Lifting Elementary Embeddings $j : V_\lambda \to V_\lambda$

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Abstract. We describe a fairly general procedure for preserving I_3 embeddings $j : V_\lambda \to V_\lambda$ via λ-stage reverse Easton iterated forcings. We use this method to prove that, assuming the consistency of an I_3 embedding, $V = \text{HOD}$ is consistent with the theory ZFC + WA where WA is an axiom schema in the language $\{\in, \mathbf{j}\}$ asserting a strong but not inconsistent form of "there is an elementary embedding $V \to V$". This improves upon an earlier result in which consistency was established assuming an I_1 embedding.

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§1. Introduction

The axiom $\mathrm{I}_3(\kappa)$ was introduced by Kunen in [Ku2] and asserts that there is a limit ordinal $\lambda > \kappa$ and an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point κ. In this paper, we are concerned with forcing extensions that preserve $\mathrm{I}_3(\kappa)$; in particular, those extensions in which the new embedding extends, or "lifts", the ground model embedding.

As is typically the case in forcing with large cardinals, the axiom $\exists\kappa\, \mathrm{I}_3(\kappa)$ is immune to small forcing. First, when the notion of forcing P is an element of V_κ (in the ground model V), j can always be lifted to $\hat{j} : V[G]_\lambda \to V[G]_\lambda$ in the standard way, by defining $\hat{j}(\dot{x}_G) = j(\dot{x})_G$. When the notion of forcing P is an element of $V_\lambda \setminus V_\kappa$, this standard way of lifting the embedding fails [Co1], but the axiom $\exists\kappa\, \mathrm{I}_3(\kappa)$ is still preserved: Let n be such that $P \in V_{\kappa_n}$, where $\kappa_n = j^n(\kappa)$. One can obtain another elementary embedding $k : V_\lambda \to V_\lambda$ definable from j and having critical point κ_n (see for example [Co2, Proposition 8.11]). But now, as before, k can be lifted in the standard way since $P \in V_{\kappa_n}$. Therefore, in the extension, the axiom $\exists\kappa\, \mathrm{I}_3(\kappa)$ continues to hold, though we have not in this case lifted the original embedding. Arguments of this kind are studied and elaborated in a more general context in [Co1].

In this paper our concern is with forcing notions $P \subset V_\lambda$ that do not belong to V_λ. We will use a forcing of this kind to establish GCH or $V = \mathrm{HOD}$ inside $V[G]_\lambda$ in a model in which we also lift the ground model embedding to $V[G]_\lambda \to V[G]_\lambda$. We will use this general technique to obtain an improved result about the consistency of $V = \mathrm{HOD}$ with an axiom WA, studied in previous work.

To state our target theorem, we first recall that Kunen [Ku2] showed there is no nontrivial elementary embedding from the universe to itself (where the argument is formalized in Kelley-Morse or Gödel-Bernays set theory). In [Co2] and [Co5], we began a finer axiomatic analysis of elementary embeddings from a model of ZFC to itself by studying such an embedding as an interpretation of a function symbol $\mathbf{j}$ in the language $\{\in, \mathbf{j}\}$. For this analysis, we used as axioms the usual ZFC axioms (for $\in$-formulas), an axiom schema Elementarity asserting that $\mathbf{j}$ is an elementary embedding, and an axiom Critical Point asserting that a least ordinal is moved by $\mathbf{j}$. In [Co2], these axioms collectively were given the name the *Basic Theory of Elementary Embeddings*, or BTEE. We showed that ZFC + BTEE has consistency strength somewhat less than the existence of $0^\#$. By adding to BTEE all instances of Separation for $\mathbf{j}$-formulas (formulas in which there is an occurrence of the symbol $\mathbf{j}$) — we call this schema $\mathrm{Separation}_{\mathbf{j}}$ — we obtain a much stronger theory. The axioms of BTEE together with those of $\mathrm{Separation}_{\mathbf{j}}$ were named collectively the *Wholeness Axiom* or WA in [Co2] and [Co5]. The axioms of WA are listed below:

The Wholeness Axiom

$(1)_\phi$ (*Elementarity Schema for $\in$-formulas*). Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula,

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right);$$

(2) (*Critical Point*). "There is a least ordinal moved by **j**".

$(3)_\phi$ (*Separation Schema for* **j**-*formulas*). Each instance of the usual Separation schema involving ϕ is an axiom (where ϕ is a **j**-formula).

The theory ZFC + WA is known [Co5] to have consistency strength strictly between that of I_3 (which asserts the existence of a nontrivial elementary embedding $V_\lambda \to V_\lambda$, λ a limit) and the existence of a cardinal that is super-n-huge for every $n \in \omega$.

In [Co5], the theory ZFC + WA was used as a background theory for studying generalized Laver sequences. A technical question related to generalized Laver sequences that later arose in [Co4] was whether $V = \mathrm{HOD}$ is consistent with ZFC + WA. In [Co3], we showed that if ZFC + "there is an I_1 embedding" is consistent, then so is ZFC + WA + $V = \mathrm{HOD}$. Seeking to weaken the I_1 hypothesis in this result, Hamkins [Ha1] showed that an axiom WA_0, slightly weaker than WA, is consistent with $V = \mathrm{HOD}$. (Hamkins' axiom WA_0 is obtained from BTEE by including just the Σ_0 instances of $\mathrm{Separation}_j$ as additional axioms.) More precisely, Hamkins showed that if ZFC + WA_0 is consistent, so is ZFC + WA_0 + $V = \mathrm{HOD}$.

