X-Laver Sequences.

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In these notes, we give some partial results concerning the following question:

1.1 Open Problem. For each class X, find a class of embeddings $\mathcal{E}_\kappa^\theta$ for which there exists a function $g : \kappa \to V_\kappa$ such that $X = \{i(g)(\kappa) : i \in \mathcal{E}_\kappa^\theta\}$.

The notation $\mathcal{E}_\kappa^\theta$ is introduced in Corazza, *The wholeness axiom and Laver sequences*, Annals of Pure and Applied Logic (2000) 105 pp. 157-260. The problem was originally motivated by an earlier effort to obtain a variant of Laver sequences for strongly compact cardinals.

As a quick observation, we note that it is consistent for a strongly compact cardinal to fail to have a Laver sequence in the usual sense: Starting with a model of "κ is strongly compact," Magidor built a forcing extension in which κ remains strongly compact, and the least strongly compact is equal to the least measurable. In that model, if the least strongly compact had a (strongly compact) Laver sequence, then κ would have to be at least a strong cardinal; but strong cardinals have many measurables below them, so this is impossible. On the other hand, there are models, as Magidor shows in the same paper, in which the least strongly compact is the least supercompact; in that model, an ordinary supercompact Laver sequence would be a strongly compact Laver sequence over the least strongly compact.

If we consider the functions $f : \kappa \to V_\kappa$ and the elementary embeddings $i \in \mathcal{E}$ that define κ as as strongly compact cardinal, one can ask: For which classes X is it possible for $X = \{i(f)(\kappa) : i \in \mathcal{E}\}$? Our discussion in these notes lays some groundwork for answering this question.

1.2 Definition. Let X be a class and $\mathcal{E}_\kappa^\theta$ a class of embeddings (not necessarily regular). We shall say that a function $g : \kappa \to V_\kappa$ is

$\mathcal{E}_\kappa^\theta$-$X^+$-*Laver* if	for every $x \in X$ there is $i \in \mathcal{E}_\kappa^\theta$ for which $i(g)(\kappa) = x$;
$\mathcal{E}_\kappa^\theta$-$X^-$-*Laver* if	for every $i \in \mathcal{E}_\kappa^\theta$, $i(g)(\kappa) \in X$;
$\mathcal{E}_\kappa^\theta$-$X$-*Laver* if	$X = \{i(g)(\kappa) : i \in \mathcal{E}_\kappa^\theta\}$.

First of all, for any class $\mathcal{E}$ defined by a suitable formula, and for each $x \in V_\kappa$, there is a $g : \kappa \to V_\kappa$ that is $\{x\}$-$\mathcal{E}$-Laver at κ: let g be the constant function with value x.

We can also modify $\mathcal{E}_\kappa^{sc}$-Laver sequences so that for any $X \in V_\kappa$, we can obtain $g : \kappa \to V_\kappa$ which is $V \setminus X$-$\mathcal{E}_\kappa^{sc}$-Laver: Given such an X, and assuming κ is supercompact, let $z \in V_\kappa \setminus X$, and let f be an $\mathcal{E}_\kappa^{sc}$-Laver sequence. Let $B = \{\alpha < \kappa : f(\alpha) \in X\}$. Define g by

$$g(\alpha) = \begin{cases} z, & \text{if } \alpha \in B; \\ f(\alpha), & \text{otherwise.} \end{cases}$$

Given $x \in X$, note that if $i(g)(\kappa) = x$ for some $i \in \mathcal{E}_\kappa^{sc}$, then $\{\alpha < \kappa : g(\alpha) = x\} \in D$, where D is the normal ultrafilter derived from i, and this is impossible.

If $x \notin X$, we can find i such that $i(g)(\kappa) = x$ as follows: Let $i \in \mathcal{E}_\kappa^{sc}$ be such that $i(f)(\kappa) = x$ and let D be the normal ultrafilter derived from i. Note that if $B \in D$, then $x \in i(X) = X$, which is impossible. Thus f and g agree on a set in D. It follows that $i(g)(\kappa) = x$.

