts, there have been numerous attempts to bypass Kunen's theorem; we review some of these efforts. Using Maharishi Vedic Science as motivation, we offer another such attempt, formuated as the Wholeness Axiom, which also bypasses Kunen's theorem, and which at the same time introduces new dynamics in V that correspond remarkably well to the dynamics of wholeness described by Maharishi. After observing the new character of mathematical proofs that arises from using the Wholeness Axiom, we develop the analogies between the wholeness of V in the presence of the Wholeness Axiom and the wholeness of pure consciousness. We then present proofs that, from the Wholeness Axiom, virtually all large cardinals can be derived. Finally, we observe that our analogy between V and the wholeness described by Maharishi extends even further than previously suggested: Using the technical language of Maharishi's Vedic Science, we suggest that the eightfold collapse of infinity to a point within wholeness, in its three phases corresponding to *rishi, devata* and *chhandas*, are actually mirrored in eight fundamental large cardinal axioms that increasingly approximate the Wholeness Axiom.

§2. The Need for a Theory of Sets

In Cantor's time it was believed that a set is simply any collection of objects that can be defined by some property. For example, the even numbers 0, 2, 4,... form a set, namely, the set of all those natural numbers having the property of being divisible by 2. As another example, the collection {2, 5, 7} is also a set; in this case the defining property of its elements is that of being equal to either 2, 5, or 7.

For most purposes, this concept of a set does not cause any problems; but technically, it is seriously flawed because, as Russell [1906] showed, one can invent very innocent-looking properties that determine collections which cannot rightly be considered sets. In particular, Russell showed that if we attempt to form the set of all sets having the property that each is not a member of itself, we arrive at a paradox.[13]

For this and other reasons, at the turn of the century a number of mathematicians focused on the task of developing a theory of sets. The idea was to set forth the most intuitively evident properties that sets ought to have, formulate them in the formal language of first-order logic, and use these formal statements as a set of axioms from which, hopefully, all the properties of sets, and hence of every other mathematical object, could be derived.

To be more concrete, we consider now a couple of the most widely accepted of these basic properties of sets. One basic property that sets ought to have is that two sets should be considered the same if they have the same elements. This property is known as Extensionality. Another basic property is that if A and B are sets, there ought to be another set $\{A, B\}$ which contains both A and B. This property is known as *Pairing*.

A number of different sets of axioms emerged from this early research. The most widely accepted axiom system, which has served as an excellent foundation for more than 50 years, is known as Zermelo-Fraenkel Set Theory with the Axiom of Choice, or ZFC for short.

§3. Zermelo-Fraenkel Set Theory with the Axiom of Choice

The axioms of ZFC, given below, detail the essential properties that sets should have and describe implicitly a procedure for building a universe of sets. The axioms do not explicitly tell us what a set is; rather, they list the essential properties sets must have in relationship both to

13. Russell's paradox is the following: Let $T(x)$ be the property of sets x that asserts, "x is not an element of x." Let S denote the collection of all sets x that satisfy $T(x)$. By the intuitive notion of 'set,' S is a set. Thus, given any set y, we should be able to use the property T to decide whether or not y belongs to S (if $T(y)$ is true, then $y \notin S$; if $T(y)$ is false, then $y \in S$). However, if we attempt to decide whether S itself belongs to S, we arrive at the logical absurdity that "S belongs to S if and only if S does not belong to S:" If S does belong to S, then $T(S)$ is false, whence S does not belong to S; conversely, if S does not belong to S, then $T(S)$ is true, whence S does belong to S. See (Weinless 1987) for further discussion of Russell's paradox.

themselves and to other sets. The intent is that if we can conceive of a vast aggregation of collections such that the collections in this aggregation obey the axioms of ZFC, then each collection in the aggregation may be called a set, and the aggregation itself may be called a universe of sets. The universe V mentioned above is such an aggregation, known as the *standard universe* and its members are *standard sets* (or just *sets*).

Axioms of Set Theory

- (*Empty Set Axiom*) There is a set with no element

- (*Axiom of Infinity*) There is an infinite set.

- (*Axiom of Extensionality*) Two sets are equal if and only if they have the same elements.

- (*Pairing Axiom*) If X and Y are sets, so is the collection $\{X,Y\}$.

- (*Union Axiom*) If X is a set whose members are also sets, then $\cup X$ is also a set.

- (*Power Set Axiom*) If X is a set, so is $P(X)$, the collection of all subsets of X.

- (*Axiom of Choice*) If X is a set whose member are nonempty pairwise disjoint sets, then there is a set Y which contains exactly one element of each member of X.

- (*Axiom of Foundation*) Every nonempty set X has a member y such that no member of y is in X (y is called an ϵ-minimal element of X).

- (*Axiom of Separation*) For every set X and every property R, the collection of all members of X which satisfy the property R is itself a set.

- (*Axiom of Replacement*) Suppose X is a set and we replace each member x of X with some set y_x, according to some well-defined rule. Then the resulting collection $\{y_x : x \in X\}$ is a set.

We take a moment here to discuss the meaning of these axioms; see (Weinless 1987) for a more detailed discussion relating the axioms to Maharishi Vedic Science. The first two axioms guarantee that certain sets actually exist. The empty set, the set with no element, is usually denoted $\varnothing$. The Axiom of Infinity asserts that there is an infinite set. It is not surprising that we require our universe of sets to include an infinite set since such sets are the most commonly used in actual mathematical practice; a familiar example of such a set is the set of natural numbers $\{0, 1, 2, \ldots\}$.[14] The Axiom of Extensionality provides a criterion for testing when two sets are equal. The Axiom of Foundation is a technical axiom which guarantees that any universe of sets that satisfies the axioms must unfold sequentially so that each set emerges only after all its members have emerged. This axiom implies that there is no "circular" set, i.e., no set which is a member of itself.

The Axiom of Foundation and Circular Sets

We consider here how the Axiom of Foundation proscribes sets containing themselves as elements: Suppose there were a set x which contained itself as an element; we show that the set $\{x\}$ would then violate the Axiom of Foundation: Since the only member of $\{x\}$ is x, and since there is a member of x (namely, x itself) which is also in $\{x\}$, the set $\{x\}$ has no ϵ-minimal element. Similar reasoning can be used to establish the result given in the following exercise:

Exercise Show that the Axiom of Foundation implies that there do not exist sets x and y such that $x \epsilon y$ and $y \epsilon x$. (Hint: If such sets x and y did exist, show that $\{x,y\}$ would violate the Axiom of Foundation.)

The Pairing, Union, and Power Set Axioms say that if certain simple operations are performed on sets, new sets are produced. Pairing, as mentioned earlier, asserts that from any two given sets, a third set can be formed having as its only elements the given two sets. The Union Axiom tells us that given any set whose elements are themselves sets,

14 It can be shown that the Axiom of Infinity is equivalent to the assertion that the collection of all natural numbers is a set.

say, X_0, X_1, X_2, . . . , a new set can be formed, called the union of X_0, X_1, X_2, . . . , which consists of all the elements of each of the given sets X_0, X_1, X_2, ... Finally, the Power Set Axiom guarantees that when we form the collection of all subsets of a given set, this new collection is itself a set. The set of all subsets of a given set is called its power set.

Applications of the Union Axiom

As an example of the Union Axiom, consider the sets $X = \{1,2,4\}$ and $Y = \{3,4,9\}$. The union of X and Y, written $X \cup Y$, is the collection $\{1,2,3,4,9\}$. The Union Axiom asserts that the collection $X \cup Y$ is itself a set. A precise definition can be given as follows: the *union* of a collection of sets is the collection formed by including as members those (and only those) objects which are members of at least one of the sets in the original collection. As another example, consider the sequence of sets $X_0 = \{0\}$, $X_1 = \{0, 2\}$, $X_2 = \{0, 2, 4\}$, ... The union of this infinite collection of sets is the set $\{0,2,4,6,...\}$ of all even numbers. We could write this union in either of the following ways:

$$X_0 \cup X_1 \cup \ldots \cup X_n \cup \ldots = \{0,2,4,\ldots\}$$

$$\bigcup \{X_0, X_1, \ldots X_n, \ldots\} = \{0,2,4,\ldots\}$$

The second of these notations is used in the statement of the Union Axiom, where $X = \{X_0, X_1, \ldots, X_n, \ldots\}$.

Applications of the Power Set Axiom

As indicated in the Power Set Axiom itself, the power set of a given set X is the collection of all subsets of X. As an example, consider the set $X = \{1, 2, 4\}$. The subsets of X can be listed: $\varnothing$, $\{1\}$, $\{2\}$, $\{4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, $\{1, 2, 4\}$. The *power set* of X, denoted $P(X)$, is the collection of all subsets of X. Thus

$$P(\{1, 2, 4\}) = \{\varnothing, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}\}.$$

The Power Set Axiom asserts that the power set of any set is again a set. As another example, notice that

$$P(\varnothing) = \{\varnothing\}.$$

The final two axioms tell us that if we are given a set and some property or rule, then the given set can be transformed using the property or rule into a new set. The Axiom of Separation asserts that if we collect together all members of a given set which satisfy a given property, the resulting collection is a set. Thus, for example, we could start with the set E of even numbers and form the collection of all members of E that have the property of being multiples of the number 7. The Axiom of Separation guarantees that this collection is a set.

The Axiom of Separation may remind the reader of the naive notion of a set that we mentioned earlier, prevalent in Cantor's time. In fact, the Axiom of Separation historically arose as a deliberate weakening of this naive notion, designed to avoid inconsistency.

Finally, the Replacement Axiom asserts that replacing elements of any set with other sets—according to some rule—produces a set. As an example, suppose we start with the set of natural numbers, $X = \{0,1,2...\}$, and we replace members of X according to the following rule: Replace each number in X by the set which contains both it and the number ½. Thus, we replace 0 by $\{0,½\}$, 1 by $\{1,½\}$, and so forth. By the Axiom of Replacement, the resulting collection

$$\{\{0,½\},\{1,½\},...\}$$

is a set.

Collectively, these axioms about sets are very powerful; every theorem in mathematics can be translated into a statement in the language of sets, and virtually all such statements can be derived directly from the list of axioms given above. This fact provides powerful conceptual unification of the entire range of mathematics. In addition, as we have said before, the axioms give rise to a very natural universe in which all mathematical objects—circles, lines, functions, numbers, groups, topological spaces, and so on—can be located.

§4. V: The Universe of Sets

The universe that can be built using ZFC proceeds in stages. The zeroth stage, denoted V_0, is the empty set itself; of course, the Empty Set Axiom guarantees that this stage is an allowable set. The next stage, V_1, is the set $\{\varnothing\}$ whose only element is the empty set; $V_2 = \{\varnothing, \{\varnothing\}\}$. These two stages can also be proven to be allowable sets by using the Pairing Axiom. The pattern of unfoldment is that each later stage is obtained by collecting together all *subsets* of the previous stage. After we have built up V_n, for every natural number n, the axioms tell us that we can continue building if we extend our number system beyond the natural numbers.

Ordinal numbers allow mathematicians to continue long constructions which extend beyond the indexing capabilities of the natural numbers. The ordinals extending past the natural numbers are given the following names, in increasing order: ω, $\omega + 1$, $\omega + 2$, . . . , $\omega + \omega$, $\omega + \omega + 1$, . . . , ω_1, $\omega_1 + 1$, . . . , ω_2, $\omega_2 + 1$, ... The Axioms of Infinity, Power Set, Union, and Replacement in combination guarantee that this long sequence of infinite numbers exists. The axioms allow us to continue defining new, larger stages of our universe: V_ω, $V_{\omega+1}$,..., $V_{\omega+\omega}$ and so forth. V_ω is obtained[15] by forming the union of all the preceding stages V_0, V_1,.... Then $V_{\omega+1}$ is the set of all subsets of V_ω, $V_{\omega+2}$ is the power set of $V_{\omega+1}$, and $V_{\omega+\omega}$ is the union of all previous stages; proceeding beyond $\omega + \omega$, we continue taking powersets and unions. Finally, we can declare our universe of sets to be the collection of all sets that can be found in at least one of the stages.

Ordinal Numbers and the Stages of the Universe

There are two basic types of ordinal numbers that are used in different ways in the build-up of the universe through the stages V_0, V_1, V_2, ... A successor ordinal is an ordinal number that has an immediate predecessor; the familiar numbers 3, 5, and 393 are examples of successor ordinals (since they have predecessors 2, 4, and 392, respectively).

15. In actual fact, the Axiom of Replacement is needed to form the sequence $<V_0, V_1, \ldots>$; once this sequence has been formed, the Axiom of Union may be applied to it (actually, to its range).

The ordinal ω+3 is also a successor ordinal, having predecessor ω+2. On the other hand, 0 and ω are examples of ordinals without immediate predecessors; such ordinals are called *limit ordinals*. The reader will notice that the stages of the universe are formed according to what kind of ordinal number is being used to index the stage: for instance, V_ω is defined to be the union of previous stages while $V_{\omega+1}$ is defined to be the power set of the stage immediately prior to it, V_ω. The formal definition of the stages of the universe is given by:

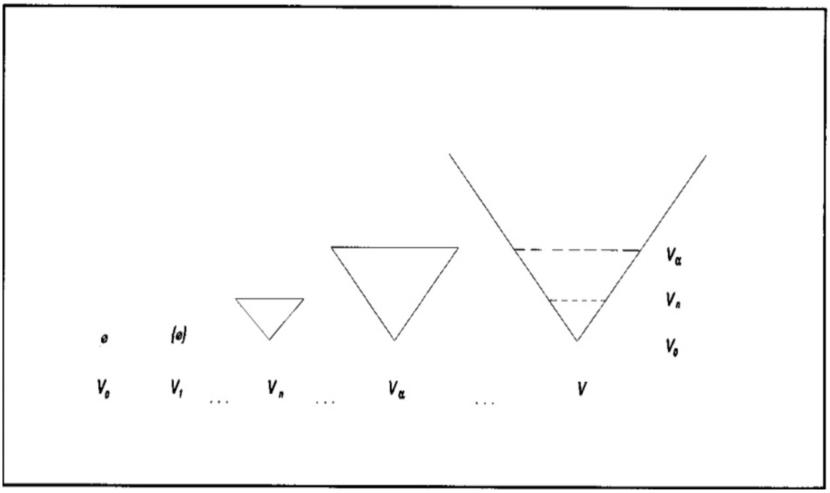
$$V_0 = \varnothing$$

$$V_{\alpha+1} = P(V_\alpha)$$

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \qquad \alpha \text{ a limit ordinal}$$

$$V = \bigcup_\alpha V_\alpha$$

Figure 1 - Sequential Unfoldment of the Universe of Sets



Locating the Set of all Fractions in *V*

We examine here how to locate the set of all fractions *a*/*b*—where *a* and *b* are positive natural numbers—withi n the universe *V*. This exercise illustrates how any set can be formally located inside *V*. First, let us see where each natural number can be found in *V*. In set theory, the natural numbers are defined as follows:

$$
\begin{array}{ccc}
0 & = & \varnothing \\
1 & = & \{0\} \\
2 & = & \{0, 1\} \\
. & & . \\
. & & . \\
. & & . \\
n+1 & = & \{0, 1, 2, \ldots, n\} \\
. & & . \\
. & & . \\
. & & .
\end{array}
$$

Notice that for each natural number n, n is a subset of V_n and is in fact a member of $V_{n+1} \setminus V_n$. It follows that the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers is a subset of V_ω, and so $\mathbb{N}$ is a member of $V_{\omega+1}$. To locate the fractions within *V*, we must find a way to code up fractions, represented in the form *a*/*b*, as sets, just as the natural numbers have been canonically identified with sets. The usual way to do this is to represent *a*/*b* as the ordered pair (a,b). Different fractions are thereby identified with different ordered pairs and each ordered pair stands for a unique fraction. The final step is to code ordered pairs of sets as other sets. Again, the usual way to do this is to represent the pair (a,b) as the set $\{\{a\}, \{a, b\}\}$. It is an interesting exercise to verify that a set of this kind successfully separates the components of the ordered pair; i.e., that two sets $\{\{a\}, \{a, b\}\}$ and $\{\{c\}, \{c, d\}\}$ are equal if and only if $a = c$ and $b = d$. Now, to locate the fractions within *V* notice that since $a \in V_{a+1}$ and $b \in V_{b+1}$ then $\{a, b\} \in V_{a+b+1}$. Thus $a/b = \{\{a\}, \{a, b\}\} \in V_{a+b+2} \subseteq V_\omega$. It follows that the set of all fractions is included in V_ω as a subset, and is therefore a member of $V_{\omega+1}$.

As we have mentioned before, the universe V is highly successful as a unifying background for mathematical research. Virtually all mathematical objects and structures can be formally located in the universe; yet paradoxical sets have been successfully excluded.

Because this mathematical universe plays the role of the fundamental wholeness underlying mathematics, it is natural at this stage to use our source of intuitive motivation, Maharishi's Vedic Science, to see to what extent the fabric of V reflects that of the wholeness of pure consciousness. Recall that we are seeking to modify this construction if possible because we wish to provide an account for large cardinals in mathematics. Thus, our plan is to bring the construction of V into closer relationship with the structure of pure consciousness, as far as this is possible.

§5. Application of Maharishi Vedic Science to Mathematics

We provide here an overview of how we plan to use Maharishi Vedic Science in our treatment of modern set theory. Our plan, as outlined briefly in the Introduction, is to use the qualities and dynamics of wholeness, pure consciousness, as principles for guiding our intuition concerning the "right" structure of the universe of sets, considered as a wholeness. We will observe that on the one hand, many of the qualities of pure consciousness have natural correlates in the standard universe V; on the other hand, those qualities concerned with *self-interaction* of pure consciousness—specifically, *fully awake within itself, self-referral,* and *bliss*—as well as the quality of *infinite correlation,* appear to be entirely absent from the universe (using a reasonable interpretation of these qualities within the context of set theory). We will also observe that the fundamental dynamics of pure consciousness, by which its infinitely expanded value collapses to its own point value, are not reflected in the structure of V. In order to enrich the universe so that these missing qualities and dynamics are more fully expressed, we will suggest that three features should be introduced to the structure of V: Some kind of truth-preserving embedding should naturally accompany the structure of V (corresponding to self-interacting dynamics of consciousness); elementary (set) submodels of V—structures which fully reflect all first order properties of V—should permeate the universe (corresponding to the quality of *infinite* correlation); and the dynamics of wholeness embodied in the truth-preserving embedding ought to reflect the collapse of infinity to a point in some natural way.

We will observe that these three criteria are met in a strikingly direct way by postulating the existence of a nontrivial, elementary embedding from the universe to itself—a transformation which preserves all first order properties of the universe and implies that nearly every set is itself an elementary submodel of V. It so happens that such an embedding represents the natural culmination of all large cardinal axioms—the very things that set theory has had such difficulty accounting for and that our new approach is designed to explain.

We will address one apparently serious technical difficulty in our approach: There is a well-known proof that seems to say the existence of such an embedding is inconsistent with set theory! To address this issue, we will show that certain assumptions (which are often not mentioned in discusions about this theorem) are required for the proof of this result to go through; and we will indicate how our approach explicitly avoids these assumptions. Having addressed the problem of inconsistency, we will assert in an axiom the existence of a certain kind of elementary embedding of the universe to itself, and add this axiom to the usual axioms of set theory. From this new expanded theory, we will indicate many of its strong consequences, among which is the fact that virtually all large cardinal axioms are derivable from this expanded set theory.

We will then discuss at some length the new dynamics that arise in the universe as a result of this new postulate. We will see how, just as the infinitely expanded value of pure consciousness collapses to a point value in the unfoldment of the Ved and creation, so, if we view sets from the perspective of V as a whole, the creation of sets can be seen to arise when the large cardinal-like properties of V become focused on the first point κ to be moved by the undefinable embedding. Then, just as the Ved sequentially emerges from the collapse of **A** to **K**, so a special sequence within the κth stage of the universe emerges from the embedding that contains essentially all the information about the location of every set in the universe. This sequence, called a *Laver magic sequence*, can be shown to "give rise to" every set, much as the Veda gives rise to every detail of creation. We will pursue the analogy further by describing an analogue to the eight stages that are involved in the collapse of

A to **K**; in particular, since the axioms defining the larger large cardinal axioms provide increasingly close approximations to our new postulate, we search for eight prominent large cardinal axioms to correspond to these eight stages described in Maharishi Vedic Science. Our search uncovers eight especially significant large cardinal axioms that represent landmarks in any investigation of the structure of V. As a further step in the analogy, we show how just as the eight stages of collapse are given expression in Rik Veda in a threefold manner, in terms of Ṛishi, Devatā, and Chhandas (elaborated in the 24 syllables of the first richa of Rik Veda), so we shall observe how these eight large cardinal axioms can be expressed in terms of the structure of V (corresponding to Ṛishi), in terms of elementary embeddings (corresponding to Devatā), and in terms of properties of a specific point in the universe, i.e., a specific large cardinal (corresponding to Chhandas).

Therefore, by introducing this new axiom, which states in mathematical terms that wholeness by its nature moves within itself and knows itself, we will find that on the one hand, the structure of V is enriched to the point of displaying nearly all the qualities and dynamics of pure consciousness, and, on the other hand, the previously mysterious large cardinal properties can be accounted for very naturally as the properties of the first point moved by our postulated undefinable elementary embedding, which represents the "unmanifest self-interacting dynamics" of the wholeness embodied in V.

§6. Qualities of Pure Consciousness and the Universe V

In this section, we will examine the universe V, looking to see which of the qualities ascribed to the field of pure consciousness in Maharishi Vedic Science find expression in this foundational structure. As we shall see, some of these qualities will seem to capture the very intent behind the cumulative hierarchy, while others may not seem quite so relevant. Since we are attempting to use Maharishi Vedic Science as a source of intuitive guidance, our plan is to look for ways of enriching the construction of V so that qualities which originally seemed irrelevant will become as fully embodied as the other qualities. As a starting point, we give a quick summary of some of the main qualities of pure consciousness [Since the time this paper was first written, the list of qualities specified by Maharishi Vedic Science has grown considerably, but this partial list still provides an excellent sampling—Ed.]

Table 1. Qualities of pure consciousness.

all possibilities All activity begins from the field of pure consciousness; all laws of nature begin to operate from this level; the point **K** represents the point of all possibilities within this field.	***infinite silence*** The infinitely silent quality of pure consciousness is expressed in the first letter **A** of Rk Ved. This quality quietly nourishes the infinitely dynamic unfoldment of pure consciousness.
ominiscience The self-interacting dynamics of pure consciousness constitures that pure knowledge on the basis of which all knowledge and existence arise. Knowing this level of life, all else is known.	***infinite dynamism*** Being awake to itself, pure consciousness undergoes an infinity of transformations within itself; the infinite organizing power inherent in these dynamics structures the infinite diversity of creation.
freedom Remaining ever uninvolved in its own self-referral dynamics, pure consciousness is a state of eternal freedom.	***pure knowledge*** Being awake to itself, pure consciousness knows itself. This self-knowing, a sequential flow within the unmanifest, is called pure knowledge.
unmanifest The self-referral dynamics of pure consciousness form the unseen government of nature. All manifest life is governed by these unmanifest dynamics.	***infinite organizing power*** "Knowledge has organizing power. Pure knowledge has infinite organizing power."
simplicity Pure consciousness is known when that which is foreign to the nature of the knower drops away. "The simplest form of awareness is a state of perfect order, the ground state of all the laws of nature." (Maharishi 1991b, p. 283)	***evolutionary*** The pure intelligence inherent in the infinite organizing power at the basis of creation directs life toward ever-increasing levels of progress and fulfillment.

omnipotence Pure knowledge has infinite organizing power. Pure consciousness knows no limitation in its creative expression as it unfolds sequentially within itself.

ominipresence The self-referral dynamics of consciousness are present at every point in creation.

total potential of natural law The creation unfolds and is maintained in accordance with the most fundamental laws of existence – the laws that govern the flow of pure consciousness from the Constitution of the Universe.

discriminating The flow of pure consciousness within itself is not only highly dynamics and unrestricted, but precise and sequential in its unfoldment. Each stage of expression comes about methodically and with full awareness of all that has come before it and all that is yet to come.

fully awake within itself Pure consciousness is, by its very nature, pure wakefulness.

harmonizing The basis of harmony is enlivenment of the infinitely harmonizing quality of pure consciousness in which extreme opposite values are

perfect orderliness The laws governing the precise sequential flow of pure consciousness are at the basis of the orderly functioning observed in nature.

self-sufficiency Pure consciousness needs nothing outside itself for its existence, creative expression, and fulfillment. Creation unfolds and dissolves within pure consciousness.

purifying Enlivenment of pure consciousness, the ultimate reality of manifest life, bring an end to unwanted tendencies, which are foreign to life.

infinite creativity The infinite organizing power inherent in pure consciousness finds unrestricted expression in the unfoldment of creation.

integrating The wholeness of pure consciousness is maintained is maintained through the integral coexistence of opposite values, such as infinite dynamism and infinite silence.

perfect balance "The balance inherent in the eternal continuum of the unmanifest nature of the Absolute is reflected in the balance that nature maintains—

simultaneously lively without conflict. ***self-referral*** Pure consciousness, through all stages of unfoldment, is awake to itself; its nature and creation are therefore self-referral. ***unboundedness*** All boundaries are structured in the boundless, unlimited value of pure consciousness. ***immortality*** Birth, death, and the field of change are the creative expression of pure consciousness. Pure consciousness itself is an immortal field, beyond the manifest field of change. ***invinicibility*** "Nothing can … disrupt the perfect balance … of this field … since everything is a part of its structure." (Maharishi 1991b, p. 281)	amidst the dynamism of evolutionary change." (Maharishi 1976, p. 148) ***bliss*** The self-interacting dynamics of consciousness form the unmanifest structure of bliss. "… the Absolute ever celebrates its own nature within its unmanifest nonchanging Self." (Maharishi 1976, p. 146) ***nourishing*** All stages of expression of pure consciousness are nourished by the infinitely silent value of pure consciousness. ***infinite correlation*** Pure consciousness "is a field of infinite correlation in which an impulse anywhere is an impulse everywhere." (Maharishi 1976, p. 150)

The alert reader will no doubt discover many ways to interpret these qualities in the context of set theory and Foundations that we have not considered here. Our account may at times assert that certain qualities are absent from the set theoretic universe which the reader, taking a slightly different approach, may find abundantly present. We feel that these different viewpoints are to be expected and mark the beginning of a healthy, rigorous research program. In our own research, we found that, as we reflected on the significance of each quality, natural ways for this quality to be expressed in a foundational context became apparent to us. We asked ourselves, for example, "What would a universe have to look like in order to embody the quality of *infinite correlation*?" It seemed apparent to us that this quality would be most clearly embodied if *all knowledge about the universe were avaiable throughout the universe*. From the perspective of set theory, we interpreted this to mean that we were looking for a universe in which a substantial proportion of sets would reflect all first-order properties of the universe. This requirement is certainly not met by the universe arising from ZFC; as we shall see, however, by suitably supplementing ZFC with an axiom about the wholeness of V, this requirement expresses one of the most appealing features about the new resulting universe.

