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Revisiting the Construction of the Real Line

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Abstract

The most common constructions of the reals from the rationals attempt to "fill in the gaps," using (equivalence classes of) Cauchy sequences of rationals or Dedekind cuts. Another technique, more familiar to logicians than to analysts, is to expand the set $\mathbb{Q}$ of rationals to the set $\mathbb{Q}^{\mathbb{N}}$ of all sequences of rationals and then collapse back down in two stages. In the first stage, produce a structure X by identifying sequences that agree on a large set; in the second stage, further collapse X by removing its infinite elements, and collapsing its infinitesimals to 0. The latter approach, though less widely known, provides, in its intermediary construction X, an example of interesting pathologies. We show here that X is an example of a complete (even open-complete) ordered field that is not Archimedean. The field X showcases the significant role of the Archimedean property in the structure of $\mathbb{R}$, as many of the fundamental properties of $\mathbb{R}$ are found to fail in X precisely because this property is absent, such as the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem. Indeed, we show that the field X is not even metrizable. We conclude with a detailed proof, accessible to non-logicians, that the second stage of the construction succeeds in producing a complete Archimedean ordered field.

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1 Introduction

The usual way to construct the real line from the set $\mathbb{Q}$ of rationals is through the use of either equivalence classes of Cauchy sequences or Dedekind cuts.[1] Another approach, which one rarely if ever finds developed fully in the literature, is based on an observation that is sometimes made ([7], [10]) in nonstandard analysis texts: If one starts with a definition of the set $\mathbb{R}$ of reals and considers some kind of nonstandard enlargement ${}^*R$ of $\mathbb{R}$, then $\mathbb{R}$ itself can be recovered from the set ${}^*Q$ of nonstandard rationals by removing infinite elements and modding out by the set of infinitesimals.[2] The mathematical logic prerequisites necessary to work through the details make this approach less popular. Moreover, to actually construct $\mathbb{R}$ from scratch using this idea, one would need to begin without assuming the existence of $\mathbb{R}$. One main purpose of this paper is to develop the details of this approach while requiring very little from the reader in the way of prerequisites; in particular, our approach will be essentially "logic-free." We hope that our effort will help this rather elegant construction find its rightful place among the better known constructions of the reals.

This particular approach brings to light an interesting combination of properties that a "complete" ordered field can have—combinations of properties that never appear in the usual constructions. For instance, as is well-known, $\mathbb{R}$ is the unique (up to isomorphism) complete ordered field (e.g. [8]). This result, however, requires the proper formulation of "completeness"—the version that is usually intended is what we will call *Dedekind-completeness*, namely, that every nonempty subset of $\mathbb{R}$ that has an upper bound has a least upper bound. Another version of completeness, which in some texts (e.g. [1]) is called the *Nested Interval Property (NIP)*, requires that any nested, decreasing sequence $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ of closed intervals has a common point. However, using this property in place of Dedekind-completeness does *not* provide us with a characterization of $\mathbb{R}$—it is not true that every ordered field satisfying the NIP is isomorphic to $\mathbb{R}$ (Theorem 3.8). To obtain the desired characterization, it is necessary to include the Archimedean property (for every x there is a natural number greater than or equal to x): Every *Archimedean* ordered field satisfying the NIP *is* isomorphic to $\mathbb{R}$ (e.g. [8], [13]). A natural question is: Does there exist a non-Archimedean ordered field that satisfies NIP?

An example of such a structure falls out of our construction automatically, in the following way: The construction proceeds in two stages. The first stage produces a non-Archimedean ordered field X; the second stage produces an

[1] The Cauchy sequence approach is carried out in detail in [13, Appendix A]. The Dedekind cuts approach is developed in [9, 12].

[2] A discussion of this kind can be found in [7].

Archimedean ordered field satisfying the NIP. We will show that this intermediate structure X, constructed in the first stage, also satisfies the NIP; X will therefore be an example of a "complete" non-Archimedean ordered field.

The structure X exhibits some additional pathologies that showcase the significance of the Archimedean property in the structure of $\mathbb{R}$. For instance, X has a subset A that is closed and bounded but not compact (showing that the Heine-Borel Theorem fails) and that has no convergent subsequence (implying the failure of Bolzano-Weierstrass as well); X admits a decreasing nested sequence of closed and bounded subsets with empty intersection, showing that, unlike the situation in $\mathbb{R}$, "closed interval" cannot be replaced by "closed and bounded set" in the statement of NIP; every increasing sequence is bounded; and finally, X itself is nonmetrizable—in fact there is no countable basis at 0 (or at any other point). Viewed as just a linear order, without the field-theoretic structure, X is also an example of an unbounded linear order that is complete in that it has the NIP, but that has no countable dense subset and, as in the algebraic case, is not Dedekind-complete.

We give an overview of the two stages of our construction. Let $\mathbb{Q}^{\mathbb{N}}$ denote the set of all sequences of rationals. Let U be a *nonprincipal ultrafilter* on $\mathbb{N}$. We will define this notion precisely in Section 1; for now, we describe U somewhat loosely as a collection consisting of only infinite subsets of $\mathbb{N}$, closed under intersection and supersets, with the property that for every $A \subseteq \mathbb{N}$, either A or its complement belongs to U. Intuitively, U allows us to partition the subsets of $\mathbb{N}$ into "big" and "small" subsets in such a way that each piece of the partition is closed in natural ways (the intersection of big sets is big; the superset of a big set is also big). The first stage of the construction recovers most of the algebraic properties that are lost by expanding to the structure $\mathbb{Q}^{\mathbb{N}}$ by declaring two sequences $s, t \in \mathbb{Q}^{\mathbb{N}}$ to be equivalent if they agree on a big set, that is, on a set in U. The resulting set of equivalence classes is denoted $\mathbb{Q}^{\mathbb{N}}/U$; in the preceding paragraphs, we have denoted this structure X. This structure will be shown to be a non-Archimedean ordered field that has the NIP and is in fact *open-complete* (any descending nested sequence of open intervals has a common point). The second stage is now necessary because, as mentioned before, $\mathbb{Q}^{\mathbb{N}}/U$ is non-Archimedean, and so is equipped with infinitesimal and infinite elements. Removing the infinite elements still leaves us with a commutative ordered ring. Then, since the set I of infinitesimals forms a maximal ideal, taking the quotient by I gives us a field, which can be shown to be an ordered field that is now Archimedean. One then argues that the resulting ordered field has the NIP and is therefore Dedekind-complete.

There are several results in the literature that are related to ours. First, we point out that an example of a "complete" non-Archimedean ordered field is already known: Let us call an ordered field F *Cauchy-complete* if every

Cauchy sequence converges (where Cauchy sequences are defined in the usual way using the natural absolute value function for F: $|a| = \max\{a, -a\}$). Hall [8, Corollary 6.6] proves that any ordered field that satisfies the NIP is Cauchy-complete. Gelbaum & Olmsted [3, p. 17] point out that the field of rational functions over $\mathbb{R}$ together with a natural ordering provides an example of an ordered field K in which the (embedded) natural numbers are bounded—hence this field is non-Archimedean. It is then observed that the collection of all equivalence classes of Cauchy sequences in K results in another ordered field in which $\mathbb{N}$ is still bounded, but which is now Cauchy-complete. This example has properties similar to our X described above; Hall's result that NIP implies Cauchy-complete suggests that our example is somewhat stronger.[3] Second, we mention that our approach to the construction of the real line has been partially pursued elsewhere. Hall [8, Section 5] takes an approach similar to ours, starting from a $\mathfrak{c}^+$-saturated nonstandard extension ${}^*Q$ of $\mathbb{Q}$; he assumes, however, that every Archimedean field has cardinality $\leq \mathfrak{c}$—a fact that relies on the existence of a Dedekind-complete ordered field (in other words, it assumes that some version of $\mathbb{R}$ already exists). Hall relies on $\mathfrak{c}^+$-saturation to argue that a strong form of NIP (one that implies the Archimedean property) holds. In our approach, we build our nonstandard extension of $\mathbb{Q}$ "from scratch" using an ultrapower (and we are assured of only $\aleph_1$-saturation as a result), but give direct but separate arguments to prove that both the Archimedean property and the NIP hold. Finally, Davis [4, p. 51] gives a purely nonstandard analytic argument for building the real line using an idea similar to ours. The arguments, however, are completely different. One reason is that our arguments have been entirely removed from the nonstandard context. A second reason is that the underlying nonstandard techniques differ in that we make use of the notion of saturation (which is closely related to the NIP), while he uses Robinson's old Concurrence Theorem [4, p. 34] to achieve a similar end. The prerequisites in the area of mathematical logic for understanding Davis's results are therefore substantial; in that sense, our proofs provide a greatly simplified approach to this style of constructing the real line.