The next theorem generalizes this technique somewhat to obtain a $\mathcal{E}$-$(V \setminus V_\kappa)$-Laver sequence, and a $\mathcal{E}$-$(V \setminus V_{\kappa+1})$-Laver sequence:

1.3 Theorem. *Assume κ is supercompact. Then there are a $g, h : \kappa \to V_\kappa$ such that g is $\mathcal{E}_\kappa^{sc}$-$(V \setminus V_\kappa)$-Laver at κ and h is $\mathcal{E}_\kappa^{sc}$-$(V \setminus V_{\kappa+1})$-Laver at κ.*

Proof. Let $f : \kappa \to V_\kappa$ be a $\mathcal{E}_\kappa^{sc}$-Laver sequence. Let

$$B = \{\alpha < \kappa : \exists x \in V_\kappa\, [f(\alpha) = x]\};$$
$$C = \{\alpha < \kappa : \exists x \in V_{\kappa+1}\, [f(\alpha) = x \cap V_\alpha]\}.$$

Define g, h by

$$g(\alpha) = \begin{cases} \alpha + 1, & \text{if } \alpha \in B; \\ f(\alpha), & \text{otherwise;} \end{cases}$$
$$h(\alpha) = \begin{cases} \alpha + 2, & \text{if } \alpha \in C; \\ f(\alpha), & \text{otherwise.} \end{cases}$$

Let $i \in \mathcal{E}_\kappa^{sc}$ and let D be the normal ultrafilter over κ derived from i. Observe that

$$\forall x \in V_\kappa \Big[i(g)(\kappa) = x \Longleftrightarrow \{\alpha < \kappa : g(\alpha) = x\} \in D \Big];$$
$$\forall x \subseteq V_\kappa \Big[i(h)(\kappa) = x \Longleftrightarrow \{\alpha < \kappa : h(\alpha) = x \cap V_\alpha\} \in D \Big].$$

Thus, if $x \in V_\kappa$ and $i(g)(\kappa) = x$, it would follow that $B \in D$, whence $i(g)(\kappa) = \kappa + 1$, which is impossible. Hence, for all $x \in V_\kappa$, $i(g)(\kappa) \neq x$. Similar reasoning shows that $i(h)(\kappa) \neq x$ for all $x \in V_{\kappa+1}$.

On the other hand, suppose $x \notin V_\kappa$ and let $i \in \mathcal{E}_\kappa^{sc}$ be such that $i(f)(\kappa) = x$. Let D be the normal ultrafilter over κ derived from i. By definition of B, $B \notin D$, and so $i(g)(\kappa) = i(f)(\kappa) = x$. Similar reasoning shows that for all $x \notin V_{\kappa+1}$, if i is such that $i(f)(\kappa) = x$, then $i(h)(\kappa) = x$. ∎

To prove there are $\mathcal{E}$-V_κ-Laver sequences, we will need two lemmas:

1.4 Lemma. *Let $\langle D_\beta : \beta < \kappa \rangle$ be an enumeration of κ distinct normal ultrafilters over κ. Then there is a $\kappa \times \kappa$ matrix $[X_{\alpha\beta}]_{\alpha > \beta}$ satisfying:*

(1) *for each $\beta < \kappa$, $X_{\alpha\beta} \in D_\beta$;*
(2) *for all $\alpha, \beta_1, \beta_2 < \kappa$, if $\beta_1 \neq \beta_2$, $X_{\beta_1} \cap X_{\beta_2} = \emptyset$;*
(3) *for all $\beta, \alpha_1, \alpha_2 < \kappa$, if $\alpha_1 < \alpha_2$, then $X_{\alpha_1} \supseteq X_{\alpha_2}$;*

(4) *for all α, β with $\alpha \le \beta < \kappa$, $X_{\alpha\beta}$ is not defined.*

Proof. We build the αth row by induction on α. Let $\langle X_{\alpha\beta} : \beta < \alpha \rangle$ denote the αth row; we build the $\alpha + 1$st row $\langle X_{\alpha+1,\beta} : \beta < \alpha + 1 \rangle$ as follows:

For each $\beta < \alpha$, let

$$Z_\beta^{(\alpha)} \in D_\alpha \setminus D_\beta;$$
$$Z_\beta \in D_\beta \setminus D_\alpha;$$
$$X_{\alpha+1,\alpha} = \bigcap_{\beta<\alpha} (Z_\beta^{(\alpha)} \setminus Z_\beta);$$
$$X_{\alpha+1,\beta} = X_{\alpha\beta} \cap Z_\beta.$$

Noting that $Z_\beta^{(\alpha)} \setminus Z_\beta \in D_\alpha \setminus D_\beta$ and $X_{\alpha+1,\beta} \in D_\beta$, it is easy to verify that properties (1)-(4) are now satisfied for the first $\alpha + 1$ rows.

Finally, assume α is a limit and a matrix $[X_{\alpha'\beta}]_{\alpha>\alpha'>\beta}$ has been defined satisfying (1)-(4). Obtain the αth row $\langle X_{\alpha\beta} : \beta < \alpha \rangle$ by taking intersections: For each $\beta < \alpha$, put

$$X_{\alpha\beta} = \bigcap_{\beta<\alpha'<\alpha} X_{\alpha'\beta}.$$

It is routine to verify that (1)-(4) continue to hold for the matrix $[X_{\alpha'\beta}]_{\alpha\ge\alpha'>\beta}$. ∎

1.5 Lemma. *Let $\langle D_\beta : \beta < \kappa \rangle$ be an enumeration of κ distinct normal ultrafilters over κ. Then there is a sequence $\langle X_\beta : \beta < \kappa \rangle$ such that for all $\beta < \kappa$, $X_\beta \in D_\beta$ and if $\beta' < \beta < \kappa$, then $X_{\beta'} \cap X_\beta = \emptyset$.*

Proof. Let $[X_{\alpha\beta}]$ be as in Lemma 1.4. For each β, let $X_\beta = \triangle_{\beta<\alpha} X_{\alpha\beta} = \{\gamma < \kappa : \gamma \in \cap_{\beta<\alpha<\gamma} X_{\alpha\beta}\}$. By normality, $X_\beta \in D_\beta$ for each β. Suppose $\beta' < \beta$ and $\gamma < \kappa$. Observe that

$$\gamma \in X_{\beta'} \cap X_\beta \Longrightarrow \gamma > \beta + 1 \text{ and } \gamma \in X_{\beta+1,\beta'} \cap X_{\beta+1,\beta},$$

which is impossible. Thus $X_{\beta'} \cap X_\beta = \emptyset$, and we are done. ∎

1.6 Theorem. *Let $\langle D_\alpha : \alpha < \kappa \rangle$ be an enumeration of κ distinct normal ultrafilters over κ. Define $\mathcal{E} = \{i_{D_\alpha} \restriction V_\beta : \alpha < \kappa < \beta\}$. Then for each $Y \subset V_{\kappa+1}$ having cardinality $\le \kappa$, there is a $\mathcal{E}$-Y-Laver sequence at κ. In particular, there is a $\mathcal{E}$-V_κ-Laver sequence at κ.*

Proof. Write $Y = \{y_\alpha : \alpha < \kappa\}$; the enumeration may have repetitions. Use Lemma 1.5 to obtain $\langle X_\alpha : \alpha < \kappa \rangle$ so that $X_\alpha \in D_\alpha$, $X_\alpha \cap X_\beta = \emptyset$ whenever $\beta \ne \alpha$, and $\cup_{\alpha<\kappa} X_\alpha = \kappa$. Define $f : \kappa \to V_\kappa$ by putting, for each $\beta \in X_\alpha$,

$$f(\beta) = y_\alpha \cap V_\beta.$$

We prove that f is $\mathcal{E}$-Y-Laver. Note that for each α, each $y \subset V_\kappa$ is represented in the ultrapower by D_α by the function $u_y : \kappa \to V : \alpha \mapsto y \cap V_\alpha$. Also observe that for each α,

$$\begin{aligned} X_\alpha \in D_\alpha &\Longrightarrow \{\beta : f(\beta) = y_\alpha \cap V_\beta\} \in D_\alpha \\ &\Longrightarrow i_{D_\alpha}(f)(\kappa) = y_\alpha. \end{aligned}$$

