Has Modern Mathematics Finally Understood The Infinite?

The Good News and the Bad News

1. Overview

- Prior to the mid-19th century, infinite sets were not studied in mathematics

- The work of Cantor led to the acceptance of infinite sets and to a rigorous theory of sets

- The new set theory led to the discovery of bizarre infinities (large cardinals) that could not be accounted for by the new set theory

- How can large cardinals be accounted for?

2. The Need For Infinity In Mathematics

- Prior to the 19th century, the natural numbers were not considered to form a **set** (or a completed collection), though it was understood that they “go on forever”.

- The need to secure the foundation of calculus led to the acceptance of infinite sets.

3. Infinite Sizes

- Sets A and B **have the same size** if there is a 1-1 correspondence between A and B.

- If a set A has the same size as $\mathbb{N}$ (the set of natural numbers), then A is said to be countably infinite

- Some countably infinite sets: $\mathbb{N}, \{\text{integers}\}, \{\text{rationals}\}$, $\mathbb{N} \times \mathbb{N}$

- An infinite set that is not countably infinite is called **uncountable**

- $A < B$ (A is **smaller than** B) if there is a 1-1 correspondence between A and a subset of B but no 1-1 correspondence between A and all of B.

- **Cantor's Theorem**. For any set X, if $\mathcal{P}(X)$ denotes the set of all subsets of X, then $X < \mathcal{P}(X)$. Therefore, there is an endless hierarchy of infinite sizes of sets.

4. The Need For Axiomatic Set Theory

- **Russell's Paradox**

- **Sets as "universal currency" of mathematics**

  1. Every mathematical object can be represented as a set (points, lines, spheres, functions, numbers)

  2. Example: ordered pairs

  3. Example: the natural numbers $0, 1, 2, 3, \ldots$

- **The universe V as a prototype** Objective: Create a universe for mathematics rich enough to include all mathematical objects (formally) but restrictive enough to exclude paradoxical collections. Use this universe to motivate the development of precise axioms about sets. Build V in stages.

5. ZFC

- The axioms of set theory form a list of properties that sets "ought to" have

- The Axioms

  Pairing For any sets A and B, there is another set $\{A, B\}$.

  Infinity There is an infinite set.

  Power Set For any set A, the collection $\mathcal{P}(A)$ of all subsets of A is also a set.

  Union If X is a set of sets, then $\bigcup X$ is a set.

  Extensionality A and B are equal if and only if A and B have the same elements.

  Choice Every set can be well-ordered.

  Replacement For any set A and any rule that associates with each element x of A a set Y_x, there is a set B that consists precisely of all Y_x, where $x \in A$.

- **Good News**: ZFC is successful as a foundation for Cantor's theory of infinite sets and for all of standard mathematics.

6. Infinite Ordinals And Infinite Cardinals

- Counting past the natural numbers: $0 \in 1 \in 2 \in \ldots \in$ n $\in \ldots \omega$
- Successor ordinals and limit ordinals.
- The first uncountable ordinal: ω_1.
- An infinite ordinal is called an **infinite cardinal** if it cannot be put in 1-1 correspondence with any of its predecessors.
- Some infinite cardinals:

$$\omega, \ldots, \omega_1, \ldots, \omega_2, \ldots, \omega_\omega \ldots$$

- Successor cardinals and limit cardinals
- **Theorem**: Every infinite set in the universe can be put in 1-1 correspondence with exactly one infinite cardinal.
- **Cardinality of a set**. The **cardinality** of a set A, denoted $|A|$, is the unique cardinal number with which A can be put in 1-1 correspondence. Examples: $|\{1, 5\}| = 2$, $|\mathbb{N}| = \omega$.

- The ordinals are used to enumerate long lists. For example, the beth numbers:

$$
\begin{aligned}
\beth_0 &= |\mathbb{N}| \\
\beth_1 &= |\mathcal{P}(\mathbb{N})| \\
\beth_2 &= |\mathcal{P}(\mathcal{P}(\mathbb{N}))| \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot \\
\beth_\alpha & \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot \\
\cdot & \quad \cdot
\end{aligned}
$$

- The ordinals are also used to construct, using the axioms of ZFC, every stage of the universe V:

$$V_0, V_1, V_2, \ldots, V_\omega, \ldots, V_\alpha, \ldots$$

- ZFC does **not** allow us to complete all the stages and form the universe V — this requires a leap of faith

7. Consistency of ZFC.

- An **axiomatic theory** is a theory T (a set of mathematical assertions) obtained from a *computable* set A of axioms.