Section 1 introduces the notation and states the background results necessary for the rest of the paper. Our intention is that anyone who has taken undergraduate courses in analysis and in abstract algebra (ring theory) will have no difficulty reading this article; Section 1, therefore, is intended to be a review of the relevant points from such courses and to introduce other concepts that will be needed but may be unfamiliar. Section 2 works through the details of the first stage of the construction, producing the structure $\mathbb{Q}^{\mathbb{N}}/U$. It is shown that $\mathbb{Q}^{\mathbb{N}}/U$ is a non-Archimedean ordered field that satisfies the NIP. We also observe that the proof that the NIP holds for $\mathbb{Q}^{\mathbb{N}}/U$ depends only on its order-

[3] In fact, the question whether every Cauchy-complete non-Archimedean ordered field satisfies the NIP seems to be open.

theoretic properties; we then show that, as a linear order, $\mathbb{Q}^{\mathbb{N}}/U$ is unbounded, has no countable dense subset, and does not satisfy Dedekind-completeness, providing thereby a counterexample in the order-theoretic context to the usual equivalence of these two notions of completeness (Theorem 4). Section 3 continues the work of Section 2 by showing some of the pathological characteristics of X, including the failure of Heine-Borel and Bolzano-Weierstrass, and also its nonmetrizability. Finally, in Section 4, we carry out the second stage of the construction, starting from an arbitrary non-Archimedean ordered field X that satisfies the NIP. In this step, infinite elements are removed and infinitesimals are collapsed to 0. We show that the resulting structure is an Archimedean ordered field that satisfies the NIP and that therefore is isomorphic to the usual set $\mathbb{R}$ of reals.

Several key steps in the proofs that we present here are "user-friendly" versions of results from nonstandard analysis. For the benefit of readers who may have some background in nonstandard analysis, we will point out the places in our reasoning that are based on techniques from this area.

The approach we have taken here has proven to be of interest to students of mathematics. For this reason, some of the steps of reasoning have been delegated to the reader in the form of exercises.

2 Preliminaries

In this paper we will use the following notation for the usual number systems:

$$\begin{aligned} \mathbb{N} &= \text{the set of natural numbers} \\ \mathbb{Z} &= \text{the set of integers} \\ \mathbb{Q} &= \text{the set of rationals} \\ \mathbb{R} &= \text{the set of reals} \end{aligned}$$

For reference, we give a standard list of axioms for the real line (e.g. [2]). In our treatment, ordered fields are structures of type $\langle 0, 1, +, \cdot, < \rangle$ (so existence of 0 and 1 is not explicitly asserted in the axioms).

Axioms for an Archimedean Ordered Field Satisfying the NIP

Field Axioms

(1) *Commutativity of addition.* For all x, y, $x + y = y + x$.

(2) *Additive identity.* For all x, $x + 0 = x$.

(3) *Associativity of addition.* For all x, y, z, $(x + y) + z = x + (y + z)$.

(4) *Additive inverses*. For each x, there is y such that $x + y = 0$.

(5) *Commutativity of multiplication*. For all x, y, $x \cdot y = y \cdot x$.

(6) *Multiplicative identity*. For all x, $x \cdot 1 = x$.

(7) *Associativity of multiplication*. For all x, y, z, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(8) *Multiplicative inverses*. For each $x \neq 0$, there is y such that $x \cdot y = 1$. (y is called the *multiplicative inverse of* x and is denoted x^{-1} or sometimes $\frac{1}{x}$.)

(9) *Distributivity*. For all x, y, z, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Ordered Field Axioms

(10) *Irreflexive order*. For all x, $x \not< x$.

(11) *Antisymmetric order*. For all x, y, if $x < y$, then y *not* $< x$.

(12) *Transitive order*. For all x, y, z, if $x < y$ and $y < z$, then $x < z$.

(13) *Total order*. For all x, y, exactly one of the following holds: $x = y, x < y, y < x$.

(14) For all x, y, z, if $x < y$, then $x + z < y + z$.

(15) For all x, y, if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Other Axioms

(16) *Archimedean property*. For any x, there is a natural number n such that $x < n$.

(17) *Nested intervals property (NIP)*. Whenever $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ is a nested sequence of closed intervals, then there is a point p that belongs to every one of these intervals.

Remark 1. Axiom (17)—the NIP—is purely a statement about linear orders, which of course makes sense equally well in the context of ordered fields.

A structure satisfying Axioms (1)–(7) and (9) is a commutative ring. In this paper, all rings (Axioms (1)–(4), (6),(7),(9)) are assumed to be commutative and are assumed to have at least two elements. The additive identity of a ring is typically denoted 0 and is sometimes described as the “zero element”; if there is a multiplicative identity, it is denoted 1 and is sometimes described as the “one element.”

An *ideal* I in a (commutative) ring R is a subset of R with the following properties:

(1) For all $a, b \in I$, $a - b \in I$.

(2) For all $a \in I$ and $r \in R$, $ar \in I$.

(3) $0 \in I$.

I is a *proper* ideal in R if $I \neq R$. A proper ideal I is a *maximal ideal* if every ideal that properly contains I is improper; equivalently, if every ideal that properly contains I contains 1.

Given a ring R and an ideal I in R, one can form the *quotient ring* $R/I = \{r + I \mid r \in R\}$ (where $r + I$ is the set $\{r + i \mid i \in I\}$), with "zero element" $0 + I$, "one element" $1 + I$, and operations defined by

$$\begin{aligned} (r+I) + (s+I) &= (r+s) + I \\ (r+I) \cdot (s+I) &= (rs) + I \end{aligned}$$

When I is a maximal ideal in R, the quotient ring R/I is a field. These basic results can be found in any text on abstract algebra (e.g. [5], [6]).

An ordered field F must have (e.g. [11]) characteristic 0, so the sequence $1, 1+1, 1+1+1, \ldots$ has no repetitions and therefore provides a copy of $\mathbb{N}$ in F. In particular, letting s_n denote the n-fold sum of 1s in F, we let $N_F = \{s_n : n \in \mathbb{N}\}$. For a copy of $\mathbb{Z}$, we let Z_F be the members of N_F together with their additive inverses and 0; in particular, $N_F \subseteq Z_F$. And the F-version of $\mathbb{Q}$ is obtained just in the way that $\mathbb{Q}$ is obtained from $\mathbb{Z}$. Therefore, we let $Q_F = \{m \cdot n^{-1} : m \in Z_F, n \in N_F, n \neq 0\}$. It is easy to show that the F-versions of these number systems are "isomorphic" to the originals in the appropriate ways (for instance, there is a natural *ring isomorphism* $\mathbb{Z} \to Z_F$). We assume further that $N_F \subseteq Z_F \subseteq Q_F \subseteq F$, and sometimes the original number systems are identified with their F-counterparts. The following is well-known [13, 2.1.4]:

Theorem 2.1 (*Denseness of the Rationals*) *In any Archimedean ordered field F, Q_F is dense and unbounded. In other words, for any $x < y$ in F, there are $q, r, s \in Q_F$ such that $q < x < r < y < s$.*

It is well-known (see [8]) that, in the presence of the other axioms, Axioms (16) and (17) together are equivalent to Dedekind-completeness. Since this fact will be important here, we outline a proof. See [8] for a fuller treatment. We begin by defining some notions and recording some facts about linear orders.

