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The Axiom of Infinity, Quantum Field Theory, Large Cardinals
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Abstract. We address the long-standing problem of finding an axiomatic foundation for large cardinals. We suggest that a compelling intrinsic justification for large cardinals should be based on a clear intuition about the meaning of "infinite." We observe that the intuition concerning the infinite that is implicit in the Axiom of Infinity of ZFC is too limited to be useful for deciding about the legitimacy of large cardinals. We propose an alternative version of the axiom having the same mathematical content but based on a much richer intuition concerning the infinite. We draw the intuition from a perspective that was common in more ancient philosophies concerning the natural numbers. These ancient philosophies considered that what "imparts" an infinite nature to the natural numbers is the character of the *source* of those numbers. The natural numbers, and indeed all diversity of existence, were seen as a side effect of processes going on within a unified substrate. We point out that this world view—that diveristy is a precipitation of the dynamics of some source—is the current intuitive model underlying quantum field theory: Particles are seen as secondary side effects, precipitations of underlying quantum fields. Seeking to represent this perspective axiomatically, we replace the usual Axiom of Infinity with an axiom asserting *There is a Dedekind self-map*—that is, an injective function $j : A \to A$ having a critical point a (a point a not in $\operatorname{ran} j$). We recall the work of Dedekind who showed how the natural numbers may be viewed as "precipitates" of the interaction of j with its critical point. We propose to use the dynamics by which the natural numbers emerge from j and a as an intuitive model for building a global Dedekind self-map $j : V \to V$ from which large cardinals may also be seen to emerge as "precipitates" of j. We observe that a Dedekind self-map on a set exhibits strong preservation properties and critical point dynamics, and leads to a kind of blueprint which, via Mostowski collapse, produces the set of natural numbers with its successor function. Using these observations, we gradually enrich a bare Dedekind self-map $j : V \to V$ by adding preservation properties and critical point dynamics in such a way that inaccessible, ineffable, remarkable, and measurable cardinals are shown to emerge. Introducing still stronger preservation and critical point requirements, we show how still stronger large cardinals emerge and ever greater portions of the universe itself emerge from a blueprint that is generated. Ultimately, based on this sequential development, we propose an axiom schema that accouts for all large cardinals through super-n-huge for every n. We conclude by arguing that the principles for generalization that have been used to arrive at this schema can be reasonably extended to account for the existence of a Reinhardt cardinal (without AC) and for large cardinals arising from the axioms I_1, I_2, I_3. Finally, we review recent work that attempts to use the axiomatic system that we propose here as a mathematical foundation for the ontological interpretation of quantum mechanics, due to D. Bohm.

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§1 Introduction Large cardinals have been a vital aspect of set theory for at least 60 years and yet there is no generally agreed upon axiomatic foundation from which some or all the known large cardinals can be derived. As far back as the 1940s, Gödel had envisioned the eventual emergence of natural axioms that would account for some or all (reasonable) large cardinals (cf. Kanamori, (1994), p. XIXff), but, though there are many candidates for such axioms and many plausibility arguments that go with them, none of these candidates has "forced themselves upon us" in the way that Gödel felt truly natural axioms do (Wang, 1996, p. 226).

The obvious problem with *not* having an axiomatic foundation for large cardinals is that it makes it impossible to establish results about large cardinals generally. One cannot even prove the following statement (in the metatheory):

It is impossible to prove the existence of large cardinals from ZFC,
unless ZFC is inconsistent.

The statement cannot be proved because we don't have axioms that specify which large cardinals exist or even what properties all large cardinals have in common; in fact, we do not even have a generally agreed upon *definition* of large cardinal.

There are different ways to go about discovering appropriate axioms that could be used to justify large cardinals. Gödel is known for suggesting a number of approaches. He classified intuitive justifications for such axioms as being either *intrinsic* on the one hand, or, on the other hand *pragmatic* (Wang, 1996, pp. 244–5) or *extrinsic* (Koellner, 2009, p. 2). An intrinsic justification appeals to the mathematical intuition as being correct in principle,[1] whereas a pragmatic or extrinsic justification is based on rich and intuitively appealing consequences that such an

[1] In the contemporary literature, one often encounters the assumption that by "intrinsic justification" Gödel means "justifiable with reference to the iterative conception of set" (see for example Koellner (2009), p. 2, and Roberts (2017), p. 657). Although Gödel did argue that axioms that are justifiable in this way—such as existence of inaccessible cardinals—are to be considered "instrinsically justified" (see for example Wang (1974), pp. 200–2, and also Koellner (2009)), it is also clear that Gödel had in mind a more inclusive concept of instrinsic justification. For instance, one reads in Wang (1996):

> In mathematics, he [Gödel] recommends [in the quest for new axioms] the path of cultivating (deepening) our knowledge of the abstract concepts themselves. The way to do this, Gödel asserts, is through Husserl's phenomenology—a "technique that should bring forth in us a new state of consciousness in which we see distinctly the basic concepts" (pp. 156-7).

The quote suggests that it should be possible to clarify the vision of the mathematician by some means so as to make "correct" axioms come more clearly into view. Indeed, Gödel went so far as to describe this heightened clarity concerning discovery of such axioms by saying that, when such clarity is achieved, the axioms "force themselves upon us." For instance, in Wang (1996), Wang quotes Gödel as saying,

> But despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception and, more generally, in mathematical intuition than in sense perception (p. 226).

axiom may have. A modern example of the latter is justifying existence of Woodin cardinals on the basis (in part anyway) of the extremely natural consequences of Projective Determinacy, which in turn is a consequence of the existence of infinitely many Woodin cardinals. Two examples of techniques for mining intrinsically motivated axioms, suggested by Gödel, are *generalization* and *reflection* (see Kanamori (1994) for a discussion of these). Each of these approaches has had a degree of success. Yet even to this day, there is no generally agreed upon axiom obtained using these approaches—one that is considered as natural as ZFC axioms themselves—that, when added to ZFC, allows us to prove the existence of even an inaccessible cardinal.

In this paper, we propose to take a fresh look at the problem of finding one or more natural axioms for justifying large cardinals; in the sequel, we will call this problem *The Problem of Large Cardinals*. We arrive at a "fresh look" by considering for a moment how a mathematical historian working several centuries in the future might view the present-day challenge to justify large cardinals. She might ask, "How did their logicians understand the concept of *infinite*?" reasoning that such a concept should be at least part of the intuition that would inform a decision to accept or reject large cardinals. To find an answer, she would perhaps study ZFC in search of axioms that talk about infinite sets. Having noticed that the Axiom of Infinity is the only such axiom, she might find it unsettling that there seems to be very little intuitive richness underlying this axiom—the underlying intuition is simply that the collection of natural numbers should exist as a completed set.

We take this common-sense observation as a starting point: The Axiom of Infinity, though sufficient to provide (in conjunction with the other ZFC axioms) the full landscape of transfinite cardinals, is not a source of rich intuition about the nature of the infinite. Whatever insights we may find buried in this axiom, they are not rich enough to guide our intuition regarding the existence of large cardinals.[2]

In this paper we propose an alternative form of the Axiom of Infinity that

- supplies much richer intuition about the "infinite"
- is provably equivalent to the usual Axiom of Infinity (relative to the theory ZFC − Infinity).

Historically, the reason an axiom of infinity of any kind was included among the fundamental axioms is that Cantor had convinced the mathematical world of the need for actually infinite sets; in particular, of the need for the set ω of finite ordinals. In the spirit of cultivating mathematical intuition about the mathematical infinite beyond Cantor's, we ask, in an informal way,

$$\text{What is it that imparts to } \omega \text{ the characteristic of being infinite?} \qquad (*)$$

The wording of the question is peculiar, to be sure; it reflects a worldview that was common to many cultures in antiquity. In that worldview, the natural numbers were seen to arise from a *source*.

[2] While it is true that set theorists have been able to identify properties of ω, such as inaccessibility and existence of an ω-complete nonprincipal ultrafilter, which, when generalized to uncountable cardinals, give rise in a natural way to inaccessible and measurable cardinals, respectively, these observations do not arise from a mathematical intuition about the infinite, but rather as technical insights obtained from a deeper study of ω.

For example, Pythagoras maintained (D'Olivet and Redfield, 1992, p. 137) that at the basis of all natural numbers is a "Number of numbers," an ultimate source of all numbers, something Divine in nature. Elsewhere he describes it as (Taylor, 1994, p. 17) "the ruler of forms and ideas." Moreover, according to T. Taylor's interpretation of Pythagoras's doctrine (1975), the dynamics of this ultimate Number are responsible for order in the universe. He writes, "But Pythagoras defined it [number] to be the extension and energy of the spermatic reasons contained in the monad [the One]. Or otherwise, to be that which prior to all things subsists in a divine intellect, by which and from which all things are coordinated, and remain connumerated in an indissoluble order. (p. 3)"

The Neoplatonist Diadochus Proclus (412–485 A.D.), one of the most prolific among the Neoplatonists, also described such an ultimate source at the basis of numbers (Proclus, 1994):

> . . . but the cause of all things being unically raised above all motion and division, has established about itself a divine number, and has united it to its own simplicity (p. 177).