It is still not known whether $V = \mathrm{HOD}$ is consistent with WA without additional hypotheses. In this paper, we weaken the I_1 hypothesis to I_3. That is, we prove the following:

Theorem 1.1. *If* $\mathrm{ZFC} + \exists\,\kappa\,\mathrm{I}_3(\kappa)$ *is consistent, so is* ZFC + WA + $V = \mathrm{HOD}$.

Our proof combines the techniques of [Ha1] and [Co3]. The theorem follows from a more general result about lifting an I_3 embedding $j : V_\lambda \to V_\lambda$ to $V[G]_\lambda \to V[G]_\lambda$ via certain types of reverse Easton forcing iterations of length λ. The following is a precise statement of the main result; the notions of *adequate* and j-*coherent* reverse Easton iterations will be defined in Section 2.

Theorem 1.2. *Suppose that, in a countable transitive model M of a sufficiently large finite fragment of* ZFC, *there is an* I_3 *embedding $j : V_\lambda \to V_\lambda$ with critical point κ. Suppose that in M, $P_\lambda \subset V_{\lambda+1}$ is a j-coherent, adequate, reverse Easton forcing. Then there is a filter G that is P_λ-generic over M for which j can be lifted to an embedding $k : V_\lambda^{M[G]} \to V_\lambda^{M[G]}$.*

As is usual in the approach to forcing that uses countable transitive models (see for example [Ku1]), we start with a countable transitive model of a "sufficiently large finite fragment of ZFC," which consists of enough of the axioms of ZFC for all the arguments to go through. For the rest of this paper, we will denote such a fragment ZFC^*.

In [Ha2] and [Co3], a (well-known) technique is given for lifting an I_1 embedding $V_{\lambda+1} \to V_{\lambda+1}$ to $V[G]_{\lambda+1} \to V[G]_{\lambda+1}$ for the types of forcings mentioned in Theorem 1.2. In this approach, the forcing notion P_λ is included in $V_{\lambda+1}$, and hence lies in the domain of the elementary embedding. This fact makes it possible to define a master condition that can be used in the usual way to establish that the lifting of j is well-defined. For I_3 embeddings, this convenience is lost and some of the arguments used in the I_1 case do not go through.

In [Ha1], Hamkins developed a different approach to lifting embeddings $j : M \to M$ for sufficiently closed M, having the property that $M = \bigcup_{n\in\omega} V^M_{\kappa_n}$, where $\kappa_0, \kappa_1, \kappa_2, \ldots$ denotes the critical sequence of j. (In particular, he forces over models $\langle M, \in, j\rangle$ of ZFC + WA_0.) Lifting an embedding in this context was accomplished by alternately forcing and lifting to obtain a sequence of embeddings

$$V^M_{\kappa_0}[G_{\kappa_0}] \xrightarrow{i_1} V^M_{\kappa_1}[G_{\kappa_1}] \xrightarrow{i_2} V^M_{\kappa_2}[G_{\kappa_2}] \xrightarrow{i_3} \ldots$$

which leads to the conclusion that

$$V^M_{\kappa_0}[G_{\kappa_0}] \prec V^M_{\kappa_1}[G_{\kappa_1}] \prec V^M_{\kappa_2}[G_{\kappa_2}] \prec \ldots .$$

The final model N is obtained as the union of the $V^M_{\kappa_n}[G_{\kappa_n}]$, and the final embedding $k : N \to N$ is obtained by piecing together the i_n. If, in the present context, one begins with a transitive model $\bar{M}$ of ZFC* in which there is an I_3 embedding $j : V_\lambda \to V_\lambda$, and one applies Hamkins' argument with $M = V^{\bar{M}}_\lambda$, it is not clear that the resulting N is a V_λ inside a forcing extension of M. This is because Hamkins' argument does not produce a final generic filter G; moreover, Hamkins' model N is not known to be a generic extension of the ground model in general.

In this paper, we steer a middle course to obtain Theorem 1.2. We obtain a kind of *local* master condition that allows us to build a forcing extension $M[G]$ for which each restriction of the master condition belongs to G (though the master condition itself does not). This permits us to carry out most of Hamkins' arguments in such a way that the final lifting is of the form $V^{M[G]}_\lambda \to V^{M[G]}_\lambda$.

We begin with a section on preliminaries to review known results and establish notation. Section 3 is dedicated to the proof of the main result.

§2. Preliminaries

The axiom $\mathrm{I}_3(\kappa)$ asserts that there is an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point κ, where λ is a limit. The *critical sequence* of such an embedding is the sequence $\langle \kappa_0, \kappa_1, \kappa_2, \ldots\rangle$, where κ_0 is the critical point of the embedding and $\kappa_{n+1} = j^{n+1}(\kappa_0)$ (and where j^{n+1} denotes the $n+1$st iterate of j under composition).