Thus,

$$Y = \{i(f)(\kappa) : i \in \mathcal{E}\},$$

and we are done. ■

It turns out that the hypothesis "there are at least κ normal ultrafilters over κ" is exactly what is needed for Theorem 1.6:

1.7 Proposition. *Suppose κ is an infinite cardinal. Then the following are equivalent:*

(1) *there are at least κ normal ultrafilters over κ;*

(2) *there is a class $\mathcal{E}$ of embeddings that admits a $\mathcal{E}$-Y-Laver sequence for each $Y \subset V_{\kappa+1}$ of cardinality $\leq \kappa$.*

Proof. Theorem 1.6 proves one direction. To prove (2) ⇒ (1), we show that if there is a class $\mathcal{E}$ and a function $g : \kappa \to V_\kappa$ that is $\mathcal{E}$-X-Laver for some $X \in V_\kappa$ of cardinality $\lambda < \kappa$, then there exist at least λ normal ultrafilters over κ. For each $x \in X$, let $i \in \mathcal{E}$ be such that $i(g)(\kappa) = x$ and let D_x be the normal ultrafilter derived from i. If $x \neq y$, it is easy to see that $D_x \neq D_y$ (since, for example, $\{\alpha < \kappa : g(\alpha) = x\} \in D_x \setminus D_y$), and we are done.■

The next theorem shows how to obtain an X-Laver sequence for some X that properly contains all sets in $V_{\kappa+1}$ and the constructible sets in V_{2^κ}, but no sets of rank $> 2^{\kappa^+}$. We do this by applying the canonical construction to the class of embeddings determined by a measurable cardinal κ (assuming the existence of a $P^2(\kappa)$ embedding; see [23] for related information).

Let $\theta_{ms}(i, \kappa, \beta, M)$ be the following suitable formula:

$$\begin{aligned} \exists U\, [&\text{"}M \text{ is transitive"} \wedge \text{"}i : V_\beta \to M \text{ is elementary with critical point } \kappa\text{"} \\ &\wedge \text{"}U \text{ is a normal ultrafilter over } \kappa\text{"} \wedge i = i_U \restriction V_\beta]. \end{aligned}$$

1.8 Theorem. *Suppose there is an elementary embedding $j : V \to N$ such that $cp(j) = \kappa$ and $P(P(\kappa)) \in N$. Then there exists an $X \in V_{(2^\kappa)^+} \setminus V_{2^\kappa}$ such that*

(1) *$X \supseteq V_{\kappa+1}$;*

(2) *$X \supseteq V_{2^\kappa} \cap L$;*

(3) *$\mathcal{E}_\kappa^{ms}$ admits an $\mathcal{E}_\kappa^{ms}$-X-Laver sequence at κ.*

Proof. We begin with $j : V \to N$ as in the hypothesis; note that N contains all normal ultrafilters U over κ and therefore correctly computes $i_U(\kappa)$ and $i_U(V_\kappa)$ for each such U. Let

$$\lambda_\kappa = \min\bigl(\{(2^{\kappa^+})^M : \exists i \in \mathcal{E}^{ms}_\kappa \,[\text{codomain}(i) = M]\}\bigr).$$

Clearly, $2^\kappa < \lambda_\kappa < (2^\kappa)^+$. Set

$$X_\kappa = \bigcap\{V^M_{\lambda_\kappa} : \exists i \in \mathcal{E}^{ms}_\kappa \,[\text{codomain}(i) = M]\}.$$

Clearly, $X_\kappa \in V_{(2^\kappa)^+} \setminus V_{2^\kappa}$; $X_\kappa \in N$; $X_\kappa \supseteq V_{\kappa+1}$; and $X_\kappa \supseteq V_{2^\kappa} \cap L$. Also, notice that for every $i : V_\beta \to M \in \mathcal{E}^{ms}_\kappa$, $X_\kappa \subseteq M$.