We find similar insights in ancient Chinese philosophy. Here, we also find the view that diversity of manifest existence, embodied in the diversity of the natural numbers, originates from a unified source to which all diversity remains connected. *I Ching* scholar Carol Anthony (1998) writes:

> The ancient Chinese, like the ancient Greek Pythagoras, saw numbers as mirroring the order of the universe. The number one represented the undifferentiated whole. . . . Within this whole existed two primary forces, called the Creative and the Receptive. . . that by interacting with each other brought about the creation of all things (p. 1).

The *source* of all number—the source even of "one"—was called by 6th century (BCE) scholar Laozi *the Tao*; in his ancient classic, the *Tao Te Ching*, he wrote (Feng and English, 1997):

> The *Tao* begot One.
> One begot Two.
> Two begot Three.
> And Three begot the ten thousand things (v. 42).

In this quote, we see that all things arise from this unfoldment within *Tao* from One to Two to Three, and so on. Laozi emphasizes in other ways as well that it is by virtue of the dynamics of *Tao* that all things arise. For example (Mitchell, 1992):

> The great *Tao* flows everywhere. All things are born from it (v. 34).

A similar insight is conveyed in the ancient Vedic approach to knowledge. In that approach, the source of all diversity—including the diversity of natural numbers—is *pure consciousness* (*cidākāša*). In the *Yajur Veda* one finds:

> *Ekā cha me tisraschcha me ...*
> One is in me, two is in me, etc.
>
> – Yajur-Veda 18.24[3]

Here, "me" is a way of referring to the ultimate reality, pure consciousness (Mahesh, 1978, p. 347).

In the Vedic classic *The Yoga Vasishtha*, the sage Vasishtha describes pure consciousness as the source of *all* diversity (Valmiki, 1993):

> Thus the pure consciousness brings into being this diversity with all its names and forms, without ever abandoning its indivisibility, just as you create a world in your dream (p. 638).

In these philosophies, the natural numbers are seen as byproducts or, using more modern terminology, *epiphenomena*, of the dynamics of a fundamental substrate.

Another philosopher of antiquity, the Neoplatonist Plotinus (204–270), makes a similar point but elaborates considerably further. Plotinus says, "Multiplicity comes after unity; it is number while unity is the source of number; multiplicity as such has as its source The One as such" (*Enneads* III.8.9).[4]

He explains further that the world of multiplicity is generated from the hidden dynamics of The One: "The originating principle [The One] is not the totality of things, but from it all things proceed" (*Enneads* III.8.9).[5]

Here again we see the idea expressed that *everything* arises from the dynamics of the source. Plotinus also holds that these hidden dynamics of the source unfold automatically and without any impact on the source itself. In other words, the source *preserves itself* in its own transformational dynamics:

> What, then, is the One? It is what makes all things possible.... What is above life is the cause of life. The activity of life, being all things, is not the first principle. It [life] flows from it as from a spring. Picture a spring that has no further origin, that pours itself into all rivers without becoming exhausted of what it yields, and remains what it is, undisturbed (*Enneads* III.8.9).[6]

Yet, as transcendent as the One is in the philosophy of Plotinus, it is at the same time present everywhere, in every grain of manifest existence: "The One is absent from nothing" (*Enneads* VI.9.4).[7]

It is often claimed that infinite multitude was considered in ancient Greek philosophies to be opposed to order and all that is good.[8] To whatever extent this may be true, what is significant in the philosophy of *Plotinus* concerning the Infinite

[3] Translation by Mahesh (1978), see p. 347.

[4] In this article, translation of this passage and all others from the works of Plotinus, unless otherwise indicated, is by O'Brien (1964). For this passage, see p. 171.

[5] See O'Brien (1964), p. 172.

[6] See O'Brien (1964), p. 173.

[7] See O'Brien (1964), p. 79.

[8] Plato's philosophy is considered by scholars to be an example of this attitude. For instance the Internet Encyclopedia of Philosophy (2013) explains, "Plato did not envision God (the Demi-urge) as infinite because he viewed God as perfect, and he believed anything perfect must be limited and thus not infinite because the infinite was defined as an unlimited, unbounded, indefinite, unintelligible chaos."

is not the multiplicity that is often associated with the concept of infinity but other aspects of it. The One is declared by him to be beyond even Being itself and therefore beyond everything that could be described as *finite*; at the same time, the One embodies unlimited power: "This All is universal power, of infinite extent and infinite in potency (*Enneads V.8.9*)." Plotinus scholar J.M. Rist (1967) remarks:

> Within recent years there has been a long and learned discussion on the infinity of the Plotinian One. . . . The chief participants are now in basic agreement that the One is infinite in itself as well as infinite in power (p. 51).

It is especially clear, therefore, that, in Plotinus's philosophy, what is significant about the notion of "infinite" is the underlying power and dynamics that could produce an infinite multitude, and not that multitude itself.

For later use, we give names to principles that we have identified in Plotinus. We will take these as guidelines for our intuition as we seek to replace the usual Axiom of Infinity with an equivalent version that is based on a richer insight about the nature of the Infinite.

Plotinian Principles of The One.[9]

(1) *Multiplicity As Epiphenomenon.* Multiplicity arises as a side effect of the internal dynamics of The One.
(2) *Preservation.* The transformations that lead from The One to multiplicity do not modify the nature of The One in any way.
(3) *Everywhere Present.* Though transcendent, The One is present in every grain of manifest existence.
(4) *Everything from the Dynamics of the Source.* Every existent thing arises from the dynamics of the source.

Our question (*) makes more sense and is more easily answered in the context of these classical worldviews. Once we assume that the natural numbers have a source, the question itself starts to make sense. In that context, the obvious answer to (*) is then, "It is by virtue of the instrinsic nature of the *source* of natural numbers that the characteristic of 'being infinite' is imparted to that collection."

This line of thinking suggests to us that, what is essential about the set of natural numbers, and what needs to be captured by an axiom of infinity, is the fact that

[9] We wish to use these principles, and those outlined earlier from the traditions of Chinese and Vedic philosophy, as intuitive precepts to inform our search for foundational axioms. It is perhaps noteworthy that in each of these traditions of knowledge, the point is made that seeing the truth of these principles requires the seeker to *cultivate* himself in certain ways, in a manner analogous to Gödel's advice to make use of Husserl's technique of phenomenological reduction (see the footnote on p. 2) to enhance the ability of the mathematician to apprehend first principles. For this purpose, in the *Tao Te Ching*, Laozi advises *pŭ sàn*—return to the state of the uncarved block (v. 28); the Vedic literature advises *nivartadhwam*—retire from the field of sense experience (*Kishkindha Kand, Ramayana, 30.16*); and Plotinus explains, "You ask, how can we know the Infinite? I answer, not by reason....You can only apprehend the Infinite by a faculty superior to reason, by entering into a state in which you are your finite self no longer" (extracted from a letter to Flaccus, available online at http://www.sacred-texts.com/eso/cc/cc13.htm).

the *source* of natural numbers has special characteristics from which the infinitude of individual natural numbers is derivable, as suggested by the *Multiplicity As Epiphenomenon* principle.

This way of looking at the natural numbers and the effort to axiomatize the intuition may seem unfamiliar and perhaps even unnatural since it runs counter to contemporary perspectives. However, even in modern times, a very similar intuition has been a central element of a revolutionary paradigm shift in the field of particle physics. We take a moment to summarize this paradigm shift; a more detailed study may be found in Hobson (2013) and Brooks (2016).

One of the challenges in the history of physics has been to identify the ultimate constituents of the physical universe. For centuries it was believed that the answer had something to do with finding an ultimate particle, or fundamental set of particles, that everything in the material world is made of. However, what was discovered as the ultimate foundational constituents of material existence are not particles at all. What was found instead, by physicists in the area of Quantum Field Theory (QFT), was that the source of all particles is *unbounded quantum fields*. Every particle has a corresponding quantum field—for instance, each electron is related to the electron quantum field. And, in fact, each electron is a *precipitation* of this quantum field.