Definition 2.2 *Suppose $(L, <)$ is any linear order (possibly an ordered field). L is Dedekind-complete if every nonempty subset of F with an upper bound has a least upper bound.*

Definition 2.3 *Suppose $(L, <)$ is any linear order. A collection $\mathcal{C}$ of closed intervals in L is said to have the finite intersection property (fip) if every finite subcollection of $\mathcal{C}$ has nonempty intersection. L is said to be Cantor-complete if every countable collection of closed intervals having the fip has nonempty intersection.*

Theorem 2.4 (*Equivalence of NIP and Cantor Completeness*) *For a linear order L, L is Cantor-complete if and only if L satisfies the NIP.*

Proof. One direction is obvious. For the other direction, assume L satisfies the NIP; we show L is Cantor-complete. Let $\mathcal{C} = \{[a_1, b_1], [a_2, b_2], \ldots\}$ be a countably infinite collection of closed intervals satisfying the fip. For each n, let

$$[c_n, d_n] = [\max\{a_1, \ldots, a_n\}, \min\{b_1, \ldots, b_n\}].$$

Clearly $[c_1, d_1] \supseteq [c_2, d_2] \supseteq \cdots$ and for each n, $[c_n, d_n] \subseteq [a_n, b_n]$. By the NIP, there is a point p that belongs to each $[c_n, d_n]$, and hence also to each element of $\mathcal{C}$.■

Theorem 2.5 (*Completeness Equivalents Theorem I*) *Suppose F is an ordered field. Then the following are equivalent:*

(1) *F is Archimedean and satisfies the NIP.*

(2) *F is Dedekind-complete.*

Outline of Proof. For (1) ⇒ (2), let $A \subseteq F$ be nonempty with an upper bound b. If $b \in A$, b is the least upper bound. So assume $b \notin A$. Assume that A does not have a least upper bound. Let $\mathcal{C}$ be the collection of all closed intervals of the form $[q, r]$, where $q, r \in Q_F$, $q \leq a$ for some $a \in A$, and $r \leq b$ is an upper bound of A. $\mathcal{C}$ is nonempty because Q_F is dense and unbounded. Moreover, $\mathcal{C}$ is countable and has the fip (since for any finite subcollection, some point of A must belong to each member of the collection). Since F is Cantor-complete, $\mathcal{C}$ has nonempty intersection. We let x be an element of the intersection.

If x is not an upper bound for A, we can find elements $q, r \in Q_F$ so that for some $a \in A, x < q \leq a$, and r is an upper bound for A not belonging to A. Then the interval $[q, r]$ belongs to $\mathcal{C}$ but does not contain x, which is impossible. Therefore, x is an upper bound of A.

Since x is not the *least* upper bound of A, there is $r < x$, $r \in Q_F$, that is also an upper bound of A. Since A has no least upper bound, $r \notin A$. Let $a \in A$ and let $q \in Q_F$ be $\leq a$. Now $[q, r] \in \mathcal{C}$, but $x \notin [q, r]$, which is impossible. Therefore, the assumption that A has no least upper bound is false, and the result follows.

For (2) ⇒ (1), we first prove the Archimedean property: Suppose $b \in F$ is larger than every natural number. Then $\mathbb{N}$ has a least upper bound r. By leastness of r, $r-1 < n$ for some $n \in \mathbb{N}$, but then $r = (r-1)+1$ is less than $n+1$, which contradicts the fact that it is an upper bound for $\mathbb{N}$. To prove the NIP, if $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ is a nested sequence of closed intervals, observe that $\sup\{a_1, a_2, \ldots\}$ belongs to each interval. ∎

Essentially the same result holds in the context of linear orders, without the algebraic structure:

Theorem 2.6 (*Completeness Equivalents Theorem II*) *Suppose L is an unbounded linear order with a countable dense subset D. Then the following are equivalent:*

(1) *L satisfies the NIP.*

(2) *L is Dedekind-complete.*

Proof. The proof of (2) ⇒ (1) is the same as for Completeness Equivalents Theorem I, which used only the order-theoretic properties of F (note that in this case, the part of the proof that established the Archimedean property is not needed here).

For (1) ⇒ (2), we first note that the proof of Theorem 2.4 goes through for L since only order-theoretic properties of F were used in the proof. Similarly, the proof of (1) ⇒ (2) in Theorem 2.5 goes through for the same reason, replacing the set Q_F in that proof with the countable dense set D in the present context. ∎

Remark 2. The proof of (2) ⇒ (1) in the Completeness Equivalents Theorem II did not make use of the countable dense set in the hypothesis, whereas the proof of (1) ⇒ (2) did. In Section 2 we show that this hypothesis is indeed necessary (Theorem 3.10).

We state without proof the usual uniqueness theorems for $\mathbb{R}$. See [8, 13] for a proof of (1) and [9] for a proof of (2). If F, G are ordered fields, an *ordered field isomorphism from F to G* is a bijection $h : F \to G$ that is order-preserving and at the same time a field homomorphism. If L_1, L_2 are linear orders, an *order isomorphism from L_1 to L_2* is a bijection from L_1 to L_2 that is order-preserving.

Theorem 2.7 (*Uniqueness Theorem for* $\mathbb{R}$)

(1) *Any two Dedekind-complete ordered fields are ordered field isomorphic.*

(2) *Any two Dedekind-complete unbounded linear orders with a countable dense subset are order-isomorphic.*

Definition 2.8 (*Infinitesimals and Infinite Elements*) *Suppose F is an ordered field. An element $x \in F$ is a positive infinitesimal if $x < \frac{1}{n}$ for every $n \in \mathbb{N}$ and a negative infinitesimal if $-\frac{1}{n} < x < 0$ for every $n \in \mathbb{N}$. Similarly, x is a positive infinite element if $x > n$ for every $n \in \mathbb{N}$ and a negative infinite element if $x < -n$ for every $n \in \mathbb{N}$. Then, for every $x \in F$, we define:*

x is an infinitesimal $\Leftrightarrow$ $x = 0$ or x is either a positive or negative infinitesimal

x is an infinite element $\Leftrightarrow$ x is either a positive or negative infinite element.

We omit the straightforward proof of the following useful criterion for testing whether an ordered field is Archimedean.

Theorem 2.9 (*Archimedean Criterion*) *For any ordered field F, the following are equivalent:*

(1) *F is non-Archimedean.*

(2) *F contains an infinitesimal.*

(3) *F contains an infinite element.*

(4) *There is an open interval (a, b) in F that misses Q_F.*

We define some notions from set theory that will be needed. For any set A, we let $A^{\mathbb{N}}$ denote the set of all sequences of elements of A. Such a sequence is typically denoted $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$.

Definition 2.10 (*Ultrafilters*) *A collection U of subsets of $\mathbb{N}$ is a nonprincipal ultrafilter on $\mathbb{N}$ if*

(1) *(Nonempty) $\mathbb{N} \in U$.*

(2) *(Closed under intersections) Whenever $A, B \in U$, $A \cap B \in U$.*

(3) *(Closed under supersets) Whenever $A \in U$ and $A \subseteq B$, then $B \in U$.*

(4) *(Maximal) For all $A \subseteq \mathbb{N}$, either $A \in U$ or $\mathbb{N} - A \in U$.*

(5) *(Nontrivial) For all $n \in \mathbb{N}$, $\{n\} \notin U$.*

Intuitively, elements of a nonprincipal ultrafilter U are “big” and sets not in U are “small.” Nonprincipal ultrafilters exist and contain only infinite sets. Existence can be proven using Zorn’s Lemma or the Hausdorff Maximal Principle. Finite sets never belong to U because of the Nontrivial property: If a finite set $S = \{s_1, s_2, \ldots, s_n\} \in U$, then for some i, $\{s_i\} \in U$. (To see this, assume that $\mathbb{N} - \{s_i\} \in U$ for each i. It follows that the intersection $\mathbb{N} - \{s_1\} \cap \cdots \cap \mathbb{N} - \{s_n\} = \mathbb{N} - S$ is in U, which is impossible since $S \in U$ and since two disjoint sets cannot both belong to U.) A fuller treatment can be found in [12].