In the discussion below, we list most of the qualities from this table and suggest the ways in which ZFC set theory, as a foundation for mathematics, exhibits these qualities. Several qualities from out table are not mentioned below; these, in our view, do not find natural expression in set theory as it is presently understood. We will discuss these at greater length later in this section, outlining the ways in which we might expect to find these qualities displayed in an enriched set theory and our reasons for believing they are absent from the present foundation.

all possibilities All models of every mathematical theory are located in V; all sets needed for the development of any mathematical theory are located in V.

omniscience Every mathematical fact is true in the model V; thus, if one could view mathematics from the vantage point of V, the wholeness underlying mathematics, every mathematical truth could be known.

freedom The power set axiom freely generates the set of all subsets of a given set; since no restriction is placed on the sets generated in this way, the continuum may consistently be taken to have arbitrarily large cardinality.

unmanifest V is too large to be an individual set; although all properties of sets can be rigorously determined and demonstrated using the axioms of set theory, nothing can be directly proven about V.

simplicity A single elegant recursive rule is at the basis of the sequential and simultaneous unfoldment of all stages of the universe.

omnipotence Any mathematical truth that has ever been demonstrated can be seen as a derivation from the axioms of set theory using rules of logic, and all of these can be found in coded form within the structure of the universe itself.

total potential of natural law The laws governing a mathematical theory are expressed by axioms; the content of every axiom of set theory is fully realized in the universe of sets.

discriminating The sets which emerge in the cumulative construction of V do not lead to any known paradox.

bountiful All mathematical knowledge and its applications arise from the "interactions" of the axioms of ZFC (i.e., from logical derivations from the axioms).

infinite silence At limit stages of the construction of the universe, no new sets are added; this silent phase of the construction creates smoothness and uniformity in the unverse.

infinite dynamism In the construction of V, each new stage produced by the power set operator is larger than the previous stage; in particular, the power set operator produces an endless sequence of ever larger infinities.

pure knowledge The information content in ZFC is the basis for virtually all known mathematical theorems.

infinite organizing power The organizing power of a mathematical theory is expressed by its models;[16] the models of set theory are infinite, complete, and all-inclusive.

evolutionary Set theory has stimulated progress in a wide range of mathematical fields.

16. Weinless [1987] discusses at some length this notion that the organizing power of a set of axioms is expressed in its models.

perfect orderliness All theorems of set theory, and hence of virtually all of mathematics, can in principle be generated automatically by a computer once sufficiently many axioms have been input.

self-sufficiency All the information needed to construct the stages of the universe is coded in the first few stages of the universe; the universe can therefore reproduce itself.

purifying The recursive construction of V systematically prevents the entry of paradoxical sets.

infinite creativity All the creativity of the brightest mathematicians of recorded history can be coded up as formal theorems derivable from the simple axioms of set theory.

integrating All mathematical theories, with their own special mathematical languages, find a common basis in set theory; the interrelationships between theories are thereby more easily identified.

harmonizing Superficial differences in style between different theories are stripped away when the formal content of these theories is expressed in the language of set theory.

perfect balance Despite the differences in style and content between different theories and their models, all such models naturally emerge in the uniform and simply defined unfoldment of the stages of the universe.

unboundedness The sequence of stages of the universe V unfold without bound; the resulting universe V is so vast that it cannot be considered a set.

nourishing Every mathematical theory has a basis in set theory; as a result, each theory can make use of the tools of set theory within its own context.

immortality The conceptual reality developed by pure mathematicians, and uniformly codified in set theory, is time independent.

omnipresence All mathematical structures can be located inside V.

As our list indicates, set theory with its universe *V* exhibits a wide range of the qualities attributed to pure consciousness in Maharishi Vedic Science. In the table below, we provide the reader additional information about our point of view concerning the presence of these qualities in set theory by considering one such quality—*self-sufficiency*—in greater detail.

Our main concern here is with the five qualities, present in the table given earlier, that do not appear on our list. These aspects of wholeness, described by Maharishi Vedic Science, are, in our view, missing from set theory and the structure of *V*; we shall argue later that the difficulties set theory faces as a foundation are intimately tied to these omissions. The omitted qualities are *infinite correlation, invincibility, fully awake within itself, bliss,* and *self-referral.*

As we mentioned at the beginning of this section, the first of these qualities, *infinite correlation*, would be exhibited in a universe in which a significant proportion of its sets satisfied all the first-order properties of the universe itself. Using Gödel's Incompleteness Theorem, however, one can easily show that it is impossible to prove from ZFC that there are *any* sets in *V* which are even models of ZFC, what to say of sets which reflect all first-order properties of the universe!

As indicated in the table, the quality of *invincibility* is the characteristic of pure consciousness by which it maintains its connection to its unbounded source through all stages of expression, and therefore is not foreign or antagonistic to any aspect of its creation. In our view, this quality could be ascribed to the universe if, as in the case of *infinite correlation*, "nearly all" sets in the universe satisfied all first order properties of *V* itself. In that case, clearly, the nature of wholeness would not be lost at any stage of the unfoldment of *V*.

The next three qualities have one common property that leads us to declare that they are absent from the structure of *V*: All three arise from a fundamental self-interaction of wholeness, of pure consciousess. According to Maharishi Vedic Science, being fully awake within itself, pure consciousness is fully awake to itself; its own wakefulness results in its own self-knowing and self-interaction. This dynamic state in which pure wakefulness is awake to itself represents the eternal nature of pure consciousness to be ever in a state of self-knowing; this unchanging condition of self-knowing is called *self-referral*, and is another fundamental instance of *self-interaction*. Finally *bliss* is a description of the *experience* of this self-referral flow of consciousness. At this level, the experience and that which is experienced are the same (see the table, and recall "The experience of pure Being and the state of Being mean the same thing," Maharishi 1966, p. 295). Thus, the subjective experience of self-referral consciousness as bliss is no different from the *reality* of self-referral consciousness as bliss. Again, this quality arises from the self-interaction of pure consciousness.[17]

What does *self-interaction* mean in the context of the universe V? At the very least, we would expect a self-interacting universe to have some sort of transformation associated with it that would move its elements. One observation that many category theorists and physicists have made regarding V both in publications (see for example Lawvere, 1979, and McLarty, 1990) and in lectures and discussions is that it is unduly static; even the central concept of a *function*—the very essence of mathematical transformation—is formalized as a *set* of ordered pairs, on a par with other sets, like the rationals or integers, which exhibit no essentially dynamic features. In short, the mathematical intuition of dynamism embodied in the concept of a function is not well expressed in V either on the local scale (particular functions are mere sets of ordered pairs) or on the global scale (V is not naturally associated with any map[18] from V to itself that would transform its elements). Thus, in order for us to declare that the universe exhibits self-interacting dynamics comparable to those of pure consciousness, we would expect that some natural transformation of V into itself should accompany the construction of V.

Thus, our viewpoint about these five basic qualities suggests to us that a universe more in accord with our objective, more in accord with the nature of the wholeness set theorists wish to capture, will display *infinite correlation and invincibility* through the widespread presence of sets embodying all first-order properties of V, and *self-interacting*

17 In [1983], Hagelin identifies the quality of *bliss* in pure consciousness as a quality of the unified field of natural law, as described by quantum field theory, because of this unified field's "continuous effervescence of topological fluctuations"—a fundamental interaction of the field with itself.

18 Interestingly, nearly two decades after this article was first written (1993), it was discovered that the Axiom of Infinity is provably equivalent to the existence of a certain kind of structure-preserving map from V to itself (Corazza, 2010). The naturalness of this phenomenon led the author to argue in favor of the Wholeness Axiom—as discussed in this paper—as a new axiom to be added to ZFC—Eds.

dynamics expressed perhaps by some natural map from V to itself.

Notice that if we are successful in our efforts to give expression to these qualities in an enriched set theory, we should expect to find that many of the other qualities on the list above will be expressed in a manner even more in accord with their expression within pure consiousness. For instance, the fact that pure consciousness can be described as *pure knowledge* or as *omniscient* arises from the nature of its self-interacting dynamics: Pure consciousness, being awake to itself, is eternally engaged in the act of self-knowing, and all knowledge emerges from the sequential unfoldment of this process. Our use of these qualities as descriptions of the foundation of mathematics differs from the pattern we find within pure consciousness, and this difference stems from the fact that the universe V, as it is presently understood, does not exhibit any fundamental self-interaction from which "knowledge" could be said to emerge. Thus, even though we feel these qualities are exhibited to some extent in the present universe, once we have invested V with a fundamental form of self-interaction, we shall expect to find the qualities of *pure knowledge*, omniscience, and many others, arising from these new dynamics.

In this section, our aim has been to identify qualities of pure consciousness that appear to be absent from the structure of V so that our intuition concerning the "right" structure for V could be suitably guided. In the next section, when we compare the *dynamics* of pure consciousness with those of V, the difference between these two wholenesses—pure consciousness and the present foundation of mathematics —will become even more apparent.

Self-Sufficiency in the Universe *V*

Here we show how all the information needed to build V can be located within V itself. The basic idea is that set theory is formalized within a symbolic language; the symbols of this language can be identified with sets, as can the basic rules of proof. This means that the informal reasoning we used earlier to build up V using the axioms of ZFC can be formalized in symbolic logic and coded as a set. We now investigate some of the details of this coding.

The symbolic language in which formal set theory is expressed is called first-order logic. The symbols which are used in first-order logic are listed below:

Variables	$x_0, x_1, x_2, \ldots$
Logical symbols	$\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$(if... then), $\forall$ (for all), $\exists$ (there exists)
Parentheses and comma	( ) ,
Membership relation	ϵ
Equality relation	=

These symbols are put together according to simple rules of formation to obtain the formal sentences of set theory. Here is an example:

$$\forall x_0 \exists x_1 (x_0 \epsilon x_1).$$

This sentence symbolically represents the assertion, "Every set is contained in some other set" (or more precisely, "for every set x_0, there exists a set x_1 such that x_0 is a member of x_1").

The axioms of set theory can be expressed in this formal language. For instance, the Empty Set Axiom has the following symbolic form:

$$\exists x_0 \forall x_1 \neg (x_1 \epsilon x_0).$$

With our formal language in place, formal rules of proof can also be developed which give precise criteria for deriving theorems from the ZFC axioms. Using these, one can, for example, give a formal proof of the formal sentence given above that asserts every set belongs to some other set. We now identify the basic symbols of first-order logic with sets according to the following scheme:

Symbol	Set	Symbol	Set
ϵ	0 (= ∅)	$\exists$	7
=	1 (= {0})	(	8
$\neg$	2 (= {0, 1})	)	9
$\wedge$	3 (= {0, 1, 2})	,	10
$\vee$	4	x_0	11 (=11•2^0)
$\rightarrow$	5	x_1	22 (=11•2^1)
$\forall$	6	x_n	11•2^n

We can now translate any formal sentence of set theory into a set by translating symbols directly into sets using the key above. In order to preserve the order in which the symbols occur in a given sentence, we attach to the first symbol the number 0, to the second the number 1, and so forth. As an example, we can translate the formula $\forall x_0 \exists x_1 (x_0 \epsilon x_1)$ as the following set:

{(0,6), (1,11), (2,7), (3,22), (4,8), (5,11), (6,0), (7,22), (8,9)}.

In this way, all the axioms of set theory can be located within the universe V (in fact, within V_ω). Moreover, it can be shown that all the rules of proof can also be identified with sets; hence, all provable statements and their proofs can also be located in V. In particular, all the reasoning needed to construct V from the axioms of ZFC can be coded up as a single set, which can actually be found in V_ω!

§7. Dynamics of Pure Consciousness and the Universe *V*

Continuing with our program to compare the structure of the universe V with the wholeness of pure consciousness described by Maharishi

Vedic Science, in this section we seek to determine to what extent the *dynamics* ascribed to pure consciousness are displayed in the structure of the universe. We will find a rather unmistakable difference between *V* and pure consciousness in this portion of our comparison. We first summarize in a table important principles of these dynamics, described in Maharishi's Vedic Science:

Table 2. The dynamics of pure consciousness.

Existence The first truth about pure consciousness is that it exists. ***Nature*** The nature of pure existence is pure wakefulness or pure intelligence. ***Three-in-one structure*** Being awake to itself, pure existence is conscious of itself and assumes the roles of *rishi* (knower), *devata* (process of knowing), and *chhandas* (that which is known). Put another way, the pure intelligence of pure existence distinguishes a three-in-one structure within pure existence, the samhita of rishi, devata, and chhandas. ***All possible transformations*** As each of *samhita, rishi, devata*, and *chhandas* is fully awake within itself, each is awake to each of the others. Being awake to each other transforms each. These transformed values of *samhita, rishi, devata,* and *chhandas* are th-	***Collapse and expansion with infinite frequency*** In the unfoldment of pure knowledge, the point, embodied in **K**, expands to infinity. The pro¬cess of collapse and expansion occurs with infi¬nite frequency and is the theme of unfoldment of the *Ved* and all of creation. ***Apaurusheya Bhasya*** Maharishi's Apaurusheya Bhasya asserts that the *Ved* provides its own commentary on itself. The structure of total knowledge is found in its most concentrated form in **A**, and in successively more elaborated forms in **AK**, in the first pad, the first richa, the first sukt, and the first mandal of *Rk Ved*, and finally in its most elaborated form in the entire *Ved*. ***Eightfold collapse*** The collapse of **A** to **K** is like a whirlpool that contracts to a point in eight stages. These eight stages correspond to the five *tanmatras and*

emselves fully awake to themselves and each other, and the process of transformation continues. An infinity of transformations—all possible transformations—of pure consciousness emerge in this unfoldment.

Pure knowledge and infinite organizing power These transformations of pure consciousness within itself constitute a sequential process by which pure consciousness knows itself. This sequential unfoldment is called pure knowledge. The Ved is pure knowledge together with the infinite organizing power contained within it. This organizing power gives rise to the whole creation and all the laws of nature.

Constitution of the Universe The laws govern¬ing the sequential unfoldment of the Veda are known collectively as the Constitution of the Universe. The self-interacting dynamics of consciousness is the primary administrator of the universe.

***Collapse of* A *to* K** Pure knowledge emerges in the collapse of the infinitely expanded value of wholeness to the fully contracted point value of wholeness; fullness, infinite silence, embodied in the three subjective principles--mind, intellect, and ego. These eight stages unfold from three perspectives: from the point of view of rishi, devata, and chhandas.

Coexistence of infinite silence and infinite dynamism The fabric of pure knowledge is composed not only of infinite dynamism and the tendency to give rise to creation, but also infinite silence by which pure consciousness remains forever uninvolved in its creation. Prakriti unfolds within Purusha; pure consciousness is both pure samhita and samhita of rishi, devata, and chhandas.

Maintaining unity, wholeness In its sequential unfoldment, the self-interacting dynamics of consciousness always remains infinitely correlated with its source, the samhita value of pure consciousness.

Present at every point in creation The self-interacting dynamics of consciousness, the Veda, is unmanifest and present at each point in creation.

A, the first letter of Rik Veda, collapses to empti¬ness, the point value, the point of all possibilities and infinite dynamism, embodied in the second letter of Rik Veda, **K**.	

In Maharishi's description of the dynamics of pure intelligence, he observes that pure intelligence, being pure wakefulness, becomes aware of itself. This process of becoming aware involves a move of the fully expanded aspect of its nature, represented by the letter **A**, toward the fully contracted, point value of its nature, represented by the letter **K**. In this collapse of infinity to a point—all within pure intelligence occurring as a fundamental flow of its own nature—all possible transformations of its nature take place. In this collapse, the full unmanifest power of unbounded silence is imparted to the point value which then is impelled to expand its contracted nature to the fully expanded infinite value. This expansion of the point to infinity gives rise in sequential fashion to the entire blueprint of creation, the Ved, which emerges as an elaboration of the transformations occurring within the original collapse.

This description suggests to us that the construction of V emphasizes only one half of the dynamics of wholeness, namely, the expansion of the point (represented by the empty set) to infinity (represented by the ever larger stages of the universe).[19] Again we note that none of the ZFC axioms actually attempts to describe the *nature of wholeness*; instead they focus on the *nature of sets*. Thus the construction of the universe necessarily proceeds in a one-sided way. From the point of view of Maharishi's Vedic Science, we would expect that the unfoldment of parts in any foundational system that does not maintain a connection with the nature of the whole is doomed to fall short of its goal (Maharishi 1991):

> If the expansion of *rishi, devata,* and *chhandas* into the infinite universe does not remain in contact with the source, then the goal of expansion

19 See (Weinless 1987) for an excellent detailed treatment of the relationship between the expansion of the universe of sets from the empty set and the dynamics of the point expanding to infinity in Maharishi Vedic Science.

will not be achieved.

In our view, this one-sided development of the universe V has created an unnecessary mystery at the root of set theory: Where do large cardinals come from? Using Maharishi Vedic Science as motivation for the intuition that the wholeness that V represents should by nature move within itself and "know" itself through its own self-interaction, we will introduce an axiom (actually, an *axiom schema*), which we call the *Wholeness Axiom*, which asserts in mathematical terms that V is moved within itself via a truth-preserving embedding and that these dynamics are present "at every point". We will see that the mystery of large cardinals vanishes in the presence of this new axiom.

We should mention here that ours is not the first attempt to locate the unmanifest dynamics of pure consciousness in the structure of V. In [1987] Weinless suggests that the "collapse of infinity to a point"—which we have said is absent from the structure of V as it is presently defined—is to be found in the *Reflection Principle*, which states that properties true of V as a whole ("infinity") should also hold true of certain sets ("point") in the universe because the universe as a whole should be conceived as structurally undefinable. We shall discuss this principle at greater length in Section 10; we shall see that the introduction of a truth-preserving endomorphism of the universe is, on the one hand, motivated by considerations such as the Reflection Principle, and yet on the other hand, accomplishes in a somewhat cleaner way many of the same things as the Reflection Principle in Weinless' treatment.[20]

We also mention here that there have been other attempts to extend ZFC by including axioms about V itself (see for example (Gödel 1940), (Kelly 1955, Appendix), (Quine 1951) and more recently (Maddy 1983)). However, these experts in foundations have been thwarted by a lack of reliable intuition about such a vast wholeness; moreover, such extended axiom systems have neither met with wide acceptance among set theorists nor resolved in the smallest way the issue about the origin of large cardinals.

20. We hasten to point out that, in the same paper, Dr. Weinless [1987, p. 174] himself suggests that an embedding of the sort we are describing would "provide an ideal mathematical expression of the self-interacting dynamics of the samhita," but did not pursue this direction because of the well-known limitative result of Kunen; see Section 15 of the present work.

Before introducing the Wholeness Axiom, we shall offer a brief introduction to the theory of large cardinals and the foundational challenges which accompany them; we shall see that large cardinals themselves suggest the very foundational solution we are seeking. Since large cardinals are vast infinite sets, we begin with a discussion of mathematical infinity.

§8. Mathematical Infinity

The whole material creation is just a sequence of quantified values of infinity.

(Maharishi 1990)

Prior to the work of Cantor, mathematicians viewed the concept of infinity as a kind of unreachable ideal which various mathematical sequences could approximate. The sequence 0, 1, 2,... of natural numbers, for example, was viewed as continuing indefinitely, but was never conceived as a completed collection. In studying certain problems in mathematical analysis, Cantor found it useful to consider certain infinite collections as completed wholes which could be further manipulated using techniques commonly used on finite collections. His work was at first met with skepticism but by now has come to be considered one of the great achievements of modern mathematics.

Once the concept of sets having infinite size is in place, it is natural to ask, as Cantor himself did, whether all infinite sets have the same size. In order to answer the question, Cantor first needed to describe a way of comparing two infinite sets. Certainly, the familiar method of comparing the sizes of two finite sets—namely, by counting the number of elements in each—would not apply to the case of two infinite sets (how many elements does an infinite set have?). However, another method of comparing finite sets does turn out to be useful in the context of infinite sets: Consider two fairly large finite sets A and B and arrange each set's elements in a row, aligned as in the diagram below:

elements of A: ....

elements of B: ....

In the diagram, it is clear that *B* has more elements than *A* even before we attempt to count the number of elements in each set. This is because, without concern for the actual numbers of elements involved, we can see that there is no way to match up the elements of *A* with those of *B* in a one-to-one way. The same sort of reasoning shows that the sets *C* and *D* below do have the same size:

elements of *C*: ....

elements of *D*: ....

Thus, Cantor reasoned, two sets, whether finite or infinite, can be said to have the same size if their elements can be matched up one for one; moreover, a set *A* can be said to be smaller than a set *B* if the elements of *A* can be matched with those of a subset of *B*, but not conversely.

Using this method of comparing infinite sets, Cantor showed that the even numbers 0, 2, 4, . . . form a set that has the same size as the entire set of natural numbers, whereas the set of real numbers is strictly bigger than the set of natural numbers.[21]

Cantor went on to make an even more startling discovery about the infinite: For any set *A*, the collection of subsets of *A* is strictly bigger than A itself. Using notation from set theory, we can say that $P(A)$ is bigger than *A*, or more briefly, $A < P(A)$, for any set *A*, where $P(A)$ stands for the set of all subsets, or power set, of *A*. In particular, we have the following endless sequence of infinite sets, each one bigger than the previous one:

$$\mathbf{N} < P(\mathbf{N}) < P(P(\mathbf{N})) < \ldots$$

Cantor hoped in his time that the sequence of infinities given above would include all possible infinite sizes of sets. (Nowadays, it is known that this hypothesis about the sequence of infinite sizes, known as Cantor's *Generalized Continuum Hypothesis*, is consistent with ZFC, but

21. See (Rucker 1982) for a popular treatment of this famous result; (Hallett 1988) for a historical treatment; (Roitman 1990) for a pedagogical treatment; and (Weinless 1987, Section I.5) for a treatment that interfaces with principles from Maharishi Vedic Science.

not provable from it.) Since he was unable to prove his conjecture, he devised a hierarchy of numbers—which he called *transfinite numbers* and which in contemporary language are called infinite *cardinals*—that were intended to represent all possible infinite sizes. In modern-day notation, Cantor's infinite cardinals form a subclass of the ordinal numbers discussed above; in the context of ordinals, a cardinal number can be defined to be any ordinal which does not have the same size as any of its predecessors. Every finite ordinal (i.e., every natural number) is also a finite cardinal; the first few infinite cardinals are listed below:

$$\omega, \omega_1, \omega_2, \ldots, \omega_\omega, \omega_{\omega+1}, \ldots$$

In particular, if Cantor's Generalized Continuum Hypothesis happens to be true, we have the following neat correspondence:

the size of $\mathbf{N}$ is ω
the size of $P(\mathbf{N})$ is ω_1
the size of $P(P(\mathbf{N}))$ is ω_2
.
.
.

It is helpful for our mathematical intuition to view the progression of the infinities of set theory from ω through Cantor's hierarchy as a growing approximation to a full description of the ultimate nature of the Infinite. We shall see that as we climb the hierarchy of infinities, more and more of the qualities of the field of pure self-referral consciousness, as described by Maharishi Vedic Science, become embodied in these cardinals. Thus, for example, the smallest infinite cardinal, ω, simply embodies the quality of *unboundedness* in that for every number n less than ω, $n+1$ is also less than ω. The qualities of c*ompleteness, indescribability, self-referral, all-inclusiveness, self-sufficiency* and others, which we find present in the ultimate Infinite are absent from ω; however, as we will see, these qualities begin to be expressed by cardinals higher up in the hierarchy. Climbing to the level of large cardinals, we will find that deep properties of the universe V as a whole begin to be reflected into sets having large cardinal size; thus, it is natural to study large cardinals to gain an intuitive sense of the nature of V as a whole.

§9. Large Cardinals

A *large cardinal* is a cardinal which cannot be obtained using any conceivable[22] set-theoretic operation on the cardinal numbers below it. Each of the first few infinite cardinals (see the list above) can be obtained by applications of the axioms of set theory to cardinals which occur earlier in the list, and hence are not large. For instance, ω is obtained explicitly from one of the axioms (the axiom says, essentially, "ω exists"). ω_ω is obtained as the union of the cardinals which are below it: $\omega_\omega = \cup\{\omega_n : n<\omega\}$. The cardinality of the stage $V_{\omega+1}$ is the size of the power set of the previous stage V_ω. Each of the cardinals whose existence is derivable in set theory is obtained in a similar way, building up from below. But a large cardinal does not arise in this way. A famous theorem due to Kurt Gödel shows that it is *impossible* to prove that large cardinals exist at all![23]

If large cardinals cannot be proven to exist, why haven't mathematicians discarded the concept altogether? One major reason is that large cardinals are a central part of a number of basic results in mainstream mathematics. There are problems in measure theory, topology, algebra, and logic whose solutions involve large cardinals in an indispensable way.

To get a feeling for large cardinals, we consider the smallest of the large cardinals, *inaccessible* cardinals. One definition of an inaccessible cardinal is the following: a cardinal κ is inaccessible if $\kappa>\omega$ and the stage V_κ has the following two properties:

1. V_κ is not the union of fewer than κ many of the earlier stages V_α.
2. The size of any previous stage V_α is less than κ.

22. By "conceivable set-theoretic operation," we mean an operation that can be formalized in ZFC.

23. For an introduction to large cardinals for the nonmathematician, see (Rucker 1982). For more formal treatments of this subject, including a discussion of all results mentioned in this section of the paper, see (Roitman 1982), (Kanamori, A. and Magidor, M. 1978), (Drake 1974), (Jech 1978), and (Corazza 2000). For an excellent discussion of large cardinals and their relationship to the principles of Maharishi Vedic Science, see (Weinless 1987, Section II.6).