This solution to the problem of finding what is at the "root" of physical reality has been so successful that by now the physics community is in agreement that the *truth* about particles is their underlying quantum fields; the particles themselves are simply *side effects*. Summarizing this insight, Art Hobson (2013), in an *American Journal of Physics* article, "There Are No Particles, There Are Only Fields," writes,

> Quantum foundations are still unsettled, with mixed effects on science and society. By now it should be possible to obtain consensus on at least one issue: Are the fundamental constituents fields or particles? As this paper shows, experiment and theory imply that unbounded fields, not bounded particles, are fundamental. . . . Particles are epiphenomena arising from fields (p. 211).

In the QFT solution, a class of discrete particles are seen to be a side effect of the dynamics of an underlying quantum field. Considering the fact that the natural numbers are, in a mathematical way, a discrete collection of quantities, we might conjecture that they too are the expression of the dynamics of some sort of unbounded field. The question becomes, how can these dynamics be expressed in the form of a foundational axiom?

A candidate to represent these dynamics has been known for a long time in mathematics and precedes historically the formulation of the Axiom of Infinity that we have today. This candidate is the concept of a *Dedekind self-map*,[10] a special kind of self-map $j : A \to A$, for an arbitrary set A, having the following properties:

(1) j is 1-1.
(2) j has a *critical point*—an element $a \in A$ that is not in the range of j.

[10] We have coined this terminology, but the concept was discovered and elaborated by Dedekind in Dedekind (1888).

The element a that we have called a critical point was called by Dedekind the *base point* for the *chain* $W = \{a, j(a), j(j(a)), \ldots\}$. (A chain for a self-map $j : A \to A$ is a subset of A that is closed under j.) The map j can be seen as a kind of "dynamics," and, as Dedekind observed (and his argument is formalizable in ZFC − Infinity), in order for j to have properties (1) and (2), A must be *unbounded*, that is, *infinite*.[11]

In his research, Dedekind sought to derive a form of the natural numbers from the existence of a Dedekind self-map. He did not believe it was necessary to obtain the precise values $1, 2, \ldots$ of natural numbers; he considered the "natural numbers" to be any chain of the form W, as just described, with the number '1' being identified with the critical point ("base point") a. He believed that giving expression to natural numbers in this way, without tying them to a particular representation, made it possible to view them as a "free creation of the human mind" (p. 15). He justified this way of defining the natural numbers by showing that any two such chains are isomorphic.

Following this QFT-inspired intuition, we plan to carry Dedekind's argument one step further, to observe that application of the Mostowski collapsing map (defined without reliance on the usual set of natural numbers) to a Dedekind self-map collapses W to ω and $j \upharpoonright W$ to the usual successor function $s : \omega \to \omega$. In this way, we can give expression to the intuition that the sequence of natural numbers arises as a "precipitation of the dynamics (embodied in j) of an unbounded field." We will call any Dedekind self-map whose Mostowski collapse is the usual successor function an *initial* Dedekind self-map.[12]

Though Dedekind did not propose a mathematical intuition for the set of natural numbers exactly like ours—with the natural numbers arising as side effects of some underlying dynamics—his view that the natural numbers are the "free creation of the mind" places the mind in a role of "underlying dynamics"; the difference is that in our approach, we wish to represent these underlying dynamics as a mathematical object (namely, j).

We propose, then, to "rewrite" the Axiom of Infinity to obtain the following:

There is a Dedekind self-map.

Though this new version adds no new mathematical content to the original Axiom of Infinity, it does suggest a direction for generalization, for scaling to much bigger kinds of infinities, and for moving toward a solution to the Problem of Large Cardinals.

The intuition that the new axiom suggests is that, just as the natural numbers themselves should, according to the analogy with QFT, be viewed as precipitations of an unbounded field, realized mathematically as a Dedekind self-map interacting with its critical point, so likewise should we expect large cardinals to arise as precipitations of some larger-scale unbounded field, realized once again as the

[11] In Dedekind (1888), Dedekind defined a set A to be infinite if a self-map j with properties (1) and (2) could be defined on A; he also showed, via an application of the Axiom of Choice, that this definition of infinite set is equivalent to the standard definition, namely, that a set is infinite if it cannot be put in 1-1 corresondence with a finite ordinal; his argument is formalizable in ZFC − Infinity.

[12] We resort to the more complicated category-theoretic definition in the main body of the paper (Theorem 3.20). It is a straightforward exercise to check that the two definitions are equivalent.

interaction of a generalized Dedekind self-map with its critical point. Since large cardinals in many cases are *global*, we conjecture that our generalized Dedekind self-maps will need to map *the universe* V *to itself*. Therefore, justifying large cardinals should amount to finding a natural kind of Dedekind self-map from V to V, whose interaction with its critical point ultimately gives rise to particular large cardinals.

Our plan for the paper is as follows. In Section §2, we develop some preliminaries, recalling well-known theorems and definitions, and fixing notation. In Section §3, working in ZFC − Infinity, we recall Dedekind's results, reframed in a more modern context and better suited to generalization, and then formulate a suitable version of the Mostowski Collapsing Theorem that allows us to derive from such a self-map the set ω and the successor function $s : \omega \to \omega$; as mentioned before, we view this derivation as a realization of the intuition that the natural numbers should be viewed as "precipitations of the dynamics of an unbounded field."

In Section §4, we reflect on the work in Section §3 in order to list properties of Dedekind self-maps that are amenable to generalization. One key example of such a property is the fact that any Dedekind self-map $j : A \to A$ with critical point a exhibits an important type of *preservation*: If $B = \operatorname{ran} j$, then $j \restriction B : B \to B$ is also a Dedekind self-map, now with critical point $j(a)$.

We will use this list of properties to formulate a conjecture (the *Dedekind Self-Map Conjecture*) concerning the kind of Dedekind self-map that we would expect to be strong enough to generate large cardinals. Later in the paper we will test various forms of Dedekind self-maps against the criteria set forth in this conjecture in the hope that the forms of Dedekind self-map that reflect most fully the properties in this list will provide our best candidates to account for large cardinals.

Also in Section §4 we will provide a rigorous account of the concept of a *blueprint*. Intuitively, a Dedekind self-map $j : A \to A$ with critical point a generates a "blueprint" for ω and the successor function (namely, $W = \{a, j(a), j(j(a)), \ldots\}$ together with $j \restriction W : W \to W$); we seek to formalize the notion of a blueprint in anticipation of generalizations to the context of Dedekind self-maps on V.

In Section §5, we begin our study of Dedekind self-maps $j : V \to V$. Guided by our observations concerning *set* Dedekind self-maps, represented in the Dedekind Self-Map Conjecture, we begin our search for Dedekind self-maps $V \to V$ that satisfy the criteria of the conjecture. Simply stated, our conjecture proposes that large cardinals should "precipitate out" of the dynamics of an appropriately defined self-map if the self-map has the right preservation (and other) properties, by analogy with *set* Dedekind self-maps.

Our first indication of success in this program will be the observation that, working in ZFC − Infinity, whenever $j : V \to V$ is a Dedekind self-map that preserves disjoint unions, the empty set, and singletons (j *preserves singletons* if, for any set x, $j(\{x\}) = \{j(x)\}$), there must exist an infinite set. We go on to discuss several results of this kind; these will lay the foundation for a study of stronger preservation properties.

In Section §6, we obtain infinite sets from Dedekind self-maps $j : V \to V$ by generalizing another property discovered earlier regarding *set* Dedekind self-maps. We expect from those earlier observations that important types of infinite sets should arise from the interaction between j and its critical point. One result along these lines is that if j has critical point a and there is a set A for which $a \in j(A)$, then, when j satisfies certain preservation properties (in particular, when j preserves disjoint unions, intersections, and the empty set, and takes one-element sets to one-element sets, and also has $\{a\}$ as a second critical point), the set $D = \{X \subseteq A \mid a \in j(X)\}$ is a nonprincipal ultrafilter. Here, an infinite set is seen to arise from j's interaction with a.

In Section §7, we pause to clarify the relationship between the different notions of "critical point" that arise in earlier sections—namely *critical points* and *strong critical points*. Examples show that the notions are different, and conditions are given under which the notions coincide.