- Consistency and the statement $\mathrm{Con}(T)$

- (*Gödel's Completeness Theorem*). To prove consistency of a collection of axioms, it suffices to obtain a model of all the axioms.

- **Important Question:** Is ZFC consistent?

- If you believe that V exists, then ZFC must be consistent. However,

- **Gödel's Second Incompleteness Theorem**. If ZFC is consistent, then there is no proof from ZFC that ZFC is consistent. Moreover, if T is any consistent axiomatic theory that includes ZFC, T cannot prove $\mathrm{Con}(T)$.

- Therefore, ZFC alone is not enough to build a model of ZFC — in particular, V can't be constructed from within ZFC alone.

8. **Large Cardinals**

- Large cardinals are cardinal numbers that have certain combinations of properties that make them very potent (but not inconsistent).

- Example

  1. ω as an example of regularity
  2. A cardinal κ is regular if κ cannot be "reached" in fewer than κ steps
  3. ω_ω is an example of an uncountable limit cardinal
  4. A cardinal is **weakly inaccessible** if it is an uncountable regular limit cardinal.
  5. An example of a beth fixed point
  6. A cardinal that is both regular and a beth fixed point is **inaccessible** (first beth fixed point is not inaccessible).

- **Bad News:** If κ is an inaccessible cardinal, then V_κ is a model of ZFC. Therefore:

    · ZFC cannot prove the existence of an inaccessible

    · ZFC cannot even prove the *consistency* of an inaccessible

    · Large cardinals lie outside the unifying framework provided by ZFC

- **Some Large Cardinals**. Large cardinals are listed from weakest to strongest:

  weakly inaccessible

  inaccessible

  measurable

  strong

  Woodin

  superstrong

  supercompact

  huge

  superhuge

  super-n-huge for every n

9. Large Cardinals In Mathematics.

- The Measure Problem – measurable cardinals
- The Normal Moore Space Conjecture – strongly compact cardinals
- Axiom of Determinacy – infinitely many Woodin cardinals
- Existence of an ω_2-saturated ideal on ω_1 — approximately a superstrong
- The theory ZFC + Martin's Maximum — approximately a supercompact
- Woodin's Program To Settle The Continuum Hypothesis — starting point: a proper class of Woodin cardinals

10. The Problem of Large Cardinals.

- **The Problem:** Which large cardinals exist and is there a natural axiomatic foundation for these large cardinals?

- Large cardinals are defined in many different contexts. Are they just a bunch of unrelated assertions, or is there a unifying theme? Consider the origins of ZFC.

- Any axiom that is to be added to ZFC must meet the standard of **naturalness**

- By now, most set theorists believe that there is nothing suspect about any of the well-known large cardinals — still, no generally agreed upon framework for deriving all large cardinals has emerged.

- A unifying theme: elementary embeddings of the universe

$$\frac{\text{sets}}{\text{mathematics}} = \frac{\text{elementary embeddings of the universe}}{\text{large cardinals}}$$

11. Elementary Embeddings Of The Universe

- Structure-preserving maps in mathematics:
  - · linear transformations
  - · group homomorphisms
  - · continuous functions

- Elementary embeddings $j : V \to M$:
  1. j is a map from the universe V to a (possibly) smaller universe M
  2. j preserves **all** (first-order) relationships that hold among sets in V — a kind of super-morphism
  3. j is not the identity map — there is some set x for which $j(x) \neq x$.

- The stronger large cardinals are defined in terms of elementary embeddings $j : V \to M$

- As the strength of the large cardinal increases, M starts to resemble V more and more closely.

- The "limit" of (virtually) all large cardinal notions:

  There is an elementary embedding $j : V \to V$.

- **Good News:** This assertion is the simplest possible form of an elementary embedding, yet it is the strongest — perhaps a candidate for a "natural" axiom to add to ZFC.

- **Question:** Is there an elementary embedding $j : V \to V$?

- **Bad News: Theorem** (Kunen, 1970) There is no elementary embedding $j : V \to V$

- **Good News:** Kunen's proof involved certain assumptions about j that do not need to be made. Eliminating the one assumption that leads to inconsistency, we have

  The Wholeness Axiom: There is an elementary embedding $j : V \to V$

- **Fact** The Wholeness Axiom is consistent with ZFC plus very strong large cardinals (that are not known to be inconsistent).

- **Theorem** From ZFC + Wholeness Axiom, all known large cardinals can be derived up through super-n-huge for every n.