We can see rather quickly that the ordinary infinite cardinals we have described so far could not possibly be inaccessible. For instance, if we consider the cardinal ω_1, it can be shown that property (2) fails because the size of $V_{\omega+1}$ is at least as big as ω_1.[24] On the other hand, if we consider the cardinal ω_ω, we can see that property (1) fails; in fact V_{ω_ω} is the union of just ω many previous stages:

$$V_{\omega_\omega} = V_\omega \cup V_{\omega_1} \cup V_{\omega_2} \cup \ldots$$

We have established that if an inaccessible cardinal exists at all, it must be extremely big. One indication of the enormity involved is the fact that if κ is inaccessible, κ must have the property that

$$(*) \qquad \omega_\kappa = \kappa.$$

Our experience tells us that the phenomenon indicated by (*) is very unusual: $1 < \omega_1$; $2 < \omega_2$; $\omega < \omega_\omega$; and so forth. To find a κ with the property (*) would require a very long journey through the hierarchy of cardinal numbers (and using ZFC alone, even in an endless journey, a large cardinal would never turn up!).[25]

As we indicated earlier, bigger infinities in the universe can be understood to be sets which embody more of the qualities of the ultimate nature of the infinite. This point can be illustrated especially well with inaccessible cardinals: Properties (1) and (2) above indicate not only that an inaccessible cardinal embodies a very strong form of unboundedness, but also that an inaccessible is truly transcendental, beyond intellectual apprehension—and these are well-known qualities of pure consciousness (Maharishi, 1969):

The senses, they say, are subtle; more subtle than the senses is mind; yet finer

24. Note that each natural number lies in V_ω; thus each (possibly infinite) subset of the set of natural numbers is a member of $V_{\omega+1}$. But there are at least ω_1 such subsets. Hence the size of $V_{\omega+1}$ is at least ω_1.

25. It is interesting to note that a cardinal having property (*) will indeed turn up after a sufficiently long climb, but such cardinals will not be large in the technical sense. The least such cardinal can be obtained by taking the supremum of the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ defined by $\alpha_0 = \omega$, $\alpha_{n+1} = \omega_\beta$, where $\beta = \alpha_n$. On the other hand, if there is a sufficiently large cardinal κ (such as a measurable), nearly all cardinals below κ have property (*)!

than mind is intellect; that which is beyond even the intellect is he.

—*Bhagavad-Gita*, 3.42

Yato vāho nivartante aprāpya manasā saha

From where speech returns, even with the mind it is unapproachable.

—*Taittiriya Upanishad* 2.4.1

Another quality of the infinite which is embodied in inaccessible cardinals is indicated by property (∗) above: if κ is inaccessible, it is its own index, and hence in a sense is known and verified only at its own level. This property of the infinite is brought out in (Maharishi, 1991b, p. 190) in comparing the field of pure consciousness with the structure of the unified field discovered by modern physics:

> Ultimately, because the unified field is completely holistic in its nature and interacts with itself alone, it can be verified only at its own self-referral level.

We find the same theme expressed more succinctly in Maharishi's commentary to the *Bhagavad-Gita* (Maharishi 1967, p. 120):

> Realization is not something that comes from outside: it is the revelation of the Self, in the Self, by the Self.

Adding large cardinals to set theory increases the power of the theory to decide a wide variety of mathematical questions and also serves to unify apparently antagonistic theories and views of foundations. When we speak of "adding large cardinals" to set theory, what we mean is "adding a large cardinal axiom to the list of ZFC axioms." A large cardinal axiom is an assertion of the form "A cardinal number having property P exists," where property P is some combination of properties which (consistently) imply (1) and (2) above. Adding to ZFC the axiom "There exists an inaccessible cardinal" (known as the Axiom of Inaccessibility) tremendously increases the power of set theory; new and interesting results can be proven which could not be proven in ZFC alone.

Below we give the names of many of the better known large cardinals in increasing order of strength: For instance, adding the axiom

"There exists a huge cardinal" to ZFC is much stronger than adding the axiom "There is a measurable cardinal."

inaccessible
weakly compact
0#
Ramsey
measurable
strong
Woodin
supercompact
extendible
huge
n-huge for every natural number n

The procession of large cardinals given in the list above comes to a dramatic halt when we arrive at "*n*-huge for every *n*," for if we attempt to strengthen this axiom even the least bit—say by considering the concept of ω-huge—we run into logical contradictions. We will see that, by carefully formulating a large cardinal axiom that is slightly stronger than "*n*-huge for every *n*," but not quite as strong as "ω-huge," riding, so to speak, on the edge of inconsistency, we can account for all other large cardinal axioms. We will see that this new axiom will be strongly motivated by principles in both Maharishi Vedic Science and the theory of large cardinals.

§10. Early Attempts to Justify Large Cardinals

Modern set theory has been rather helpless in trying to explain the peculiar phenomenon of large cardinals. A naive solution to the dilemma would be simply to *declare* that large cardinals exist; we could simply add large cardinal axioms to the axioms of ZFC and the worries would be resolved. However, all the axioms of set theory have strong intuitive motivation; each axiom is a simple property of sets that really *ought* to be true about sets. Why should an axiom of the form "There exists a large cardinal" be true?

One of the most successful efforts to motivate such large cardinal axioms involves a concept known as the *Reflection Principle*. The Reflection Principle asserts that any property which is true of the universe *V* as a whole should be true of some set; in addition, any property which is true of the class of ordinals as a whole should be true of some particular ordinal. The reason the Reflection Principle is reasonable is that the universe and the class of all ordinals represent a kind of absolute infinity[26] which is too vast to be captured by a single property; if for some property *R*, the universe *V* were the only collection which had the property *R*, i.e., no set had this property, then *V* could actually be defined as the unique collection which satisfies the property *R*. This sort of conclusion is intuitively unappealing; *V* ought to be somewhat more rich than the property *R* is able to express. In some sense, being so vast, *V* ought to be "indescribable." Thus, the Reflection Principle makes sense, and appeals to the intuition of many set theorists.

Although the universe *V* was not formally defined in Cantor's time, Cantor had an intuitive conception that, beyond all sets of all possible infinite sizes, there must lie an Absolute Infinite beyond which no larger infinity could be conceived, an Infinite whose properties no mere set could ever begin to approximate (Hallett, 1988, p. 13):

> The Absolute, says Cantor, is the `veritable infinity' whose magnitude is such that it `...cannot in any way be added to or diminished, and it is therefore to be looked upon quantitatively as an absolute maximum. In a certain sense, it transcends the human power of comprehension, and in particular is beyond mathematical determination.'

Further (Hallett, loc. cit.):

> What surpasses all that is finite and transfinite...is the single completely individual unity in which everything is included...

Cantor's intuition about the Absolute Infinite was the original motivation for the work done in the last quarter century on the Reflection Principle. Moreover, recent research by Jensen [∞], Friedman [1993], and others, provides impressive ways of demonstrating the transcendent vastness of *V*; they show that it is impossible to prove that *V* can be obtained by expanding any of the known highly structured well

26. See (Hallett, 1988) for a historical discussion of Cantor's original notion of the Absolute Infinite. See (Reinhardt, 1974) for a justification of the Reflection Principle based on Cantor's notion of the Absolute Infinite. See (Weinless, 1987) for a detailed study of the Reflection Principle and its relationship to the principles of Maharishi Vedic Science.

understood models of set theory by means of standard expansion techniques.[27]

Before explaining how the Reflection Principle is used to justify large cardinals, let's consider a simpler application of this principle. Consider the following property:

$$R(x): \quad x \text{ has infinitely many predecessors.}$$

R is a property which is true of the class ON of all ordinals: if we think of ON as the largest of all ordinals, then $R(ON)$ is true because ON does indeed have infinitely many predecessors. The Reflection Principle then tells us that there must be an actual ordinal number α(i.e., an ordinal not equal to ON) which also has the property R. The simplest example of such an α is the least infinite ordinal ω.

Let us now turn to the justification of the existence of inaccessibles offered by the Reflection Principle. The property $R(x)$ we wish to consider is

$R(x)$: (1) x is a cardinal $> \omega$, and V_x is a stage of the universe which is not the union of fewer than x previous stages, and
(2) the size of V_α is less than x for all $\alpha < x$.

If we assume for the moment that $V = V_{ON}$, as if V were obained as its own last stage, then V is not the union of fewer than ON many stages, so part (1) of $R(V)$ holds. As for (2), it is also clear that the size of any stage is an actual cardinal number, hence less than ON itself. Thus (2) holds as well. By the Reflection Principle, there must be an ordinal κ such that $R(\kappa)$ holds. Hence, there is an inaccessible cardinal in the universe.

The Reflection Principle goes a long way toward justifying the presence of large cardinals in mathematics, but is not entirely successful.[28] First, as far as anyone knows, the very largest of the large cardinals

27. In particular, they show that V cannot be proven to be a locally Cohen generic extension over L, nor, assuming large cardinals, over any known form of the core model K. See Section 20 of this paper or (Weinless, 1987, pp. 181-4) for a discussion of L and of forcing extensions of a model of set theory.

28. See (Weinless, 1987) for a justification of ineffable cardinals using the Reflection Principle, and (Reinhardt, 1974) for a justification of measurables and extendibles.

cannot be justified with these reflection arguments. Second, and more importantly, the Reflection Principle is not entirely precise in its formulation—which "properties" $R(x)$ are we allowed to use? If, for example, we try the property

$$R(x) : \text{Every set is a member of } x,$$

we are faced with the undesirable fact that although $R(V)$ is true, $R(A)$ is false for every set A. To fully understand the origin of large cardinals, we need a deeper and more exact principle than the Reflection Principle.

Although the Reflection Principle does not give a complete solution to the problem of large cardinals, it does give us a significant hint: Large cardinals exhibit properties of the universe *as a whole*.[29] As we observed earlier, the most noticeable omission in the development of modern set theory—at least from the point of view of Maharishi's Vedic Science—is the lack of an axiom describing the nature of the universe V as a whole. The Reflection Principle tells us that the information about the nature of the wholeness of V is revealed ever more fully in the properties of ever larger large cardinals.

Our plan is therefore to consider the very largest of the large cardinals and see what properties they exhibit; these properties should suggest to us what V's "nature" is, as a wholeness. We will see that the strongest of these large cardinal axioms assert that the universe V can be embedded in another model of set theory in a highly coherent way. To grasp the subtleties involved, we must introduce the concept of a model of a theory.

§11. Mathematical Theories and Their Models

A mathematical theory, like set theory, is a collection of all the mathematical statements that can be proven from a set of basic axioms for the theory. Models of a theory are the concrete structures in which the dynamics inherent in the theory are fully realized. Using terminology from Maharishi Vedic Science, Weinless (1987, pp. 180-1) observes

29. Here we are not intending to make precise use of the converse of the Reflection Principle ("large cardinal properties which hold true of κ (V_κ) also hold true of ON (V)). We intend only to assert that experience with the Reflection Principle suggests that larger large cardinals reflect ever more fully the properties of the universe as a whole.

that the axioms for a theory correspond to the concept of "pure knowledge," and its models are the expression of its organizing power. At the same time, as we shall see, models, their elements, and a fundamental logical relationship between them provide a direct parallel to principles in Maharishi's theory of knowledge.

Models of a theory, even though concrete realizations of exactly the same set of axioms, may differ radically in their structure. To illustrate this crucial point, we consider a simple set of axioms for the operation of addition and note the wide range of models admitted by the theory. The basic properties of the operation of addition are that it is commutative, associative and has an additive identity which is usually denoted by '0'. Let us assume we have in our formal language the symbol 0; we can state these three properties as basic axioms:

Axioms for Addition

(1) (Additive Identity) for all x, $x + 0 = x$
(2) (Commutativity) for all x, y,$x + y = y + x$
(3) (Associativity) for all x, y, z, $(x + y) + z = x + (y + z)$

The model we have in mind when we set forth these axioms is the structure **N** of the natural numbers, that is, the set {0, 1, 2, . . .} together with the usual operation of addition. And it is certainly true that all three axioms hold in the model **N**; we say that N is the *intended interpretation* of the theory. Often, however, much of the richness of a theory is discovered in studying the models of the theory that are quite different from the intended one; this is certainly the case in set theory.

Examples of other models of this "theory of addition" abound; consider the set **Z** = { . . . -2, -1, 0, 1, 2, . . . } of integers or the set **Q** of all rational numbers, each with its usual operation of addition. Both these structures satisfy all three axioms; yet each satisfies unique properties that do not hold in our intended interpretation **N**. For instance, **Z** has the property:

(4) There are x, y, both not equal to 0, such that $x + y = 0$;

and **Q** has the property:

(5) For any $z > 0$, there are $x, y > 0$ such that $z = x + y$.

Neither property holds in the structure **N**.[30] Property (4) makes use of the presence of the negative integers in **Z**; property (5) makes use of the fractions in **Q**. Yet, both **Z** and **Q** are perfectly valid models of our theory of addition even if they exhibit properties that may be unexpected from the point of view of our intended interpretation.

§12. Models of Set Theory

The simple situation described in the last section parallels the state of affairs in set theory. We have a set of intuitively appealing axioms for describing the behavior of sets, namely, the axioms of ZFC; and we have an intended interpretation of ZFC, namely, the universe *V*. As we mentioned earlier, *V* is often called the standard universe of sets. Nonetheless, ZFC admits other interpretations; that is, it is quite possible to have a wide range of models of ZFC. Each model, since it satisfies all the basic axioms of sets, is a suitable universe of sets in its own right; each can be considered an adequate background for all of mathematics. Yet, models of ZFC may be radically different in many respects. There are many mathematical statements which cannot be either proven or disproven from the axioms of ZFC; these will hold true in some models of set theory and be false in others. Such statements are said to be independent of ZFC.

The most famous independent statement is known as Cantor's *Continuum Hypothesis*. Cantor's Continuum Hypothesis is a conjecture that states that the size of the set of real numbers is precisely the cardinal number ω_1, just one level of infinity greater than the size of the set of natural numbers.[31] He was never able to prove this result, and some 70 years later, Paul Cohen demonstrated why: Cohen produced models

30 Property (4) fails because the only solution to the equation $x + y = 0$ in **N** is $x = 0$, $y = 0$. Property (5) fails because there do not exist two positive integers having a sum of 1.

31 Cantor was able to prove unequivocally that the set **R** of real numbers is larger than the set of natural numbers; the question was, *how much* bigger? See (Hallett, 1988) for the historical development leading to Cantor's Continuum Hypothesis; see (Kunen, 1980) and (Jech, 1978) for a modern-day formal treatment of statements that are independent of ZFC; see (Weinless, 1987) for a discussion of independence results in the context of Maharishi Vedic Science.

of set theory in which Cantor's Continuum Hypothesis was false (the size of the real line turns out to be ω_2 in one model, ω_{17} in another, and $\omega_{\omega+1}$ in yet another); some years earlier, Kurt Gödel had shown that the Continuum Hypothesis was true in certain other models. The two results together demonstrate that the Continuum Hypothesis is independednt of the axioms of ZFC set theory. (See Jech, 1978, for a modern-day treatment of these results.)

Set theorists look at this multiplicity of possible universes for mathematics as different points of view about the nature of sets; if you look through the glasses provided by one model of set theory, sets appear to have one set of properties; through glasses provided by another, a different set of properties emerge. In each universe, the basic laws given by the ZFC axioms remain true, but independent statements like the Continuum Hypothesis will be settled in different ways.

This attitude that different models of ZFC represent different views of the universe represents an important approach to research among set theorists. To illustrate this approach, consider the following perplexing fact: If ZFC has a set model, then one can actually find (using techniques familiar to logicians) a model *M* of ZFC which has the same size as the set of natural numbers. Yet, being a model of ZFC, it must contain the set of all real numbers, a set which is bigger than the model itself! The paradox is resolved by observing that the model *M* "doesn't know" that it has the same size as the set of natural numbers; the one-to-one match-up between *M* and the natural numbers is not available to *M*'s "world-view". Being a model of ZFC, *M* "knows" that the set of reals is bigger than the set of natural numbers, and "believes" that its own structure is much vaster than that of the reals, expansive enough to include all cardinal numbers. From the perspective of the real world *V*, *M*'s version of the real line is only countable and so *M*'s view of the world is somewhat distorted.

This manner of ascribing the subjective qualities of "knowing" and "believing" to models of set theory is very natural and parallels to a high degree Maharishi's principle that knowledge is different in different states of consciousness. Recall that according to Maharishi's Vedic Science, as more of the value of pure consciousness becomes available to the experiencer, the nature of his knowledge of any object of knowing also changes, reflecting a more holistic and comprehensive appreciation of whatever is known. Now, just as varying the state of consciousness produces different knowledge about the world, so varying the model of set theory produces different truths about sets. And, just as there is a level of awareness[32] which automatically perceives the ultimate truth of the world, so the universe V may be seen as an absolute reference frame in which truth is absolute truth.

We can take the analogy further, and it will be useful to do so to lay the groundwork for later work. As the reader will recall, Maharishi points out that there are three components to the emergence of knowledge: the knower (Ṛishi), the object of knowledge (Chhandas), and the process of knowing (Devatā).[33] So far, we have identified models of set theory as analogous to the Ṛishi aspect and individual sets as analogous to the Chhandas value (interestingly, Weinless [1987] observes the same analogy with somewhat different motivation). What corresponds to the Devatā value? The way that statements about sets are determined to be true or false in a given model M is by means of the satisfaction relation ($\models$), a logical relation that is designed to systematically[34] determine the truth or falsity of a given statement relative to M. Consider for instance the statement

The real line $\mathbf{R}$ has size ω_5.

Since this statement depends on the two sets $\mathbf{R}$ and ω_5, we can name the above statement $\phi(\mathbf{R}, \omega_5)$. ($\phi$ is called a *formula* with parameters $\mathbf{R}$ and ω_5.) If M is a model of set theory in which $\mathbf{R}$ does indeed have size ω_5, we would write:

$$M \models \phi(\mathbf{R}, \omega_5).$$

In this type of expression, common in the literature in set theory, we see the analogues to Ṛishi, Devatā, and Chhandas clearly displayed: M corresponds to Ṛishi; the sets in M—in this case $\mathbf{R}$ and ω_5—correspond to Chhandas; and the satisfaction relation $\models$ corresponds to Devatā. Recall that in Maharishi's theory of knowledge, knowledge is what

32. Maharishi calls this level of consciousness *unity consciousness*; see (Maharishi 1972).

33. Cf. (Chandler 1987, 5-26).

34. The word "systematically" is not intended to mean "algorithmically" here.

emerges in the relationship, or "togetherness," of Ṛishi, Devatā, and Chhandas; likewise, the relationship of M, $\models$, and the sets $\mathbf{R}$ and ω_5 results in *M's* (partial) "knowledge" of these sets.

To sum up, we see that models of set theory are closely related to Maharishi's theory of knowledge: A model of set theory together with its satisfaction relation ($\models$) and its members, i.e., sets, correspond to the basic principles of Ṛishi, Devatā, and Chhandas; moreover, just as different levels of consciousness provide the knower with different truths and knowledge about reality, so different models of set theory provide different views of sets and their relationships.

We next turn to a consideration of the relationship between different models of set theory; elementary embeddings offer the most interesting of these possible relationships by providing a natural analogue to self-knowledge. As a preliminary to the notion of elementary embeddings, we introduce its conceptual components: elementary submodels and isomorphisms.

§ 13. Elementary Submodels and the Concept of Isomorphism

Whenever mathematicians encounter a proliferation of differences, like the variety of models of set theory, one question that naturally arises is, "In what ways are these different objects actually the same?" In the context of models of set theory, this question is answered in two ways: Models are basically the same either if they are *isomorphic* or if one is an *elementary submodel* of another.[35] We will see later that both these concepts are intrinsic features of the fundamental nature of V as it moves within itself and "knows" itself.

First, let's consider elementary submodels. Suppose we have two models of set theory, M and N, where M is a subset of N. It is conceivable that, concerning the sets that both models know about, namely, the sets inside M, both models could have exactly the same knowledge, and believe exactly the same statements. When this phenomenon occurs, M is said to be an *elementary submodel* of N and we write

35. Actually, there are other notions of sameness between models. One notion of equivalence that is at least as prevalent in the work of model theorists as elementary submodels and isomorphism is that of *elementary equivalence*; two models are elementarily equivalent if they satisfy the same first order sentences. If two models are isomorphic, or if one is an elementary submodel of the other, then the models are elementarily equivalent. See (Chang and Keisler, 1973).

$$M \prec N.$$

Stated more precisely, we say M is an *elementary submodel of N* if $M \subseteq N$ and for any sets $A_1, A_2,..., A_n$ in M and any relationship $\phi(A_1, . . . , A_n)$ between them, ϕ is true in M if and only if ϕ is true in N. To state the matter another way, M's knowledge of the sets $A_1, . . . , A_n$ is exactly the same as N's knowledge of them; borrowing terminology from Maharishi Vedic Science, we could say that M and N exhibit "infinite correlation" in terms of their knowledge of the sets in M, even though in many ways, M and N may appear different (for instance, the two models might have radically different sizes).

Next we consider isomorphic models. Suppose again that M and N are two models of set theory. M and N are said to be *isomorphic* if there is a way to transform M into N so that all relationships and differences among sets in M are preserved in N. More precisely, M and N are isomorphic if there is a way to match up each set A in M one-for-one with a set B in N so that for any formula $\phi(A_1, . . . , A_n)$ is true of the sets $A_1, . . . , A_n$ in M if and only if $\phi(B_1, . . . , B_n)$ is true of the sets $B_1, . . . , B_n$ in N. Intuitively, M and N exhibit the same properties and dynamics qua universes of sets; each has its own version of the empty set; each has its own version of the real line **R** and of the cardinal numbers. If, for example, M thinks that its own version of **R** has size equal to its own version of ω_5, then N will believe that the object in N matched with the **R** in M has size equal to the cardinal in N matched with the ω_5 of M.

Elementary submodels and isomorphic models exhibit a preservation of fundamental structure in the face of certain types of transformation. An elementary submodel of a model can be said to have maintained the structure of the large model in the face of "miniaturization." An isomorphic copy of a model M can be said to have maintained the structure of M in the face of a "name reassignment."

These kinds of preservation are central in the study of set theory. Moreover, they exhibit a basic property of the dynamics of pure consciousness itself, as described by Maharishi: the many stages of expression that emerge from the self-interacting dynamics of pure consciousness always remain connected with the holistic *samhita* value of consciousness; this feature is the basis for the qualities of *infinite correlation* and *invincibility* that are ascribed to pure consciousness.

In the next section, we will encounter the surprisingly powerful consequences that result from combining the concepts of elementary submodel and isomorphism in the context of models for ZFC.

§14. Elementary Embeddings of the Universe

Having considered the concept of a model of set theory and the possible structure-preserving relationships between models, we can return to our study of the strongest large cardinal axioms to see what conclusions can be drawn about the nature of V as a whole.

In the hierarchy of large cardinals, those at the upper end, like measurable, strong, supercompact, and huge cardinals, are defined in terms of a special kind of transformation j called a *nontrivial elementary embedding of the universe*. A typical embedding of this kind is given by an expression like the following:

$$j : V \to M,$$

where M is a transitive model[36] of set theory containing all the ordinals. The behavior of j can be considered in two steps. First, j isomorphically transforms V into another model V' where V' forms a subcollection of M. Secondly, the model V' is an elementary submodel of M. See Figure 2.

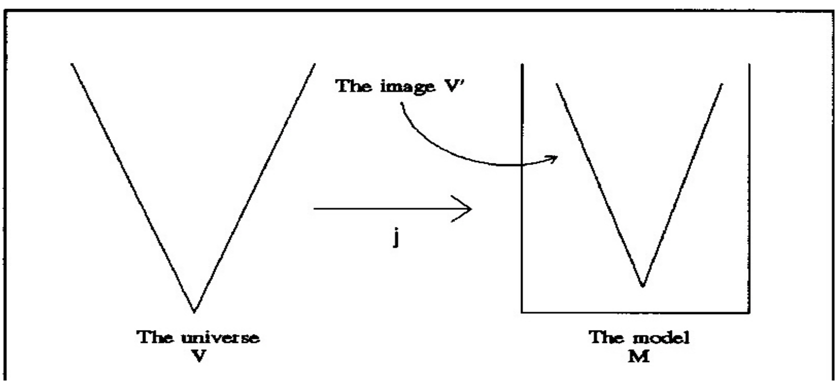


Figure 2. An elementary embedding of the universe V.

36 M is *transitive* if for any $y \in M$ and any $x \in y$, we have $x \in M$. The transitive models tend to be easiest to understand because their elements are "normal" sets.

From our discussion in the last section, it follows that V, V' and M are all extremely similar in their structure. In fact, V and M bear the following relationship:

> For all sets $A_1, \ldots, A_n$, and for any relationship $\phi(A_1, \ldots, A_n)$ between them, $\phi(A_1, \ldots, A_n)$ is true (in V) if and only if $\phi(j(A_1), \ldots, j(A_n))$ holds true in M.

This property of j is simply a combination of the fact that V and V' are isomorphic via the transformation j and that V' is an elementary submodel of M.

To illustrate the property, suppose it is true that in V, the size of **R** is ω_5. Then given j as above, it follows that in M, the size of $j(\mathbf{R})$ is $j(\omega_5)$. Moreover, since V believes **R** is the real number line, M will believe that $j(\mathbf{R})$ is the real number line. And since V believes ω_5 is the fifth uncountable cardinal, M will believe that $j(\omega_5)$ is the fifth uncountable cardinal. See Figure 3.

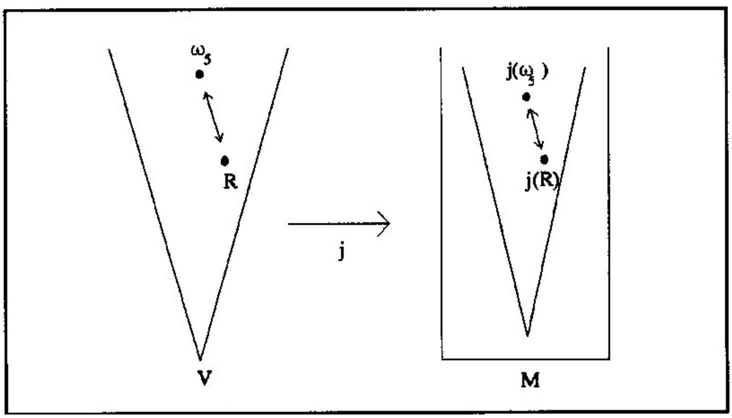


Figure 3. Preservation of relationships under elementary embeddings.