In Section §8, we pursue the theme that important infinite sets should arise from the interaction between j and its critical point by considering an altogether different way of constructing j. We review an old result of W. Lawvere, which states that (working in ZFC − Infinity), if j is formed as a composition $j = G \circ F$, where $G : \mathbf{Set}^{\circlearrowleft} \to \mathbf{Set}$ is the *forgetful functor* defined on the category $\mathbf{Set}^{\circlearrowleft}$ of self-maps $t : B \to B$ (here, "forgetful" means that $G(t) = \text{dom } t = B$), and $F : \mathbf{Set} \to \mathbf{Set}^{\circlearrowleft}$ is *left adjoint* to G, then 1 is a critical point of j and $j(1)$ is infinite. (Definitions and background results are discussed in Section §2.) Here, we think of the category **Set** as being simply a ZFC − Infinity universe V so that j is a self-map on V. We extend Lawvere's work a bit further to show that there is a natural way to define from j a Dedekind self-map $k : j(1) \to j(1)$. Generalizing, we provide sufficient conditions for any Dedekind self-map $j : V \to V$ with critical point a, obtained as a composition of adjoint functors, to yield a Dedekind self map $k_a : j(a) \to j(a)$. In particular, we show, working in ZFC − Infinity, that whenever $j : V \to V$ is a Dedekind self-map that arises as a composition of adjoint functors and that has a critical point a with the property that $|a| < |j(a)| = |j(j(a))|$ (such a j is called a *Dedekind monad*), then there is a Dedekind self-map $k_a : j(a) \to j(a)$ that arises "naturally" from j.

Section §9 combines the strategies for obtaining an infinite set from a $j : V \to V$, mentioned in Sections §5–§6, to show how inaccessible, ineffable, and measurable cardinals can be obtained when j is endowed with suitable preservation properties. We then consider what happens when a Dedekind self-map $j : V \to V$ is required to preserve *all* first-order properties; that is, when j is an elementary embedding. Because of K. Kunen's well-known inconsistency result (1971), such elementary embeddings need to be handled in an appropriate way to ward off inconsistency. We achieve this aim by working in a language having as its extralogical symbols both $\in$ (as usual) and a unary function symbol $\mathbf{j}$. We introduce an axiom schema BTEE (Basic Theory of Elementary Embeddings) which asserts that, for each $\in$-formula ϕ, and all $\vec{x}$, $\phi(\vec{x}) \leftrightarrow \phi(\mathbf{j}(\vec{x}))$ holds, and also that $\mathbf{j}$ has a least ordinal moved, typically denoted κ. If (M, E, j) is a model of ZFC + BTEE, $j : M \to M$ is a nontrivial elementary embedding. From the theory ZFC + BTEE, one can prove that the critical point of $\mathbf{j}$ is n-ineffable for each $n \in \omega$.

Starting from a BTEE-embedding $j : V \to V$ with critical point κ, we argue that the class $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ "ought to be" a set, and thereby motivate an additional axiom, the *Measurable Ultrafilter Axiom* (MUA) which simply asserts that this collection D is a set.

Section §10 takes a deeper look at the theory ZFC + BTEE + MUA. In this theory, we begin to observe more of the characteristics originally discovered for *set* Dedekind self-maps, studied earlier. In addition to strong preservation properties, we see that the interaction between an MUA-embedding j with its critical point κ

produces a blueprint (in the formally defined sense) of $V_{\kappa+1}$ and a *strong* blueprint of $V_{\kappa+1} - V_\kappa$.

In attempting to locate analogues to other characteristics of a set Dedekind self-map, we bring to light in this section undesirable limitations of the theory ZFC + BTEE + MUA, which suggest (as we will argue) the naturalness of replacing MUA with a stronger axiom, called *Amenability*$_{\mathbf{j}}$, which asserts that the restriction of $\mathbf{j}$ to any set is itself a set. The axioms BTEE + Amenability$_{\mathbf{j}}$ are collectively known as WA_0, the *weak* Wholeness Axiom.

In Section §11, we study the theory ZFC + WA_0, and its strengthened version ZFC + WA (WA is an abbreviation for *Wholeness Axiom*; see Section 2 for the definition). We review some of the results known about these theories, such as the fact that the critical point κ of any WA_0-embedding j is the κth cardinal that is super-n-huge for every $n \in \omega$. Reworking known results, we also observe that interaction between a WA_0-embedding and its critical point produces a blueprint of the entire universe V (and a strong blueprint of $V - V_\kappa$). We show how the limitations encountered in the theory ZFC + BTEE + MUA are overcome in the theories ZFC + WA_0 and ZFC + WA.

We conclude the section by observing that, in the theory ZFC + WA (and for the most part, even in ZFC + WA_0), we have obtained a reasonably natural generalization of the properties that we originally noted concerning *set* Dedekind self-maps and have satisfied the criteria in the Dedekind Self-Map Conjecture. In the end, what distinguishes our approach to arriving at reasonable axioms to supplement ZFC is a deliberate effort to identify first principles and an adherence to these principles in the process of generalization.

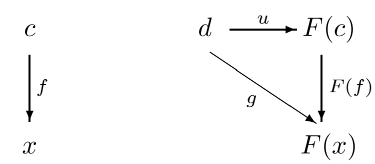
In the final section, we suggest ways to apply the techniques developed in this paper to provide justification for large cardinal notions that are stronger than the Wholeness Axiom—in particular, Woodin's weak Reinhardt cardinals, as well as the original Reinhardt cardinals, devised by Reinhardt, in a choiceless universe. We also review some research that has been done to build up quantum physics using the theory ZFC + WA as a mathematical foundation.

§2 Preliminaries

2.1 Category Theory In this section we give a concise review of category-theoretic concepts that are used in the paper. We assume the reader knows the definition of a category and the assortment of standard constructions that are done in categories: (finite) products and limits, (finite) coproducts and colimits, equalizers, coequalizers, terminal objects, initial objects, and exponentials. For any category $\mathcal{C}$ and $\mathcal{C}$-objects A, B, the collection of all $\mathcal{C}$-arrows $A \to B$ is denoted $\mathcal{C}(A, B)$. For any $\mathcal{C}$-object A, we also have a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ defined on objects by $B \to \mathcal{C}(A, B)$ and on arrows by $\mathcal{C}(A, f)(g) = f \circ g$ whenever $f : B \to C$ and $g \in \mathcal{C}(A, B)$. We assume familiarity with the definitions of *monic, epic* and *iso* arrows, and of *functors* and *natural transformation*. See Mac Lane (1978) or Awodey (2011) as necessary.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *left (right) exact* if F preserves all finite limits (colimits). F is *exact* if it is both left exact and right exact.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor and $c \in \mathcal{C}, d \in \mathcal{D}$. A $\mathcal{D}$-arrow $u : d \to F(c)$ is a *universal arrow* if for any $x \in \mathcal{C}$ and any $g : d \to F(x)$ in $\mathcal{D}$, there is a unique $f : c \to x$ in $\mathcal{C}$ such that $g = F(f) \circ u$.



Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. Then F is *left adjoint to* G, and we write $F \dashv G$, if there is, for each object c in $\mathcal{C}$ and object d in $\mathcal{D}$, a bijection $\theta_{cd} : \mathcal{D}(F(c), d) \to \mathcal{C}(c, G(d))$ that is natural in c and d. In this case (F, G, θ) is said to be an *adjunction*.

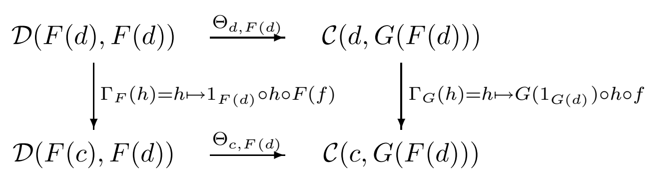
For each object $c \in \mathcal{C}$, let η_c denote $\theta_{c,F(c)}(1_{F(c)})$. One shows that $\eta_c : c \to G(F(c))$ is a universal arrow for each c and that the collection $\eta_c, c \in \mathcal{C}$, are the components of a natural tranformation $\eta : 1_{\mathcal{C}} \to G \circ F$. η is called the *unit of the adjunction*.

A folklore result concerning the unit η of an adjunction (F, G, θ) is the following:

LEMMA 2.1 *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors with $F \dashv G$. Let Θ be the natural bijection for the adjunction and let η be the unit of the adjunction. Suppose $f : c \to d$ is a $\mathcal{C}$-arrow. Let $\overline{F(f)} : c \to G(F(d))$ be the transpose of $F(f)$; that is, $\overline{F(f)} = \Theta_{c,F(d)}(F(f))$. Then the following holds:*

$$\overline{F(f)} = \eta_\beta \circ f.$$

Proof. We use the following diagram which is commutative because of the naturalness of the bijection θ:



Tracing through the diagram in one way, we have

$$\Gamma_G(\Theta_{B,F(B)}(1_{F(B)})) = \Gamma_G(\eta_B) = \eta_B \circ f.$$

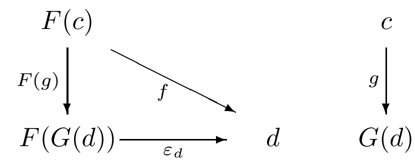
Tracing through in the other way, we have

$$\Theta(A, F(B))(\Gamma_F(1_{F(B)})) = \Theta(A, F(B))(F(f)) = \overline{F(f)}.$$

It follows that $\overline{F(f)} = \eta_B \circ f$, as required. □

Dually, for each object $d \in \mathcal{D}$, let ε_d denote $\theta^{-1}_{G(d),d}(1_{G(d)})$. One shows that the $\varepsilon_d, d \in \mathcal{D}$, are the components of a natural transformation, and that each $\varepsilon_d : F(G(d)) \to d$ is a *co*-universal arrow, in the following sense: For each $f : F(c) \to d$

in $\mathcal{D}$ there is a unique $f : c \to G(d)$ in $\mathcal{C}$ so that $\varepsilon_d \circ F(g) = f$.