A forcing iteration P_λ is a *reverse Easton iteration* if direct limits are taken at all inaccessible cardinal stages and inverse limits are taken at all other limit stages. A reverse Easton iteration is *adequate* if it satisfies the following: Suppose the inaccessible cardinals are unbounded in λ. Let $in = in_\lambda : \lambda \to \lambda$ be defined by $in(\alpha) =$ least inaccessible $> \alpha$. Then P_λ is adequate if for all $\alpha < \lambda$ we have:

(1) $|P_\alpha| < in(\alpha)$;
(2) $\Vdash_{P_\alpha} Q_\alpha$ is α^+-directed closed.

Furthermore, given a transitive model N of ZFC* for which $P_\alpha \in N$ for all $\alpha < \lambda$ and an elementary embedding $j : N \to N$ with critical point $\kappa < \lambda$, P_λ is said to be *j-coherent* if, whenever $\kappa \leq \alpha < \lambda$, $j(P_\alpha) = P_{j(\alpha)}$. (This is a slight weakening of the definition as it appears in [Co3] and [Ha2] — here we do not require P_λ itself to be included in N — but the change does not affect the truth of the theorems in that paper.) In this paper, the model N that we have in mind will be of the form V_λ (inside another transitive model of ZFC*).

The significance of adequate reverse Easton iterations lies in the following:

Proposition 2.1. *Suppose λ is a limit of inaccessibles and P_λ is an adequate reverse Easton iteration. Then for all $\alpha < \lambda$*

$$\Vdash_\alpha P_{\alpha\lambda} \text{ is } \alpha^+\text{-directed closed.}$$

Proposition 2.2. *Suppose λ is a limit of inaccessibles and P_λ is an adequate reverse Easton iteration. Then forcing with P_λ preserves inaccessibles $\leq \lambda$.*

The proof of Proposition 2.1 is like [Ba, Theorem 5.5] though the hypotheses are somewhat different. Proposition 2.2 is proved in [Co3].

For completeness, we review arguments from [Ha1], [Ha2] and [Co3], tailored slightly to the present context, which show how we may obtain the result that $V_\lambda^{M[G]} \models V = \text{HOD}$ after forcing with an appropriately defined j-coherent adequate reverse Easton iteration. These results will allow us to focus in the next section on j-coherent iterations without the added concern about how to ensure $V = \text{HOD}$ in the final model. We give outlines of proofs here; more details can be found in [Ha1] and [Ha2] (for the GCH argument) and in [Co3] (for the $V = \text{HOD}$ argument).

Theorem 2.3. *Suppose M is a transitive model of* $\text{ZFC}^* + \exists\kappa\, \text{I}_3(\kappa)$*, with witness $j : V_\lambda \to V_\lambda$ having critical point κ. Then there is, in M, a j-coherent adequate reverse Easton iteration $P_\lambda \subseteq V_{\lambda+1}$ such that for any filter G that is P_λ-generic over M,*

$$V_\lambda^{M[G]} \models \text{ZFC} + \text{GCH}.$$

Proof. (*Outline*) Define the usual reverse Easton iteration P_κ for forcing GCH up to κ: For each $\alpha < \kappa$ that is a cardinal in $M[G_\alpha]$, let $\dot{Q}_\alpha$ be a P_α name of least rank for the forcing that adds a Cohen subset to α^+. (Leastness of rank assists in the computation of the size of each P_α, to establish adequacy.) In the usual way, the αth stage P_α forces "$\dot{Q}_\alpha$ is α^+-directed closed". A straightforward computation shows that $|P_\alpha| < in(\alpha)$.

Having defined P_κ, inductively define P_{κ_n}, $n \in \omega$, using the embedding j:

$$P_{\kappa_{n+1}} = j(P_{\kappa_n}).$$

Finally, define P_λ to be the inverse limit of $P_\gamma, \gamma < \lambda$. It follows that P_λ is an adequate, j-coherent reverse Easton iteration, and

$$M[G_\lambda] \models \forall \alpha < \lambda\, (2^\alpha = \alpha^+).$$

It follows that

$$V_\lambda^{M[G_\lambda]} \models \text{ZFC} + \text{GCH}.\blacksquare$$

Now, combining Theorem 2.3 with Theorem 1.2, we can find a generic filter G so that GCH holds in $V_\lambda^{M[G]}$ and j lifts to $V_\lambda^{M[G]} \to V_\lambda^{M[G]}$ in $M[G]$.

