FORCING OVER THE LANGUAGE

$\{\in, \mathbf{j}\}$

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I. Introduction

1. Kunen's Theorem: Formalizing in KM Set Theory, there is no elementary embedding $j : V \to V$.

2. Replace KM Set Theory with axiomatic approach in the language $\{\in, \mathbf{j}\}$.
   Axioms:
   - ZFC (for $\in$-formulas)
   - Elementarity schema (asserts $\mathbf{j}$ is elementary)
   - Critical Point axiom (asserts there is a least ordinal moved by $\mathbf{j}$)
   - Separation$_{\mathbf{j}}$ schema (asserts usual Separation holds for $\mathbf{j}$-formulas)

3. The **Wholeness Axiom** (WA) = Elementarity+Critical Point+ Separation$_{\mathbf{j}}$

   WA_0 = Elementarity+Critical Point+ Σ_0-Separation$_{\mathbf{j}}$

4. Standard models of ZFC+WA (ZFC+ WA_0) are of the form $\langle M, \in, j\rangle$ where $j : M \to M$ is elementary, and j-fmlas satisfy Separation (Σ_0-Separation).

$$\kappa_0 = \mathrm{crit}(j) \qquad \kappa_{n+1} = j(\kappa_n)$$

$$V^M_{\kappa_0} \prec V^M_{\kappa_1} \prec V^M_{\kappa_2} \prec \ldots \prec M.$$

5. Strength of these theories:
   - ZFC + $\mathrm{WA}_0 \vdash \kappa$ is super-n-huge for "every" n
   - if $i : V_\lambda \to V_\lambda$ is an I_3 embedding,

$$\langle V_\lambda, \in, i \rangle \models \mathrm{ZFC} + \mathrm{WA}$$

   - so:

$$\begin{aligned} \mathrm{Con}(I_3) &\implies \mathrm{Con}(\mathrm{WA}) \\ &\implies \mathrm{Con}(\mathrm{WA}_0) \\ &\implies \mathrm{Con}(\text{super-}n\text{-huge}) \end{aligned}$$

6. Natural Question:
   - Is GCH or $V = HOD$ consistent with $\mathrm{ZFC} + \mathrm{WA}$ (or $\mathrm{ZFC} + \mathrm{WA}_0$)?

7. Fundamental Problem:

   Starting from a model $\langle M, \in, j\rangle$ of ZFC + WA (or ZFC + WA_0), after forcing with a **typical** iteration, can j be extended to $\hat{j} : M[G] \to M[G]$ so that $\langle M[G], \in, \hat{j}\rangle$ is also a model of ZFC + WA (or ZFC + WA_0)?

8. The typical iteration P_λ: Suppose inaccessibles are unbounded in λ. Let

   $$in_\lambda : \lambda \to \lambda$$
   $$: \alpha \mapsto \text{least inaccessible above } \alpha$$

   - reverse Easton iteration
   - adequate ($|P_\alpha| < in(\alpha)$; $\Vdash_\alpha \dot{Q}_\alpha$ is α^+-directed closed)

9. Fact: If P_λ is a λ-stage adequate reverse Easton iteration and inaccessibles are unbounded in λ, then for every $\alpha < \lambda$,

$$P_{\alpha\lambda} \text{ is } \alpha^+\text{-directed closed.}$$

II. A Coarse Solution

1. Start with an I_1 embedding $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ inside a countable transitive model V of ZFC.

2. Let $P_\lambda \subseteq V_{\lambda+1}$ be adequate reverse Easton iteration.

3. Lift j to $\hat{j} : V[G]_{\lambda+1} \longrightarrow V[G]_{\lambda+1}$ by

$$\hat{j}(\sigma_G) = (j(\sigma))_G$$

(for a particular choice of G)

4. Result:

$$\langle V[G]_\lambda, \in, \hat{j} \restriction V[G]_\lambda \rangle \models \mathrm{ZFC} + \mathrm{WA}$$

5. Why it works: Main issue is to verify that $\hat{j}$ is well-defined.

6. Verifying "well-defined": Show

   if $p \Vdash \sigma = \tau$ and $p \in G$

   then $j(p) \Vdash j(\sigma) = j(\tau)$ and $j(p) \in G$

7. Requires:
   - $\Vdash$ is definable in $V_{\lambda+1}$ (holds because $P_\lambda \subseteq V_{\lambda+1}$)
   - existence of "master condition" q — i.e. q satisfies:

     $$\forall p \in G_{\kappa_n}\ q \restriction \kappa_{n+1} \leq j(p).$$

     (holds because of adequacy of P_λ)
   - Separation$_{\mathbf{j}}$ automatically holds

III. A Refinement

1. Start with a transitive model $\langle M, \in, j_0 \rangle$ of ZFC + WA (or ZFC + WA_0).

2. Try taking limit of **models** instead of iterating forcing

3. Inductively build

$$M \subseteq M[G_{\kappa_0}] \subseteq M[G_{\kappa_1}] \subseteq \dots;$$
$$N = \bigcup_{n \in \omega} M[G_{\kappa_n}],$$

   and embeddings $j_1, j_2, \dots$ so that

$$M[G_{\kappa_0}] \xrightarrow{j_1} M[G_{\kappa_1}] \xrightarrow{j_2} M[G_{\kappa_2}] \xrightarrow{j_3} \dots$$

4. The Induction:

$$\left(P_{\kappa_0} \subseteq V_{\kappa_0}\right)^M$$

let G_{κ_0} be P_{κ_0}-generic

FIRST STEP:

- $\left(P_{\kappa_1} = j(P_{\kappa_0}) \sim P_{\kappa_0} * P_{\kappa_0\kappa_1}\right)^M$
- let $G_{\kappa_0\kappa_1}$ be $P_{\kappa_0\kappa_1}$-generic
- $G_{\kappa_1} = G_{\kappa_0} * G_{\kappa_0\kappa_1}$
- lift j_0 to $j_1 : M[G_{\kappa_0}] \to M[G_{\kappa_1}]$ by

$$j_1(\sigma_{G_{\kappa_0}}) = \big(j_0(\sigma)\big)_{G_{\kappa_1}}$$

- well-defined since $p \in G_{\kappa_0} \Rightarrow p \leq j(p)$

SECOND STEP:

- $\left(P_{\kappa_2} = j(P_{\kappa_1}) \sim P_{\kappa_1} * P_{\kappa_1\kappa_2}\right)^M$
- pick $G_{\kappa_1\kappa_2}$ $P_{\kappa_1\kappa_2}$-generic

  — $G_{\kappa_1\kappa_2}$ contains master cond'n
- $G_{\kappa_2} = G_{\kappa_1} * G_{\kappa_1\kappa_2}$
- lift j_1 to $j_2 : M[G_{\kappa_1}] \to M[G_{\kappa_2}]$ by

  $j_2(\sigma_{G_{\kappa_1}}) = \left(j_0(\sigma)\right)_{G_{\kappa_2}}$
- well-defined since $p \in G_{\kappa_1} \Rightarrow j(p) \in G_{\kappa_2}$

  (master condition argument)

5. As usual

$$j_1 \restriction (V_{\kappa_0})^{M[G_{\kappa_0}]} \text{ is the identity}$$

So

$$(V_{\kappa_0})^{M[G_{\kappa_0}]} \prec (V_{\kappa_1})^{M[G_{\kappa_1}]}$$

and we get an elementary chain

$$(V_{\kappa_0})^{M[G_{\kappa_0}]} \prec (V_{\kappa_1})^{M[G_{\kappa_1}]} \prec \ldots \prec N \models \text{ZFC}.$$

6. Show that restrictions of the j_n are elements of their next higher ranks, and piece these together to obtain elementary embedding $k : N \to N$.

7. We no longer may conclude that final model satisfies Separation$_{\mathbf{j}}$. But since restriction of k to each rank belongs to N, Σ_0-Separation$_{\mathbf{j}}$ does hold.

8. We have

$$\langle N, \in, k \rangle \models \mathrm{ZFC} + \mathrm{WA}_0$$

9. Question. Can a variation of this argument prove consistency of $\mathrm{ZFC} + \mathrm{WA}$?

10. A review of the argument shows that, to begin, we only need $\langle M, \in, j \rangle$ to be a transitive model of $\mathrm{ZFC} + \mathrm{WA}_0$.

IV. Do We Have A Relative Consistency Result?

1. When forcing over ZFC, we may assume the ground model is transitive. Reason: For any finite number of ZFC axioms $\phi_1, \ldots, \phi_n$, ZFC proves there is a ctm of $\phi_1 \wedge \ldots \wedge \phi_n$. Argument uses Reflection, hence Replacement.

2. In starting with a model of ZFC+WA_0, in the language $\{\in, \mathbf{j}\}$, can't assume Replacement for **j**-formulas. Also, the theory ZFC + WA_0 is finitely axiomatizable. Therefore: cannot assume transitive ground model.

3. **Question.** Starting with a possibly nonstandard model $\langle M, E, j \rangle$ of ZFC + WA_0, how to proceed with forcing?

   - Work in Boolean-valued model M^B? But $[\![\phi]\!]_B$ is not well-defined when ϕ is a **j**-formula. Also, how to handle iteration of **models** used above instead of forcing iterations?

V. A Solution

1. Develop forcing for nonstandard models of ZFC — given M, obtain M^B and collapse with a generic ultrafilter $M_U = M^B/U$.

2. Because E_U may be nonwellfounded, usual collapsing function (giving $M[U]$) is not available.

3. Usual results about forcing extensions now hold "up to canonical isomorphism."

4. Now attempt to copy earlier proof:

- Given a model $\langle M, E, j_0 \rangle$ of ZFC+ WA_0, force over $\langle M, E \rangle$ to obtain $\langle M_{U_0}, E_{U_0} \rangle$
- Force again to get $\langle M_{U_1}, E_{U_1} \rangle$
- "Lift" j_0 to $j_1 : M_{U_0} \to M_{U_1}$
- Continue induction over the **real** natural numbers (rather than using the possibly nonstandard integers of any of the models)
- Obtain model N as a direct limit of models rather than union of elementary chain.

- N may not be a model of ZFC; however, images in N of diagonal ranks form an elementary chain whose union $\tilde{N}$ is desired model.

5. Some consistency results using this method:

   A. $\mathrm{Con}(\mathrm{ZFC}+\mathrm{WA}_0) \implies \mathrm{Con}(\mathrm{ZFC}+\mathrm{WA}_0+\mathrm{GCH})$

   B. $\mathrm{Con}(\mathrm{ZFC}+\mathrm{WA}_0) \implies \mathrm{Con}(\mathrm{ZFC}+\mathrm{WA}_0+V=\mathrm{HOD})$

VI. Other Results

1. Small forcing cannot destroy WA_0 or WA.

2. Set forcing from a model of ZFC + WA cannot destroy WA_0.