The embedding is called *nontrivial* to eliminate the possibility that j is merely the identity function from V to itself, the function which assigns to each set A in V the set A itself; the identity function, though important in its own way, does not have any powerful mathematical consequences. In particular, if j is a nontrivial elementary embedding, some set in V must be sent to a set different from itself in M; we say that some set is *moved* by *j*. See Figure 4.

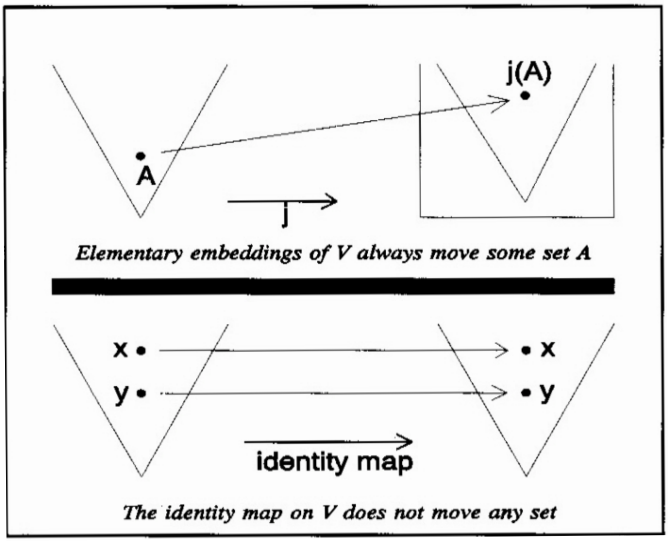


Figure 4. Nontrivial elementary embeddings vs the identity map.

As we mentioned above, elementary embeddings of the universe *V* give rise to large cardinals. It can be shown that if a set is moved by *j*, some ordinal must also be moved. The least ordinal moved is called the *critical point of j*. This ordinal is typically denoted by a Greek letter; in this paper, we use the letter κ (pronounced "kappa"). Moreover, this critical point is necessarily a large cardinal, in fact a rather large large cardinal (at least a measurable cardinal). (See Jech, 1978.) Thus, the first ordinal moved by this very natural-seeming transformation of *V* into another universe is infused with extraordinary properties of infinity.

The stronger large cardinal axioms assert the existence of nontrivial elementary embeddings of various kinds. As we remarked earlier, experience with the Reflection Principle suggests that the larger large cardinals reveal properties of the universe as a whole. We suggest therefore that *V* "tends" to move within itself: It is a characteristic of the structure of *V* to be moved into a universe of sets (this universe could be *V* itself or some other universe) via an elementary embedding. In this way, large cardinals are generated that allow us to see, in the realm of sets, what is true about a wholeness (namely V) that is beyond our ability to know.

Moreover, it turns out that the stronger the large cardinal generated by an embedding (or class of emebeddings) of the form $j : V \to M$, the more closely M must resemble V in its structure. To illustrate this pattern, let us contrast the definition of a measurable cardinal with that of the much stronger concept of a *strong* cardinal. We begin with the formal definitino of a measurable cardinal:

Definition (*Measurable Cardinals*). κ is a *measurable cardinal* if there is a nontrivial elementary embedding $j : V \to M$ having critical point κ.

Figure 5 displays the behavior of such an embedding. No special extra properties have been imposed on j or M; the presence of any nontrivial elementary embedding of the universe is sufficient to give rise to a measurable cardinal.

As Figure 5 indicates, in addition to resembling V in all the ways that an elementary embedding requires, M also resembles V in that its first κ stages are the same as the first κ stages of V. One writes:

$$V_\kappa = V_\kappa^M.$$

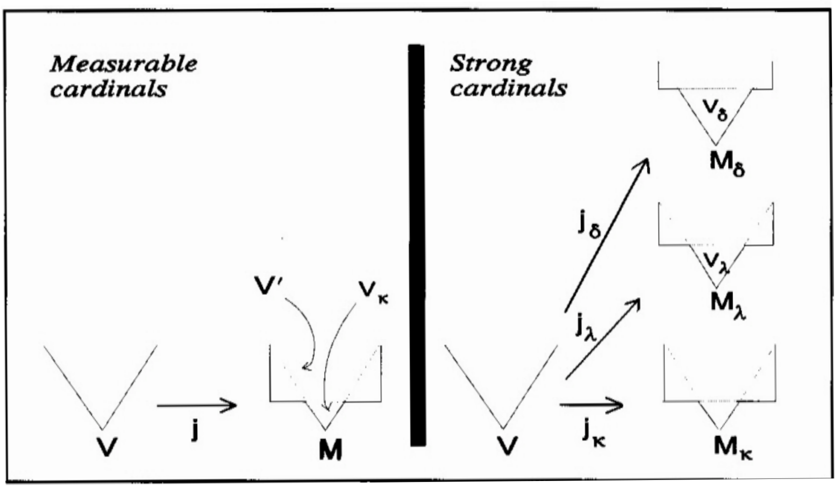


Figure 5. Measurable cardinals and strong cardinals

Definition (*Strong Cardinals*). κ is a *strong cardinal* if for every cardinal number $\lambda \geq \kappa$, we can find a universe M_λ and a nontrival elementary embedding $j_\lambda : V \to M_\lambda$ having critical point κ, such that

$$V_\lambda = V_\lambda^{M_\lambda} .$$

What is new here is that an entire class of embeddings is guaranteed to exist and the correponding universes M_λ resemble V more and more completely as λ increases; as λ increases, more and more stages of V are required to be identical to those in the image model. In particular, for each λ, the first λ stages of V are identical to the first λ stages of M_λ.

Still stronger large cardinal axioms require, sometimes in subtler ways, that the image models M resemble V even more closely.

If stronger and stronger large cardinal axioms assert the existence of embeddings $j : V \to M$ with M resembling V more and more closely, could it happen that $M = V$? An axiom that asserts the existence of a nontrivial elementary embedding $j : V \to V$ would represent the natural culmination of all previous large cardinal axioms; one would expect that the large cardinal that would emerge from such an embedding as its critical point would have the strongest properties of all.

Moreover, let us consider the implications of such an embedding $j : V \to V$ in light of our earlier analogy between models of set theory and Maharishi's theory of knowledge. As the reader will recall, V "knows" about the properties of the sets it contains via the satisfaction relation; here, V plays the role of knower; sets, the known; and the satisfaction relation, the process of knowing. These dynamics parallel the familiar process of gaining knowledge of the outer world. However, this same process of knowing can be applied to consciousness itself and the result, as described in Maharishi Vedic Science is the dynamics of self-knowing that constitute the eternal unmanifest activity of pure consciousness at the basis for all activity in the manifest world.

Likewise, in the presence of j, a fundamental dynamism is introduced that places V—representing the rishi or knower—in relationship *with itself*. First, V is transformed within itself to V', the image of V under j. This transformation of V is completely structure-preserving: All truths about the structure of V are preserved under this transformation. Then, although in certain respects V' appears different from V, V' remains infinitely correlated with V in the sense that V' is an elementary submodel of *V*. Thus, as *V* interacts with itself via the embedding *j*, the structure of *V* remains intact throughout the phases of transformation.

We find the dynamics embodied in *j* quite similar to the dynamics of self-knowing attributed to the wholeness of pure consciousness or Samhitā: Recall from Maharishi's Vedic Science that in order for wholeness to know itself, the fundamental unity of Samhitā appears as three; Samhitā must assume the roles of knower, object of knowledge, and the relationship between them in order for Samhitā to know itself. Samhitā, remaining ever the same, yet exhibits these divisions within its own nature. Likewise, the nature of *V* as a whole becomes known when *V* moves within itself via the embedding *j*. Without *j*, *V* remains a transcendental wholeness beyond the realm of sets; the embedding *j*, however, dynamically relates *V* to itself, placing it in different roles in relationship to itself while preserving its fundamental structure (on the one hand it plays the role of pure Samhitā, the unified value of wholeness; on the other hand, it assumes the roles of Ṛishi—"knowing" as it does the various truths about its own structure—and Chhandas, as that which is being known). We shall see in the next section that new knowledge about the *V*'s fundamental structure emerges from this interaction.

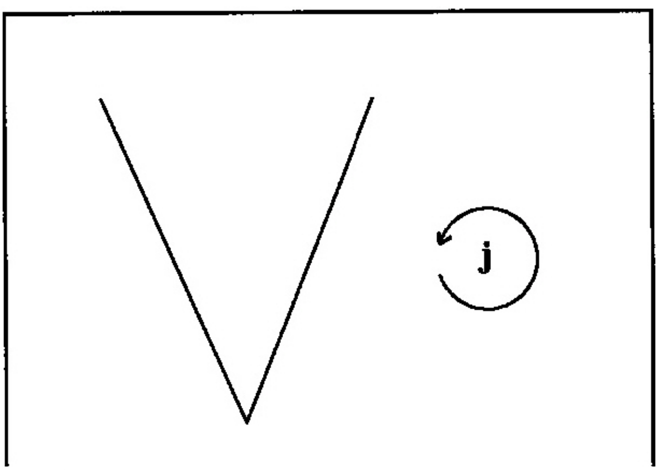


Figure 6. An elementary embedding from the universe to itself.

Despite the naturalness of the large cardinal axiom "There is a nontrivial elementary embedding from V to V," K. Kunen [1971] proved, under certain assumptions, that the existence of such an embedding would lead to an inconsistent set theory. In the next section, we discuss this dilemma at some length and suggest an attractive solution. The starting point is the intuition, motivated by Maharishi Vedic science, that the dynamics of V, which represents wholeness in mathematics, should mirror the dynamics of wholeness itself, the dynamics by which creation itself emerges. This intuition suggests that some kind of truth-preserving embedding from V to itself ought to exist. This intuition will motivate us to look more carefully at the assumptions underlying Kunen's theorem and its proof.

§15. Attempts to Bypass Kunen's Theorem

The naturalness of a nontrivial elementary embedding $j : V \to V$ has not gone unnoticed by set theorists. Kunen's proof that such embeddings do not exist has been studied from a number of different angles to see if some weaker form of elementary embedding retaining the flavor of $j : V \to V$ could still be consistent.

One line of thought that has resulted in deep work by H. Woodin (1989) begins with the observation that Kunen's proof relies heavily on the Axiom of Choice (see Weinless for a discussion of this axiom)—so much so that the proof will not work if the Axiom of Choice is replaced by any of the better known weakenings of this axiom. Consequently, it is quite likely that there is a universe V in which Choice fails but one of these weakenings of choice still holds, and there is a nontrivial elementary embedding $j : V \to V$. Woodin has shown that such a "choiceless" embedding is still strong enough (in the presence of a weaker choice principle) to consistently imply all known large cardinal axioms; in particular, he has shown how, starting from such a j, to build another model M in which the Axiom of Choice holds and all known large cardinal axioms are true.

Woodin's result is a masterpiece of mathematics, but we do not feel that a universe in which the Axiom of Choice fails is the right starting point for mathematics; nor is it intuitively desirable to have to step into the relativized world of Woodin's forcing model to gain access to large cardinals.

Another lesser known angle has been to weaken the definition of elementary embedding, the insight being that perhaps "elementary embedding" is simply too powerful a concept to be the "right" notion. Work is still underway in this area by a handful of researchers. The most enticing result so far has involved weakening "elementary embedding" to "exact functor"; an exact functor from the universe to itself is one that preserves certain simple functional relationships between finite collections of sets;[37] it is an especially natural concept in the context of the geometry of sheaves. A. Blass (1976) showed that there is a nontrivial[38] exact functor from the universe[39] to itself if and only if there is a measurable cardinal.

This line of research is very promising; so far however, functors of this kind have not produced cardinals even as large as extendible. Moreover, if such a functor could be devised, it remains to be seen if its properties will be as geometrically natural as "exactness."

Yet another observation has been that Kunen's proof does not forbid elementary embeddings from a stage V_λ to itself when λ is a limit; such an embedding forces V_λ to be a model of set theory. One could then ask if such a V_λ would be the right foundation for all of mathematics. One might expect the answer to be "no" because V_λ fails to include most of the stages of V (namely, those that come after V_λ), but this problem is not so serious as one might expect. This approach has interesting parallels with Maharishi Vedic Science and plays an important role in the approach that we propose in this paper; we therefore postpone further discussion for a later section (see Section 19).[40]

§16. The Wholeness Axiom

A closer look at Kunen's proof reveals another assumption implicit in the reasoning: In order to arrive at an inconsistency, it must be assumed

37. Cf. (Mac Lane 1971) for a precise definition of *exact functor*.

38. In this context, "nontrivial" means "not naturally isomorphic to the identity functor".

39. In this context, the universe is understood in the context of category theory; V is taken to be the category of all sets together with all functions. See (Weinless, 1987) and (Mac Lane 1971) for further discussion.

40. A nontrivial elementary embedding from a stage $V_{\lambda+1}$ to itself for some limit λ is also not known to be inconsistent, but Kunen's proof forbids such an embedding from $V_{\lambda+2}$ to itself. See (Kunen 1971).

that the elementary embedding j is *weakly definable*[41] in V. Intuitively, this means that V "knows about" the embedding in much the same way it "knows about" sets. If our intuition about j is to be guided by the principle that j corresponds to the fundamental dynamics of wholeness moving within itself, as described by Maharishi Vedic Science, and that sets correspond to manifest existence, we would not expect the dynamics of j to be on a par with the dynamics of sets. Moreover, we would expect that, if possible, j ought to be in some way *unmanifest*, hidden from the more "expressed" activity of sets. Here is what Maharishi says about the dynamics within wholeness:

> In the state of one-being-three we have the state of complete unified wakefulness. In this is the first value of transformation in the unmanifest value. When we say 'transformation', we still mean this level is unmanifest. Samhita in terms of rishi, devata, and chhandas, and rishi, devata, and chhandas in terms of samhita: this is the fundamental transformation, the fundamental relationship. (Maharishi 1990)

> Supreme intelligence does not partake of activity. It is so exalted and powerful that by virtue of its very being it is the field of all possibilities, the source of all action. It is so unlimited that it can function without functioning—its very presence regulates activity so that it is spontaneously right. (Maharishi 1976, p. 131)

Guided by this intuition, we suggest that the "right" axiom for describing the fundamental dynamics of V should involve an elementary embedding j which is not (weakly) definable in V. Technically, j, as a "function" from V to itself, is a subcollection of V, but there is no first-order formula which defines this subcollection.[42] We say that j ought to

41. Kunen's proof forbids more than just *definable* elementary embeddings from V to itself; a class $\mathbf{C}$ in V is weakly definable if, treating $\mathbf{C}$ as an extra predicate in the language, all instances of Replacement in the expanded language hold true.

42. The definition of "function" must be modified somewhat to be applicable in this context. The most obvious problem is that, because j is not a set, it cannot properly be called a function either. One may still consider j to be a vast subcollection of V consisting of ordered pairs, $j = \{(x,y): y = j(x)\}$, but this formulation is also not correct because the definition of j as an elementary embedding requires that the codomain of j be specified. Thus, to be precise, if we let $j_0 = \{(x,y): y = j(x)\}$, then we may formally define j to be the *disjoint union* of the collections j_0 and V.

be a *transcendental*[43] *elementary embedding*.

Definability and Weak Definability

Intuitively speaking, the sets, functions, and other mathematical concepts that mathematicians typically work with are definable. More precisely, if the objects constituting a set (or the ordered pairs comprising a function) are precisely those objects which satisfy some formula (in a given model M), then that set (or function) is said to be definable (in M).

For example the set $\{0, 2, 4, \ldots\}$ of even natural numbers is definable in the model $\mathbf{N}$ of natural numbers: Consider the formula

$$\phi(x)\text{: } \textit{there exists } n \textit{ such that } x = n + n.$$

The natural numbers which can be used to replace the variable x to obtain a true sentence are precisely the even numbers.

Next, we consider an example of a definable function in the context of sets rather than natural numbers. Let F be the function defined on sets which assigns to each set A the singleton set $\{A\}$, so $F(A) = \{A\}$. To see that F is definable (in V), consider the formula

$$\phi(x, y)\text{: } \textit{the only member of } y \textit{ is } x.$$

Now, we can see that for all sets A, B, $F(A) = B$ if and only if $\phi(A, B)$ holds in V.

The function F provides a typical, though simple, example of definable functions: The definability of such a function guarantees that there is a uniform procedure for obtaining the output

43. Sometimes such embeddings are called "external"; we have chosen not to use this terminology because it incorrectly suggests that j lies outside of V. Certainly j is not an element of V and is not definable in V, but, as we have seen, j does form a subcollection of V. The point is that j lies within V but is not "graspable" within V in the usual ways; Maharishi Vedic Science provides excellent intuition for this phenomenon.

given any input.

What about weak definability? The difference between these two notions is subtle. For most purposes, the concepts are the same and so we will loosely proceed in this paper as if they were the same.

From the point of view of Maharishi Vedic Science, we would not expect that the behavior of a function intended to represent the unmanifest dynamics of existence could be uniformly described with a single formula; therefore, it is natural to expect that, if a nontrivial embedding $j : V \to V$ exists at all, it must be undefinable (indeed, not even weakly definable).

We must be careful, however, not to remove j too radically from the world of sets in V. From the mathematical point of view, to insist that j be transcendental without any other conditions would significantly weaken the axiom—so much so that the resulting axiom would be weaker than a measurable cardinal!

From the perspective of Maharishi Vedic Science, we need to consider somewhat more deeply our analogy between j, V, and sets on the one hand, and the dynamics of pure intelligence, wholeness, and manifest existence, respectively, on the other hand. Maharishi explains that the fundamental self-interacting dynamics of pure intelligence form the blueprint of creation itself, the Ved, and that this field of life is the primary administrator of all of creation (Maharishi 1976, p 123):

> ...consciousness is the prime mover of life and administrator of all action, and...anyone who develops in himself the full potential of consciousness enjoys a natural authority over the whole field of action and achievement.

This fundamental field of life has two important attributes:

1. Its activity is hidden from view, unmanifest.
2. Its activity is intimately integrated with creation; in fact it is present at every point in creation.

Maharishi elaborates on this second point in the *Science of Being* (Maharishi 1966, p. 29):

> It has been said that Being is the ultimate reality of creation and that It is present in all strata of creation. It is present in all forms, words, smells, tastes and objects of touch; in everything experienced; in the senses of perception and organs of action; in all phenomena; in the doer and the work done; in all directions—north, south, east and west; in all times past, present and future; It is uniformly present. It is present in front of man, behind him, to left and right of him, above hime, below him and in him. Everywhere and in all circumstances Being, the essential constituent of creation, permeates everything.

Maharishi explains that this twofold reality of pure consciousness—that it is both unmanifest and present at every point in creation—is not merely an abstract truth of transcendental existence, but can be made a living reality in individual life through the development of consciousness (Maharishi 1976, p. 132):

> When consciousness is so developed that it can make everything its own, it flows into the channels of relative life while at the same time maintaining its own transcendental, absolute state, non-channelled and all-pervading.

Using these two points to guide the development of our axiom about *j*, we see that our decision to require *j* to be transcendental corresponds to the first of these points (that the activity within pure consciousness is unmanifest), but that we need to formulate another condition corresponding to the second point (that this activity is present at every point in creation).

A very natural way to frame the second point in mathematical terms emerges when we look to see why a bare transcendental elementary embedding is so weak: The problem is that when we attempt to form sets in the universe using *j*, the collections we form turn out not to be sets at all (since *j* is transcendental) but, like *j*, remain "hidden from view," forbidden from interacting with other sets in the usual way.

We may eliminate this discoordinating effect of having an undefinable embedding by requiring that *V* be *fully j-closed*. This means that whenever we define a subcollection of a given set using *j* (or one of its iterates j^n), the subcollection turns out to be a real set in the universe. This requirement corresponds very nicely to our second point: When *V* is fully *j-closed*, *j* is permitted to participate in set formation as a parameter in formulas in exactly the same way individual sets can; since sets represent the point-value of the universe V, we can say that full *j-closedness* permits j to play the role of a point-value in the universe.

We can now state our axiom. For completeness, we first give a the following formulation of the concept of *j-closedness*:[44]

Defintion: Suppose $j : V \rightarrow V$ is a transcendental elementary embedding. Then V is fully *j-closed* if for every set A, every natural number m, every formula $\phi(x_1, \ldots, x_n, x_{n+1}, x_{n+2})$ and all sets $B_1,\ldots, B_n$, the collection

$$\{a \in A\colon \phi(B_1,\ldots,B_n, A, j^m)\}$$

is a set, where j^m is m-fold composition of j with itself.

Wholeness Axiom

> There is a transcendental (nontrivial) elementary embedding $j : V \rightarrow V$ such that V is fully *j-closed*.

Before exploring the interesting consequences of the Wholeness Axiom, let us summarize what we have accomplished so far. We began our study by observing that, while set theory with its universe of sets has been extremely successful as a foundation for mathematics, the fact that large cardinals, which arise naturally in many areas of mainstream mathematics, cannot be accounted for by ZFC impels one to search for a satisfactory intuition by which to strengthen the present axiom system and thereby determine which large cardinals should be allowed in the universe. We chose to use principles of Maharishi Vedic Science to clarify our intuition about the nature of wholeness, believing that the wholeness set theorists are attempting to express in the concept of the universe of sets has been examined thoroughly in the Vedic tradition of knowledge. In reviewing the basic qualities of wholeness, as described by Maharishi Vedic Science, we found that a few such qualities–*infinite correlation, awake within itself, self-referral and bliss*– were not adequately expressed in the construction of V; in particular, we observed that in order for a universe of sets to exhibit these

44 The definition and formulation of the concept of *j*-closedness and of the Wholeness Axiom given here are made more technically precise in the mathematical literature (Corazza 1994, 2000, 2006).

qualities adequately, it should have an abundance of sets that reflect all the first order properties of the universe, and there should be, associated with it, some natural kind of transformation so that it exhibits a form of self-interaction. In addition, analysis of the dynamics of pure consciousness revealed that, using ZFC alone, sets emerge from the empty set in a way that is strongly analogous to just one-half of the dynamics of wholeness moving within itself—namely, those dynamics concerned with the expansion of the point value of wholeness to the fully expanded infinite value of wholeness. Thus, we were led to seek a universe exhibiting several new qualities and displaying new dynamics such as "infinity collapsing to a point." From the point of view of mathematics, we took a hint from the Reflection Principle about where to look for new axioms that talk about the nature of the universe as a whole. This principle suggested to us that large cardinal properties are powerful properties of the infinite that actually ought to be considered properties of the wholeness of V itself. This intuition suggested that, since all the strongest large cardinal axioms are framed in terms of the existence of elementary embeddings of the universe with image models increasingly similar to V itself, we should consider it to be the very nature of the wholeness of V to be moved by such an embedding, and the most natural of such embeddings should have codomain V. In light of Kunen's theorem, we clarified the requirements of the embedding so that it would not be definable (or even weakly definable) but, at the same time, remain coordinated with the structure of V (requiring that V be fully j-closed).

We proceed now to show that our efforts have been successful. We will indicate how the creation of sets in the universe, in light of our new axiom, embodies the new qualities and dynamics we have been seeking. We will also see that virtually all known large cardinals can be accounted for in our new set theory.

To begin the discussion, we will first gain a feel for the new dynamics that the Wholeness Axiom introduces.

§17. Simple Consequences of the Wholeness Axiom

The Wholeness Axiom, as we have indicated, has many powerful consequences. However, in this section we focus instead on the new style of reasoning that arises in applications of the axiom. We shall see how proofs from the axiom involve repeated swings between particulars about individual sets on the one hand, and awareness of the nature of V as a whole on the other hand. This continual calling of attention to the fact that V is our underlying model and that j is moving V within itself produces a new dimension of profundity to the mathematical arguments.

In order to appreciate the new feature that arises in reasoning with j, we first observe that statements in mathematics are assertions that certain properties hold with respect to certain sets (even a mathematical computation can be viewed in this way, where the property involved is "equality"). In the language of Maharishi Vedic Science, the particular sets that we reason about in a mathematical argument are "point values"—specific points in the universe. Our attention is focused on manipulating and relating points in the universe; it is *not* focused on the underlying wholeness in which these activities are being carried out. The new feature that emerges in working with j is that our attention must swing between properties about points to awareness of V as a whole, and then back again to points. As we discussed earlier, the creative activity within pure consciousness unfolds through the repeated collapse of infinity to a point and expansion of point to infinity. The fact that a similar phenomenon occurs in working with j provides further confirmation that addition of the Wholeness Axiom to ZFC introduces dynamics into the foundation of mathematics that mirror those of Nature's functioning.

To illustrate this new feature, let us first recall the characteristic feature of our embedding j: Being an elementary embedding from V to itself means that for any property P that talks about sets $A_1, A_2, \ldots, A_n$, two things must happen:

(1) if P is a *true* property of the sets $A_1, A_2, \ldots, A_n$, then P must also be a true property of the image sets $j(A_1), j(A_2), \ldots, j(A_n)$;
(2) if P is false with respect to $A_1, A_2, \ldots, A_n$, then P must also be false with respect to $j(A_1), j(A_2), \ldots, j(A_n)$.

In this way, the embedding j can be seen to be a *truth-preserving and truth-reflecting map*; every conceivable relationship between sets is maintained both by j and its "inverse".

We will now apply these properties of j in a simple example and point out the new features of the argument. Let us prove that $j(\varnothing) = \varnothing$ (where, recall, $\varnothing$ denotes the empty set). First, let us observe that $\varnothing$ is the unique set that has no element. We have just described a property P:

$$P(x): \quad x \text{ has no element;}$$

moreover, this property P is true with respect to the empty set $\varnothing$; that is, the formula becomes true when $\varnothing$ is substituted for x. In other words,

"$P(\varnothing)$: $\varnothing$ has no element" is true.

Now we can apply (1) to conclude that P is also true with respect to $j(\varnothing)$; in symbols:

"$P(j(\varnothing))$: $j(\varnothing)$ has no element" is true.

Of course now, since $\varnothing$ is the *only* set that has no element, it follows that $j(\varnothing) = \varnothing$.

