When Are Closed and Bounded Sets Compact?

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Abstract. By the Heine-Borel Theorem, closed and bounded subsets in $\mathbb{R}^n$ are compact, but this fact does not hold in metric spaces—even complete metric spaces—generally. We give necessary and sufficient conditions on a metric space for closed and bounded subsets to be compact.

Let us call a metric space *CBC* if each of its closed and bounded subspaces is compact. By the Heine-Borel theorem, $\mathbf{R}^n$ is CBC. But there are many metric spaces, even complete metric spaces, that fail to have this property. The simplest example is the set $\mathbf{N}$ of natural numbers with the discrete metric: Notice that any infinite subset is closed, has diameter ≤ 1, but is not compact (since it contains no limit points).

This particular space reveals an exotic feature that some metrics have, which will suggest a condition that will characterize CBC spaces. The space has a decreasing chain of closed sets whose diameters converge to a $\rho > 0$ but whose intersection is empty; in particular, if we let $F_n = \mathbf{N} - \{0, 1, \ldots, n-1\}$, then the diameter $\delta(F_n)$ of each F_n is 1, but $\bigcap_{n \geq 0} F_n = \emptyset$. On the other hand, because the space is a *complete* metric space, whenever we have a decreasing chain of nonempty closed sets with diameters tending to 0, the intersection is *nonempty*. (In this example, this is because the only set with diameter 0 is the empty set.) Notice that this phenomenon could not occur in $\mathbf{R}^n$: Whenever we have decreasing F_n with diameters converging to some $\rho > 0$, we can find $G_n \subseteq F_n$ that are also closed and decreasing and whose diameters tend to 0; by completeness, the intersection is nonempty.

Our candidate, therefore, for a characterizing property of CBC spaces is the following:

Q: For all decreasing chains $F_0 \supseteq F_1 \supseteq \ldots$ of nonempty closed sets for which $\delta(F_n) \to \rho \geq 0$, we have $\bigcap_{n \geq 0} F_n \neq \emptyset$.

Notice that if a metric space has property **Q**, it must be complete, and that compact metric spaces necessarily have property **Q**. (To see this, one can use the characterization of compactness that states that whenever a nonempty collection of closed sets has the finite intersection property, its intersection is nonempty (e.g. [1, 5.1]).)

We prove that property **Q** characterizes CBC spaces:

Theorem 1 (*Equivalence of CBC and* **Q**). *A metric space is CBC iff it satisfies property* **Q**.

Proof. Suppose the metric space (X, d) is CBC. Let $F_0 \supseteq F_1 \supseteq \ldots$ be nonempty closed sets and let $\rho \geq 0$ be such that $\delta(F_n) \to \rho$. Let n be such that $\delta(F_n) < \infty$. Then F_n is a compact subspace. Now we can apply the compactness property to the collection $\{F_k : k > n\}$ to conclude that $\bigcap_{k \geq 0} F_k = \bigcap_{k>n} F_k \neq \emptyset$.

Conversely, assume (X, d) satisfies property **Q.** We will use the fact that in a metric space, compactness is equivalent to the property that every infinite subset has a limit point (see [1, 5.5]). First we observe that each infinite and bounded subset of X has a limit point: Assume $A \subseteq X$ is infinite and bounded with no limit point; without loss of generality, assume A is countable, $A = \{a_0, a_1, \ldots\}$. For $i = 0, 1, \ldots$, let $A_i = A - \{a_0, a_1, \ldots, a_{i-1}\}$. The A_i are closed and decreasing. Since $A_i \subseteq A$ and $\delta(A) < \infty$, it follows that for some $\rho \geq 0$, $\lim_{i\to\infty} \delta(A_i) = \rho$. By property **Q**, $\bigcap_{i \geq 0} A_i \neq \emptyset$, which is impossible. Thus A does have a limit point.

To complete the second half of the proof, suppose $F \subseteq X$ is closed and bounded. If A is an infinite subset of F, since A is bounded, A has limit point in F. It follows that F is compact. ∎

As an application (pointed out to me by a colleague), we can show that the real Hilbert space ℓ^2 with the metric $d(\bar{x}, \bar{y}) = [\sum_{i=1}^{\infty}(x_i - y_i)^2]^{1/2}$ is not CBC, using reasoning similar to our simple example of **N**with the discrete metric. Given any $\rho > 0$ we can construct a decreasing sequence $F_0 \supseteq F_1 \supseteq \ldots$ of nonempty closed sets such that $\delta(F_n) \to \rho$ and $\bigcap_{n \geq 0} F_n = \emptyset$ as follows: Setting $\bar{x}_1 = \langle \rho/\sqrt{2}, 0, 0, 0, \ldots \rangle$, $\bar{x}_2 = \langle 0, \rho/\sqrt{2}, 0, 0, \ldots \rangle$, $\bar{x}_3 = \langle 0, 0, \rho/\sqrt{2}, 0, \ldots \rangle$, and so forth, let $F_n = \{\bar{x}_i : i > n\}$. Now the F_n form a decreasing chain of closed sets for which $\delta(F_n) \to \rho$ and $\bigcap_{n \geq 0} F_n = \emptyset$. By the Theorem, ℓ^2 is not CBC.

References

[1] J. Kelley. *General topology.* Springer-Verlag, 1955.