Theorem 2.4. *Suppose M is a transitive model of* $\text{ZFC}^* + \exists\kappa\, \text{I}_3(\kappa)$*, with witness $j : V_\lambda \to V_\lambda$ having critical point κ. Then there is, in M, a j-coherent adequate reverse Easton iteration $P_\lambda \subseteq V_{\lambda+1}$ such that for any filter G that is P_λ-generic over M,*

$$V_\lambda^{M[G]} \models \text{ZFC} + V = \text{HOD}.$$

Proof. (*Outline*) Given that M is a transitive model of $\text{ZFC}^* + \exists\kappa\, \text{I}_3(\kappa)$ with witness $j : V_\lambda \to V_\lambda$, we may assume, by the previous theorem, that GCH holds up to λ. We again define a reverse Easton iteration P_κ, except here the forcing at each coordinate is a particular kind of Easton forcing. To describe this forcing, we set up some initial notation: Let $\pi : ON \times ON \to ON$ be the definable bijection given by Gödel's definable well-ordering of $ON \times ON$, having the property that for every cardinal ν, $\pi \restriction \nu \times \nu$ is a bijection from $\nu \times \nu$ onto ν. For each beth fixed point ν and each set $A \subseteq \nu$, we write $A \sim V_\nu$ if there is a bijection $t : \nu \to V_\nu$ such that for all $(\beta, \alpha) \in \nu \times \nu$, $t(\beta) \in t(\alpha)$ iff $\pi(\beta, \alpha) \in A$. (For any beth fixed point ν, one may always find a set A such that $A \sim V_\nu$.) Given a beth fixed point ν and a set $A \subseteq \nu$ with $A \sim V_\nu$, we define our Easton function $f = f_{\nu,A}$ on $S = S_{\nu,A} = \{\gamma :\ \gamma \text{ is a successor cardinal and } \nu = \omega_\nu < \gamma < \omega_{\nu+\nu}\}$ by setting, for each $\alpha < \nu$,

$$f(\omega_{\nu+\alpha+1}) = \begin{cases} \omega_{\nu+\alpha+3} & \text{if } \alpha \in A \\ \omega_{\nu+\alpha+2} & \text{if } \alpha \notin A \end{cases}$$

Our Easton forcing, relative to (ν, A), will then be defined to be $E(f)$ — the Easton forcing defined from f (see [Ku1] or [Co3]). In particular, we define the nontrivial coordinates of the iteration P_κ: For each inaccessible $\alpha < \kappa$, let $\dot{Q}_\alpha$ be a P_α name of least rank for a partial order defined in $M[G_\alpha]$ as follows: Pick $A \sim V_\alpha$ with $A \subset \alpha$, and let $f = f_{\alpha,A}$ be the Easton function defined above. Finally, let $Q = E(f)$.

Having defined P_κ, we extend to P_λ as in the GCH argument by inductively defining $P_{\kappa_{n+1}} = j(P_{\kappa_n})$, and letting P_λ be the inverse limit of $P_\gamma, \gamma < \lambda$. P_λ is easily shown to be an adequate, j-coherent reverse Easton iteration.

Suppose G is P_λ-generic over M and let α be inaccessible in M. By Theorem 2.2, α must still be inaccessible in $M[G_{\alpha+1}]$ and so GCH must still hold on the interval $[\alpha, \lambda)$. Therefore the Easton function $f = f_{\alpha,A}$ determines values of the continuum on its domain; in particular, for each $\gamma < \alpha$,

$$\gamma \in A \text{ iff } 2^{\omega_{\alpha+\gamma+1}} = \omega_{\alpha+\gamma+3}.$$

Thus, A is seen to be ordinal definable, and therefore, so is each element of V_α. By closure properties guaranteed by adequacy, ordinal definability of the sets having rank $< \alpha$ is preserved in $M[G]$. It follows that $V_\lambda^{M[G]} \models \text{ZFC} + V = \text{HOD}$. $\blacksquare$.

Now we can prove Theorem 1.1 from Theorem 1.2 and Theorem 2.4:

Proof of Theorem 1.1. Suppose M is a transitive model of ZFC* containing an I_3 embedding $j : V_\lambda \to V_\lambda$ with critical sequence $\kappa_0, \kappa_1, \kappa_2, \ldots$. In M, let P_λ be the j-coherent adequate reverse Easton iteration guaranteed by Theorem 2.4. By Theorem 1.2, there is a filter G that is P_λ-generic over M for which j can be lifted to an embedding $k : V_\lambda^{M[G]} \to V_\lambda^{M[G]}$. By Theorem 2.4, $V_\lambda^{M[G]} \models V = \mathrm{HOD}$. But since k is an I_3 embedding in $M[G]$, we have that $\langle V_\lambda^{M[G]}, \in, k\rangle \models \mathrm{WA}$, as required. ∎

§3. Main Result

In this section, we prove Theorem 1.2. We begin with a countable transitive model M of ZFC* in which there is an I_3 embedding $j : V_\lambda \to V_\lambda$ with critical sequence $\kappa_0, \kappa_1, \kappa_2, \ldots$.

Let $P_\lambda \subseteq V_{\lambda+1}$ be, in M, an adequate, j-coherent reverse Easton iteration. Unlike the situation in [Co3], P_λ is not included in the domain of j. We obtain a condition $q \in P_\lambda$ in essentially the same way in which a *master condition* is defined in [Co3] and [Ha1]. In particular, we obtain the condition q inductively, satisfying the following:

Let $\dot{G}$ be a name for a filter that is P_λ-generic over M, and let $\dot{G}_{\kappa_n}$ be a P_{κ_n}-name for $\dot{G}$ up to κ_n.