The flow of the argument develops in three stages: First, there is an assertion concerning the empty set; this assertion has the same characteristic as most other mathematical statements in that it is concerned about localized sets and not at all about V. Second, we take stock of the property asserted to hold of $\varnothing$ and formulate it precisely; this property comes to be viewed as a point of knowledge (that V has concerning one of its sets) to which the global functioning of j may be applied. When we make the move from "P($\varnothing$) holds" to "P($j(\varnothing)$) holds," applying j in this way to the parameter $\varnothing$ of the formula, we are engaging the dynamics of the universe's move within itself; our focus has expanded to a global one. Finally, having applied j, we return to the localized world of small sets and evaluate the character of the new set $j(\varnothing)$, discovering that because "emptiness" uniquely characterizes $\varnothing$, $j(\varnothing) = \varnothing$.

A similar pattern is evident *whenever* we attempt to use j in our arguments: At certain points in our reasoning, we must expand our local context to the strictly unlocalized behavior of j; having applied j, we again return to our local context for further reasoning. In practice, the contrast between local and global can be quite startling and often leads to elegant proofs.

Recall that in set theory, the natural number 0 is identified with the empty set $\varnothing$; thus the argument above shows that $j(0) = 0$. Similar reasoning shows that for each natural number n, $j(n) = n$. It turns out that for every set A that commonly occurs in mathematical practice, $j(A) = A$. As a second example of this latter fact (omitting certain details), let us consider the circle C with radius 1 and origin (0,0), and compute $j(C)$. Of course, if j were an ordinary function, $j(C)$ ocould be anything—certainly there would be no reason to expect $j(C)$ to have any of the characteristics of a circle. But since j "preserves all properties," we should expect to find a strong resemblance between C and $j(C)$. Let's begin the computation by stating precisely how C has been defined:

"C is a circle with radius 1 and center (0,0)."

This is a statement involving the set parameters C, 1, and (0,0); applying j tells us that $j(C)$ is a circle with radius $j(1)$ and center $j((0,0))$. We have already seen that $j(1) = 1$. Let's evaluate $j((0,0))$ by considering the property $P(x,y,z)$ that asserts that z is the ordered pair with components x and y:

$$P(x,y,z):\ z = (x,y).$$

Then $P(0,0,(0,0))$ is true. Applying j, we conclude that $P(j(0),j(0),j((0,0)))$ is also true; in other words, $j((0,0))$ is the ordered pair with components $j(0)$ and $j(0)$. Since $j(0) = 0$, $j((0,0)) = (0,0)$.

We may now conclude that $j(C)$ is a circle whose radius is 1 and center is (0,0); it follows, therefore, that $j(C) = C$.

In this argument, we expanded to an unlocalized view at least twice. The first time involved applying j to C, 1, and (0,0) in order to find out what properties $j(C)$ would have. The second time involved applying j to 0 in order to evaluate $j((0,0))$.

As a final example, we shall prove a proposition that will be quite useful later on: Not only is it true that the critical point κ is the first ordinal number moved by j, but in fact, no *set* (ordinal or otherwise) that occurs in any of the first κ stages of V is moved by j. In other words:

Proposition *For all sets A in V_κ, $j(A) = A$.*

Proof First, let us observe that any set $A \in V_\kappa$ is actually in some V_α, $\alpha < \kappa$. We first show that A and $j(A)$ must lie in exactly the same stages V_α for $\alpha < \kappa$. First let us notice that, for such α, $j(V_\alpha) = V_\alpha$: Notice that "V_α is the αth stage" is a property true in V; applying j, we conclude that "$j(V_\alpha)$ is the $j(\alpha)$th stage" is also true. Thus, $j(V_\alpha) = V_{j(\alpha)}$. But now since $\alpha < \kappa$, it follows that $j(\alpha) = \alpha$, and so

$$j(V_\alpha) = V_{j(\alpha)} = V_\alpha.$$

We wish to show that for any A and any $\alpha < \kappa$, $A \in V_\alpha$ if and only if $j(A) \in V_\alpha$. So, suppose $A \in V_\alpha$. Notice that "$A \in V_\alpha$" is a property true in V. Applying j (by (1) above), we have that $j(A) \in j(V_\alpha)$; now since $j(V_\alpha) = V_\alpha$, we conclude that $j(A) \in V_\alpha$.

Conversely, suppose $j(A) \in V_\alpha$. We wish to show that $A \in V_\alpha$. Since $j(V_\alpha) = V_\alpha$, it follows that $j(A) \in j(V_\alpha)$. Therefore, "$j(A) \in j(V_\alpha)$" is a property that is true in V. Applying (2) above to this property, we conclude that $A \in V_\alpha$. Summing up, for any set A and any $\alpha < \kappa$, we have seen that $A \in V_\alpha$ if and only if $j(A) \in V_\alpha$.

We have completed the first phase of the proof by showing that A and $j(A)$ must lie in the same stages of V. To show that the two sets are equal, we use an inductive argument. We begin with the definition of the *rank* of a set: the rank of a set X in V is the least ordinal γ for which $X \subseteq V_\gamma$. We will argue by induction on the rank of A. Let $\beta = \text{rank}(A)$, and (arguing by induction), let us assume that for any set B, if the rank of B is less than β, then $B = j(B)$. We show that $A = j(A)$ by showing that $A \subseteq j(A)$ and $j(A) \subseteq A$. Given a set $B \in A$, since B is of lower rank than A, $j(B) = B$. But "$B \in A$" is a property true in V; applying j gives us the true property "$j(B) \in j(A)$." Therefore, since $j(B) = B$, we conclude that $B \in j(A)$. We have shown $A \subseteq j(A)$.

For the other direction, assume $B \in j(A)$. Since $j(A)$ has the same rank as A (as we showed in the first phase of the proof), B must be of rank lower than that of A, and so, again, $B = j(B)$. Now, the property

"$j(B) \in j(A)$" is a property that holds in V; applying rule (2) again leads to "$B \in A$." We have therefore shown $j(A) \subseteq A$, and we are done.
End of Proof

With this glimpse of the new character of proofs using the Wholeness Axiom, we move on to examine the new features that arise in the structure of V as a result of postulating this axiom.

§18. The New Dynamics of V

We recall that one of the shortcomings of the universe V (as constructed from ZFC) as an analogy for wholeness as described by Maharishi Vedic Science was that the unfoldment of V exhibited only "one half" of the dynamics found within pure intelligence: The principle of expansion of the point to infinity is actualized in the unfoldment of all sets from the empty set, but the collapse of infinity to a point does not appear to have a parallel in the dynamics of V.[45] In this section, we will see how the addition of the Wholeness Axiom to ZFC results in a new way to view the unfoldment of sets, now from the perspective of the *wholeness* of V as it gives rise to each set individually; we will see that these new dynamics provide a striking parallel to the dynamics of the unfoldment of the Ved emerging from the collapse of infinity to a point within pure intelligence.

We begin with a discussion of several points on Maharishi Vedic Science. Maharishi explains that being wholeness, pure intelligence has within it both the fully expanded infinite value of wakefulness and the fully contracted point value of wakefulness (Maharishi 1991):

> **A** will not be fully awake without its own point.

By the nature of these opposite values, the fully expanded value of wholeness is drawn to the fully contracted value and there is a move of pure intelligence within itself. Maharishi (1974a) explains that this fully expanded value of wholeness is embodied by the Sanskrit letter **A**, the first letter of the Rik Veda; pronunciation of this letter is done with the throat fully open, without stops or modifications of any kind. **A** also represents infinite silence. The fully contracted value of whole-

45 Actually, one can argue, as Weinless does (1987), that this direction of *Akshara* is expressed, to some extent at least by the Reflection Principle.

ness is embodied in the letter **K**, the second letter of the Rik Veda, pronounced with a fully closed throat—the ultimate value of "stop."

According to Maharishi, in the move of the fully expanded value **A**, to the fully contracted value **K**, awareness of the move within pure intelligence happens when the point value **K** is reached; prior to the emergence of **K**, awareness of the move is not available. When **A** is stopped at **K**, then infinite dynamism is imparted to this point value **K** in preparation for the full unfoldment of the Veda and the creation.

> It is in the experience of stop that we gain knowledge of the move. If we continue **A** we wouldn't know that it is moving. So the move which is the concentration of all principles is from the experience of stop.
>
> (Maharishi 1974a)

> In **K** is the awakening of knowledge of move. This knowledge of move is that package of knowledge which is the fountainhead of all principles of knowledge and creation.
>
> (Maharishi 1974a)

> *Richo akshare parame vyoman*
>
> *The hymns of the Ṛk Veda emerge in the collapse of A.*
>
> (*Ṛk Veda* 1.164.39)

Another feature of this collapse of **A** to **K** is that all possible transformations of pure intelligence within itself occur in this transition; all transformations in the unfoldment of creation can be located in the collapse of **A** to **K**. These transformations can be understood as arising from the infinite silence of wholeness by virtue of the fact that pure intelligence, Samhitā, is by its nature pure wakefulness; being awake within itself it assumes the roles of knower, known and process of knowing (Ṛishi, Chhandas, Devatā respectively). As each of these phases of pure intelligence is fully awake within itself, each can become awake to the other, and new values of each emerge. In this way, an infinity of transformations of pure intelligence emerge by virtue of its fundamental nature to be awake. Thus, the entire range of self-interacting dynamics of consciousness can be seen to be the process of wholeness knowing itself.

Moreover, this move of **A** to **K** and the consequent dynamics of unfoldment of the Veda constitute the foundation of creation; Maharishi calls the laws governing these dynamics the *Constitution of the Universe*. The administration of all the affairs of Nature is conducted by means of these unmanifest dynamics (Maharishi 1992):

> The laws governing the self-interacting dynamics of the Unified Field can therefore be called the Constitution of the Universe—the eternal, nonchanging basis of Natural Law and the ultimate source of the order and harmony displayed throughout creation.

Overview of the New Dynamics of V. With these dynamics of wholeness in mind, let us turn to the unfoldment of the sets in the universe, now from the perspective of the wholeness of V itself using our new Wholeness Axiom. The Wholeness Axiom tells us that V moves within itself and, in a sense, "knows" itself, by virtue of the existence of $j : V \to V$.

The first phase of the behavior of j is silent: until its critical point κ is reached, j behaves like the identity, and no set is moved. Now the identity map $i : V \to V$ represents the nonactive, silent, unmoving value of V; hence we may say that, prior to moving the first point κ, j embodies the absolute silence represented by the identity map $i : V \to V$.

The significance of j as a *nontrivial* elementary embedding arises when the κth stage is reached and j assigns a value to κ (see Figure 7).

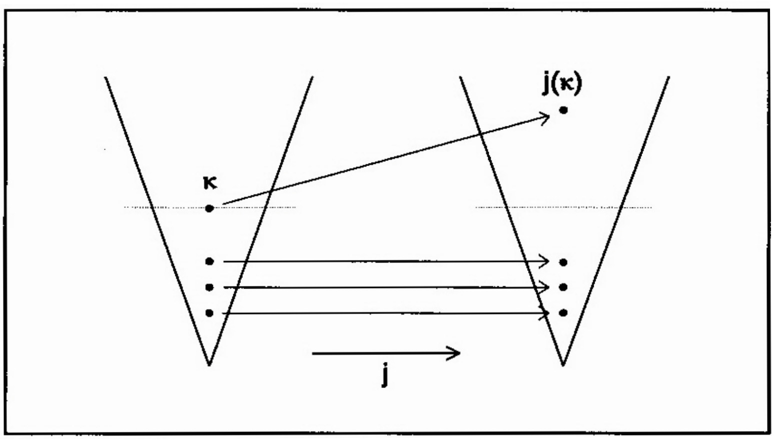


Figure 7. The embedding j: silent below κ, dynamic at κ.

As we remarked earlier, a great deal of theoretical power is generated by the existence of this type of embedding, and this power is as if directed into a single point in the universe—the critical point κ. By being the first ordinal moved by j, κ inherits many of the powerful properties of V (in fact, as we shall see, the stage V_κ inherits *all* first-order properties of V).

Our analogy so far associates V with the quality of fully expanded wholeness; the embedding j with the basic move of wholeness within itself; and the critical point κ with the fully contracted value of wholeness, represented by **K**.

All Possible Transformations Coded into a Magic Sequence. After κ is moved by j, several stages of unfoldment occur, eventually giving rise to the Laver magic sequence. First, if we follow the behavior of j past κ, we find that, once κ has been moved, every set that occurs past κ in the universe is also moved higher up in the universe. In particular, $\kappa < j(\kappa) < j(j(\kappa)) < \ldots$. In the process of moving κ, $j(\kappa)$, and so on, we find that j "interacts" with itself through *composition*. For instance, $j(j(\kappa))$ is obtained by applying $j \circ j$ to κ. We use the notation j^2 to denote $j \circ j$, j^3 to denote $j \circ j \circ j$, and so forth.

Meanwhile, as j moves sets stage by stage through the universe, a number of special sets are created. For each $\alpha \geq \kappa$, let us denote the set of all subsets of α of size less than κ by $P_\kappa\alpha$. One can define, using j, $j^2, j^3, \ldots$, a certain "measure," which partitions the subsets of $P_\kappa\alpha$ into "large" and "small" subsets. The large subsets form a set denoted U_α which is known as the *supercompact ultrafilter over* $P_\kappa\alpha$. This canonical collection of ultrafilters thereupon gives way to an explosion of additional supercompact ultrafilters throughout the universe. In fact, once we have U_α for $\alpha \geq |PP_\kappa\lambda|$, one can show that $P_\kappa\alpha$ bears the maximum possible number of supercompact ultrafilters. Each such supercompact ultrafilter over $P_\kappa\alpha$ gives rise to a new canonical elementary embedding i (called a *supercompact embedding*) having critical point κ, domain V, and codomain a new model of set theory M; all possible supercompact embeddings with these properties must have one of the canonical embeddings as a factor.[46] These canonical embeddings are selectively

46. If $h : V \to N$ is any supercompact embedding with critical point κ with $h(\kappa) > \lambda$, then if we let U consist of all sets $A \subseteq P_\kappa\lambda$ such that $h''\lambda \in h(A)$, then U is a supercompact ultrafilter and gives rise to a canonical supercompact embedding

coded into a special sequence $S = \langle X_0, X_1, \ldots, X_\alpha, \ldots \rangle_{\alpha < \kappa}$ of subsets of the stage V_κ.

The sequence S is known as a "magic sequence" because of its unusual properties. One of these properties is that every set in the universe can be located using S: Given any set A in the universe, there is an ordinal $\lambda \geq \kappa$ such that A can be located as the κth term of the image of S under a canonical supercompact embedding i obtained from a supercompact ultrafilter U over $P_\kappa\lambda$; in symbols,

$$A = i(S)(\kappa).$$ [47]

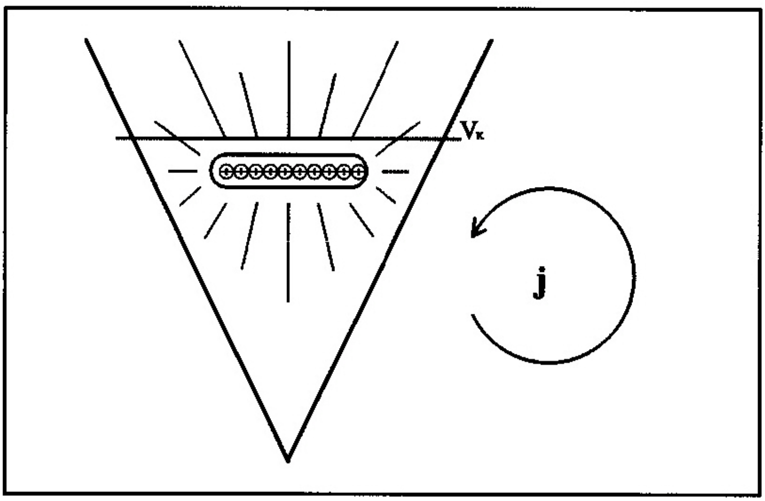


Figure 8. *j* gives rise to a Laver magic sequence from which all sets in *V* can be located.

In these dynamics, the vast collection of all possible supercompact embeddings (with critical point κ) emerging from the iterates of j corresponds to the infinity of transformations of pure intelligence within itself in the collapse of **A** to **K**, in which every possible transforma-

$i : V \rightarrow M$; one can then show that there is an elementary embedding $k : M \rightarrow N$ such that $k \circ i = h$. Hence every supercompact h has a canonical i as a factor.

47. In fact, for each set A and each λ at least as large as the transitive closure of A (the smallest transitive set that includes A), there is a supercompact ultrafilter U on $P_\kappa\lambda$ such that if i is the corresponding embedding, $i(S)(\kappa) = A$.

tion of Samhitā, Ṛishi, Devatā, and Chhandas, one into another, takes place in sequential fashion. Notice here that the knowledge about these embeddings does not arise until the critical point κ has been moved; likewise, knowledge about the infinity of transformations from **A** to **K** emerges as a commentary to this fundamental collapse. As Maharishi explains, the entire Veda, and the creation itself, serve as an elaborated commentary on these fundamental dynamics.[48] Then, just as an infinity of transformations in the collapse of **A** to **K** structures the sequential unfoldment of the Ved which gives rise to every detail of manifest creation, so we find that these supercompact embeddings structure a compact sequence from which every set in the universe can be located.

The Analogy Between Laver's Magic Sequence and the Veda. The similarity between the magic sequence defined above and the Veda goes much further. Not only does the Veda give rise to every detail of manifest creation, but it does so, according to Maharishi Vedic Science, by virtue of its nature as simultaneously infinite dynamic and infinitely silent; all opposite values find their lively integration within this field. This high degree of integration is due to the fact that at each stage of unfoldment, the Ved remains completely self-referral and united with itself; the parts of expression never dominate the original underlying wholeness. This integration within the Veda is responsible for all unity and coherence displayed in manifest existence.

A striking feature of the structure of the Veda is the fact that the totality of Veda is fully present in increasingly elaborated "packets" as the Veda expands from its first letter **A** to its first word, hymn, and mandala. Maharishi describes this structure of the Ved as a "self-created commentary" (*Apaurusheya Bhasya*) since later stages of unfoldment express in ever greater detail the totality of knowledge inherent in the earlier stages.

We have seen that our magic sequence S gives rise to every set in the universe by way of canonical supercompact embeddings. We shall see that this extraordinary fact is due to the internal structure of S which exhibits to a high degree the same qualities that are characteristic of the Veda, including the stage-by-stage unfoldment that is central to the Veda's structure.

48. See (Maharishi 1991a).

If we peer into S—which, as the reader will recall, is a κ-sequence of elements of V_κ—we first notice repeated occurrences of familiar sets. We find, for example, that the number 0, the set $\mathbb{N}$ of natural numbers, the real number line **R**, and the set of all mathematical structures ever used in physics all occur κ many times in the sequence S. In fact, every set $A \in V_\kappa$—and such sets account for all mathematical objects used in ordinary mathematics—occurs *stationarily often* (this is even stronger than saying that each occurs κ many times). This phenomenon directly accounts for the fact that every $A \in V_\kappa$ occurs as the κth term of the image sequence $i(S)$ for some canonical supercompact embedding i. As we discuss below, similar though more complex dynamics are responsible for the full Laver property, that *every* set A is $i(S)(\kappa)$ for some i. This phenomenon is also reminiscent of the fact that the Veda, rather than being separate from or external to creation, is in fact the very dynamics and life breath of creation. The creation can be located in the Veda just as a tree can be located in a seed: if one sees clearly enough the fine mechanics of transformation within the seed, the tree in all its detail can said to be fully present within the seed.

The unifying character of S is more completely revealed when we attempt to locate within the structure of S the dynamics which allow us to capture not only every set in V_κ, but *all* sets. To understand these dynamics, let us start with a set A that we wish to capture. To find the right supercompact embedding i to capture A as the κth term of the sequence $i(S)$, we need to be sure that $i(\kappa)$ is larger than the transitive closure[49] of A; this will guarantee that the image model M contains A. Once we have the model M, the set A is associated in M with a function g defined on $P_\kappa\lambda$ (where λ is at least the size of the transitive closure of A). Now the magical trait of S is that, stationarily often, the value of g (on a set P) is the same as the value of S (on the set $P \cap \kappa$); this guarantees that $i(S)(\kappa) = A$. Thus, the magic sequence serves to "harness" and coordinate the great dynamism of the huge collection of supercompact embeddings acting on and transforming V. We can say that every set is captured by S because the internal, infinitely diverse, highly coherent structure of S focuses the actions of these embeddings so that each point in the universe eventually occurs as an output.

The sequence S not only unifies the dynamism of these super-

49. The transitive closure of a set A is $A \cup (\cup A) \cup (\cup \cup A) \cup \ldots$

compact embeddings, but displays infinite dynamism within its own structure. Dynamism in mathematics and in nature is often expressed through rapidly growing functions, such as exponential functions. For instance, the exponential function which takes a real number x to 2^x eventually dominates every polynomial. On a stationary set, a Laver sequence exhibits similar behavior in a more dramatic way. To set up a revealing example, let us define, for any ordinals α and β, the cardinal number $\beth(\alpha, \beta)$ by induction: $\beth(0, \beta) = \beta$; $\beth(\alpha + 1, \beta) = 2^{\beth(\alpha, \beta)}$; $\beth(\lambda, \beta) = \cup_{\alpha < \lambda}\beth(\alpha, \beta)$ for λ a limit. Hence, for example,

$$\beth(0, \omega) = \omega; \beth(1, \omega) = 2^{\omega}; \beth(2, \omega) = 2^{2^{\omega}}$$

Now, as an example of a rapidly growing function, consider $g: \kappa \rightarrow \kappa$, defined by letting $g(\alpha) = \beth(\alpha, \alpha)$ for each $\alpha < \kappa$. Thus, for example,

$$g(\omega) = \cup\{\omega, 2^{\omega}, 2^{2^{\omega}}, \dots\}$$

The dynamism of a magic sequence S becomes apparent when we consider the fact that there is a stationary set $B \subseteq \kappa$ such that for all $\alpha \in B$, $|S(\alpha)| > g(\alpha)$. In fact, it can be shown that such a stationary set B can be found for virtually any[50] function definable in V_κ. Such results show that if we use rapidly growing functions as a measure of dynamism, a magic sequence exhibits a strong type of dynamic behavior within its own structure.

At the same time, S exhibits infinite silence in the following way: Stationarily often, the αth value of S is α itself; such α are not moved by S. It is interesting to note that this feature of the internal structure of S comes into view when we attempt to capture κ itself by S; that is, attempt to find i such that $i(S)(\kappa) = \kappa$. It can be shown that in order for such an i to exist, S must not move stationarily many α. Therefore, the truth of the self-referral expression "$i(S)(\kappa) = \kappa$" depends on the pervasiveness of "silence" in S.

The magic sequence S that we have defined using the Wholeness Axiom[51] also has an internal structure in which earlier "stages" of the

50. The result is given precise formulation in (Corazza, 2009).

51. The existence of a magic sequence S at κ with the property that the set of all $\alpha < \kappa$ at which there is another magic sequence has normal measure 1 does not follow from the existence of a supercompact cardinal—although a single supercompact *is* sufficient to

sequence have the same essential properties as the sequence as a whole, reminiscent of the fact that the Veda unfolds in discrete stages, each elaborating in greater detail the same totality of knowledge inherent within earlier stages. It can be shown that for stationarily many α, the restriction $S\restriction\alpha$ is itself a magic sequence at α. Thus, the special properties which define S as a whole are found everywhere permeating the structure of S.

As a final point linking S to the structure of the Veda, we point out that a magic sequence exhibits as one of its properties a kind of *incorruptibility* in that we may alter as many as κ many terms of S (as long as those κ many terms form a “”thin” enough subset of κ) without changing its status as a magic sequence. If, for example, we decide to replace the first ω_ω terms of S with the number 0, this altered sequence still gives rise to every set in the universe exactly as before. This suggests that the reality expressed by a particular magic sequence may still be accurately expressed even if many of the details of expression are changed. Likewise, the eternal reality of the Ved does not lie at the level of the various interpretations of the Vedic Literature that may be possible; rather, as the Ṛk Veda itself declares, the Ṛichas or hymns of the Veda exist in the immutable transcendental field; recall the verse *richo akshare parame vyoman*, mentioned earlier. The profound internal dynamics of the Ved are as if hidden from view; so likewise does the “magic” of a magic sequence reside at a more holistic level of the structure of the sequence, since changes to individual parts of the sequence do not alter its fundamental properties.

Collapse and Expansion with Infinite Frequency We recall that the theme of unfoldment of the Veda consists in the infinitely frequent oscillation of “collapse of infinity to a point and expansion of point to infinity.” By Maharshi’s *Apaurusheya Bhāshya*, these dynamics can be located in **AK**, the first syllable of Ṛk Veda, in which all possible transformations of pure consciousness occur in seed form as Ṛishi, Devatā, Chhandas, and Samhitā interact amonst themselves.

We find this theme of unfoldment expressed in the new dynamics of V in the following way: The vast infinity of transformations that arise from j (in the form of supercompact embeddings) are actually struc-

obtain an ordinary magic sequence. It can be shown that this stronger property actually implies that κ is the κth supercompact cardinal, and much more.

tured on the basis of an infinity of new point values that arise from κ (namely, the supercompact ultrafilters over all possible index sets $P_\kappa\lambda$) and that provide new focal points for an infinite variety of "collapses" of V to other new models of set theory. These new embeddings correspond to new values of Ṛishi, Devatā, Chhandas, and Samhitā that emerge from the original three-in-one structure represented by our embedding $j : V \to V$.

We will now explain in greater detail how the emergence of an infinity of new models of set theory in the presence of these derived supercompact embeddings can be understood as a repeated "collapse of infinity to a point and expansion of point to infinity."

As we have seen, the maximum possible number of supercompact ultrafilters over each $P_\kappa\lambda$ ($\lambda \geq \kappa$) arise from the fundamental move due to $j : V \to V$. Each such ultrafilter U may be viewed as a new "point value" that emerges from κ and the behavior of j.[52] Now U becomes the focal point for a more specialized collapse of the wholeness of V—a collapse "relative to" the index set $P_\kappa\lambda$ on which U is based.