The transformation ε is the *counit* of the adjunction.

An adjunction is completely determined by its unit. That is, given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\eta : 1_{\mathcal{C}} \to G \circ F$ such that, for each $c \in \mathcal{C}$, $\eta_c : c \to G(F(c))$ is a universal arrow, then $F \dashv G$. Likewise, an adjunction is completely determined by its counit.

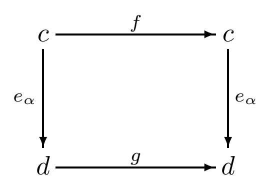
If $F \dashv G$, and if η and ε are, respectively, the unit and counit of the adjunction, we will sometimes refer to the adjunction as the quadruple $(F, G, \eta, \varepsilon)$.

A couple of facts about adjunctions that we will need are summarized in the following:

PROPOSITION 2.2 (Adjoints) *Suppose $F \dashv G$ with $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.*

(1) *F preserves all colimits of $\mathcal{C}$ and G preserves all limits of $\mathcal{D}$.*
(2) *Suppose $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are functors and suppose that each is a left adjoint of G. Then F_1 and F_2 are naturally isomorphic.*

The *category of sets*, denoted **Set**, has as objects all sets and as arrows all functions between sets.[13] For any category $\mathcal{C}$, the *category of self-maps (or endos) from $\mathcal{C}$*, denoted $\mathcal{C}^{\circlearrowright}$, has as objects all $\mathcal{C}$-arrows $c \to c$. Given $f : c \to c, g : d \to d \in \mathcal{C}^{\circlearrowright}$, an arrow $\alpha : f \to g$ is a $\mathcal{C}$-arrow $e_\alpha : c \to d$ that makes the following diagram commute:



Suppose $F : \mathcal{C} \to$ **Set** is a functor and $c \in \mathcal{C}$. An object $u \in F(c)$ is a *weakly universal element for* F if for each $d \in \mathcal{C}$ and each $y \in F(d)$ there is an $f_d : c \to d$ in $\mathcal{C}$ so that $F(f_d)(u) = y$; more verbosely, F is said to be *weakly represented by c with weakly universal element u*. Moreover, if f_d is unique for each choice of d, then u is a *universal element* for F; again, one also says in this case that F is *represented by c with universal element u*, or more simply that F is *representable*. Any such universal element u induces (by the Yoneda Lemma) a natural isomorphism $\mathcal{C}(c, -) \to F$,

[13] Any model (M, E) of ZFC − Infinity can be turned into a cartesian closed category $\bar{M}$ (i.e. a category that has all finite limits and exponentiation) as follows: The objects are the elements of M. Given $a, b \in M$, $a \xrightarrow{f} b$ is an arrow in the category if and only if $M \models$ "$f : a \to b$ is a function". Since, internal to M, the usual set-theoretic product $a \times b$ and exponentiation a^b operations can be carried out, $\bar{M}$ is cartesian closed. For convenience, we will denote this category M instead of $\bar{M}$. We will refer to a model of this kind as *a category of sets*.

and, conversely, any such natural isomorphism yields a universal element $u \in F(c)$. Therefore, F is representable if and only if there is a $\mathcal{C}$-object c for which there is a natural isomorphism $C(c, -) \to F$. See Mac Lane (1978).

Suppose $\mathcal{C}$ is a category. $\mathcal{C}$ is said to be *closed under small copowers* if, for any object c in $\mathcal{C}$ and any index set I, there is a $\mathcal{C}$-object d such that $d = \amalg_I c$ and a $\mathcal{C}$-arrow $c \to d$. We will make use of the following lemma (see Mac Lane (1978), Exercise V.8.1).

LEMMA 2.3 *Suppose $\mathcal{C}$ is closed under small copowers and $G : \mathcal{C} \to$ **Set** is a functor. Then the following are equivalent:*

(1) *G has a left adjoint;*
(2) *G is representable.*

From a foundational point of view, one significant feature of a (weakly) universal element u for a functor F is that it provides a way of reaching a vast expanse of sets from a single "seed" u. For our purposes, it will be useful to know whether *every* set in **Set** can be reached in this way. We will declare that a functor $F : \mathcal{C} \to$ **Set** is *cofinal* if for every $x \in$ **Set** there is $c \in \mathcal{C}$ such that $x \in F(c)$. One easily verifies that if u is a weakly universal element for a functor $F : \mathcal{C} \to$ **Set**, then every set is expressible as $F(f)(u)$ for some arrow f in $\mathcal{C}$ if and only if F is cofinal.[14]

2.2 The Theory* ZFC−Infinity *and Transitive Containment. Appropriately enough, many of the proofs in this paper take place in the theory ZFC−Infinity as we attempt to discover dynamics by which an infinite set arises in the universe (see Section §5). Occasionally, a proof will need the property Trans[15] (transitive containment), provable in ZFC:

Trans: Every set is contained in a transitive set.

However, Trans does not follow from ZFC − Infinity (for a proof, see Kaye and Wong (2007)). Trans is needed to prove (without the Axiom of Infinity) that every hereditarily finite set belongs to some finite rank V_n (see Corazza (2016), Theorem 78, for a proof of this).

Because Trans is a theorem of ZFC, we view the addition of Trans to the theory ZFC − Infinity to be well-motivated. For convenience of notation in this paper, we will adopt a modified version of the ZFC axioms in which Trans is included explicitly. Then, the theory ZFC − Infinity will be understood to include Trans among its axioms.

For the most part, Trans does not play a role in our arguments. It does appear twice in the proof of Theorem 5.43.

***2.3 Elementary Embeddings* $j : V \to V$.** In Corazza (2000), (2006), (2010), elementary embeddings $j : V \to V$ are studied. Kunen (1971) showed that if we take j to be a class in a class theory, the only such elementary embedding that can

[14] From the category-theoretic point of view, this definition of "cofinal" is rather unnatural because it is not preserved by natural transformations. This notion and its (set-theoretic) connection to a universal element has turned out to be conceptually useful, so we have used it advisedly.

[15] It is more common to denote this property TC but, following another notational convention, we have reserved "TC" for the transitive closure operator.

exist is the identity function. It turns out that assuming j is a class of this kind is a strong assumption and masks the true strength of "elementarity"; in particular, assuming j is a class in a class theory implies that all instances of Separation and Replacement must hold when j is treated as a predicate. In Corazza (2000), (2006), (2010), we examine the strength of theories in which j is assumed to be elementary (and having additional properties) without assuming outright that Separation or Replacement hold for j; without Separation or Replacement for formulas that talk about j, neither Kunen's inconsistency argument nor any of the other arguments known that demonstrate inconsistency can be carried out.

We approach the subject of elementary embeddings $V \to V$ by working in the expanded language $\mathcal{L}_{\mathbf{j}} = \{\in, \mathbf{j}\}$, where $\mathbf{j}$ is a unary function symbol. Formulas in which $\mathbf{j}$ does not occur will be called $\in$-formulas whereas formulas having at least one occurrence of $\mathbf{j}$ will be called $\mathbf{j}$-formulas. Including the function symbol $\mathbf{j}$ means that we need to consider $\mathcal{L}_{\mathbf{j}}$-terms (which we will call $\mathbf{j}$-terms from now on). As usual, terms are defined by the clauses: (a) a variable is a term, and (b) if t is a term, so is $\mathbf{j}(t)$. The terms are of the form $\mathbf{j}^n(x)$ for variables x, where $\mathbf{j}^0(x)$ is taken to be x. (The enumeration $\langle \mathbf{j}^n(x) : n \in \omega \rangle$ lives in the metatheory.)