Theorem 2.11 (*Ultrafilter Theorem*) *A nonprincipal ultrafilter exists. Moreover, for every such ultrafilter U, if $A \in U$, then A is infinite.*

3 A "Complete" Non-Archimedean Ordered Field

The first step in our plan to complete the set $\mathbb{Q}$ of rationals is to "expand" the context by considering the set $\mathbb{Q}^{\mathbb{N}}$ of all sequences of rationals. We equip this set with natural operations $+, \cdot$ and an order relation $<$:

$$\begin{aligned} r+s &= \langle r_1, r_2, \ldots\rangle + \langle s_1, s_2, \ldots\rangle \\ &= \langle r_1+s_1, r_2+s_2, \ldots\rangle \\ r\cdot s &= \langle r_1, r_2, \ldots\rangle \cdot \langle s_1, s_2, \ldots\rangle \\ &= \langle r_1\cdot s_1, r_2\cdot s_2, \ldots\rangle \end{aligned}$$

Also, we will say $r < s$ if and only if $r_n < s_n$ for all n. We map $\mathbb{Q}$ into $\mathbb{Q}^{\mathbb{N}}$ by associating with each $q \in \mathbb{Q}$ the sequence $c^q = \langle q, q, q, \ldots\rangle$; this map gives us natural choices for the "zero" and "one" elements, namely, $z = c^0$ and $u = c^1$.

The structure $\mathbb{Q}^{\mathbb{N}}$ is certainly a commutative ring, but is neither a field nor an ordered ring.

Exercise 1. Show that $(\mathbb{Q}^{\mathbb{N}}, c^0, c^1, +, \cdot)$ is a commutative ring, but neither a field nor an ordered ring. Hint: Let $o = \langle 0, 1, 0, 1, \ldots\rangle$ and $e = \langle 1, 0, 1, 0, \ldots\rangle$. Show that $o \cdot e$ is the zero element, but neither factor is zero.

From this starting point we recover some of the lost structure of $\mathbb{Q}$ by identifying elements in a certain way. The idea in this step is to identify sequences that agree on a "big" set. So, for example, if it turns out that the set E of even numbers is one of our big sets, then, since for $n \in E$, $z_n = e_n$ (sequences are indexed starting with 1) and $u_n = o_n$, we identify e with the zero element and o with the unity element. Then the product $o \cdot e$ is still the zero element, but now one of the factors is also zero.

To be more precise, let U be a nonprincipal ultrafilter on $\mathbb{N}$. For $s, t \in \mathbb{Q}^{\mathbb{N}}$, we write $s \sim_U t$ if and only if $\{n \mid s_n = t_n\} \in U$ (so s and t agree on a big set). When the context makes the meaning clear, we write $s \sim t$ instead of $s \sim_u t$. We write

$$[s]_U = \{t \in \mathbb{Q}^{\mathbb{N}} \mid s \sim_U t\}.$$

Finally, we let

$$\mathbb{Q}^{\mathbb{N}}/U = \{[s]_U : s \in \mathbb{Q}^{\mathbb{N}}\}.$$

We write $[s]$ for $[s]_U$ when the context makes the meaning clear.

We define the distinguished elements "zero" and "one," the operations, and the order relation in the obvious way: The "zero" element is $[z]$; the "one" element is $[u]$; and, given $r, s \in \mathbb{Q}^{\mathbb{N}}$, we define addition and multiplication as follows:

$$\begin{aligned}[r] + [s] &= [r + s] \\ [r] \cdot [s] &= [r \cdot s]\end{aligned}$$

Finally, we declare $[r] <_U [s]$ if and only if $\{n \mid r_n < s_n\} \in U$, and we write $[r] < [s]$ instead of $[r] <_U [s]$ when the meaning is clear. Also, in the usual way, we write $[r] \leq [s]$ as shorthand for "$[r] < [s]$ or $[r] = [s]$."

Exercise 2. Show that these definitions do not depend on the choice of representatives. In particular, if $r' \sim r$ and $s' \sim s$, show that $r + s \sim r' + s'$ and $r \cdot s \sim r' \cdot s'$, and also that $[r] < [s]$ if and only if $[r'] < [s']$. Hint. For verification that multiplication is well-defined, show $S = \{n : r_n \cdot s_n = r'_n \cdot s'_n\} \in U$, and for this, show that $S \supseteq T = \{n : r_n = r'_n \text{ and } s_n = s'_n\}$.

We now wish to claim that $\mathbb{Q}^{\mathbb{N}}/U$ is an ordered field; proving the claim requires that we verify axioms (1)–(15) given in the Preliminaries section. Most of the verifications are straightforward. We discuss here multiplicative inverses and leave the rest as an exercise for the reader.

Theorem 3.1 *Axiom (8) holds in $\mathbb{Q}^{\mathbb{N}}/U$. In other words, every nonzero element of $Q^{\mathbb{N}}/U$ has a multiplicative inverse.*

Proof. Suppose $[s] \in \mathbb{Q}^{\mathbb{N}}/U$ and $s \not\sim z$. To compute the inverse, we first define another sequence s' of rationals as follows: For each n for which $s_n = 0$, we define $s'_n = 1$; for all other n, $s'_n = s_n$. Since $\{n : s_n = 0\} \notin U$, it follows that $\{n : s_n = s'_n\} \in U$, and so $[s] = [s']$.

Now define $[s]^{-1}$ to be the class $[\langle {s'_n}^{-1} : n \in \mathbb{N}\rangle]$. It is straightforward to verify that $[s]^{-1}$ is well-defined. We verify that $[s] \cdot [s]^{-1} = [u] = [\langle 1, 1, 1, \ldots\rangle]$. Let $t = [s]^{-1}$. Let $A = \{n \mid s_n \neq 0\}$; recall $A \in U$. For each $n \in A$, $s_n \cdot t_n = 1$. We have shown that $A \subseteq \{n \mid s_n \cdot t_n = u_n\}$, from which we may conclude $[s] \cdot [s]^{-1} = [u]$.■

Exercise 3. Verify that $\mathbb{Q}^{\mathbb{N}}/U$ satisfies axioms (1)–(7) and (9)–(15).

We show next that $\mathbb{Q}^{\mathbb{N}}/U$ is non-Archimedean. For this purpose, and for future arguments, it is helpful to have convenient representations of the number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ in $\mathbb{Q}^{\mathbb{N}}$. In the Preliminaries section, we reviewed the fact that these number systems are faithfully represented in any ordered field, denoted respectively by N_F, Z_F, and Q_F (where, in this context, $F = \mathbb{Q}^{\mathbb{N}}/U$).

We also have seen how these number systems are naturally mapped into $\mathbb{Q}^{\mathbb{N}}/U$ by $q \mapsto [c^q]$. It is helpful to observe that this mapping produces the same representatives for these number systems:

Exercise 4. Define N', Z', Q' by

$$\begin{aligned} N' &= \{[c^n] \mid n \in \mathbb{N}\} \\ Z' &= \{[c^i] \mid i \in \mathbb{Z}\} \\ Q' &= \{[c^q] \mid q \in \mathbb{Q}\} \end{aligned}$$

Show that $N_F = N', Z_F = Z', Q_F = Q'$.

In the sequel, we will freely identify $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ with these different representations in $\mathbb{Q}^{\mathbb{N}}/U$. We will now use the Archimedean Criterion mentioned in the Preliminaries section to show that $\mathbb{Q}^{\mathbb{N}}/U$ is non-Archimedean. In fact, if $s = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$ then $[s]$ is an infinitesimal: Given any $k \in \mathbb{N}$, $\{n \mid s_n < \frac{1}{k}\} \in U$. It follows that $[s] < [c^{1/k}]$. Since the result holds for all $k \in \mathbb{N}$, $[s]$ is an infinitesimal. We have shown that $\mathbb{Q}^{\mathbb{N}}/U$ is non-Archimedean.

Exercise 5. Give an example of an infinite element in $\mathbb{Q}^{\mathbb{N}}/U$.

Our final step is to show that the NIP holds in $\mathbb{Q}^{\mathbb{N}}/U$. We will actually prove something stronger: that $\mathbb{Q}^{\mathbb{N}}/U$ is *open-complete*—a notion that we define next.