(1) Let $q \restriction \kappa_1 = \langle \dot{1}, \dot{1}, \ldots \rangle$.
(2) For each $n \geq 1$, let $q(\kappa_n) \in P_{\kappa_n, \kappa_n+1}$ be a name for a condition lying below all $j(p)(\kappa_n)$ for which $p \in \dot{G}_{\kappa_n}$.
(3) For each $n \geq 1$, let $q \restriction (\kappa_n, \kappa_{n+1}) \in P_{\kappa_n, \kappa_{n+1}}$ be a name for a condition lying below all $j(p) \restriction (\kappa_n, \kappa_{n+1})$ for which $p \in \dot{G}_{\kappa_n}$

Parts (2) and (3) of the inductive construction can be done because each $P_{\kappa_n, \lambda}$ is κ_n^+-directed closed in $V[G_{\kappa_n}]$, by Proposition 2.1. See [Co3] and [Ha2] for more details. One slight difference in our approach here from that found in [Co3] and [Ha2] is that we have been careful to define q in terms of conditions p that lie in an initial part of P_λ, rather than referring to arbitrary elements of P_λ. This care is necessary because in the present context, elements of P_λ do not lie in the domain of j, since an inverse limit is taken at the λth stage of the iteration. In particular, q itself is not in the domain of j, and so it follows that q is technically not a master condition in the sense of [Co3] and [Ha2]. However, as we will show, q retains the local properties of a master condition, and these are enough to carry out the necessary argument.

Let G be P_λ-generic over M with $q \in G$. We may assume that, for each n, $P_{\kappa_n} \subseteq V_{\kappa_n}^M$. Note that for each n, G_{κ_n} is P_{κ_n}-generic over $V_{\kappa_n}^M$. We define, in $M[G]$, a sequence $\langle i_1, i_2, i_3, \ldots, \rangle$ of elementary embeddings

$$V_{\kappa_0}^M[G_{\kappa_0}] \xrightarrow{i_1} V_{\kappa_1}^M[G_{\kappa_1}] \xrightarrow{i_2} V_{\kappa_2}^M[G_{\kappa_2}] \xrightarrow{i_3} \ldots$$

as follows: For $\sigma \in M^{P_{\kappa_n}}$,

$$i_{n+1}(\sigma_{G_{\kappa_n}}) = (j(\sigma))_{G_{\kappa_{n+1}}}.$$

We will show that i_{n+1} is well-defined; the usual arguments (see [Je] for example) can then be used to conclude that i_{n+1} is elementary. First observe that for any $p \in P_{\kappa_n}$ and any $\sigma, \tau \in M^{P_{\kappa_n}}$,

$$p \Vdash \sigma = \tau \text{ implies } j(p) \Vdash j(\sigma) = j(\tau),$$

because $\Vdash \; = \; \Vdash_{\kappa_n}$ is definable in V_λ^M. Therefore, it suffices to show that $p \in G_{\kappa_n} \longrightarrow j(p) \in G_{\kappa_{n+1}}$.

Because, in M, $P_{\kappa_{n+1}} \sim P_{\kappa_n} \otimes P_{\kappa_n,\kappa_{n+1}}$, we may write

$$j(p) = (p', \dot{r}),$$

where

$$(3.1)_n \qquad p' \in P_{\kappa_n} \quad \text{and} \quad 1 \Vdash \dot{r} \in P_{\kappa_n,\kappa_{n+1}} \text{ and } \dot{r} = j(p) \restriction [\kappa_n, \kappa_{n+1}).$$

Claim 1. If $M \models j(p) = (p', \dot{r})$ and $p \in G_{\kappa_n}$, then $p' \in G_{\kappa_n}$ and $\dot{r}_{G_{\kappa_n}} \in G_{\kappa_n,\kappa_{n+1}}$.

Proof. Working in M, we proceed by induction on $n \geq 0$. For $n = 0$, there is $\alpha < \kappa_0$ such that $\mathrm{supt}(p) \subseteq \alpha$. It follows that $\mathrm{supt}(p) = \mathrm{supt}(j(p))$. Also, since for each $\beta < \alpha$, $p(\beta) \in V_{\kappa_0}^M$, it follows that $j(p) \restriction \kappa_0 = p \restriction \kappa_0$, and so

$$j(p) = (p, \langle \dot{1}, \dot{1}, \dot{1}, \ldots \rangle),$$

and this completes the proof for $n = 0$.

Assume the result holds for $n \geq 0$. Let $p \in G_{\kappa_{n+1}}$ and write $j(p) = (p', \dot{r})$ where $(p', \dot{r})$ satisfies $(3.1)_{n+1}$; that is, $p' \in P_{\kappa_{n+1}}$ and $1 \Vdash \dot{r} \in P_{\kappa_{n+1},\kappa_{n+2}}$ and $\dot{r} = j(p) \restriction [\kappa_{n+1}, \kappa_{n+2})$. We first show that $p' \in G_{\kappa_{n+1}}$. Let $p_n = p \restriction \kappa_n \in P_{\kappa_n}$. Since $p \in G_{\kappa_{n+1}}$, $p_n \in G_{\kappa_n}$. By the induction hypothesis, $j(p_n) = (p'_n, \dot{r}_n)$ where $p'_n \in G_{\kappa_n}$ and $(\dot{r}_n)_{G_{\kappa_n}} \in G_{\kappa_n,\kappa_{n+1}}$. Therefore, we have

$$\begin{aligned} p' &= j(p) \restriction j(\kappa_n) \\ &= j(p \restriction \kappa_n) \\ &= j(p_n) \in G_{\kappa_{n+1}}. \end{aligned}$$