Let $Y = V_{(2^\kappa)^+}$. To prove the theorem, we will construct $g : \kappa \to V_\kappa$ that is both $\mathcal{E}^{ms}_\kappa$-X^+_κ-Laver and $\mathcal{E}^{ms}_\kappa$-Y^--Laver. We will only need to concern ourselves with the first of these two conditions because, as is easily verified, *every* $h : \kappa \to V_\kappa$ is $\mathcal{E}^{ms}_\kappa$-Y^--Laver.

Let R be a well-ordering of V_κ. Define $g : \kappa \to V_\kappa$ in the structure $\langle V_\kappa, \in, R\rangle$ by

$$g(\alpha) = \begin{cases} \emptyset, & \text{if } \alpha \text{ is a cardinal and } g \restriction \alpha \text{ is a } \mathcal{E}^{ms}_\alpha\text{-}X^+_\alpha\text{-Laver sequence at } \alpha; \\ x \in X_\alpha, & \text{if } \alpha \text{ is a cardinal and } g \restriction \alpha \text{ is not } \mathcal{E}^{ms}_\alpha\text{-}X^+_\alpha\text{-Laver at } \alpha, \\ & \text{where } x \text{ is } R\text{-least such that } \psi(g \restriction \alpha, x); \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\psi(h, x)$ is the following formula:

"there exists a cardinal α with $h : \alpha \to V_\alpha$, and for all $i \in \mathcal{E}^{ms}_\alpha$, $i(h)(\alpha) \neq x$."

Assume that g is not $\mathcal{E}^{ms}_\kappa$-X^+_κ-Laver at κ, and let $x \in X_\kappa$ be a witness. Let D be the normal ultrafilter over κ derived from j.

If $\{\alpha < \kappa : \alpha \text{ is a cardinal and } g \restriction \alpha \text{ is a } \mathcal{E}^{ms}_\alpha\text{-}X^+_\alpha\text{-Laver sequence at } \alpha\} \in D$, then

$$N \models \langle V_{j(\kappa)}, \in, j(R)\rangle \models \text{ "}g \text{ is an } \mathcal{E}^{ms}_\kappa\text{-}X^+_\kappa\text{-Laver sequence at } \kappa\text{"}.$$

So, in $\langle V_{j(\kappa)}, \in, j(R)\rangle^N$, there is a normal ultrafilter over κ such that $i_U(g)(\kappa) = x$, which is impossible.

Thus, $\{\alpha < \kappa : \psi(g \restriction \alpha, g(\alpha))\} \in D$. Hence, if $x = j(g)(\kappa)$, then for all normal ultrafilters U over κ,

$$i_U(g)(\kappa) \neq x. \tag{1.1}$$

Let $i_D : V \to M$ be the canonical embedding defined from D, and let $k : M \to N$ be the usual embedding such that $k \circ i_D = j$. Since $j(g)(\kappa) \in X_\kappa$, $j(g)(\kappa) \in M$. But $rank(x) < \text{cp}(k) = (2^{\kappa^+})^M$ (see [16]); thus $k(x) = x$ and $i_D(g)(\kappa) = x$, contradicting (1.1). This completes the proof of the theorem. ■

To conclude the discussion, we mention one of Barbanel's results that can be cast in the present setting:

1.9 Proposition (Barbanel [2 ,p.16])**.** *For each 2-huge embedding $j : V \to N$, there is an $\mathcal{E}^{sh}_\kappa$-X_j-Laver sequence at κ, where $X_j = \{x \in V_{j(\kappa)} : |x| \geq \kappa\}$.* ■

References

[2] Barbanel, J., *Making the hugeness of κ resurrectable after κ-directed forcing,* **Fundamenta Mathematicae,** 136, 1991, pp. 9-24.

[3] ———, *Flipping properties and huge cardinals,* **Fundamenta Mathematicae,** 13, 1989, pp. 161- 188.

[4] Barbanel,J., DiPrisco, C., Tan,I., *Many-times huge and superhuge cardinals*, Journal of Symbolic Logic, vol. 49, no. 1, 1984, pp. 112-122.

[23] Mitchell, W., *Sets constructible from sequences of ultrafilters,* **Journal of Symbolic Logic**, Vol. 39, 1, 1964, pp. 56-66.