Definition 3.2 *Suppose F is an ordered field. Then F is said to be open-complete if any nested sequence $(u_1, v_1) \supseteq (u_2, v_2) \supseteq \cdots \supseteq (u_n, v_n) \supseteq \cdots$ of open intervals has a common point.*

Hall [8] calls this property *algebraic saturation* because it follows from the "saturation property" of nonstandard models. We have chosen a more suggestive term for our present context.

We begin with some preliminaries that will facilitate our work with open intervals in $\mathbb{Q}^{\mathbb{N}}/U$. We will not treat open intervals as "open balls," as one does in a metric space context; we restrict ourselves to the order-theoretic meaning. The open intervals do, however, form a basis for a topology, known as the *interval topology*, as is the case for any linear order. We will not be too concerned with this topology here, except to note that, relative to this topology, the terms "open" and "closed" have their usual meanings.

Remark 3. The proofs and discussion for the rest of this section, up to Theorem 3.8, make use only of the order-theoretic properties of $\mathbb{Q}^{\mathbb{N}}/U$; this observation will be important in our proof of Theorem 3.10.

Suppose $[a], [b] \in \mathbb{Q}^{\mathbb{N}}/U$, where U is a nonprincipal ultrafilter on $\mathbb{N}$, and $a = \langle a_1, a_2, \ldots \rangle$, and $b = \langle b_1, b_2, \ldots \rangle$. Suppose also that $[a] \leq [b]$. The definitions of *open interval* and *closed interval* in $\mathbb{Q}^{\mathbb{N}}/U$ derive from $\mathbb{Q}^{\mathbb{N}}/U$'s property of being a linear order. We spell out these definitions here: We denote an open interval from $[a]$ to $[b]$ by $I_{a,b}$; it is defined by

$$I_{a,b} = \{[x] \in \mathbb{Q}^{\mathbb{N}}/U \mid [a] < [x] < [b]\}.$$

We denote the *closed interval from* $[a]$ *to* $[b]$ *in* $\mathbb{Q}^{\mathbb{N}}/U$ by $C_{a,b}$; it is defined by

$$C_{a,b} = \{[x] \in \mathbb{Q}^{\mathbb{N}}/U \mid [a] \leq [x] \leq [b]\}.$$

Exercise 6. Show that the notations $I_{a,b}$ and $C_{a,b}$ do not depend on the choice of a, b. In other words, show that if $[a] = [c]$ and $[b] = [d]$, then $I_{a,b} = I_{c,d}$ and $C_{a,b} = C_{c,d}$. Hint. To show that whenever $[x] \in I_{a,b}$, $[x] \in I_{c,d}$, note that $A \cap B \in U$, where $A = \{n \mid a_n < x_n < b_n\}$ and $B = \{n \mid a_n = c_n \text{ and } b_n = d_n\}$. Then show that $A \cap B \subseteq \{n \mid c_n < x_n < d_n\}$.

Lemma 3.3 *Suppose $I_{a,b}$ is an open interval in $\mathbb{Q}^{\mathbb{N}}/U$. Then $I_{a,b}$ is nonempty. In particular, there is $[x] \in \mathbb{Q}^{\mathbb{N}}$ with $[a] < [x] < [b]$.*

Proof. By definition of $I_{a,b}$, $[a] < [b]$, so there is a set $A \in U$ such that $A = \{n \mid a_n < b_n\}$. Since the a_n and b_n are rationals, we can find, for each $n \in A$, a $c_n \in \mathbb{Q}$ with $a_n < c_n < b_n$. For $n \notin A$, let c_n be an arbitrary rational, and let $c = \langle c_1, c_2, \ldots, c_n, \ldots \rangle$. Since $A \subseteq \{n \mid a_n < c_n < b_n\}$, it follows that $[a] < [c] < [b]$.■

Lemma 3.4 *Suppose $I_{a,b}$ and $I_{c,d}$ are open intervals in $\mathbb{Q}^{\mathbb{N}}/U$. Then $I_{c,d} \subseteq I_{a,b}$ if and only if $[a] \leq [c]$ and $[d] \leq [b]$.*

Proof. Let $a, b, c, d \in \mathbb{Q}^{\mathbb{N}}$, where

$$\begin{aligned} a &= \langle a_1, a_2, \ldots, a_n, \ldots \rangle \\ b &= \langle b_1, b_2, \ldots, b_n, \ldots \rangle \\ c &= \langle c_1, c_2, \ldots, c_n, \ldots \rangle \\ d &= \langle d_1, d_2, \ldots, d_n, \ldots \rangle \end{aligned}$$

and $[a] < [b]$ and $[c] < [d]$.

For one direction, assume $I_{c,d} \subseteq I_{a,b}$. For a contradiction, we assume $[a] > [c]$ or $[d] > [b]$. We show that assuming $[a] > [c]$ leads to a contradiction; the proof that $[d] > [b]$ also leads to a contradiction is basically the same, so we omit it.

Note that if $[a] \geq [d]$, it follows easily that $I_{a,b} \cap I_{c,d} = \emptyset$, so we assume $[a] < [d]$. Using Lemma 3.3 twice, we can find c' so that

$$[c] < [c'] < [a] \text{ and } [c] < [c'] < [d].$$

It follows that $[c'] \notin I_{a,b}$ but $[c'] \in I_{c,d}$, contradicting the fact that $I_{c,d} \subseteq I_{a,b}$.

For the other direction, suppose

$$(*) \qquad [a] \leq [c] \text{ and } [d] \leq [b].$$

Let $[x] \in I_{c,d}$, so $[c] < [x] < [d]$. Now it is clear that $(*)$ implies that $[a] < [x] < [b]$, as required.∎

The next lemma shows that arguments about open intervals in $\mathbb{Q}^{\mathbb{N}}/U$ can be reduced to arguments about sequences of rational open intervals. This is a crucial piece of machinery from nonstandard analysis that yields the surprising conclusion that $\mathbb{Q}^{\mathbb{N}}/U$ is open-complete. The nonstandard analyst will recognize this proposition as saying that *open intervals are internal.*

Suppose $a < b$ belong to $\mathbb{Q}$. The *rational open interval* (a, b) is the set $\{x \in \mathbb{Q} \mid a < x < b\}$.

Lemma 3.5 *Suppose a, b, c, d are sequences, as defined above. Then*

$$I_{c,d} \subseteq I_{a,b} \textit{ if and only if } \{n \mid (c_n, d_n) \subseteq (a_n, b_n)\} \in U,$$

where (a_n, b_n) and (c_n, d_n) denote the rational open intervals from a_n to b_n and from c_n to d_n, respectively.

Proof. Let $C = \{n \mid (c_n, d_n) \subseteq (a_n, b_n)\}$. Let $S = \{n \mid a_n \leq c_n\}$ and let $T = \{n \mid d_n \leq b_n\}$.

For one direction, suppose $I_{c,d} \subseteq I_{a,b}$. By Lemma 3.4, $[a] \leq [c]$ and $[d] \leq [b]$, and so each of S, T is in U. It follows that $S \cap T \in U$. To complete this direction, it is enough to show that $S \cap T \subseteq C$. For this, suppose $n \in S \cap T$, so that $a_n \leq c_n$ and $d_n \leq b_n$. It follows from this that, as open intervals, $(c_n, d_n) \subseteq (a_n, b_n)$, as required.

Conversely, suppose $C \in U$. Then, since $C \subseteq S$ and $C \subseteq T$, we have $S \in U$ and $T \in U$. It follows that $[a] \leq [c]$ and $[d] \leq [b]$, whence $I_{c,d} \subseteq I_{a,b}$.∎

We now turn to the proof that a given nested sequence of open intervals has a common point. For the nonstandard analyst, our proof is simply a special case of the *saturation property* that nonstandard models always have; the saturation property states that *the intersection of countably many nested internal sets is nonempty.*

Before obtaining an element that belongs to each open interval of a given sequence, we need to establish that expected relationships between the corresponding rational intervals hold almost everywhere. Suppose

$$I_{a^1,b^1} \supseteq I_{a^2,b^2} \supseteq \cdots \supseteq I_{a^k,b^k} \supseteq \cdots$$

is a nested sequence of open intervals in $\mathbb{Q}^{\mathbb{N}}/U$, where, for each k, $a^k = \langle a_1^k, a_2^k, \ldots, a_n^k, \ldots\rangle$ and $[a^k] < [b^k]$. For each k let

$$S^k = \{n \mid (a_n^1, b_n^1) \supseteq (a_n^2, b_n^2) \supseteq \cdots \supseteq (a_n^k, b_n^k) \text{ and } a_n^k < b_n^k\}.$$

Let $\hat{S}^k = \{n \geq k \mid n \in S^k\}$.