To effect this collapse, we first form a new "variable" universe $V^{P_\kappa\lambda}$; the elements of this "universe" consist of all functions from the index set $P_\kappa\lambda$ to V. We call this vast collection of functions a variable universe because each function $f \in V^{P_\kappa\lambda}$ may be thought of as representing a set that varies according to a parameter that ranges through all elements in $P_\kappa\lambda$. For instance, suppose $A, B \in P_\kappa\lambda$. Then we could consider f to be the set $f(A)$ relative to A and $f(B)$ relative to B.[53]

Now, technically, $V^{P_\kappa\lambda}$ is not a universe of sets as it does not satisfy all the axioms of ZFC (indeed, the natural way of thinking of $V^{P_\kappa\lambda}$ as a model at all is to treat is as having infinitely many truth values in addition to the usual values of "true" and "false"). We obtain an ordinary universe of sets by "collapsing" this variability (and "collapsing" the infinity of truth values) by way of the point U in the following way: Two functions f, g in $V^{P_\kappa\lambda}$ will be called *equivalent mod*

52. Weinless [1987] discusses at length why individual sets in the universe may be viewed as reasonable analogues to "point values" relative to the whole universe V; the main insight is that, compared to the vastness of V, any particular set must be negligibly small.

53. The notion of variable sets that depend on a parameter in this way is due to Lawvere who has used it to great advantage in creating category-theoretic models of a variety of dynamical phenomena; see (Lawvere, 1976) and (Bell, 1988, Epilogue).

U if they agree pointwise on a set in U. The equivalence classes that arise from this equivalence relation form the elements of a new class $V^{P_\kappa\lambda}/U$. By amalgamating functions into their equivalence classes, the variable quality of $V^{P_\kappa\lambda}$ is eliminated. Because we use an ultrafilter U to define the equivalence classes, which partitions all sets in $P_\kappa\lambda$ into just two classes ("large" and "small"), the number of truth values associated with the new model is reduced to the two usual values: "true" and "false." Since the formation of $V^{P_\kappa\lambda}/U$ from $V^{P_\kappa\lambda}$ occurs by treating equivalence classes as points—relative to the reference point U—in a new model, the process of formation of the new model is clearly analogous the "collapse of infinity to a point."

The class $V^{P_\kappa\lambda}/U$ turns out to be a new model of set theory (new in that it is not identical to V); unfortunately, since its elements are equivalence classes, its membership relation must be nonstandard. This minor inconvenience can be corrected by "reshuffling" the elements of the model using a standard technique that is based on the *Mostowski Collapsing Theorem*. The result is a standard (transitive) model M of set theory that is isomorphic to $V^{P_\kappa\lambda}/U$. In this form, the unfoldment of sets from the empty set closely parallels the unfoldment of sets in V itself; the difference is that in the unfoldment within M, *M's* version of the power set operator is used. As we saw for V, the resulting sequential unfoldment of sets provides an analogue for "expansion of point to infinity." Thus, the formation of the model M can be seen as the expression of "collapse of infinity to a point" as well as "expansion of point to infinity."

We can summarise the entire process of building the model M by describing the supercompact embedding i that naturally arises from the construction. The embedding i is a composition $\pi \circ \eta \circ e$ where

(1) $e(x)$ expands the set x to the constant function $c_x : P_\kappa\lambda \to V$ where $c_x(A) = x$.
(2) $\eta : V^{P_\kappa\lambda} \to V^{P_\kappa\lambda}/U$ maps f to $[f]$
(3) π is the Mostowski collapsing isomorphism that transforms $V^{P_\kappa\lambda}/U$ to the model M.

Thus, the construction of M may also be understood as the outcome of a fundamental transformation of the wholeness V, represented by the embedding i.

Next, we examine how the emergence of these supercompact embeddings also provides a parallel to the infinity of transformations of Ṛishi, Devatā, Chhandas, and Samhitā amongst themselves in the sequential unfoldment of the Veda. Recall that the flow of the Veda may be considered a flow of an infinite variety of frequencies or sounds. As an analogy, each supercompact embedding i may be considered as a particular frequency extracted from its original source in the embedding j on the basis of the point value U. All such embeddings i taken together may be considered again by analogy to express the entire range of "frequences" contained in seed form within the embedding j (where again we consider j as an analogue to the flow of the Veda). In the Veda, frequencies arise in the interaction of Ṛishi, Devatā, and Chhandas within Samhitā, which produce an infinity of derived values of Ṛishi, Devatā, Chhandas, and Samhitā. Recall from earlier sections that we naturally associate the fundamental embedding j with the Devatā value and that, since V is interacting with itself, V is associated both with Ṛishi and Chhandas. Moreover, since the embedding behaves entirely within V, V plays the role of the wholeness that unites the three as well, namely, Samhitā. Now we can further observe that the vast collection of supercompact embeddings display "derived" values of Ṛishi, Devatā, Chhandas, and Samhitā in that each embedding emerges from j (thus, is "derived" from j) and each embedding $i : V \to M$ displays a three-in-one structure of knowledge, where V plays the role of Ṛishi, i the role of Devatā, and M the role of Chhandas, and V itself also plays the role of Samhitā since all the dynamics of the embedding occur within V.

Thus, the unfoldment of supercompact embeddings from j corresponds not only to the repeated collapse of infinity to a point and expansion of point to infinity (the theme for the expansion of the Veda), but also to the infinity of transformations of Ṛishi, Devatā, Chhandas, and Samhitā, which constitute the very fabric of the Veda.

Summary. We summarize the correspondences discussed so far in a chart followed by a diagram illustrating the stages of unfoldment under the Wholeness Axiom:

Dynamics of j Moving V Under the Wholeness Axiom	**Dynamics of Pure Intelligence Moving Within Itself**
V, the universe of sets as a whole	Fully expanded value of Wholeness, represented by **A**
$j : V \to V$	The move of pure intelligence within itself
κ, the critical point of j	The fully contracted value of Wholeness, represented by **K**
The class of all possible supercompact embeddings (having critical point κ) arising from j and its iterates	All possible transformations emerging in the collapse of **A** to **K**
The magic sequence $S = \langle X_0, X_1, \ldots, X_\alpha, \ldots \rangle_{\alpha<\kappa}$	The Veda
The repeated collapse of the infinite dynamism of V, embodied in classes of the form $V^{P_\kappa \lambda}$, focused at new point values $P_\kappa\lambda$ (for all $\lambda \geq \kappa$) and U derived from κ, resulting in the unfoldment of fully expanded universes M, with all dynamics embodied in the canonical supercompact embeddings—all giving rise to Laver's magic sequence	Collapse of infinty to a point and expansion of point to infinity occurring with infinite frequency, giving rise to the self-interacting dynamics of pure intelligence
Sets	Manifest creation

The initial embedding $j : V \rightarrow V$ gives rise to all possible supercompact embeddings each of which in turn gives rise to a canonical sequence of additional embeddings that can be seen as natural and inevitable modification of the original	Samhitā assuming the roles of Ṛishi, Devatā and Chhandas, interacting with themselves to give rise to all possible transformations

Table 3. The analogy between the dynamics of the universe V and the dynamics of wholeness.

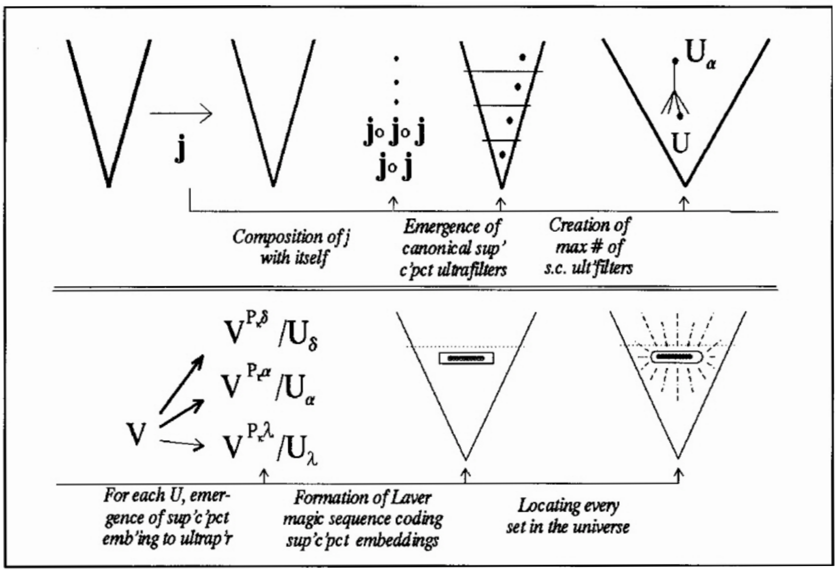


Figure 9. Emergence of a Laver magic sequence from the wholeness operator *j*.

New Qualities in V. Our analysis suggests that adding the Wholeness Axiom to ZFC brings the fundamental dynamics of the universe in much closer accord with the dynamics of pure intelligence, as described by Maharishi Vedic Science. Notice that, from this new perspective, V very naturally exhibits the qualities of pure intelligence that seemed to be missing before: Now, V is understood by its very nature to move within itself and "know" itself through its own self-interaction; hence, by our earlier discussion, V can be said to more fully exhibit the qualities of *self-referral, awake within itself,* and *bliss.*

To see how the fourth quality, *infinite correlation*, is enlivened by the presence of the Wholeness Axiom, we need to mention yet another new feature of the structure of V. Starting with our wholeness operator $j : V \rightarrow V$ and its critical point κ, we will use Kunen's inconsistency proof in a new way to show that the sequence of ordinals we obtain by repeatedly applying j to κ,

$$\kappa, j(\kappa), j(j(\kappa)), \ldots,$$

(called the *critical sequence of j*) is unbounded in the universe! In other words, we will show that there is no ordinal in the universe which is simultaneously larger than every term of the critical sequence. To see this, first, notice that the terms of the sequence are strictly increasiing:

$$\kappa < j(\kappa) < j(j(\kappa)) < \ldots.$$

This can be shown using the elementarity of j (since $\kappa < j(\kappa)$, applying j yields that $j(\kappa) < j(j(\kappa))$, and so forth). Now suppose there actually is an ordinal which exceeds all the ordinals $\kappa, j(\kappa), j(j(\kappa)), \ldots$; let λ denote the least such. (λ is called the *supremum* of the sequence.) It can be shown that j takes each set in the stage $V_{\lambda+2}$ back into $V_{\lambda+2}$. Thus the restriction of j to $V_{\lambda+2}$ is an elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. But now, as we mentioned in an earlier section, we can carry out Kunen's proof and arrive at a contradiction! Thus, assuming that there is an ordinal lying above $\kappa, j(\kappa), j(j(\kappa)), \ldots$ leads to an inconsistent set theory. For ordinary sequences of ordinals, the assumption that the sequence has a supremum would be warranted; but here, the sequence is defined from our *transcendental* elementary embedding j. Consequently, the sequence has no supremum.[54] We give this new principle

54. Intuitively, the fact the ordinals $\kappa, j(\kappa), j(j(\kappa)), \ldots$ lacks a supremum seems odd, especially because this sequence is so short: notice that these ordinals are matched one-for-one with the natural numbers 0, 1, 2,... (simply match 0 with κ, 1 with $j(\kappa)$, 2 with $j(j(\kappa))$, and so on). Thus, they form the shortest possible infinite sequence, and yet they extend all the way through the universe. One possible intuition, taken from Maharishi

the following name:

Principle of Countable Unboundedness

The sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ has no supremum.

By elementarity of j, one may show (Corazza 2006) that the sequence

$$V_\kappa \prec V_{j(\kappa)} \prec \ldots V_{j^n(\kappa)} \prec \ldots$$

forms a chain of elementary submodels whose union is *V.* Once we have this elementary chain, it can be shown that "for almost all" ordinals α,

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Vedic Science, which may help to explain this phenomenon comes from considering the nature of evolution to enlightenment. We can think of climbing upward through the universe *V*, stage by stage, as analogous to a pathway to the ultimate realization of wholeness in life. Without a proper technique, such a path will be truly endless. In the language of ZFC, any path passing all the way through *V* must necessarily be of greater length than any infinite cardinal. In the language of Maharishi Vedic Science, the vastness of the creation simply cannot be fathomed; attaining the knowledge of the field of action that brings the ultimate knowledge of life requires a technique. Commenting on the discourse of Lord Krishna in verse 17, Chapter 3, of the *Bhagavad-Gita*, Maharishi (1967, p. 278) remarks:

> The Lord has said that knowledge of action is necessary and yet, the course of action, being unfathomable, knowledge of it must remain incomplete. He therefore brings to light a technique by which the effects of knowledge will be gained without the necessity for gaining the knowledge.

With a suitable technique, an individual may begin at whatever level of involvment in relative existence he may find himself, and quickly awaken to the wholeness of life within his own awareness (Maharishi 1966, pp. 55-56):

> This practice [Transcendental Meditation] is pleasant for every mind. Whatever the state of evolution of the aspirant, whether he is emotionally developed or intellectually advanced, his mind, by its very tendency to go to a field of greater happiness, finds a way to transcend the subtlest state of thinking and arrive at the bliss of absolute Being.

By directly enlivening within individual awareness the dynamics of pure intelligence—and recall that these dynamics, in our analogy, correspond to the action of *j*—the path to full enlightenment becomes relatively short. Likewise, although no ordinary sequence of sets is sufficiently long to pass through the entire universe, still, with reference to the embedding *j,* an extremely short path $\kappa, j(\kappa), j(j(\kappa)), \ldots$ passing beyond every stage, is as if carved out of the vastness of *V.*

V_α is an elementary submodel of V.[55] This result is extremely powerful; it says that full information about the nature of V as a whole is available throughout the universe. Recall from our discussion of elementary submodels that one model is an elementary submodel of another, the two models are "infinitely correlated" in the sense that they satisfy exactly the same properties. Thus, the fact that we find such a pervasive occurrence of elementary submodels of the universe suggests that the Wholeness Axiom has enlivened the quality of infinite correlation in the structure of V.

Totality of Mathematical Knowledge in One Step We can make one final point about the dynamics of V under our new axiom, motivated by a remark by Maharishi (Hagelin 1992). Maharishi points out that the ideal expression of the totality of knowledge should involve no steps, as in the first letter, **A**, of Ṛk Veda. But, to express this totality of knowledge through a discipline based on steps, the most compact expression that could be hoped for would be some analogue to the transformation from **A** (the first vowel of Ṛk Veda) to the expression **AA** (the last vowel) capturing the totality of knowledge in *one* step. To some extent at least, this goal is realized in the statement of our Wholeness Axiom, which can be represented in the following diagram:

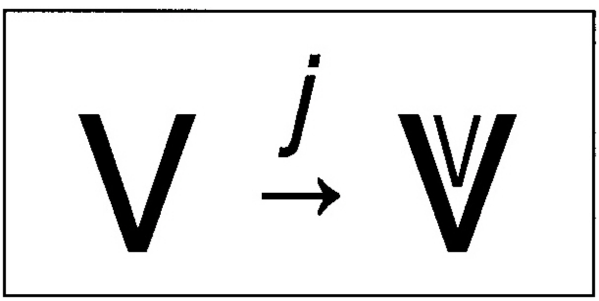

Figure 10. The totality of knowledge in one step.

As the diagram suggests, the fundamental move of the wholeness represented by V involves a single step in which V moves in a very coherent way within itself. From this single move, all the dynamics of

55. For each n, it can be shown that the set of all ordinals α such that V_α is an elementary submodel of $V_{j^n(\kappa)}$ is closed and unbounded in $j^n(\kappa)$.

sets, and hence all mathematics, is generated.

We have seen that new qualities and dynamics in the universe arise in the presence of the Wholeness Axiom and that these mirror, to a great extent, those of pure consciousness as it moves within itself. Thus, our attempt to include more of the properties of our intuitive model from Maharishi Vedic Science into our construction of V has been successful. In the next section, we show that our efforts have useful mathematical consequences; most importantly, we will show how we can derive large cardinals from our new axiom.

§19. The Origin of Large Cardinals

With the addition of the Wholeness Axiom to ZFC, we are now in a position to understand the origin of large cardinals. Since we obtained this axiom as the culmination of the strongest known large cardinal axioms, it may seem almost obvious that our axiom will succeed in providing the necessary derivations. However, the fact that we have insisted that the elementary embedding j be transcendental and that V be fully j-closed often introduces a twist in the expected reasoning.

More interesting than the proofs, however, is the realization that all large cardinal properties can be understood as the properties of the critical point of this basic elementary embedding j; that large cardinal properties arise as the properties of the focal point of the fundamental move of the wholeness of the universe within itself. This perspective provides not only an account of the origin of all large cardinal properties but, as well, an intuition about why these properties have proven to be so powerful.

In this section, we will carry out three of the derivations of large cardinals from the Wholeness Axiom; in particular, we will show that the Wholeness Axiom implies the existence of inaccessible, measurable, and extendible cardinals. We will also consider in this section the relationship between our axiom and the existence of an elementary embedding from a stage V_λ to itself; we will see that this extremely strong axiom is naturally related to ours. The reader who wishes to avoid mathematical technicalities may wish to skip to the next section.

The proof of the existence of inaccessibles will use the same sort of techniques we used in Section 17. It will also make central use of the Principle of Countable Unboundedness, introduced in the last section.

The proof of the existence of measurables will illustrate the mathematical impact of our axiomatic assumptions about j. The verifications that other large cardinals can be derived from the Wholeness Axiom have a similar flavor.

The proof of the existence of extendibles, on the other hand, is trivial and illustrates how large cardinal axioms may very naturally be viewed as approximations to an elementary $j : V \to V$. Because it is so brief, we proceed with the proof here: A cardinal κ is said to be *extendible* if, for each ordinal β, there are an ordinal ζ and an elementary embedding $i_\beta : V_{\kappa+\beta} \to V_\zeta$ with critical point κ, satisfying

$$(*) \qquad \kappa + \beta < i_\beta(\kappa) < \zeta.$$

To obtain the needed embeddings i_β, we can simply use the iterates of j: Given β, let n be least such that $j^n(\kappa) > \kappa + \beta$; if we let $i_\beta = j^n \restriction V_{\kappa+\beta}$, then $i_\beta : V_{\kappa+\beta} \to V_{j(\kappa+\beta)}$ is elementary with critical point κ and satisfies $(*)$. We proceed to the other proofs:

Theorem. *The Wholeness Axiom implies that there exists an inaccessible cardinal.*

Proof. To begin, let us recall the defintion of inaccessible cardinal given earlier: a cardinal γ is inaccessible if $\gamma > \omega$ and the stage V_γ has the following two properties:

(1) V_γ is not the union of fewer than γ many of the earlier stages V_α.
(2) The size of any previous stage V_α is less than γ.

We also need to recall that in Section 17 we showed that

(a) $j(\omega) = \omega$;
(b) if κ is the critical point of j, then j does not move any set in V_κ; that is, for all sets $A \in V_\kappa$, $j(A) = A$.

Therefore, we begin the proof by letting $j : V \to V$ be the wholeness embedding having critical point κ. We show κ is inaccessible. From (a) we may infer that $\kappa > \omega$; thus, the critical point of j is already showing signs of being quite big. To establish (1), suppose on the contrary that V_κ is the union of δ many of the earlier stages V_α, where $\delta < \kappa$. A chart will be useful here:

Formula true in V	Formula after applying j (also true in V)
V_κ is the union of the following list of δ many earlier stages: $\{V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots\}$	$j(V_\kappa)$ is the union of the following $j(\delta)$ many earlier stages: $\{j(V_{\alpha_0}), j(V_{\alpha_1}), j(V_{\alpha_2}), \ldots\}$

The lower right box can be simplified: As we showed before (Section 17), for any ordinal γ, $j(V_\gamma) = V_{j(\gamma)}$. Also, since $\delta < \kappa$, $j(\delta) = \delta$. Likewise, $j(\alpha_0) = \alpha_0$, $j(\alpha_1) = \alpha_1$, and so forth. The expression in the lower right box becomes:

$V_{j(\kappa)}$ is the union of the following δ many earlier stages:

$$\{V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots\}$$

Thus, both V_κ and $V_{j(\kappa)}$ are the union of the stages $\{V_{\alpha_0}, V_{\alpha_1}, V_{\alpha_2}, \ldots\}$ and so $V_\kappa = V_{j(\kappa)}$. But this is impossible because $\kappa < j(\kappa)$. Hence, our assumption that V_κ could be expressed as the union of fewer than κ many earlier stages has proven to be incorrect, and we have thereby established (1).

We proceed to (2): Let us assume that some stage V_α, with $\alpha < \kappa$, has size greater than or equal to κ. The Principle of Countable Unboundedness guarantees that the size of V_α cannot simultaneously exceed every one of the cardinals $\kappa, j(\kappa), j(j(\kappa)), \ldots$. Let n be large enough so that the nth iterate $j^n(\kappa)$ of j applied to κ is greater than the size of V_α; assume further that n is least for which $j^n(\kappa)$ is greater than the size of V_α. We may use a chart to arrive at a contradiction:

Formula true in V	Formula after applying j (also true in V)
The size of V_α is $\geq j^{n-1}(\kappa)$	The size of $j(V_\alpha)$ is $\geq j(j^{n-1}(\kappa))$

Simplifying the right box again, we see that $j(V_\alpha) = V_{j(\alpha)} = V_\alpha$ and $j(j^{n-1}(\kappa)) = j^n(\kappa)$. Thus, the right box says:

$$\text{the size of } V_\alpha \text{ is } \geq j^n(\kappa),$$

and we have a contradiction. Thus, for all $\alpha < \kappa$, the size of V_α is less than κ. **End of Proof**

We now turn to the proof of the existence of a measurable cardinal:

Theorem. *The Wholeness Axiom implies there is a measurable cardinal.*

Proof. To begin, we need to recall that κ is measurable if κ is the critical point of a nontrivial elementary embedding $i : V \to M$ which is definable in V. As a first attempt to prove the theorem, we might try using j as our embedding; but because j is not definable in V, we are forced to proceed along less direct lines.

Still, there are fairly standard procedures one can follow to obtain i. If we can find a way to divide up all the subsets of κ into two classes, "big" and "small," then a well-known[56] procedure called the *ultrapower construction* (which was described briefly in the previous section) will produce the model M and the embedding i: $V \to M$ that are required. The hard part, then, is to divide up the subsets of κ into these two classes. Once "big" has been properly defined, the meaning of "small" will be clear (namely, a set will be small if it is not big!). Thus, our task is to delineate the "big" subsets of κ. In order for our construction to work, our notion of "big" must meet the following requirements:

(1) κ, as a subset of itself, is "big";
(2) all "big" subsets must have size κ;
(3) if a subset A includes a "big" set B as a subset, then A must itself be "big"; and
(4) the intersection of fewer than κ many "big" sets is again "big."

Elegantly enough, if we consider all subsets A of κ with the property that $j(A)$ contains κ as an element, these sets satisfy all four require-

56. See (Jech 1978, Chapters 27-28).

ments, where $j : V \to V$ is a wholeness operator. The reason that this method works, intuitively speaking, is that κ is not actually a member of the image V' of V by j and plays the role of a random point among the ordinals below $j(\kappa)$. In order for the image $j(A)$ of the set A to contain this random point, A must be quite large as a subset of κ.

We will denote by U the collection consisting of all these "big" sets; more precisely,

$$U = \{A \subseteq \kappa : \kappa \in j(A)\}.$$

The next step is to define i using U, following standard procedures. However, in order to do so, we need to know that U is a set. Notice that U has been defined using j; if V were not j-closed, there would be no hope of demonstrating that U is set. However, because the Wholeness Axiom tells us that V is j-closed, and because U is a subcollection of a known set (namely, the power set of κ) defined using j, we may conclude that U is indeed a set.

Thus the standard ultrapower construction can be carried out to produce the model M and the required elementary embedding $i : V \to M$. Therefore, the critical point of a wholeness operator is a measurable cardinal. **End of Proof**

Finally, let us return to some remarks we made earlier about various weakenings of the inconsistent notion of a nontrivial definable elementary embedding of the universe to itself. We mentioned that there were two possible consistent weakenings which took the form "there is an elementary embedding from some V_λ to itself"; these axioms have been given the names I_1 and I_3 in the literature:[57]

I_1:	There are ordinals $\kappa < \lambda$ and an elementary embedding $i : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ such that $\lambda = \sup\{\kappa, i(\kappa), i(i(\kappa)), \ldots\}$.

57. The 'I' in I_1 and I_3 stands for "inconsistent." Kunen discovered these axioms as a corollary to his proof that, in Kelley-Morse set theory, there is no nontrivial elementary embedding $j : V \to V$; Kunen observed that the axioms $I_1 - I_3$, so close to inconsistency, could not quite be proved inconsistent using the techniques of his paper. See (Kunen, 1971).

I_3:	There are ordinals $\kappa < \lambda$ and an elementary embedding $i : V_\lambda \rightarrow V_\lambda$ with critical point κ such that $\lambda = \sup\{\kappa, i(\kappa), i(i(\kappa)), \ldots\}$.

Each of these axioms is strong enough to imply the consistency of virtually all large cardinal axioms, just as the Wholeness Axiom does.[58] It is natural to wonder about the relationships among these powerful axioms.

Before giving an answer to this natural question, we first notice that there is one feature of our Wholeness Axiom which deserves clarification: If $j : V \rightarrow V$ is supposed to be *transcendental* to V, and V includes "everything," where is j supposed to exist? As we mentioned before, j can be coded as a subcollection of V which does not happen to be definable. Using our model of wholeness from Maharishi Vedic Science for intuition, this picture of the universe makes sense: The embedding j, representing the *unmanifest* dynamics of wholeness interacting with itself, is neither an element of V nor definable within V even though it lies within V as a subcollection.

On the other hand, it is possible to picture the situation in another way: we may wish to think of both j and V as existing as elements within a greater wholeness, a wholeness which includes them both but does not participate directly in their activities. The axiom I_3 suggests this intuition.

In the diagram, if we think of V_λ—which can be shown from I_3 to be a model of set theory—as being the actual universe V, and the real universe as representing some sort of *superuniverse*, then we have a concrete realization of the intuition just described. In this context, both j and the universe are mere sets in this vaster superuniverse.

58. The critical points of I_1 and I_3 are quite strong, but do not have all the large cardinal properties that the critical point of the wholeness operator have. This is because these axioms have a restricted range of influence; they say nothing about the structure of the universe above λ; there may not even be an inaccessible above λ. Nevertheless, these axioms do imply the existence of *models* of all the strongest large cardinal notions, and for this reason they are considered stronger as axioms than other large cardinal axioms.