Next, we show that $\dot{r}_{G_{\kappa_{n+1}}} \in G_{\kappa_{n+1},\kappa_{n+2}}$. There is $s \in G_{\kappa_{n+1}}$ with $s \leq p'$ and $s \leq q \restriction \kappa_{n+1}$ (since both p' and $q \restriction \kappa_{n+1} \in G_{\kappa_{n+1}}$). It follows that

$$s \Vdash q \restriction [\kappa_{n+1}, \kappa_{n+2}) \leq j(p) \restriction [\kappa_{n+1}, \kappa_{n+2}) = \dot{r}.$$

The result follows. ∎

Claim 2. $V_{\kappa_0}^M[G_{\kappa_0}] \prec V_{\kappa_1}^M[G_{\kappa_1}] \prec V_{\kappa_2}^M[G_{\kappa_2}] \prec \ldots$.

Proof. We first show that $V^M_{\kappa_0}[G_{\kappa_0}] \prec V^M_{\kappa_1}[G_{\kappa_1}]$. Notice that the embedding $i_1 : V^M_{\kappa_0}[G_{\kappa_0}] \to V^M_{\kappa_1}[G_{\kappa_1}]$ extends $j \restriction V^M_{\kappa_0} : V^M_{\kappa_0} \to V^M_{\kappa_1}$. In particular, i_1 is an elementary embedding with no critical point. It follows that $V^M_{\kappa_0}[G_{\kappa_0}] \prec V^M_{\kappa_1}[G_{\kappa_1}]$.

Next we make two observations. First, since each κ_n is inaccessible, it follows that

$$\text{for all } n \in \omega,\ V^{M[G_{\kappa_n}]}_{\kappa_n} = V^M_{\kappa_n}[G_{\kappa_n}].$$

Secondly, by directed closure as in Theorem 2.1, we have

$$V^{M[G_{\kappa_n}]}_{\kappa_n} = V^{M[G]}_{\kappa_n}. \tag{3.2}$$

To complete the proof, we now show that $V^{M[G_{\kappa_n}]}_{\kappa_n} \prec V^{M[G_{\kappa_{n+1}}]}_{\kappa_{n+1}}$ by induction on n. We have taken care of the $n = 0$ case. Assume the result for n; we therefore have, by (3.2),

$$M[G] \models V_{\kappa_n} \prec V_{\kappa_{n+1}}. \tag{3.3}$$

Since i_{n+2} is defined from initial segments of G and from j, $i_{n+2} \in M[G]$. Applying, in $M[G]$, the embedding i_{n+2} to the formula "$V_{\kappa_n} \prec V_{\kappa_{n+1}}$" yields "$V_{\kappa_{n+1}} \prec V_{\kappa_{n+2}}$" in $M[G]$. It follows that

$$V^{M[G_{\kappa_{n+1}}]}_{\kappa_{n+1}} = V^{M[G]}_{\kappa_{n+1}} \prec V^{M[G]}_{\kappa_{n+2}} = V^{M[G_{\kappa_{n+2}}]}_{\kappa_{n+2}},$$

and this completes the induction and the proof of the Claim. ∎

Claim 3. $V^{M[G]}_\lambda = \bigcup_{n\in\omega} V^{M[G_{\kappa_n}]}_{\kappa_n}$. Therefore, for each n, $V^{M[G_{\kappa_n}]}_{\kappa_n} \prec V^{M[G]}_\lambda$.

Proof. We have

$$\bigcup_{n\in\omega} V^{M[G_{\kappa_n}]}_{\kappa_n} = \bigcup_{n\in\omega} V^{M[G]}_{\kappa_n} = V^{M[G]}_\lambda. \blacksquare$$

Finally, we define the required elementary embedding $k : V^{M[G]}_\lambda \to V^{M[G]}_\lambda$: Define k to be the union of the i_n. Elementarity of k is verified in a straightforward way: If $\phi(x_1, \ldots, x_n)$ is a formula and $a_1, \ldots, a_m$ are parameters all lying in $V^{M[G]}_\lambda$, then by Claim 3 we have that for some n, these parameters lie in $V^{M[G_{\kappa_n}]}_{\kappa_n}$ and $\phi(a_1, \ldots, a_n)$ holds in $V^{M[G_{\kappa_n}]}_{\kappa_n}$. By elementarity of i_{n+1}, $\phi(i_{n+1}(a_1), \ldots, i_{n+1}(a_n))$ holds in $V^{M[G_{\kappa_{n+1}}]}_{\kappa_{n+1}}$. The result now follows by another application of Claim 3.

Our work shows that we have a lifting of the original embedding $j : V_\lambda \to V_\lambda$ to $k : V^{M[G]}_\lambda \to V^{M[G]}_\lambda$. This completes the proof of Theorem 1.2.