Lemma 3.6 *For each k, $S^k \in U$ and $\hat{S}^k \in U$.*

Proof. We proceed by induction on $k \geq 1$ to prove $S^k \in U$. For $k = 1$, this follows from the assumption that $[a^1] < [b^1]$. Assuming the result for k, notice

$$S^{k+1} = S^k \cap W_k, \text{ where } W_k = \{n \mid (a_n^k, b_n^k) \supseteq (a_n^{k+1}, b_n^{k+1})\}.$$

But $W_k \in U$ since $I_{a^k,b^k} \supseteq I_{a_n^{k+1},b_n^{k+1}}$. This completes the proof that $S^k \in U$. The proof that $\hat{S}^k \in U$ follows from the fact that $\{n \mid n \geq k\} \in U$. ∎

For each n for which $a_n^1 < b_n^1$ (that is, for each $n \in S^1$), we compute the *max index for n*, denoted $\kappa(n)$, as follows:

$$\kappa(n) = \text{largest } k \leq n \text{ for which } (a_n^1, b_n^1) \supseteq \cdots \supseteq (a_n^k, b_n^k) \text{ and } a_n^k < b_n^k.$$

In other words, $\kappa : S^1 \to \mathbb{N}$ is defined so that, for all $n \in S^1$,

$$\kappa(n) = \max\{k \leq n \mid n \in S_n^k\}.$$

Now we obtain an element $[x] \in \mathbb{Q}^{\mathbb{N}}/U$ which we will claim lies in each of the open intervals. For $n \in S^1$, x_n is chosen to be any element of $(a_n^1, b_n^1) \cap \cdots \cap (a_n^{\kappa(n)}, b^{\kappa(n)})$. For $n \notin S^1$, $x_n = 0$. For each k, let $T_x^k = \{n \mid a_n^k < x_n < b_n^k\}$.

Lemma 3.7 *For each k, $T_x^k \in U$. Hence $[x]$ belongs to each of the intervals I_{a^k,b^k}.*

Proof. We show that, for each k, $\hat{S}^k \subseteq T_x^k$. Let n be such that $n \geq k$ and $n \in S^k$. We show that $n \in T_x^k$. Since $k \leq \kappa(n)$, $x_n \in (a_n^1, b_n^1) \cap \cdots \cap (a_n^k, b_n^k)$. In particular, $a_n^k < x_n < b_n^k$. Therefore $x_n \in T_x^k$. This completes the proof that $T_x^k \in U$ for each k, and hence that $[x]$ belongs to each of the intervals I_{a^k,b^k}. ∎

We now show that the NIP follows from open-completeness. Suppose $C_{a^1,b^1} \supseteq C_{a^2,b^2} \supseteq \cdots \supseteq C_{a^k,b^k} \supseteq \cdots$ is a nested sequence of closed intervals in $\mathbb{Q}^{\mathbb{N}}/U$. If for some k, either $[a_k]$ or $[b_k]$ occurs as an endpoint infinitely many times, this endpoint must in fact be a common point for all the closed intervals. Otherwise, we may conclude that, for all k, $[a^k] < [b^k]$ (if $[a^k] = [b^k]$ for some k, then for all $m \geq k$, $[a^m] = [a^k]$ and $[b^m] = [b^k]$), and we consider the corresponding open subintervals $I_{a^1,b^1} \supseteq I_{a^2,b^2} \supseteq \cdots \supseteq I_{a^k,b^k} \supseteq \cdots$. By Lemma 3.7, this sequence of open intervals has a common point $[x]$, and, clearly, $[x]$ is also a common point for the original sequence of closed intervals. This completes the proof that $\mathbb{Q}^{\mathbb{N}}/U$ is a non-Archimedean ordered field satisfying the NIP.

We have proven the following:

Theorem 3.8 *The structure $\mathbb{Q}^{\mathbb{N}}/U$ is a non-Archimedean ordered field that satisfies NIP. Indeed, $\mathbb{Q}^{\mathbb{N}}/U$ is open-complete.* ∎

Exercise 7. Identify $Q' \subseteq \mathbb{Q}^{\mathbb{N}}/U$ with the set of rationals $\mathbb{Q}$. Show that there is a nonempty rational open interval (a,b) in $\mathbb{Q}^{\mathbb{N}}/U$, with $0 < a$, so that (a,b) is a subset of every one of the following intervals in $\mathbb{Q}^{\mathbb{N}}/U$:

$$(0,1), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \ldots, \left(0, \frac{1}{n}\right), \ldots$$

As was pointed out in Remark 3, the discussion following Remark 3 up to Theorem 3.8 does not make use of the algebraic properties of $\mathbb{Q}^{\mathbb{N}}/U$—only its order-theoretic properties are used. These arguments therefore allow us to conclude that $\mathbb{Q}^{\mathbb{N}}/U$ is open-complete (hence, satisfies the NIP), as a linear order. We establish several other facts about $\mathbb{Q}^{\mathbb{N}}/U$ as a linear order as preparation for Theorem 3.10.

Lemma 3.9 *The linear order $\mathbb{Q}^{\mathbb{N}}/U$ has the following properties:*

(1) *It is unbounded.*

(2) *It has no countable dense set.*

(3) *It is not Dedekind-complete; indeed, $Q' = \{[c^q] \mid q \in \mathbb{Q}\}$ is a set with an upper bound but with no supremum.*

Proof. For (1), notice that if $[x] \in \mathbb{Q}^{\mathbb{N}}/U$, we can find, for each $n \in \mathbb{N}$, a rational y_n so that $y_n > x_n$. Letting $y = \langle y_1, y_2, \ldots \rangle$, it is clear that $[x] < [y]$.

For (2), let $\langle [a^1], [a^2], \ldots, [a^n], \ldots \rangle$ be an enumeration of a countable set A. It suffices to obtain an upper bound $[x]$ for this set since then we may conclude that A misses the open interval $I_{x,x+1}$.

Reasoning as in Lemma 3.7, we first show that every sequence of nested open intervals of the form $I_{a^1,\infty}, I_{a^2,\infty}, \ldots$ has a common point. As in the remarks preceding Lemma 3.6, define, for each k,

$$S^k = \{n \mid (a_n^1, \infty) \supseteq (a_n^2, \infty) \supseteq \cdots \supseteq (a_n^k, \infty)\},$$

where, in this case, we let $S^1 = \mathbb{N}$. A proof by induction shows each S^k belongs to U. As before, we let $\kappa : \mathbb{N} \to \mathbb{N}$ compute the max index of n: $\kappa(n) = \max\{k \leq n \mid n \in S^k\}$. Then define $[x]$ by letting x_n be an element of $(a_n^1, \infty) \cap (a_n^2, \infty) \cap \cdots \cap (a_n^{\kappa(n)}, \infty)$. We observe that, for each k and each $n \geq k$, if $n \in S^k$, then $x \in (a_n^1, \infty) \cap (a_n^2, \infty) \cap \cdots \cap (a_n^k, \infty)$ since $k \leq \kappa(n)$. Since $S^k \cap \{n \mid n \geq k\} \in U$ and

$$S^k \cap \{n \mid n \geq k\} \subseteq T_x^k = \{n \mid x_n \in (a_n^k, \infty)\},$$

it follows that $T_x^k \in U$ for each k, and so $[x]$ is a common point of $I_{a^1,\infty}, I_{a^2,\infty}, \ldots$. Clearly $[x]$ is the required upper bound on A.