When working with $\mathcal{L}_{\mathbf{j}}$ theories, we take as our background theory either ZFC or ZFC $-$ Infinity; in the usual way, the axioms of these theories are $\in$-formulas. Also part of our background theory is the first-order logic for $\mathcal{L}_{\mathbf{j}}$, which does include $\mathbf{j}$-sentences. We will denote these theories $\mathrm{ZFC}_{\mathbf{j}}$ and $\mathrm{ZFC}_{\mathbf{j}} - \mathrm{Infinity}$. Using the usual facts about first-order logic together with the Completeness Theorem (Corazza, 2006) the formula $\mathbf{j}(x) = y$ defines a class function in all extensions of $\mathrm{ZFC}_{\mathbf{j}} - \mathrm{Infinity}$ and we have

$$\mathrm{ZFC}_{\mathbf{j}} \vdash \forall x \, \exists! y \, \mathbf{j}(x) = y.$$

When $\mathcal{L}_{\mathbf{j}}$-axioms are added to one of the theories $\mathrm{ZFC}_{\mathbf{j}}$ or $\mathrm{ZFC}_{\mathbf{j}} - \mathrm{Infinity}$, we adopt the following convention: If σ is an $\mathcal{L}_{\mathbf{j}}$-sentence having an occurrence of $\mathbf{j}$, then we shall denote the extended theory $\mathrm{ZFC}_{\mathbf{j}} + \sigma$ by simply $\mathrm{ZFC} + \sigma$ (and similarly for $\mathrm{ZFC}_{\mathbf{j}} - \mathrm{Infinity}$), with the understanding that our language is $\mathcal{L}_{\mathbf{j}}$ and we are using the first order logic for $\mathcal{L}_{\mathbf{j}}$.

In order to establish results about elementary embeddings $j : V \to V$, we work in $\mathrm{ZFC}_{\mathbf{j}}$ (or $\mathrm{ZFC}_{\mathbf{j}} - \mathrm{Infinity}$) supplemented with axioms asserting collectively that $\mathbf{j}$ is an elementary embedding $V \to V$ having a critical point. These axioms are known (Corazza, 2006) as the *Basic Theory of Elementary Embeddings*, or BTEE. In working in the theory ZFC + BTEE, no other axioms regarding $\mathbf{j}$ are assumed (in particular, no instances of Separation or Replacement for $\mathbf{j}$-formulas are assumed).

Definition 2.4 **Axioms of BTEE** (Corazza, 2006)

(1)$_\phi$ (Elementarity Schema for $\in$-formulas). *Each of the following* $\mathbf{j}$*-sentences is an axiom, where* $\phi(x_1, x_2, \ldots, x_m)$ *is an* $\in$*-formula:*

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m)) \right);$$

(2) (Critical Point). *"There is a least ordinal moved by* $\mathbf{j}$*".*

Introducing the critical point κ of $\mathbf{j}$ as a constant obtained by definitional extension, it is shown in Corazza (2006) that ZFC + BTEE $\vdash$ "κ is totally indescribable" and, for each particular natural number n, ZFC + BTEE $\vdash$ "κ is n-ineffable". A transitive set model for ZFC + BTEE can be obtained from an ω-Erdös cardinal (Corazza, 2006).

A number of properties of class maps $j : V \to V$ or class embeddings of the form $j : V \to M$ (where M is an inner model of ZFC) that are easy to prove in ZFC become more problematic in the theory $\mathrm{ZFC}_{\mathbf{j}}$—even in ZFC + BTEE. For instance, working in ZFC + BTEE, one cannot prove that the critical sequence $\kappa, \mathbf{j}(\kappa), \mathbf{j}(\mathbf{j}(\kappa)), \ldots$ is strictly increasing, nor that, for every ordinal α, $\mathbf{j}(\alpha) \geq \alpha$. Two axioms that are often added to the theory to add familiarity are $\text{Induction}_{\mathbf{j}}$ and the Least Ordinal $\text{Principle}_{\mathbf{j}}$.

Induction$_{\mathbf{j}}$: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\left[\phi(0, \vec{a}) \wedge \forall n \in \omega \left[\phi(n, \vec{a}) \Longrightarrow \phi(n+1, \vec{a})\right]\right] \Longrightarrow \forall n \in \omega\, \phi(n, \vec{a}).$$

We let Σ_n-*Induction*$_{\mathbf{j}}$ (Π_n-*Induction*$_{\mathbf{j}}$) denote $\text{Induction}_{\mathbf{j}}$ restricted to Σ_n (Π_n) $\mathbf{j}$-formulas.

A sample application of $\text{Induction}_{\mathbf{j}}$ is the following: While one may prove from ZFC+BTEE (using induction in the metatheory) that for each particular n, κ is n-ineffable, one proves from ZFC+BTEE+$\text{Induction}_{\mathbf{j}}$ the following formal statement:

$$\forall n \in \omega\, (\text{“}\kappa \text{ is } n\text{-ineffable”}).$$

It is often useful to refer to the critical sequence of $\mathbf{j}$, namely, $\kappa, \mathbf{j}(\kappa), \ldots, \mathbf{j}^n(\kappa), \ldots$, within some extension of ZFC + BTEE (or even of $\mathrm{ZFC}_{\mathbf{j}}$). Although we may legitimately consider this enumeration in the metatheory, *Induction*$_{\mathbf{j}}$ is needed to refer to it formally within one of these theories. For this purpose, we define in Corazza (2006) a formula $\Theta(f, n, x, y)$ which asserts that $f = \langle \kappa, \mathbf{j}(\kappa), \ldots, \mathbf{j}^n(\kappa)\rangle$:

$$\begin{aligned}&\text{“}f \text{ is a function”} \wedge \mathrm{dom}\ f = n+1 \wedge f(0) = x \wedge \\ &\quad \forall i \left(0 < i \leq n \Longrightarrow f(i) = \mathbf{j}(f(i-1))\right) \wedge f(n) = y.\end{aligned}$$

Then the informal statement $y = j^n(x)$ is captured by $\Phi(n, x, y) : \ \exists f\, \Theta(f, n, x, y)$. It is shown in Corazza (2006), Proposition 4.4(2), that it is provable from ZFC + BTEE + Σ_1-$\text{Induction}_{\mathbf{j}}$ that Φ is a (total) class function; that is

$$\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_1\text{-Induction}_{\mathbf{j}} \vdash \forall n \in \omega \forall x \exists! y\, \Phi(n, x, y).$$

We remark that, in the absence of Σ_1-$\text{Induction}_{\mathbf{j}}$, it is possible for there to exist, in the formal theory, a finite ordinal N for which $\mathbf{j}^N(\kappa)$ is not defined.[16]

To prove in an extension of ZFC + BTEE that, for every ordinal α, $\mathbf{j}(\alpha) \geq \alpha$, even $\text{Induction}_{\mathbf{j}}$ does not suffice. What is needed is the following:

Least Ordinal Principle$_{\mathbf{j}}$: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\begin{aligned}&\exists \alpha \left[\text{“}\alpha \text{ is an ordinal”} \wedge \phi(\alpha, \vec{a})\right] \Longrightarrow \\ &\quad \exists \alpha \left[\text{“}\alpha \text{ is an ordinal”} \wedge \phi(\alpha, \vec{a}) \wedge \forall \beta \in \alpha \left(\neg \phi(\beta, \vec{a})\right)\right].\end{aligned}$$

The axiom says that, whenever there is an ordinal that satisfies the $\mathbf{j}$-formula ϕ, there is a least such ordinal. The Σ_n-*Least Ordinal Principle*$_{\mathbf{j}}$ (Π_n-*Least Ordinal Principle*$_{\mathbf{j}}$) is the Least Ordinal Principle restricted to Σ_n (Π_n) $\mathbf{j}$-formulas.

An application of this principle yields the following:

[16] See pp. 351, 377 in Corazza (2006) for a discussion of this point.

THEOREM 2.5 ZFC+BTEE+Σ_0-*Least Ordinal Principle*$_{\mathbf{j}}$ $\vdash \forall\alpha($"α is an ordinal" $\rightarrow \mathbf{j}(\alpha) \geq \alpha)$.

The anomalies that arise in the absence of Induction$_{\mathbf{j}}$ and Least Ordinal Principle$_{\mathbf{j}}$ vanish when working within a transitive model of ZFC$_{\mathbf{j}}$ (Corazza, 2006, Propositions 4.1 and 6.2):

THEOREM 2.6 *Suppose* $\mathcal{M} = \langle M, \in, j\rangle$ *is a transitive model of ZFC*$_{\mathbf{j}}$*. Then both Induction*$_{\mathbf{j}}$ *and Least Ordinal Principle*$_{\mathbf{j}}$ *hold in* $\mathcal{M}$*.*

Transitive models of ZFC + BTEE can be obtained (Corazza, 2006) under the assumption of an ω-Erdös cardinal. Moreover, in the most naturally occurring models of this kind, obtained by the *canonical construction* (Corazza, 2006, Remark 3.1), the critical sequence of j is cofinal in the ordinals:

PROPOSITION 2.7 *Suppose* $\mathcal{M} = \langle M, \in, j\rangle$ *is a transitive model of ZFC + BTEE obtained from an* ω*-Erdös cardinal by the canonical construction. Then the critical sequence* $\langle j^n(\kappa) \mid n \in \omega\rangle$ *is cofinal in* $\mathrm{ON}^{\mathcal{M}}$*.*

Large cardinal strength is added to ZFC + BTEE by adding more instances of the Separation axiom for $\mathbf{j}$-formulas (denoted Separation$_{\mathbf{j}}$). Formally, an *instance* of Separation$_{\mathbf{j}}$ is a sentence

$$\forall A\, \forall \vec{a}\, \exists z\, \forall u\, [u \in z \longleftrightarrow u \in A \wedge \phi(u, A, \vec{a})],$$

where ϕ is a $\mathbf{j}$-formula; in particular, this is the instance of Separation$_{\mathbf{j}}$ that is *determined by* ϕ. The notions Σ_n (Π_n) Separation$_{\mathbf{j}}$ have the obvious meaning.