Our work establishes the consistency of $V = \mathrm{HOD}$ with WA assuming an I_3 embedding. As we mentioned in the Introduction, it is not known whether the assumption "$\exists\kappa\, \mathrm{I}_3(\kappa)$" can be replaced by "WA". We are left with the following open question:

Open Question. Does consistency of ZFC + WA imply consistency of ZFC + WA + $V = \mathrm{HOD}$?

We conclude the paper with a refinement suggested by the referee. With a slight modification of the arguments and results mentioned in Section 2, we can extend the iteration P_λ through the ordinals so that the forcing extension in which the embedding j has been lifted satisfies $V = \mathrm{HOD}$. Our work so far guarantees only that, in the final model, $V = \mathrm{HOD}$ holds in the rank V_λ. For the remainder of the paper, we outline the proof of the following:

Theorem 3.1. *If* $\mathrm{ZFC} + \mathrm{I}_3$ *is consistent, so is* $\mathrm{ZFC} + \mathrm{I}_3 + V = \mathrm{HOD}$. *Moreover, given a countable transitive model* M *of* $\mathrm{ZFC}^* + \mathrm{I}_3$, *with witness* $j : V_\lambda \to V_\lambda$, *there is a class forcing in* M *and a generic (class) filter* G *such that* j *lifts to an elementary embedding* $V_\lambda^{M[G]} \to V_\lambda^{M[G]}$ *and* $M[G] \models V = \mathrm{HOD}$.

Before discussing the proof, we remark that, because of the non-absoluteness of $V = \mathrm{HOD}$, we cannot establish Theorem 1.1 from Theorem 3.1 simply by relativizing down: There is no guarantee in the statement of Theorem 3.1 that, in $M[G]$, for each set $x \in (V_\lambda)^{M[G]}$, the ordinal parameters of a formula that witnesses ordinal definability of x in $M[G]$ all lie in $(V_\lambda)^{M[G]}$.

We also observe that the work of Roguski in [Ro] (see also [Za, Section 4]) already implies the first part of Theorem 3.1, namely, that

$$\mathrm{Con}(\mathrm{ZFC} + \exists\kappa\, \mathrm{I}_3(\kappa)) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\, \mathrm{I}_3(\kappa) + V = \mathrm{HOD}).$$

He shows that, in fact, for any Σ_2-sentence σ, $\mathrm{Con}(\mathrm{ZFC} + \sigma) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \sigma + V = \mathrm{HOD})$. However, his result is not sufficient to guarantee that the embedding in the extension is a lifting of the embedding in the ground model. And, for the same reasons given in the previous paragraph, his result does not imply Theorem 1.1.

To prove Theorem 3.1, we would like to continue the iteration P_λ through the ordinals. However, our approach to establishing the crucial property

$$\text{for all } \alpha, \beta \text{ with } \alpha < \beta \ \Vdash_\alpha P_{\alpha\beta} \text{ is } \alpha^+\text{-directed closed}$$

(a property we will call *strong closure* for the remainder of the paper) has relied so far on the notion of *adequacy* of a reverse Easton iteration, and this notion no longer makes sense if there happen to be very few (or no) inaccessibles above λ. However, using arguments from Menas [Me], we can establish strong closure without reliance on adequacy. We will prove the following modified version of Theorem 1.2 and use this in combination with a modified $V = \mathrm{HOD}$ argument to obtain the final result.

Theorem 3.2. *Suppose that, in a countable transitive model* M *of a sufficiently large finite fragment of* ZFC, *there is an* I_3 *embedding* $j : V_\lambda \to V_\lambda$ *with critical point* κ. *Suppose that in* M, P *is a strongly closed, reverse Easton class forcing that is* j*-coherent up to* λ. *Then there is a (class) filter* G *that is* P*-generic over* M *for which* j *can be lifted to an embedding* $k : V_\lambda^{M[G]} \to V_\lambda^{M[G]}$.

The only modification to the statement of Theorem 1.2 here is replacement of adequacy of the iteration by strong closure of the iteration. To prove Theorem 3.1, we revisit the proofs of Theorems 2.3, 2.4, and 1.2. and mention the necessary modifications.

First, we can prove the analog to Theorems 2.3 simply by replacing adequacy with strong closure in the argument, and continuing the iteration past λ, through the ordinals. The fact that the iteration is strongly closed and that the resulting class forcing produces a model of ZFC+GCH is well-known in this case without reference to adequacy (see for example [Fr, Chapter 3]). As before, the iteration is j-coherent up to λ. Now, assuming Theorem 3.2, we can establish, in addition, that the ground model embedding can be lifted to $(V_\lambda)^{M[G]} \to (V_\lambda)^{M[G]}$, as before.