Finally, to prove (3), we show Q' is bounded but has no least upper bound. Since $\mathbb{Q}^{\mathbb{N}}/U$ has infinite elements, Q' is bounded; let $[x]$ be an upper bound. Since Q' is unbounded (in itself), $[x] \notin Q'$. Now consider the collection $\mathcal{O} = \{I_{q,x} \mid q \in \mathbb{Q}\}$. Clearly $\mathcal{O}$ is countable, so, by open-completeness, there is a $[y]$ in its intersection. Now, clearly, $[y]$ is an upper bound of Q' that is smaller than $[x]$. ∎

Note that Theorem 3.8 already implies that $\mathbb{Q}^{\mathbb{N}}/U$ is not Dedekind-complete. We have given a different argument here that does not depend on the notion of an Archimedean ordered field.

We may summarize our results on $\mathbb{Q}^{\mathbb{N}}/U$ as a linear order as follows:

Theorem 3.10 *The linear order $\mathbb{Q}^{\mathbb{N}}/U$ is an unbounded linear order that satisfies the NIP, has no countable dense subset, and is not Dedekind-complete.*∎

4 Special Properties of $\mathbb{Q}^{\mathbb{N}}/U$

Many of the pathological properties of $\mathbb{Q}^{\mathbb{N}}/U$ can be found in the representative of $\mathbb{N}$ in $\mathbb{Q}^{\mathbb{N}}/U$, namely, the set $N' = \{[c^n] \mid n \in \mathbb{N}\}$; we identify N' with $\mathbb{N}$. One consequence is that we can write an interval of the form I_{c^m,c^n} simply as $I_{m,n}$.

Lemma 4.1 *N' is closed discrete. In other words, for every $[x] \notin N'$ there is an open interval $I_{a,b}$ such that $[x] \in I_{a,b}$ and $I_{a,b} \cap N' = \emptyset$.*

Proof. If $[x] < [c^n]$ for some n, let n_0 be the least such n. If $n_0 = 1$, then let $[y] < [x]$ and observe that $[x] \in I_{y,1}$ and $I \cap N' = \emptyset$. If $n_0 > 1$, then I_{n_0-1,n_0} contains $[x]$ and misses N'.

On the other hand, if $[x]$ lies above $[c^n]$ for all $n \in \mathbb{N}$—and hence is an infinite element—then we claim there is $[y] < [x]$ that also lies above every $[c^n]$: Suppose not. Let $[y] = [x] - [c^1]$. Since $[y]$ is not an upper bound of N', there is some $[c^n]$ such that $[y] \leq [c^n]$. Therefore, $[x] = [y] + [c^1] \leq [c^{n+1}]$, which is impossible because $[x]$ is an infinite element. We let $[y]$ be infinite with $[y] < [x]$ and let $[z] = [x] + [c^1]$. Then $[x] \in I_{y,z}$ and $I_{y,z} \cap N' = \emptyset$.∎

Corollary 4.2 *N' is closed and bounded but not compact.*

Proof. In Lemma 4.1, we showed N' is closed. Certainly $[z]$ is a lower bound and any infinite element is an upper bound for N', so N' is bounded. For each $[c^n]$, form the open interval $I(n) = I_{n-\frac{1}{2},n+\frac{1}{2}}$. For each n, $[c^n] \in I(n)$, and the $I(n)$ are pairwise disjoint. Therefore $\mathcal{C} = \{I(n) \mid n \in \mathbb{N}\}$ is a cover of N' with no finite subcover.∎

Corollary 4.3 *N' is a bounded sequence with no convergent subsequence.*

Proof. We have already shown N' is bounded. Since it has no limit points and no repetitions as a sequence, none of its subsequences converge. ∎

Corollary 4.4 *There is a decreasing nested sequence $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq$ of closed and bounded sets having empty intersection.*

Proof. Notice that every subset of N' is closed. Therefore, if we let $K_n = N' - \{[c^1], \ldots, [c^n]\}$, $\mathcal{C} = \{K_n \mid n \in \mathbb{N}\}$ has the required properties.∎

Corollary 4.5 *There is no countable base at 0. Therefore, in the interval topology, $Q^{\mathbb{N}}/U$ is not metrizable.*

Proof. The "therefore" clause follows from the fact that every metrizable space is first countable. Let $I_{a^1,b^1}, I_{a^2,b^1}, \ldots$ be a countable collection of open intervals containing 0. Consider the collection $I_{0,b_1}, I_{0,b_2}, \ldots$. By open-completeness, there is a point y common to all these intervals. Consider also the sequence $I_{a_1,0}, I_{a_2,0}, ...$; by open-completeness, there is a point x common to all these intervals. Now (x, y) contains 0 but no (a_n, b_n) is a subset of (x, y).∎

Theorem 4.6 *Every increasing sequence in $\mathbb{Q}^{\mathbb{N}}/U$ is bounded.*

Proof. This is essentially the same as the proof of Lemma 3.9(2).∎

Recalling that an ordered field is *Cauchy-complete* if every Cauchy sequence converges, the following is a consequence of Corollary 6.4 of [8]:

Theorem 4.7 ([8]) *In $\mathbb{Q}^{\mathbb{N}}/U$, every convergent sequence is eventually constant. Therefore, $\mathbb{Q}^{\mathbb{N}}/U$ is Cauchy-complete.* ∎

5 Constructing $\mathbb{R}$ from Any Non-Archimedean Ordered Field Having the NIP

In this section we complete the second stage of our construction of the real line. To simplify notation and introduce a slight generalization, we begin with an arbitrary non-Archimedean ordered field $(X, <)$ that satisfies the NIP and show how, by removing infinite elements and modding out by the set of infinitesimals, we produce an *Archimedean* ordered field satisfying the NIP. By the Uniqueness Theorem, this structure must be ordered field isomorphic to $\mathbb{R}$.

As described in the Preliminaries section, we assume $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq X$. Since X is non-Archimedean, it contains infinite and infinitesimal elements. We define:

$$\begin{aligned} I &= \{x \in X \mid x \text{ is infinitesimal}\} \\ J &= \{x \in X \mid x \text{ is an infinite element}\} \end{aligned}$$

We let $X_{fin} = X - J$. X_{fin} still satisfies most of the properties of an ordered field:

Exercise 8. Show that X_{fin} is an ordered ring and that every $x \in X_{fin} - I$ has an inverse that also belongs to $X_{fin} - I$.

In the usual way, we let $X_{fin}/I = \{r + I \mid r \in X_{fin}\}$ denote the quotient ring by I, with the usual operations of $+, \cdot$. Our plan is to show that X_{fin}/I is an Archimedean ordered field satisfying the NIP. We will first observe that X_{fin}/I is a field (since I is a maximal ideal); then we will show that it is in fact an *ordered* field, because the order relation on X_{fin} satisfies one additional property that is needed; we then observe that X_{fin}/I is Archimedean, and finally that it satisfies the NIP. This final step will rely on the fact that X itself satisfies the NIP.

To indicate two elements r, s of X_{fin} are equivalent mod I, we write $r \simeq s$, which holds if and only if $r - s \in I$.

Lemma 5.1 *In X_{fin}, I is a maximal ideal. Therefore, X_{fin}/I is a field.*

Proof. We first verify that I is an ideal. Let $a, b \in I$. Without loss of generality, assume $a - b \geq 0$. Let $n \in \mathbb{N}$. Then $a - b < \frac{1}{n}$ because each of a and b is less than $\frac{1}{2n}$ and $a - b < a + b < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$. Next, suppose $a \in I$ and $x \in X_{fin}$. Let $n \in \mathbb{N}$. If $x \in I$, then since both a and x are less than $\frac{1}{n}$, their product is less than $\frac{1}{n^2}$, which is also less than $\frac{1}{n}$. If $x \notin I$, then by Exercise 8, x has an inverse x^{-1} also not in I. Since X_{fin} contains no infinite elements, we can find $m \in \mathbb{N}$ with $m > x$. It follows that $\frac{1}{m} < x^{-1}$ and so $\frac{1}{mn} < \frac{1}{n} \cdot x^1$. Since $a < \frac{1}{mn} < \frac{1}{n} \cdot x^{-1}$, it follows that $ax < \frac{1}{n}$, as required.