An example of extending ZFC + BTEE with an instance of Separation$_{\mathbf{j}}$ that we will consider in this paper is the theory (Corazza, 2006) ZFC+BTEE+MUA, where MUA—the *Measurable Ultrafilter Axiom*—states that the collection $\{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$ is a set. Formally, MUA is the instance of Σ_0-Separation$_{\mathbf{j}}$ obtained from the formula $\kappa \in \mathbf{j}(X)$. In this context, $\mathbf{j}$ is called an *MUA-embedding.* It is shown in Corazza (2006) that the critical point of an MUA-embedding is measurable of high Mitchell order.

2.4 Compatibility Between Particular Large Cardinals and an Elementary Embedding $j : V \rightarrow V$***.*** In Corazza (2000), familiar *globally defined* large cardinal notions (like supercompactness and superhugeness) are represented as classes of set embeddings. The reason for this representation is to provide a uniform setting for discussing existence of Laver sequences for arbitrary globally defined large cardinals. Recall that if κ is supercompact, a function $f : \kappa \rightarrow V_\kappa$ is *Laver* (Laver, 1968) if for each x and each $\lambda \geq \max(\kappa, |\mathrm{TC}(x)|)$, there is a normal ultrafilter U over $P_\kappa\lambda$ such that $x = i_U(f)(\kappa)$, where $i_U : V \rightarrow V^{P_\kappa\lambda}/U \cong M$ is the canonical elementary embedding derived from U. The starting point for a more general treatment is the notion of a *suitable formula.*

Let $\theta(x, y, z, w)$ be a first-order formula (in the language $\{\in\}$) with all free variables displayed. We will call θ a *suitable formula* if the following sentence is provable in ZFC:

$$\forall x, y, z, w\big[\theta(x, y, z, w) \implies \text{"}w \text{ is a transitive set"} \wedge z \in \mathrm{ON} \wedge \text{"}x : V_z \rightarrow w \text{ is an elementary embedding with critical point } y\text{"}\big].$$

For each cardinal κ and each suitable $\theta(x, y, z, w)$, let

$$\mathcal{E}^\theta_\kappa = \{(i, M) : \exists\beta\, \theta(i, \kappa, \beta, M)\}.$$

The codomain of an elementary embedding i needs to be explicitly associated with i in the definition for technical reasons; for practical purposes, we think of $\mathcal{E}_\kappa^\theta$ as a collection of elementary embeddings $i : V_\beta \to M$ with critical point κ.

In Corazza (2000), familiar large cardinals, like supercompact and superhuge, are re-defined in terms of classes $\mathcal{E}_\kappa^\theta$ of embeddings. For example, there is a suitable formula θ_{sc} for which the following holds for each ordinal α:

$$\alpha \text{ is supercompact} \iff \forall\gamma > \alpha \exists\beta \geq \gamma \exists i \exists M \left(i : V_\beta \to M \ \wedge \ (i, M) \in \mathcal{E}_\alpha^{\theta_{\mathrm{sc}}} \right).$$

With this general concept of classes of set embeddings, we defined in Corazza (2000) a general notion of Laver sequence, which we reproduce here.

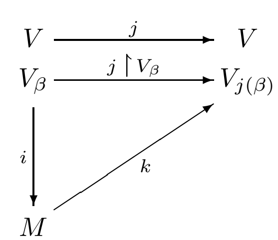
Given a class $\mathcal{E}_\kappa^\theta$ of embeddings, where θ is a suitable formula, a function $g : \kappa \to V_\kappa$ is defined to be $\mathcal{E}_\kappa^\theta$*-Laver at* κ if for each set x and for arbitrarily large λ there are $\beta > \lambda$, and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$.

It is observed in Corazza (2000) that if κ is supercompact and $f : \kappa \to V_\kappa$, then f is a Laver function at κ if and only if f is $\mathcal{E}_\kappa^{\theta_{\mathrm{sc}}}$-Laver at κ.

A key sufficient condition for Laverness, mentioned in Corazza (2000), is *compatibility* with an ambient elementary embedding $j : V \to V$ having critical point κ, relative to the language $\{\in, \mathbf{j}\}$ as described above. Suppose $\kappa < \lambda < \beta$, and $i_\beta : V_\beta \to M$ is an elementary embedding with critical point κ. Then i_β is *compatible with* j *up to* V_λ if there is an elementary embedding $k : M \to V_{j(\beta)}$ with

$$j \restriction V_\beta = k \circ i \text{ and } k \restriction V_\lambda \cap M = \mathrm{id}_{V_\lambda \cap M}.$$

If θ is suitable, then $\mathcal{E}_\kappa^\theta$ is said to be *compatible with* j if for each $\lambda < j(\kappa)$ there is a $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ that is compatible with $j \restriction V_\beta$ up to V_λ.



For the present paper, to handle cases in which the ambient embedding $j : V \to V$ does not have a global character (this will be the situation when we consider the theory ZFC + BTEE + MUA), since we do not expect to obtain a Laver function for such a cardinal, we introduce local versions of the notion of Laver function and compatibility, as follows. We develop the ideas relative to a measurable cardinal, but the definitions provided here are applicable to other locally defined large cardinals in obvious ways.

First, we can represent the concept of a measurable cardinal with a suitable formula θ_m:

$$\theta_m(i, \kappa, \beta, M) : \quad \beta = \kappa + 1 \wedge M \text{ is transitive} \wedge i : V_\beta \to M \text{ is elementary.}$$

Now κ is measurable if and only if there exist i, β, M such that $\theta_m(i, \kappa, \beta, M)$: If κ is measurable, let $e : V \to N$ be an elementary embedding with critical point κ, and consider $i = e \restriction V_{\kappa+1} : V_{\kappa+1} \to V^N_{e(\kappa)+1} = M$. Conversely, given $i : V_{\kappa+1} \to M$

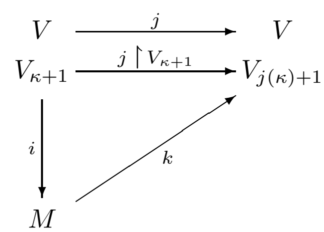
with critical point κ, define

$$U = \{X \subseteq \kappa \mid \kappa \in i(X)\}.$$

U is well-defined since $\mathcal{P}(\kappa) \subseteq V_{\kappa+1}$, and is easily seen to be a normal measure on κ; thus κ is measurable.

Next, we define the concept of an X-$\mathcal{E}_\kappa^{\theta_m}$-Laver sequence, adapting the definition of Laver sequence to a local context. First, we define the concept of an X-Laver sequence for a given set X, relative to measurable embeddings: For any set X, a function $f : \kappa \to V_\kappa$ is an X-Laver function at κ if, for each $x \in X$, there is a normal measure U on κ such that $i_U(f)(\kappa) = x$, where i_U is the canonical elementary embedding $V \to V^\kappa/U \cong M$ derived from U.[17] Next, we define the same concept relative to the class $\mathcal{E}_\kappa^{\theta_m}$: For any set X, a function $f : \kappa \to V_\kappa$ is an X-$\mathcal{E}_\kappa^{\theta_m}$-Laver function at κ if, for each $x \in X$, there is $i \in \mathcal{E}_\kappa^{\theta_m}$ such that $i(f)(\kappa) = x$. It is straightforward to show that for any $f : \kappa \to V_\kappa$ and any set X, f is X-Laver at κ if and only if f is X-$\mathcal{E}_\kappa^{\theta_m}$-Laver at κ.