For the analog to Theorem 2.4, we may assume this time that the ground model satisfies full GCH, and again we replace adequacy with strong closure of the iteration in the statement of the theorem. We attempt to extend the reverse Easton iteration beyond λ, through all the ordinals. Here, we take as nontrivial coordinates the successors of beth fixed points instead of inaccessibles. In defining the iteration up to P_κ, for each such successor $\alpha < \kappa$, let $\dot{e}$ denote a P_α-name for the increasing enumeration of the beth fixed points $\leq \kappa$ in $M[G_\alpha]$. Let $\dot{A}$ be a P_α name for a subset of $e(\alpha)$ for which $\Vdash_{P_\alpha} \dot{A} \sim V_{\dot{e}(\alpha)}$. Let $\dot{Q}_\alpha$ be a P_α name of least rank for the partial order $E(f)$, where $f = f_{e(\alpha),A}$ is defined in $M[G_\alpha]$ as before (replacing α with $e(\alpha)$ in the present context). Now extend the iteration from P_κ to P_λ as before, using j, and then continue the iteration past λ using the definitions given for the stages up to P_κ.

Menas [Me, proof of Theorem 20] has shown that this iteration has the property that each stage preserves beth fixed points and that for every α, $|P_\alpha| < e(\alpha{+}2)$. This latter property provides a bound that can be used, as in the proof of Theorem 2.1, to show that the iteration is strongly closed — as before, the proof follows closely that of [Ba, Theorem 5.5].

Verification that $V = \text{HOD}$ holds in the extension $M[G]$ is similar to the proof given for Theorem 2.4: First notice that, for each nontrivial coordinate β and set $B \sim V_{e(\beta)}$, the domain of the Easton function $f = f_{e(\beta),B}$ used in the iteration must lie below the next beth fixed point $e(\beta + 1)$. As a result, after forcing with $E(f)$, GCH will continue to hold for all $\gamma \geq e(\beta + 1)$. Now, given α that is a successor of a beth fixed point, obtain $A \sim V_{e(\alpha)}$ in $M[G_{\alpha+1}]$. As we just observed, GCH must continue to hold on all $\gamma \geq e(\alpha)$; therefore we have as before that, for all $\delta < e(\alpha)$, $\delta \in A$ if and only if $2^{\omega_{e(\alpha)+\delta+1}} = \omega_{e(\alpha)+\delta+3}$. Therefore A, and also each element of $V_{e(\alpha)}$, is ordinal definable, relative to $M[G_{\alpha+1}]$. By strong closure, this ordinal definability continues to hold in $M[G]$. This completes the proof that $M[G] \models V = \text{HOD}$.

The proof of Theorem 3.2 is exactly the same as the proof of Theorem 1.2; this follows because the only way that adequacy is used in Theorem 1.2 is in making use of the strong closure property of the iteration.

Finally, we can establish Theorem 3.1: Given a countable transitive model M of $\text{ZFC}^* + \text{I}_3$, with witness $j : V_\lambda \to V_\lambda$, obtain the class forcing P described in the last few paragraphs for obtaining $V = \mathrm{HOD}$ in the extension. By Theorem 3.2, there is a (class) generic filter G so that j can be lifted to $V_\lambda^{M[G]} \to V_\lambda^{M[G]}$, and by the discussion above, $M[G] \models V = \mathrm{HOD}$. This completes the proof of Theorem 3.1.

References.

[Ba] Baumgartner, J., *Iterated forcing*, Surveys in Set Theory (A.R.D. Mathias, ed.), Cambridge Univ. Press, 1983, pp. 1–59.

[Co1] Corazza, P., *Indestructibility of wholeness*, In preparation.

[Co2] ———, *The spectrum of elementary embeddings* $j : V \to V$, **Ann. Pure Appl. Logic**, (2006), 139, pp. 327-399.

[Co3] ———, *Consistency of* $V = HOD$ *with the wholeness axiom*, **Arch. Math. Logic** (2000), 39, pp. 219–226.

[Co4] ———, *Laver sequences for extendible and super-almost-huge cardinals*, **J. Symbolic Logic** (1999), 64, pp. 963–983.

[Co5] ———, *The wholeness axiom and Laver sequences*, **Annals of Pure and Applied Logic**, (2000), 105, pp. 157–260.

[Fr] Friedman, S., **Fine Structure and Class Forcing**, Walter de Gruyter, New York, 2000.

[Ha1] Hamkins, J.D., *The wholeness axioms and* $V =$ HOD, **Arch. Math. Logic** (2001), 40, pp. 1–8.

[Ha2] ———, *Fragile measurability,* **J. Symbolic Logic,** (1994), 59, pp. 262-282.

[Je] Jech, T., **Set Theory,** Academic Press, New York, 1978.

[Ku1] Kunen, K., **Set Theory: An Introduction To Independence Proofs**, North-Holland Publishing Company, New York, 1980.

[Ku2] Kunen, K., *Elementary embeddings and infinitary combinatorics,* **J. Symbolic Logic** (1971), 36, pp. 407–413.

[Me] Menas,T., *Consistency results concerning supercompactness,* **Trans. Amer. Math. Soc.,** Vol. 223, 1976, pp. 61-91.

[Ro] Roguski, S., *The theory of the class HOD* in **Set Theory and Hierarchy Theory**, Springer Lecture Notes in Math. (619), 1977.

[Za] Zadrożny, W., *Iterating ordinal definability*, **Annals of Pure and Applied Logic**, (1983), 24, pp. 263–310.