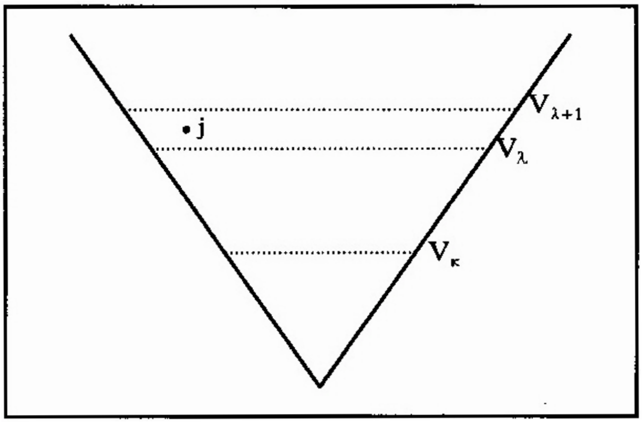

Figure 11. The external embedding $j : V_\lambda \to V_\lambda$ as a point in a super-universe.

Using the axiom I_1 yields a similar result, although now we must picture the ordinary universe as having a top layer,[59] $V_{\lambda+1} - V_\lambda$, with j occurring as an element of some superuniverse at the stage $V_{\lambda+1}$.

This picture of the universe very naturally corresponds with a basic distinction that is drawn in Maharishi Vedic Science between the two essential natures of wholeness. In this paper, we have emphasized the nature of pure intelligence to be awake to itself; this aspect is the basis for all the dynamism of existence. Maharishi has called this aspect of the nature of wholeness variously *pure intelligence* (1972, Lesson 8), Samhitā of Ṛishi, Devatā and Chhandas, and Samhitā (1990b), or the *eight-fold creative nature of* Samhitā (Weinless 1987, p. 151). However, there is another aspect to wholeness: the value of wholeness which is one without diversity, which Maharishi calls variously *pure existence* (1972, Lesson 8) or *pure* Samhitā (1990b). In the Bhagavad-Gita, the distinction is expressed as follows (Maharishi, 1969):

bhumir apo 'n alo vayuh
kham mano buddhir
evaca ahamkara itiyam me

59. Models of Bernays-Gödel set theory or Morse-Kelly set theory have this feature. See (Jech 1978, p. 76).

bhinna prakritir ashtadha

Earth, water, fire, wind,
space, mind, intellect, and ego;
this is the eightfold
division of my nature. (7.4)

apareyam itas tvanyam prakritim
viddhi me param jivabhutam
This is my lower nature.
Know my other, higher,
transcendental nature, the Self. (7.5)

It is as if the dynamism underlying the universe is just a fraction of the total reality of wholeness. Maharishi makes this observation in another way in the *Science of Being* (1966, p. 46):

> The realisation that eternal Being is the one ultimate, supreme reality of existence shows that the cause of creation, or almighty creativity, is latent in the very nature of Being and that It expresses Itself in the form of creation. So we find that absolute, attributeless, eternal Being is the ultimate reality of existence, and that by virture of Its own nature the process of creation, evolution, and dissolution continues eternally without affecting the absolute status of eternal Being. This is a complete picture of absolute, eternal Being in relation to Its own almighty, creative intelligence, or universal mind, as well as to the individual mind.

Thus, just as the move of wholeness within itself is simply a play within the vastness of pure Being, so the fundamental move of the universe of mathematics by the embedding j can be seen as the interaction of mere sets within the context of a much vaster superuniverse.

§20. Eightfold Collapse of Infinity to a Point

So far, we have used Maharishi Vedic Science to motivate a new axiom of set theory that would introduce new qualities and dynamics into the universe, aligning its structure more with that of the wholeness of pure consciousness. As we have seen, the program has succeeded at accomplishing its goals. In this section we examine the dynamics of wholeness more deeply and find that the parallels between the universe of sets and pure consciousness extend considerably farther than we have indicated so far.

As we have seen, according to Maharishi Vedic Science, all the self-interacting dynamics of pure consciousness can be located in the collapse of fullness, represented by **A**, to the point value, represented by **K**. These dynamics are elaborated in sequential fashion as the verses of the Veda unfold; in fact, the entire Veda can be seen as a commentary on the expression **AK**.[60] According to Maharishi's *Apaurusheya Bhāshya*, the Veda unfolds in packets of knowledge, with each successive stage being a commentary and elaboration of earlier stages. Thus the first syllable of the Veda contains the totality of knowledge; in progressively more elaborated form, the first word, the first Pāda (consisting of the first eight syllables of the Ṛk Veda), the first richa (the first 24 syllables of the Ṛk Veda, consisting of three Pādas), and the first Sūkta (consisting of 8 Ṛichas) all are expressions of the totality of knowledge.

Maharishi (1991) likens the collapse of **A** to **K** to a whirlpool, spiraling from its fullest value, corresponding to **A**, to its point value, correpsonding to **K**. He explains that this whirlpool effect unfolds in eight stages. These stages are separately elaborated in the eight syllables of the first Pāda; these eight syllables correspond to the eight Prakritis, or fundamental qualities of consciousness, namely, the five *Mahabhutas*—*Prithivi* (earth), *Jal* (water), *Agni* (fire), *Vāyu* (air), and *Akash* (space)—and the three subjective principles, *Manas* (mind), *Buddhi* (intellect), and *Ahamkar* (ego). The first Ṛicha, consisting of 24 syllables, provides a further elaboration of these first 8; the eight-syllable structure of the first Pāda appears three times in the first Ṛicha, the first time from the point of view of the knower (Ṛishi), the second, from the point of view of the process of knowing (Devatā), and the third, from the point of view of the object of knowledge (Chhandas).[61]

We are able to locate eight stages of elaboration of the dynamics of the collapse of **A** to **K** in the dynamics of the universe V in the presence

60. In fact, the Ved can be seen as a commentary on **A** itself since all the dynamics in **AK** are contained in **A**--see (Maharishi 1991).

61. See (Maharishi 1992) for a further elaboration of these stages of unfoldment.

of the Wholeness Axiom. We recall that an elementary embedding from the universe to itself represented a principle which was "approximated" with ever greater accuracy by the known large cardinal axioms. Among these large cardinal axioms, there are eight which stand out as milestones in the climb to the Wholeness Axiom. These eight naturally are partitioned into two groupings; the first group of five gives more "objective" information about the size of *V*; the second group of three provides information about *V* as it "reflects upon itself"—information which has a more subjective flavor. These eight can be viewed from the perspective of the knower, which, as we have seen, corresponds to the universe *V*; the process of knowing, corresponding to elementary embeddings; and the object of knowledge, which we associate here with the critical points of these elementary embeddings, i.e., the "point values."

For purposes of discussion, we summarize these eight stages, cast in three perspectives, in Table 4. Each of the large cardinal properties represented in the table can be appreciated in terms of the properties of a particular cardinal number (right column—Chhandas value); in terms of the associated elementary embedding (center column—Devatā value); and in terms of the new structural features of *V* that become apparent in the presence of the given large cardinal properties (left column—Ṛishi value). The relationship between the right and center columns is easy to understand—the cardinal number given in the right column is generally the critical point of the embedding given in the center column. The structural features of *V* that we have placed in the left column represent, on the one hand, a catalogue of deep insights into the structure of the universe and, on the other hand, a bridge which connects large cardinal properties to the eight stages of collapse of infinity to a point in Maharishi Vedic Science.

The first five large cardinal properties listed in the chart demonstrate with increasing cogency the truly unlimited nature of the structure of *V*. As these are properties of the *structure* of *V*, it is natural to connect them with the structural, or objective, principles of creation: earth, water, fire, air, and space. The stages of development of these large cardinal properties correspond to the history of models of set theory, as we shall discuss shortly. The last three large cardinal properties reveal the universe's "ability" to interact with itself in powerful ways; this self-

8–fold *Prakriti*	*rishi* (the totality *V*)	*devata* (elementary embeddings of the universe)	*chhandas* (point value of the embeddings)
prithivi (earth)	V_κ is a model of ZFC, and for nearly all ordinals $\alpha < \kappa$, V_α is also a model of ZFC	There is a $\lambda < \kappa$ such that the identity *id*: $V_\lambda \to V_\kappa$ is an elementary embedding	κ is an inaccessible cardinal
jal (water)	$V \neq L$ (not *CP(L)*)	There is an external elementary embedding $j: L \to L$	0# exists
agni (fire)	$V \neq K$ (not *CP(K)*)	There is a nontrivial elementary embedding $j: V \to M$	Measurable cardinals exist
vayu (air)	$V \neq L[A]$ for any set *A*	For each λ, there is $j_\lambda: V \to M_\lambda$ such that $V_\lambda \subseteq M_\lambda$	Strong cardinals exist
akash (space)	AD holds in *L(R)*	There is a $j: V \to M$ with critical point κ and there are cardinals $\delta_0 < \delta_1 < ... < \delta_n < ... < \kappa$ such that for each *n* and each $f: \delta_n \to \delta_n$ there is a $\lambda_n < \delta_n$ such that $f''\lambda_n \subseteq \lambda_n$ and there is $i_n: V \to M_n$ with critical point λ_n and $V_\gamma \subseteq M_n$ where $\gamma = (i_n(f))(\lambda_n)$	There are ω Woodin cardinals with a measurable above
manas (mind)	A magic sequence can be defined from which all sets can be located	For each λ, there is $j_\lambda: V \to M_\lambda$ with critical point κ such that all λ–sequences from M_λ are members of M_λ	Supercompact cardinals exist
buddhi (intellect)	Almost all cardinals are large cardinals	For each α, there is $j_\alpha: V_{\kappa+\alpha} \to V_\zeta$ with critical point κ	Extendible cardinals exist
ahamkar (ego)	There exist arbitrarily close approximations to a countable sequence unbounded in *V*	For each *n*, there is $j_n: V \to M_n$ such that M_n is closed under κ_n–sequences (where κ_n is the *n*th iterate of j_n applied to κ)	Cardinals *n*–huge for every *n* exist

Table 4. Eight-fold collapse of infinity to a point in the context of large cardinals.

interaction is suggestive of a sort of subjective quality present in the universe, and again, it is natural to connect these with the subjective principles of creation: mind, intellect, and ego.

We begin with the first five properties in the chart. These are statements that assert in an increasingly powerful way that none of the structurally complete and well-behaved models of set theory—which have been "built from below" to anticipate and resolve as many independent mathematical statements (like the Continuum Hypothesis) as possible—are simply not vast enough to be the universe V itself.

The first of these properties, arising from the existence of an inaccessible cardinal, asserts that V is much more than merely a model of ZFC: in the presence of an inaccessible, a model of ZFC is quite a *common* phenomenon in the sense that nearly every stage V_α, for $\alpha < \kappa$, is such a model.[62]

The second through fifth of these properties represent historical developments in a field that has come to be known as *inner model theory*. These developments began with Kurt Gödel's (1938) proof that the Generalized Continuum Hypothesis was consistent with ZFC. His method was to construct a model of set theory, known as the *constructible universe* and denoted L, in which each successor stage is built by collecting together only those subsets of the previous stage which are *definable* in the previous stage. (Recall that to form a successor stage in the construction of V, *all* subsets of the previous stage are used.) This method drastically limits the number of sets that are introducted at each stage—so much so that the power set of any infinite set is guaranteed to be precisely the next larger cardinal number (the smallest value such a power set could have). One of the remarkable consequences of Gödel's model is that virtually all statements known to be independent of ZFC can be decided under the assumption that $V = L$; that is, given virtually any statement in the language of set theory which is known to be independent of ZFC, a proof is also known which demonstrates that the statement is either true or false under the additional assumption that $V = L$. The reason for this remarkable empirical fact is that, built into the design of L is a wide range of powerful combinatorial tools—tools that emerge from the severe restriction on the number of sets allowed to enter at each stage and that give the mathematician extraordinary control over the behavior of mathematical objects. Hence, problems about sets, the real line, trees, infinite abelian groups, and general topological

62. In the present context, "nearly all α" means that the set of all such ordinals forms a closed unbounded set in the regular cardinal κ.

spaces that were impossible resolve using the tools available in *V* could be decided using the tools in *L*.

A natural question to entertain in light of Gödel's discovery—and which forms the basis for the second entry in the chart above—is whether our universe *V* could really be nothing other than *L*. It is consistent with ZFC for this to be so. The universe according to *L* would be extremely precise and well-formed, but extremely restrictive. This extreme restrictiveness has led set theorists, including Gödel, away from the belief that $V = L$. There is common agreement that the "absolute" universe ought to be somewhat more vast and expansive than *L*.

It turns out that there is a precise point—known as $0^\natural$ (pronounced "zero sharp")—in the ascent through large cardinal axioms at which the fundamental structural differences between *V* and *L* become apparent. R. Solovay [1967], using other large cardinals, discovered a real number which contained enough information (in coded form) to demonstrate that *L* was a model of set theory radically different from *V*; he called this real number $0^\natural$. As an example of the impact of $0^\natural$, one of its consequences is that *L*'s set of real numbers is (from the point of view of *V*) no larger than the set of natural numbers! Results of this kind immediately demonstrated that in the presence of $0^\natural$, $V \neq L$ and suggested that the world according to *L* is in fact somewhat distorted; thus, while the axiom $V = L$ has unquestionable value for establishing formal consistency results, it should not be taken as an intuitively clear assumption about the structure of *V*.

After Solovay's discovery of $0^\natural$, K. Kunen showed that the axiom "$0^\natural$ exists" could be seen as a large cardinal axiom in its own right by showing that it was equivalent to the existence of an external (to *L*) elementary embedding from *L* to *L*.

Perhaps the deepest work done in this area is due to R. Jensen. He showed (Devlin and Jensen 1975) that the axiom "$0^\natural$ exists" is the precise point in the hierarchy of large cardinal axioms at which the structures of *L* and *V* radically diverge. In particular, he showed that if $0^\natural$ does not exist, then *L* and *V* are very similar in the sense that every uncountable set (that is, every set bigger than the set of natural numbers) of ordinals lies in a set in *L* of the same size; conversely, if $0^\natural$ does exist, then some large set of ordinals in *V* is *not* contained in any set of ordinals in *L* of the same size. This result has come to be known as the *covering lemma* for *L*. When a model *M* satisfies the property that every uncountable set of ordinals in *V* is contained in a set in *M* of the same size, *M* is said to have the *covering property* and we write *CP*(*M*). Thus, Jensen's covering lemma for *L* states that the nonexistence of $0^\natural$ is equivalent to *CP*(*L*). As we shall see, efforts have been made to generalize Jensen's covering lemma to models besides *L*.

One important thread of research which emerged from Jensen's work, and which leads us to the third property listed in our chart, was the search for a more expansive model than *L* which retained the richness of *L*'s combinatorial tools, but which did not so readily diverge from the structure of *V*. The objective was to find a model with a fine structure like that of *L,* but which satisfied the covering property even in the presence of $0^\natural$ and possibly much larger large cardinals as well. T. Dodd and R. Jensen (1982) published extraordinary results achieving this objective by introducing the *core model* denoted by *K*. The core model retains all the most desirable combinatorial properties of *L* but has sufficient flexibility to satisfy the covering property in the presence of $0^\natural$ and much larger cardinals as well, such as Erdös cardinals and Ramsey cardinals. The precise point of failure of *CP*(*K*) turns out in this case to be a measurable cardinal: *CP*(*K*) holds if and only if there is no (inner) model of a measurable cardinal. In particular, in the presence of a measurable cardinal, the structures of *K* and *V* can be seen to be radically different.[63]

Another attempt to expand Gödel's *L* to include more large cardinals but retain strong combinatorial properties led to the development of the class of models *L*[*A*] for various sets *A*; these models explicitly expand *L* in its stage-by-stage definition so that at each successor stage, the new sets that are introduced are those definable from the previous stage and from the set *A* itself. This construction generally results in a model bigger than *L*. The most important application of this construction was to expand *L* so that it would not conflict with measurable cardinals (recall that *V* cannot equal *L* or *K* if there is a measurable

63. Jensen's discovery of these turning points for *L* and *K* in the large cardinal hierarchy has had a far greater significance in mathematics than simply to exhibit points of divergence between various models of set theory. A major consequence of his work is that it makes the powerful combinatorial tools present in *L* available for use in ordinary mathematics. One recent application of these tools has been in the determination of the exact strength of the Normal Moore Space Conjecture in general topology.

cardinal). The solution to the problem was simple and elegant: if κ is a measurable cardinal and U is an ultrafilter on κ which demonstrates that κ is measurable, the model $L[U]$ retains the fine structure of L and at the same time retains the knowledge that κ is measurable and that U (actually, $U \cap L[U]$) is the corresponding ultrafilter.

Because of the elegance of this discovery, a program of research emerged that sought to obtain ever richer models of the form $L[A]$ in which ever larger large cardinals could be found. One hope among some of these researchers was that if enough large cardinals could be represented in such a model, then either the model itself or some sort of corresponding core model (bearing the same relationship to the model $L[A]$ as K bears to the model $L[U]$[64]) could then be taken to be the "real" universe of sets.

In pioneering work, W. Mitchell (1974) had the idea to build models $L[A]$ where A was a sequence of ultrafilters; this technique resulted in nice inner models for large cardinals known as *hypermeasurable*—much stronger than ordinary measurable cardinals.

The first major stumbling block in this research program (corresponding to the fourth property on our chart) was that, beyond a certain point in the large cardinal hierarchy, it is no longer possible for V to equal $L[A]$ if A is a *set*. The point in the hierarchy at which this phenomenon is first encountered is a *strong* cardinal. Thus, if there is a strong cardinal, we find a radical divergence between the structure of V and the structure of all models of the form $L[A]$, A a set.

Mitchell and others (see Baldwin, 1986) were able to overcome this difficulty by considering sequences (and directed systems) which were themselves *proper classes*[65] instead of sets; using these methods, they obtained combinatorially rich inner models of strong cardinals, and other larger large cardinals.

A more significant obstacle to this program arose in a rather unexpected way through work by another group of researchers who also

64. Unlike L and $L[U]$, the core model is a highly variable model whose structure depends on what large cardinals actually exist. If $0^\#$ does not exist, then $K = L$. If $0^\#$ does exist, but $0^{\#\#}$ does not exist (0## is the real number that bears the same relationship to $L[0^\#]$ as $0^\#$ does to L) then $K = L[0^\#]$. K assumes its most expansive form in the presence of a measurable cardinal; in this case it is the intersection of all iterated ultrapowers of $L[U]$—the "core" of $L[U]$, just barely avoiding the existence of a measurable cardinal.

65. See (Weinless 1987) for a discussion of proper classes.

were developing an extension of L that could assume the role of the "real" universe. Their work began with the observation made earlier that, even in the presence of mild large cardinals, the real line in L is only countable. Since the reals are such an important part of mathematics, it was natural to try to expand L so that combinatorial properties are preserved and yet the real line $\mathbf{R}$ retains its status (as the "real" real line). The resulting model was $L(\mathbf{R})$; this model is constructed by beginning at stage 0 with $\mathbf{R}$ itself (actually, the transitive closure of $\mathbf{R}$), instead of the empty set, and then proceeding as in the construction of L. As desired, $L(\mathbf{R})$ contains the "real" real line and does indeed retain many of the nice combinatorial features of L. However, these nice combinatorial features turn out to be available only in the "upper" reaches of the universe and not in the realms of ordinary mathematics. Perhaps worse, the model (typically) failed to satisfy the Axiom of Choice. Researchers sought to replace the Axiom of Choice with another axiom that could serve to restore rich combinatorics down low in $L(\mathbf{R})$. The axiom that emerged was called the *Axiom of Determinacy* or AD for short.

A dedicated group of researchers in Descriptive Set Theory developed the mathematical theory based on the axiom $V = L(\mathbf{R})$ and the assumption that *AD* holds in $L(\mathbf{R})$. For fifteen years, this group continued working out the theory, undaunted by the rather unsettling fact that it was not known whether *AD* held in $L(\mathbf{R})$—in fact, it was not known whether *AD* was consistent at all!

Toward the end of the 1980's, Martin, Steele, and Woodin impressively demonstrated the consistency of *AD* assuming large cardinals. Woodin eventually showed that the exact large cardinal strength of the consistency of *AD* is ω Woodin cardinals. Moreover, assuming ω Woodin cardinals plus a measurable cardinal above them all, he showed that *AD* holds in $L(\mathbf{R})$.

Using hindsight, many feel that the dedication of the original group of Descriptive Set Theorists to their unproven intuition about the truth of *AD* in $L(\mathbf{R})$ is strong evidence for the naturalness of this universe, despite the conspicuous absence of the Axiom of Choice.

Woodin's result—indicated as the fifth property on our chart—represents another significant structural breakthrough in the ascent through large cardinal axioms. Not only did his result provide the missing link for an important research program in Descriptive Set Theory; not only did it serve to reveal the universe to be unexpectedly harmonizing in bringing validation to a choiceless universe within a universe in which the Axiom of Choice is valid;[66] but also, as we will now see, Woodin's result marks an upper bound to the program of inner model theory described above.

As we mentioned before, inner model theorists sought to expand Gödel's constructible universe L in such a way that combinatorial properties were preserved and yet large cardinals could be included. The combinatorial properties that researchers sought to preserve were well agreed upon and included the following:

(1) The Generalized Continuum Hypothesis
(2) The Diamond Principle[67]
(3) The existence of a "nice" (*projective*[68]) well-ordering of the reals.

But one immediate consequence of Woodin's result is that if there are ω Woodin cardinals with a measurable above, then no "nice" well-ordering of the reals exists at all! Thus, the program of inner model theory, using the above criteria, must come to an end at the point in the hierarchy of large cardinals at which there are ω Woodin cardinals with a measurable above; in the presence of such large cardinals, the structure of the universe V diverges from the structure of inner models constructed according to the general criteria given above. (In contemporary research, different notions of "canoncial inner model" have emerged that are not limited by the presence in the universe of ω Woodin cardinals with a measurable above.)

As we have seen, each of the first five properties given in our chart above marks a turning point in knowledge about the structure of V: inaccessibles mark the point at which it becomes clear that V is far

66. Cf. (Weinless 1987) for an interesting discussion

67. The Diamond Principle asserts that there is a special sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of subsets of ω such that for each $\alpha, A_\alpha \subseteq \alpha$, and for *any* subset A of ω_1, there are stationarily many α for which $A \cap \alpha = A_\alpha$.

68. A well-ordering of the reals **R** is a subset of **R x R**. The simplest subsets of **R x R** are the open sets and the closed sets. A subset of **R x R** is *projective* if it is a continuous image of a closed set, or the complement of such a set, or a continuous image of such a complement, or the complement of such a set, and so forth.

more than merely a model of set theory; $0^\natural$ marks the realization that V is radically different from L; measurable cardinals mark the realization that V is also radically different from the core model K; strong cardinals mark the stage at which V is known to be different from models of the form $L[A]$, A a set; and finally, ω Woodin cardinals with a measurable above marks the point at which the traditional program of inner model theory to capture V with a model of the form $L[\mathbf{C}]$, $\mathbf{C}$ a proper class, finally breaks down, and also marks the point in the hierarchy in which AD is seen to hold in $L(\mathbf{R})$ and, philosophically speaking, where Choice and Determinacy are found compatible.

We now turn to the last three properties on our chart which describe more "subjective" features which seem to arise in the universe in the presence of even larger large cardinals. Each of these large cardinal properties marks a stage at which the universe becomes, as if, "aware" of it own nature, structure, and "abilities."

At the level of a supercompact cardinal (corresponding to the sixth level of our chart) we find that the remarkable magic sequence, described earlier, can be defined. In the absence of a supercompact cardinal, the sets in the universe are located in the usual way: sets unfold, stage by stage, starting from the empty set, by means of the power set and union operations. But in the presence of a supercompact cardinal,[69] a magic sequence becomes available by which every set in the universe can be located from the perspective of the wholeness of V (recall that if S is a magic sequence coded by a class of embeddings $\langle i_\alpha : \alpha \in ON \rangle$, then any set X can be located as the κth term in the sequence $i_\alpha(S)$, for some ordinal α). In a certain sense, the magic sequence allows V to become "aware" of its constituent sets.

Extendible cardinals (at the seventh tier of our chart) are the first in the large cardinal hierarchy which imply that the universe is pervaded

69. One shows that the existence of a supercompact cardinal is equivalent to the existence of a magic sequence as follows: First note that, using the proof of Kunen's theorem, for any elementary embedding $i : V \to M$ having critical point κ, the set $\{\kappa, i(\kappa), i(i(\kappa)), \ldots\} \notin M$. Thus, given any set $\{i_\alpha : \alpha < \rho\}$ of elementary embeddings of the universe, each with critical point κ, the disjoint union A of the sets $A_\alpha = \{\kappa, i_\alpha(\kappa), i_\alpha(i_\alpha(\kappa)), \ldots\}$ is not in the union of any of the image models M. Thus, for any $f : \kappa \to V_\kappa$ and any $\alpha < \rho$, $i_\alpha(f)\kappa \neq A$. [It is now known that a notion of Laver sequence can be proven to exist using only a strong candinal; such Laver sequences play the same role as those described in this paper. Vedic researchers will need to find the best way to revise the sixth tier here. -Ed.]

with other large cardinals. As we have seen, large cardinals represent the universe's ability to reflect its properties of wholeness into its own sets; from the point of view of Maharishi Vedic Science, this process of reflection is reminiscent of the wholeness of pure consciousness awakening the point values of life to their fully expanded state.

Most of the large cardinals smaller than extendible imply the existence of many large cardinals *below*. For instance, if κ is a Mahlo cardinal, κ many of the cardinals below κ must be inaccessible. If κ is measurable, *nearly all* cardinals below κ are Mahlo. If κ is supercompact, *nearly all* cardinals below κ are measurable. As remarkable as these results are, none of these large cardinals implies that even a single inaccessible exists *above* them. Thus, their range of implication is restricted to a small portion of the universe.

On the other hand, an extendible cardinal implies that arbitrarily large measurable cardinals exist above it. As we just observed, nearly all cardinals below a measurable are Mahlo (and hence inaccessible). It follows that the existence of an extendible implies that *nearly* all *cardinals in the universe are large*! The range of influence of an extendible is therefore global and suggests a radical shift in our knowledge about the structure of V: An extendible tells us that at nearly every stage of the universe, we can find the lively presence of properties of V as a whole.

Finally, let us consider the eighth level of our chart, occupied by large cardinals which are n-huge for every n. Such cardinals hover at the verge of inconsistency. We start with the definition of n-huge for every n:

Definition. A cardinal κ is *n-huge for every n* if for each natural number n, there is an elementary embedding $j_n: V \rightarrow M_n$ with critical point κ such that every sequence of length $j_n^{\,n}(\kappa)$ (where $j_n^{\,n}$ is the nth iterate of j_n) lies in M_n.