To show I is a maximal ideal, suppose $J \supseteq I$ is a proper superset of I, and let $x \in J - I$. By Exercise 8 again, x has an inverse x^{-1} that is not infinite. Therefore, since J is an ideal, $1 = x \cdot x^{-1} \in J$, from which it follows that $J = X_{fin}$. We have shown that any ideal in X_{fin} that properly contains I must be an improper ideal. Hence I is maximal, and X_{fin}/I is a field.∎

Next we define an order relation on X_{fin}/I, which, by an abuse of notation, we will denote $<$: For each $r, s \in X_{fin}$,

$$r + I < s + I \text{ if and only if } r < s \text{ and } r - s \notin I.$$

For arbitrary ordered rings R with maximal ideal I, defining $<$ in this way does not make R/I an ordered field, but if the order relation on R has one additional property, we are able to draw this conclusion.

Definition 5.2 *Suppose $(R, <)$ is an ordered ring with maximal ideal I. The relation $<$ respects I if, whenever $a \in I$ and $b \leq a$, then $b \in I$.*

Lemma 5.3 *Suppose $(R, <)$ is an ordered ring with maximal ideal I and suppose $<$ respects I. If we define $<$ on R/I by $r + I < s + I$ if and only if $r < s$ and $r - s \notin I$, then $(R/I, <)$ is an ordered field.*

Proof. We first verify that $<$ is well-defined on R/I. We show that if $r \simeq r'$ and $s \simeq s'$ and $r + I < s + I$, then $r' + I < s' + I$. It suffices to show that $r' < s'$ and $r' - s' \notin I$. Let $i = r - r', j = s - s'$. If $r' \not< s'$, it is not possible for $r' = s'$ since this would imply $r \simeq s$. If $r' > s'$, then $r - i > s - j$ and so $0 < s - r < j - i \in I$, which is impossible. We have shown $r' < s'$.

We show $r' - s' \notin I$. Suppose not. Let $k = r' - s' \in I$. But now

$$r - s = r - r' + r' - s' + s' - s = i + k + j \in I,$$

which is impossible. Therefore $r' - s' \notin I$, and $<$ is well-defined on R/I.

We verify that $<$ satisfies axioms (10)–(15). The antireflexive and antisymmetric laws hold because they hold in R. We verify transitivity, which requires that $<$ respects I: Suppose $r, s, t \in R$ and $r + I < s + I < t + I$. Then $r < s < t$ implies $r < t$ by transitivity of $<$ in R. We also have $r - s \notin I$ and $s - t \notin I$. We verify that $r - t \notin I$. Suppose not. Then it follows that both $r - t$ and $t - r$ belong to I. We have, since $t - s > 0$,

$$0 < s - r \leq (s - r) + (t - s) = t - r \in I.$$

Since $<$ respects I, $s - r \in I$, and this contradicts the fact that $r + I < s + I$. Therefore, $r - t \notin I$ and $<$ on R/I is transitive.

Next we check that $<$ is a total order on R/I: Given $r+I, s+I \in R/I$, assume $r+I \neq s+I$, so $r-s \notin I$. Since $<$ is total for R, exactly one of $r<s$, $s<r$ holds; but then this implies exactly one of $r+I < s+I$, $s+I < r+I$ holds.

It is straightforward to verify axiom (14). For axiom (15), given $r+I, s+I \in R/I$ with $0+I < r+I$ and $0+I < s+I$. Note that this condition implies $r \notin I$ and $s \notin I$. By properties of $<$ in R, we have that $0 < rs$. We must prove $rs \notin I$. If $rs \in I$, since $r < rs$ and $<$ respects I, it would follow that $r \in I$, which is not possible. Thus $rs \notin I$ and we conclude that $0+I < rs+I$. ■

Lemma 5.3 implies that X_{fin}/I is an ordered field. The fact that X_{fin}/I is also Archimedean is now obvious, since X_{fin} has no infinite elements. We show that X_{fin}/I satisfies the NIP.

Theorem 5.4 *X_{fin}/I satisfies the NIP. Therefore, X_{fin}/I is a Dedekind-complete ordered field.*

Proof. Suppose $[a_1+I, b_1+I] \supseteq [a_2+I, b_2+I] \supseteq \cdots \supseteq [a_n+I, b_n+I] \supseteq \cdots$ is a nested sequence of closed intervals in X_{fin}/I. If for some n, $a_n+I = b_n+I$, then a_n+I is a common point. So, we assume that $a_n+I < b_n+I$ for each n. We would like to consider the sequence $[a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n], \cdots$ in X, or the corresponding sequence of open intervals, to obtain a common point, but this approach does not quite work since the a_i and b_i are only *representatives* of the classes a_i+I, b_i+I; it is therefore possible that $[a_2, b_2] \not\subseteq [a_1, b_1]$ or even that no two of these intervals in X are comparable under $\subseteq$. What can be shown is that they satisfy the finite intersection property—any finite subcollection has a nonempty intersection—and it can be shown that the NIP in X is sufficient to conclude that there is a common point.

For the details, we will take a different route. We will define another sequence that truly is nested and descending: $[c_1, d_1] \supseteq [c_2, d_2] \supseteq \cdots \supseteq [c_n, d_n] \supseteq \cdots$, with the additional property that, for every n, $[c_n, d_n] \subseteq [a_n, b_n]$. The new sequence of intervals is defined like this: $c_1 = a_1, d_1 = b_1$, and, for each n,

$$c_{n+1} = \max\{c_n, a_{n+1}\} \qquad d_{n+1} = \min\{d_n, b_{n+1}\}.$$

An easy induction shows that $c_n \leq c_{n+1}$ and $d_{n+1} \leq d_n$ for every n.

To prove that $c_n < d_n$ for every n, we show first that

$$(*) \qquad \text{for every } n, c_n \simeq a_n \text{ and } d_n \simeq b_n.$$

Notice that if we can establish $(*)$, it will follow that for every n, $c_n < d_n$: Assuming $(*)$, since $a_n+I < b_n+I$, we have $a_n < b_n$, $a_n \not\simeq b_n$, $a_n \simeq c_n$ and $b_n \simeq d_n$. Because $<$ is well-defined, it follows that $c_n < d_n$.

To prove $(*)$, we proceed by induction. The base case $c_1 < d_1$ follows because $a_1 + I < b_1 + I$. For the induction step, assume $c_n \simeq a_n$ and $d_n \simeq b_n$. For the proof that $a_{n+1} \simeq c_{n+1}$, there are two cases. The first is that $a_n + I = a_{n+1} + I$ and the second is $a_n + I < a_{n+1} + I$. There are two similar cases for the proof that $b_{n+1} \simeq d_{n+1}$, which we leave to the reader.

For the first case, where $a_{n+1} + I = a_n + I$, we have

$$a_{n+1} \simeq a_n \simeq c_n \simeq \max\{c_n, a_{n+1}\} = c_{n+1}.$$

For the second case, where $a_n + I < a_{n+1} + I$, we have $a_n < a_{n+1}$ and $a_n \not\simeq a_{n+1}$. Since $c_n \simeq a_n$, it follows that $c_n \not\simeq a_{n+1}$. We claim $c_n < a_{n+1}$. If not, then $c_n > a_{n+1}$ and we would have $a_n < a_{n+1} < c_n$ and

$$(**) \qquad c_n - a_{n+1} \leq c_n - a_n \in I.$$

Since $<$ in X respects I, $(**)$ implies $c_n - a_{n+1} \in I$, contradicting our earlier observation. Thus, in this case, $c_{n+1} = \max\{c_n, a_{n+1}\} = a_{n+1}$, establishing once again the desired result.

This completes the induction. We have established that, for each n, $[c_n, d_n] \subseteq [a_n, b_n]$, $c_n < d_n$, and $[c_{n+1}, d_{n+1}] \subseteq [c_n, d_n]$. Now we may use the fact that X satisfies the NIP to obtain a point $x \in X$ that belongs to all the intervals $[c_n, d_n]$, and hence to each $[a_n, b_n]$. Since $b_n \in X_{fin}$, $x \in X_{fin}$; it follows that $x + I$ belongs to each of the intervals $[a_n + I, b_n + I]$. We have shown that X_{fin}/I satisfies the NIP. ■

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