To see how close this definition takes us to inconsistency, suppose that in the definition, we require infinitely many of the j_n to be the same embedding—call it j. Then one may show that the corresponding image model—call it M—must in fact be closed under sequences of length λ, where $\lambda = \sup\{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$. But now we can use λ to carry out Kunen's inconsistency proof to conclude that 0 = 1! Thus, on one view, cardinals that are n-huge for every n approximate inconsistency arbitrarily closely.

Another way to view the impact of these large cardinals is to say that a cardinal which is n-huge for every n gives rise to arbitrarily long (definable) finite sequences of the form $\langle \kappa, j(\kappa), j(j(\kappa)), \ldots, j^n(\kappa)\rangle$ which approximate a (undefinable) countable sequence unbounded in the universe, as described by our Principle of Countable Unboundedness. Thus, an alternative viewpoint concerning the presence of these large cardinals in the universe is that they arbitrarily closely approximate the Wholeness Axiom using definable concepts.

Maharishi points out that the very structure of pure consciousness is upheld by the coexistence of opposite values; that its very nature is at once infinite dynamism and infinite silence; and that one of the distinguishing features of an enlightened individual is the ability to live and integrate opposite—even contradictory—values in a state of balance and harmony.[70]

In light of Maharishi's Vedic Science, the presence of cardinals which are n-huge for every n is reminiscent of the pinnacle of subjective development in which the full extent of contradictory values is reconciled in the full awakening of wholeness.

Thus, the last three levels of our chart suggest the unfoldment of more subjective qualities within the universe: Supercompact cardinals mark a new awareness of the orgin of sets within the universe; extendible cardinals bring with them an awareness of the omnipresence of large cardinals which in turn bring into the realm of sets central properties of the wholeness of the universe; and cardinals which are n-huge for every n provide a strong analogue to that state of awareness which is on the brink of awakening to the full value of wholeness in which even the most contradictory values are unified.

The eight stages of collapse of infinity to a point, as described by Maharishi Vedic Science, is an unchanging pattern lying within the blueprint of creation; for this reason, we expect to find this pattern at work within the foundation of every discipline. Our chart and subsequent discussion suggest that the eight large cardinal axioms we have identified represent natural "power" points in the ascent through the large cardinal hierarchy, marking the confluence of diverse and seem-

70. See (Maharishi, 1991a).

ingly unrelated results as well as radical changes in the complexion and structure of the universe. For this reason, we feel these eight, cast in the three-fold framework described in our chart and emerging in the context of the new dynamics of wholeness provided by our Wholeness Axiom, give expression to the fundamental pattern of eightfold collapse described in Maharishi Vedic Science.

Of course, it is the nature of western science to refine itself continually. While the dynamics of wholeness embodied in the Wholeness Axiom and its ramifications do appear to give expression to the fundamental dynamics of the wholeness of pure consciousness as described by Maharishi, one would expect that as the Foundations of Mathematics evolves, new, fuller expressions of these fundamental patterns of nature will inevitably emerge. We view our work here as part of an ongoing program to give ever fuller expression to the deepest dynamics of consciousness within the foundation of mathematics for the sake both of perfecting mathematics and of bringing fulfillment not only to the mathematician but to all who come in contact with the field of mathematics.

§ 21. Conclusion

We began our study of ZFC and the universe of sets with the observation that, while it succeeds in providing a foundation for most of modern mathematics, this foundational structure fails to account for the presence of large cardinals within mainstream mathematics. None of the heuristic devices used by mathematicians so far to account for these infinities (or to remove them) succeeds in accounting for all large cardinals, and none is very compelling. This state of affairs makes apparent the need for a single set-theoretic principle which at once accords with the fundamental intuition of set theorists and accounts for large cardinals.

Typically, mathematical intuition derives from observation of nature and from mathematical exprience. In the case of large cardinals, mathematical experience suggests that large cardinal properties are actually intimately tied to metatheoretic properties of the universe as a whole. The Reflection Principle uses this connection to provide justification for many of the smaller large cardinals, while nearly all the larger large cardinals can be framed in terms of the of the universe's interaction with other universes, in terms of elementary embeddings of the universe. Thus, mathematical experience with large cardinals suggests that we look for our justification for large cardinals by gaining a clear intuitive sense of the "true" nature of the universe V as a whole.

Interestingly, the same conclusion results from any reasonable attempt to look to nature's functioning for intuitive guidance concerning such foundational concerns as the origin of large infinities. Certainly, nature's behavior on the superficial level of finite collections of objects interacting with each other is of little value in motivating concepts pertaining to the infinite. On the other hand, one might expect that nature's behavior at its roots would suggest the "right" picture for the foundation of mathematics. On a parallel track, recent work in quantum field theory (Hagelin, 1987) tells us that, at small time and distance scales, all force and matter fields can be seen as expressions or precipitations of a single, unified, self-interacting superfield. Unified field theories provide an unprecedented unification of widely diverse phenomena and physical theories, and suggest to us the principle that, in the presence of a "theory of wholeness," theoretical explanations for various other phenomena become more available.

For these and other reasons, we have investigated the nature of the universe of sets as a whole, seeking a basic unifying principle. We observed that certain great mathematicians and philosophers in history—Plato, Cantor, and Gödel to name three—have claimed to have a more or less clear intuition of "mathematical reality," and that their view of their work as an attempt to give expression to this deeper reality has led to important discoveries (such as transfinite cardinals and the completeness and incompleteness theorems). We expressed the belief that this sort of intuition does not appear to be awake equally in all mathematicians and suggested the need to revitalize the intuition of all practising set theorists to gain a more uniform view of the reality glimpsed by some of the mathematical giants.

We have claimed that the basic reality that has been intuited more or less clearly by some mathematicians and which has been uncovered to some extent through objective means in research on completely unified field theories has in fact been thoroughly studied through *subjective* technologies in many traditions of knowledge throughout the world, the most ancient of which being the Vedic tradition. For this reason, we have appealed to Maharishi Vedic Science—constituting a modern-day systematic treatment of this ancient wisdom, complete with effective inner technologies for exploring the realities proclaimed in the ancient texts—as an attempt to deepen our intuition about the wholeness we are trying to capture through the concept of a universe of sets.

Reviewing the *qualities* of the field of wholeness, pure consciousness, as described in Maharishi Vedic Science, we found certain deficiencies in the structure of *V* as a model for wholeness. These deficiencies suggested new features of the universe we might wish to include in a unifying axiomatic principle, such as a natural transformation from *V* to itself to capture the quality of self-interacting, and the presence of many sets reflecting all first-order properties of *V* to capture the quality of *infinite correlation*. A study of the *dynamics* of pure consciousness revealed a more compelling deficiency, namely, that the dynamics of unfoldment of the universe is unilateral, expanding from the empty set to all sets in the universe, and not, as we find within the wholeness of pure consciousness, collapsing from the fully expanded value to a point value. This deficiency again suggested another dynamic to incorporate into our new axiomatic principle.

Appealing to mathematical experience, we also sought intuition from the statements of the strongest large cardinal axioms. These assert the existence of elementary embeddings from the universe *V* to models resembling more and more closely the structure of *V* itself. The natural limit to the large cardinal properties is the existence of an elementary embedding from *V* to itself. One easily verifies that the features of the universe suggested to us by our analysis of the qualities and dynamics of pure consciousness become evident in the presence of such an embedding. Although Kunen showed that no such embedding could be (weakly) definable in the universe, we observed—treating such an embedding as an analogue to the *unmanifest* dynamics of pure consciousness moving within itself and knowing itself—that it would be more natural to require that such an embedding *j* be *undefinable* (more precisely, not weakly definable) or *transcendental.* Realizing that pure consciousness is not only unmanifest but also present at each point in creation, we further required that the universe be *fully j-closed*. In this way we arrived at our Wholeness Axiom, asserting that such an embedding *j* exists and that *V* is *fully j-closed*. This approach avoids the contradiction produced by Kunen's proof and at the same time successfully accounts for virtually all large cardinal properties as the properties of the critical point of a single, original embedding j of the universe to itself.

Examining the dynamics arising from set theory enriched by the Wholeness Axiom, we found deep and unexpected parallels with Maharsihi Vedic Science. On the one hand, the collapse of the infinitely expanded value of wholeness, represented by **A**, to its fully contracted value, represented by **K**, imparting **K** with infinite dynamism at the basis of the formation of the Veda and all creation is paralleled, on the other hand, in the first stirring within V resulting from the action of j: the first set κ moved by j stands as a focal point, embodying the properties of the wholeness of V and imbued with the truly infinite dynamism expressed as a vast class of supercompact embeddings which are selectively coded into a compact magic sequence, which gives rise to every set in the universe. We observed that the infinity of transformations that emerge in this collapse of **A** to **K** finds its analogue in the fact that j very naturally gives rise to all possible supercompact embeddings—each occurring as a factor of one of the canonical supercompact embeddings derived from the myriad supercompact ultrafilters that unfold from j. We found that the interplay of collapse of infinity to a point and expansion of point to infinity with infinite frequency occurring at each moment as the fundamental dynamics of pure consciousness again finds a parallel in the infinitely often repeated construction, through expansion and collapse, of the canonical models via supercompact ultrafilters. We noticed also that the magic sequence that emerges naturally in the dynamics of j embodies many qualities and dynamics of the Veda: by coordinating and unifying the action of the vast class of supercompact embeddings—having j as their basis—every set can be accounted for by virtue of the internal structure of the magic sequence; yet this sequence at the same time exhibits a quality of infinite silence since it is the identity on a large subset. Finally, we observed that the eightfold structure of the collapse of **A** to **K**, in its three phases of Ṛishi, Devatā and Chhandas finds a striking parallel in eight large cardinal principles that increasingly approximate, in consistency strength, the Wholeness Axiom.

We have discovered how the dynamics of pure consciousness are a fundamental reality of Nature. In the spirit of J. Hagelin's paper "Is Consciousness the Unified Field?" (1987), we feel that just as modern unified field theories are an attempt to model this reality with the tools of question field theory, so our efforts in this paper—and so the efforts of all contributors to set theory whether intentionally or not—are an attempt to model this reality with the tools of set theory. The fact that so many strong analogies can be found between the structure of V and the structure of pure consciousness as a consequence of adding a single axiom suggests that our efforts have been successful.

By giving a unified and compelling account of the origin of large cardinals, we feel we have shown that our efforts to model the dynamics of pure consciousness within set theory have a genuine mathematical value. We anticipate that further investigations of this kind will turn up a rich assortment of mathematical results. This sort of relationship between a mathematical field and another science has always been fruitful in mathematics' long history. The study of problems in physics and biology—even gambling—has resulted in the creation of whole new branches of mathematics that have occupied the careers of many bright minds. Most recently, the tools of category theory have been brought to bear on modeling problems in computer science; the relationship between these fields in the past decade has brought advances not only in computer science—since this was the area in which researchers were seeking solutions—but in category theory as well. Category theorists, in using the tools of their trade in new ways, have encountered a whole new class of pure mathematical problems which have stimulated considerable research. We feel that a relationship of this kind is beginning to emerge between Maharishi Vedic Science and a number of sciences, including mathematics. The fact that the former provides a natural framework in which to view the origin of large cardinals, as we have discussed here, is, we feel just the beginning. When we are asked to look at the universe of sets from the point of view of attempting to model the wholeness of pure consciousness, new features of the landscape of set theory become important and new questions arise. For instance, consider the following:

1. In our study of the eightfold collapse of infinity to a point, we singled out certain large cardinal axioms as especially signifi-

cant turning points in the ascent up the large cardinal hierarchy. Are these the best choices? One criterion of "best" might be whether the properties associated with them in the Ṛishi column of the chart are in fact *equivalent* to the large cardinal property. We observed, for example, that the existence of a magic sequence is equivalent to the existence of a supercompact cardinal. But if we reword the definition of a magic sequence so that it does not appear to depend so heavily on the concept of supercompactness, do we preserve the equivalence? Suppose we call a function $f : \kappa \to V_\kappa$ a magic sequence if for each set A there is *some* elementary embedding $i : V \to M$ with critical point κ for which $i(f)\kappa = A$. Is this concept equivalent?[71]

2. Likewise, is the notion of *extendible cardinal* the proper choice in our eightfold structure? It is clear that the fact that there is a proper class of measurables above an extendible does not characterize extendibilit –but what structural property of the universe *does* characterize this large cardinal?
3. In comparing the dynamics which arise from the Wholeness Axiom with those attributed to pure consciousness, we saw that a Laver magic sequence—selectively coding information concerning the proper class of canonical supercompact embeddings arising from j—plays a role that is in many ways quite similar to the role of the Veda in the unfoldment of creation: A Laver sequence is a highly compact expression by means of which every set in the universe can be located. An ongoing aspect of our present research is to determine how far this analogy goes. We described earlier certain striking analogies between these structures; however, there are still many details to the structure of the Ved that we anticipate will be reflected in the structure of these magic sequences, or of some related structures. As an example of work in this direction, it can be shown (Corazza 2009) that the magic sequence S that we derived from j has the property that, stationarily often, the restriction $S|\alpha$ is also a magic sequence at α. We might also expect that stationarily often $S(\alpha)$ is also a magic sequence at α, but this does not appear to be true; at present we can prove only that stationarily

69. This notion is now known to be equivalent to the existence of a strong cardinal (and such cardinals are known to be much weaker than supercompact). See (Corazza 2000).

often, $S(\alpha)$ is a magic sequence at α *of degree* 2^α because of certain technical limitations imposed on us by the ultrapower construction. This may suggest that a stronger notion of "magic sequence" could be defined and proven to arise from the Wholeness Axiom.[72]

3. With our Wholeness Axiom, we have successfully accounted for all large cardinals except one. In proving that AD holds in $L(\mathbf{R})$, Woodin originally assumed the following axiom:

   $$I_0\text{: There is an elementary embedding } j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1}).$$

   Later, he was able to get by with a much weaker axiom, but no inconsistencies arose in his work with I_0. Now I_0 stands as the strongest of large cardinal axioms but, so far, we do not have a satisfactory account for this axiom using the Wholeness Axiom. Perhaps a stronger version of the Wholeness Axiom can be defined which captures even more completely the spirit of our present axiom.

Even as questions of mere technical interest, the mathematical problems that arise in this study justify further research in the connections between set theory and Maharishi Vedic Science. However, we feel that this sort of research, as it directly familiarizes the researcher with the deepest dynamics of Nature's functioning, makes a much more significant contribution to the field than simply a new set of problems to work on. We feel that once set theory—or any scientific discipline—is organized around the very fundamentals of Nature, Nature will yield its secrets far more effortlessly. And, certainly, once these dynamics are awakened not merely in the objective work of the scientists but in the inner life of the scientist as well, then the activity of the discipline will serve not only to bring the objective rewards of expanded knowledge but also to bring to the life of the scientist the much greater fulfillment that arises from aligning individual endeavors with the deeper purpose behind the creation itself and that extends beyond one's professional life to transform each area of individual and collective concern.

72. This hypothesis has been demonstrated to be correct. See (Corazza, 2000; 2009).

References

Barbanel, Diprisco, and Tan

[1984] Many-times huge and superhuge cardinals. *Journal of Symbolic Logic*. Vol. 49, No. 1, pp. 112-122.

Benaceraff, P., and Putnam, H. (editors)

[1983] *Philosophy of Mathematics,* 2nd edition, Cambridge, England: Cambrdige University Press.

Blass, A.

[1976] Exact functors and measurable cardinals. *Pacific Journal of Mathematics,* Vol. 63, No. 2, pp. 335-346.

Cantor, G.

[1883] Über unendliche, lineare Punktmannigfaltigkeiten, 5. Mathematische Annalen 21, pp. 545-86. (Cantor [1932], pp. 165-209)

[1932] Gesammelte Abhandlungen mathematischen und philosophischen Inhalts. (ed. E. Zermelo) Berlin: Springer (reprinted 1980).

Chandler, K.

[1987] Modern science and Vedic science: an introduction. In Modern Science and Vedic Science: An Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Volume 1, No. 1.

Chang, C. C. and Keisler, H. J.

[1973] Model Theory. New York: North Holland Publishing Company.

.

Corazza, P.

[1993] An Introduction To The Teachings of Maharishi Mahesh Yogi, limited edition. 2nd edition in press.

[2000] The Wholeness Axiom And Laver Sequences, Annals of Pure and Applied Logic, Vol. 105, pp. 157-260.

[2006] The Spectrum of Elementary Embeddings $j : V \rightarrow V$, Annals of Pure and Applied Logic, Vol. 139, pp. 327-399.

[2009] The wholeness Axiom. In Consciousness-Based Education, Mathematics, this volume.

[2010, March] The axiom of infinity and transformations $j : V \rightarrow V$, Bulletin of Symbolic Logic.

Cranson, R. W. [1988] Intelligence and growth of intelligence in Maharishi's Vedic Psychology and twentieth century psychology. Unpublished doctoral dissertation, Maharishi International University, Fairfield, IA.

Davies, P.C.W. and Brown, J.

[1988] Superstrings: A Theory of Everything? Great Britain: Cambridge University Press.

Devlin and Jensen

[1975] Marginalia to a theorem of Silver. In ISILC Logic Conference, Lecture Notes in Mathematics 499. New York: Springer-Verlag, pp. 115-142.

Dodd, T. and Jensen, R.

[1982] The covering lemma for K. Annals of Mathematical Logic 22. New York: North-Holland Publishing Company, pp. 1-30.

Domash, L.

[1975] The Transcendental Meditation technique and quantum physics: Is pure consciousness a macroscopic quantum state in the brain? In D.W. Orme-Johnson & J. T. Farrow (eds.) [1975], pp. 652-670.

Drake, F.

[1974] Set theory: an introduction to large cardinals. New York: American Elsevier Publishing Company.

Foreman, M.

[1986] Potent axioms, Transactions of the American Mathematical Society, Vol. 294, No. 1, pp. 1-28.

Friedman, S.

[1993] Handwritten notes.

Gödel, K.

[1938] The consistency of the axiom of choice and of the generalized continuum hypothesis, Proceedings of the National Academy of the USA, No. 24, pp. 556-7.

[1940] The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory, Annals of Mathematical Studies, Vol. 3 Princeton, New Jersey: Princeton University Press.

[1947] What is Cantor's continuum problem? [reprinted in Benacerraf and Putnam [1983], pp 52-61].

Hagelin, J.

[1983] [Chart and discussion]. In Maharishi Technology of the Unified Field: Integration of Modern Science and Ancient Vedic Science Creating Unified Field Based Perfect Civilization for all Mankind in this Scientific Age. West Germany: MERU Press, p. 11.

[1987] Is consciousness the unified field? A field theorist's perspective. Modern Science and Vedic Science, an Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Vol. 1, No. 1. pp. 28-87.

[1989] Restructuring physics from its foundations in light of Maharishi's Vedic Science. Modern Science and Vedic Science, an Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Vol 3, No. 1, pp 2-72.

[1992] [Lecture to Mathematics and Physics Departments]. Fairfield, Iowa.

[1992b] Administration through natural law [chart], In Maharishi Mahesh Yogi, Maharishi's Absolute Theory of Government. Holland: Maharishi Vedic University.

Hallett, M.

[1988] Cantorian set theory and limitation of size. Oxford, England: Clarendon Press.

Jech, T.

[1978] Set theory. New York: Academic Press.

Jensen, R.

[∞] Unpublished manuscripts.

Kanamori, A. and Magidor, M.

[1978] The evolution of large cardinals in set theory. In G. H. Mueller & D. S. Scott (Eds.), Proceedings of the Conference on Higher Set Theory, Oberwolfach, Germany, April 13-23, 1977 (pp. 99-275). New York: Springer-Verlag.

Kelley, J. L.

[1955] General Topology, New York: Van Nostrand Reinhold.

Kunen, K.

[1971] Elementary embeddings and infinitary combinatorics. Journal of Symbolic Logic, No. 36, pp. 407-413.

[1980] Set Theory: An Introduction to Independence Proofs. New York: North Holland.

Kunen, K., and Vaughn, J.E. (eds.)

[1984] Handbook of Set-theoretic Topology. New York: Elsevier Science Publishers.

Lawvere, F. W.
[1979] Categorical dynamics. In Kock, A., Topos Theoretic Methods in Geometry. Aarhus, Denmark: Aarhus Universitet Matematisk Institut.

Lester, B.

[1987] Unified field based computer science: Towards a universal science of computation, Modern Science and Vedic Science, an Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Volume 1, No. 3, pp. 266-321.

MacLane, S.

[1971] Categories for the working mathematician. New York: Springer Verlag.

Maddy, P.

[1983] Proper classes, Journal of Symbolic Logic, 48, pp. 113-139.

[1988a] Believing the axioms, I, Journal of Symbolic Logic, 53, No. 2, pp. 481-511.

[1988b] Believing the axioms, II, Journal of Symbolic Logic, 53, No. 3, pp. 736-764.

Maharishi Mahesh Yogi

[1966] The science of being and art of living. Livingston Manor, New York: MIU Press.

[1967] Maharishi Mahesh Yogi on the Bhagavad-Gita, a new translation and commentary, chapters 1-6, with Sanskrit text. Middlesex, England: Penguin Books.

[1972] Science of creative intelligence, knowledge and experience (Lessons 1-33) [Transcript of videotaped course]. Los Angeles: Maharishi International University Press.

[1973] Love and God. Los Angeles, California: MIU Press.

[1974a] Vedic studies core course. Lesson 8: The continuing stream of evolution: the fundamentals of progress unfold. [Videotaped lecture, February, 1974, Interlaken, Switzerland].

[1974b] Vedic studies. [Videotaped lecture--unedited, February, 1974, Interlaken, Switzerland].

[1974c] Vedic studies core course. Lesson 3: The reality of two fullnesses: the Absolute on the move. [Lecture notes, February, 1974, Interlaken, Switzerland].

[1976] Creating an ideal society. Rheinweiler, West Germany: MERU Press.

[1977] The scientific age rising to be the age of enlightenment. Rheinweiler, West
Germany: MERU Press.

[1978] Enlightenment to every individual, invincibility to every nation. West Germany: MERU Press.

[1980] Science, consciousness and ageing. Rheinweiler, W. Germany: MERU Press, pp. 73-80.

[1981] Education for Enlightenment: An Introduction to Maharishi International University. Fairfield, Iowa: MIU Press.

[1984] Maharishi Technology of the Unified Field: Results of Scientific Research on the Transcendental Meditation and TM-Sidhi Programme. West Germany: MERU Press.

[1985] Maharishi Vedic University, Inauguration. (January 12) [Washington, D.C.]: Age of Enlightenment Press.

[1986] Life Supported by Natural Law. Washington, D.C.: Age of Enlightenment Press.

[1986b] Maharishi Vedic University, Bulletin for 1986-1988. West Germany: MERU Press.

[1989] Maharishi on modern science and Vedic Science. Modern Science and Vedic Science, an Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Volume 3, No. 2, pp. i-iii.

[1990a] Structure of Maharishi Jyotish and its application to life. Theme 6, Block 2, Jyotish Teacher Training Course. [Videotaped lecture, August, 1990, Holland.]

[1990b] Samhita of rishi, devata, and chhandas: Science and Technology of the Unified Field. [Videotaped lecture, 23 July 1990, Holland].

[1990c] Science and technology of the unified field, a course for professionals: Mathematics [Syllabus of 7-lesson videotaped course]. Holland: MVU Press

[1991a] The unbounded scope of Maharishi's Vedic Science: investigating into the ultimate reality, pure wakefulness, and locating all diversity coexisting in the structure of unity. [Videotaped lecture, 14 January 1991, Maastrict, Holland].

[1991b] Maharishi's master plan to create heaven on earth. Holland: Maharishi Vedic University Press.

[1992] Constitution of the universe, the source of all order and harmony displayed throught the universe. Fairfield, Iowa: MIU press.

Mathias, A. R. D. (Ed.)

[1983] Surveys in Set Theory. Cambridge, Great Britain: Cambridge University Press.

McLarty, C.

[1990] The uses and absuses of the history of topos theory. British Journal of the Philosophy of Science, 41, pp. 351-375.

Mitchell, W.

[1974] Sets constructible from sequences of ultrafilters. Journal of Symbolic Logic, Vol. 39, No. 1, pp. 57-66.

Moschovakis, Y.

[1980] Descriptive Set Theory. New York: North-Holland Publishing Company.

Quine, W. V. O.

[1951] Mathematical Logic (Revised edition; first edition 1940), Cambridge, Massachusetts: Harvard University Press.

Reinhardt, W. N.

[1974] Remarks on reflection principles, large cardinals, and elementary embeddings. In T. J. Jech (Ed.), Axiomatic set theory (Part 2, pp. 189-205). Providence, RI: American Mathematical Society.

Roitman, J.

[1990] Introduction to Modern Set Theory. New York: John Wiley and Sons.

Rucker, R.
[1982] Infinity and the Mind. Boston: Birkhauser.

Russell, B.

[1906] Some difficulties in the theory of transfinite numbers and order types, Proceedinof the London Mathematical Society, Vol. 2, No. 4, pp. 29-53.

Shelah, S.

[1984] Can you take Solovay's inaccessible away? Israeli Journal of Mathematics. No. 48, pp. 1-47.

SIMS (Students' International Meditation Society)

[1969] TM, some results. Students International Meditation Society.

Solovay, R.

[1967] A nonconstructible set of integers, Transactions of the American Mathematical Society, No. 127, pp. 50-75.

[1970] A model of set theory in which every set of reals is Lebesgue measurable, Annals of Mathematics (1970), pp. 1-56.

[1971] Real-valued measurable cardinals, in Axiomatic Set Theory, Proceedings of Symposia on Pure Mathematics, 13, I (Scott, ed.), Providence, Rhode Island: American Mathematical Society, pp. 397-428.

Solovay, Reinhardt, Kanamori

[1978] Strong axioms of infinity and elementary embeddings, Annals of Mathematical Logic, No. 13, pp. 73-116.

Ulam, S.

[1930] Zur Masstheorie in der algemeinen Mengenlehre. Fundamenta Mathematicae 16, pp. 140-150.

Wang, H.

[1974] From Mathematics to Philosophy. New York: Humanities Press.

Weinless, M.

[1987] The samhita of sets: Maharishi's Vedic Science and the foundation of mathematics, Modern Science and Vedic Science, an Interdisciplinary Journal Devoted to Research on the Unified Field of All the Laws of Nature, Volume 1, No. 2, pp. 141-204.

Woodin, Hugh

[1989] Private communication.