These definitions lead to a *local* form of compatibility of $\mathcal{E}_\kappa^{\theta_m}$ with $j : V \to V$: Starting with a BTEE-embedding $j : V \to V$ with critical point κ, we declare that $\mathcal{E}_\kappa^{\theta_m}$ is *locally compatible with* j if there is $i : V_{\kappa+1} \to M \in \mathcal{E}_\kappa^{\theta_m}$ that is compatible with $j \upharpoonright V_{\kappa+1}$ up to $V_{\kappa+1}$. Note that when j is an MUA-embedding with critical point κ, $\mathcal{E}_\kappa^{\theta_m}$ is indeed locally compatible with j, since i can be obtained as the canonical embedding i_D derived from the ultrafilter $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$.



§3 Derivation of ω from a Dedekind Self-Map In this section we work in ZFC − Infinity and derive consequences of the following variation of the Axiom of Infinity:

Dedekind Axiom of Infinity

There is a Dedekind self-map

Our plan is to implement the intuition described earlier, which asks us to consider the finite ordinals as "precipitations" of the dynamics of an unbounded field; our implementation of this intuition is to represent such dynamics using a Dedekind self-map and to give a derivation of the set ω of finite ordinals and the successor function $s : \omega \to \omega$. Although the equivalence of the usual Axiom of Infinity and our Dedekind Axiom of Infinity, relative to ZFC−Infinity, is well-known, a typical proof of this fact would make free use of ordinary induction. To adhere to the intuitive

[17] The concept of an X-Laver function was introduced in Corazza (1998), with a slightly different meaning. The definition given in the present paper matches the definition of X^+-Laver function given in Corazza (1998).

principle we have in mind, however, we wish to carry out this derivation without making use of the natural numbers in any way.

Over a century ago, most of the steps of the derivation we have in mind were carried out by Dedekind himself in his paper Dedekind (1888). At the time his paper was written, there was not a clearly defined set of foundational axioms; as Kanamori (2012) points out, Dedekind makes implicit use of Replacement in his paper without recognizing that he had done so; as a consequence, his argument to prove definition by recursion in his approach has a subtle circularity. Nonetheless, if we consider the background theory for Dedekind's arguments to be ZFC − Infinity, the circularity disappears. Our approach will be to cast Dedekind's work in a more modern framework, with different notation and slightly different concepts; our proofs for the most part will simply rely on Dedekind's arguments suitably modified for our present context. Detailed proofs for this derivation, using a different approach and without reference to Dedekind's work, can be found in Corazza (2016), pp. 23–33.

We begin with an overview. Given a Dedekind self-map $j : A \to A$ with critical point a—which we sometimes denote $\mathrm{crit}(j)$—we wish to obtain first a blueprint $W \subseteq A$ for ω. The intuition is that W should be the set $\{a, j(a), j(j(a)), \ldots\}$. The fact that j is 1-1 and $a \notin \mathrm{ran}\, j$ guarantees that the elements $a, j(a), j(j(a)), \ldots$ do not repeat. We first obtain W as the intersection of all subsets of A that are closed under j and define an order relation $\sqsubseteq$ on W that, intuitively speaking, is defined by $x \sqsubseteq y$ if and only if one can obtain y from x by applying j at most finitely many times to x: $y = j(j \ldots (j(x)) \ldots)$. Relying on Dedekind's arguments to establish that $\sqsubseteq$ is a well-ordering of W, we prove a Mostowski Collapsing Lemma for the order $(W, \sqsubseteq)$ and show that the collapsing map transforms $(W, \sqsubseteq)$ to $(\omega, \in)$ and $j \restriction W$ to the usual successor function $s : \omega \to \omega$. Letting $i = j \restriction W$ and using a form of induction on W that is valid because of the fact that $\sqsubseteq$ is a well-ordering, we can define the sequence of iterations $\langle i^n \mid n \in \omega\rangle$ and with these show that in fact $W = \{i^n(a) \mid n \in \omega\}$.

We go on to show that if $\pi : W \to \omega$ is the Mostowski collapsing map, then π is a bijection and is in fact an isomorphism in the obvious sense: $\pi \circ (j \restriction W) = s \circ \pi$; we will say that π is a Dedekind self-map isomorphism. Letting $\tau = \pi^{-1}$, we observe that τ is also a Dedekind self-map isomorphism; this observation leads to the usual Definition by Recursion Theorem for ω. This theorem can then be used in the usual way to define the operations of addition and multiplication on ω, and, ultimately, to prove that the Peano axioms hold.

We turn now to a more detailed treatment. We define the concept of a j-inductive set: A set $B \subseteq A$ will be called *j-inductive* if

(1) $a \in B$,
(2) whenever $x \in B$, $j(x)$ is also in B.

Notice that A itself is j-inductive. Therefore, if

$$\mathcal{I} = \{B \subseteq A \mid B \text{ is } j\text{-inductive}\},$$

then $\mathcal{I}$ is nonempty. Let $W = \bigcap \mathcal{I}$.

Lemma 3.8 *W is a j-inductive subset of A.*

Proof. For (1), since a belongs to every j-inductive subset of A, $a \in W$. For (2), assume $x \in W$. Then x belongs to every j-inductive subset of A. For each such j-inductive subset B, since $x \in B$, $j(x) \in B$. Therefore $j(x) \in W$. □

In Dedekind's approach, W is defined in the following way. Given a Dedekind self-map $j : A \to A$ with critical point a, a subset B of A is said to be a *j-chain* if $j[B] \subseteq B$. Given any $C \subseteq A$, the j-chain *generated by* C, denoted C_D ("D" for *Dedekind*), is the intersection of all j-chains that contain C. It is an easy exercise to see that $W = \{a\}_D$. Dedekind calls any set W obtained in this way *simply infinite* and refers to the critical point a as the *base point* of W.

In addition, Dedekind defines a set A to be *infinite* if A is the domain of a Dedekind self-map. Following modern convention, we call such sets *Dedekind infinite*. We will give our definition of "infinite" later in this section.

We follow Dedekind in defining a binary relation $\sqsubseteq$ on W as follows: For all $x, y \in W$,

$$x \sqsubseteq y \text{ if and only if } \{y\}_D \subseteq \{j(x)\}_D.$$

We introduce the following notation: For each $x \in W$, let $W_x = \{u \in W \mid u \sqsubseteq x\}$.

The following is a sampling of results from Dedekind (1888) (also proved in Corazza (2016) using a different definition of $\sqsubseteq$); these particular results will be used in the sequel.

THEOREM 3.9

(1) *$\sqsubseteq$ is a well-ordering of W.*
(2) *Suppose $x, y \in W$.*

   (A) *If $x \sqsubseteq j(y)$ and $x \neq y$, then $x \sqsubseteq y$.*
   (B) *If $x \sqsubseteq y$, then $x \sqsubseteq j(y)$*
   (C) *The critical point a is the $\sqsubseteq$-least element of W.*
   (D) *The element $j(x)$ is the $\sqsubseteq$-least element of $\{z \in W \mid x \sqsubseteq z\}$.*[18]

[18] In the author's (2016), an "internal" definition of $\sqsubseteq$ is given. We indicate why the definition given in (2016) is equivalent to Dedekind's. Following (2016), let us define $\sqsubseteq'$ on W as follows:

DEFINITION 3.10 (Joining Sets) $x \sqsubseteq' y$ if and only if $\exists F \subseteq W$ satisfying the following:

(i) $x, y \in F$,
(i) for some $v \in F$, $y = j(v)$,
(iii) there is no $u \in F$ for which $x = j(u)$,
(iv) if $u \in F$ and $u \neq y$, then $j(u) \in F$,
(v) if $v \in F$ and $v \neq x$, there is $u \in F$ such that $v = j(u)$.

The set F is said to *join x to y*, and is called a *joining set*.

It is shown in (2016) that $\sqsubseteq'$ satisfies the trichotomy laws. Lemma 3 of (2016) shows that whenever $y \sqsubseteq' x$, it follows that $W_y \subseteq W_x$. We show that $y \sqsubseteq' x \Rightarrow y \sqsubseteq x$; this suffices to show that the orderings are the same since $\sqsubseteq$ also satisfies trichotomy.

$$\begin{aligned} y \sqsubseteq' x \quad &\Rightarrow \quad W_y \subseteq W_x \\ &\Rightarrow \quad \{x\} \cup \{u \in W \mid x \sqsubseteq' u\} \subseteq \{y\} \cup \{u \in W \mid y \sqsubseteq' u\} \\ &\Rightarrow \quad \{x\} \cup \{u \in W \mid x \sqsubseteq' u\} \subseteq \{u \in W \mid y \sqsubseteq' u\} \\ &\Rightarrow \quad \{x\} \cup \{x\}_D \subseteq \{y\}_D \\ &\Rightarrow \quad \{x\}_D \subseteq \{j(y)\}_D \\ &\Rightarrow \quad y \sqsubseteq x \end{aligned}$$

THEOREM 3.11 (Mostowski Collapsing Theorem for W) *There is a unique function π defined on W that satisfies the following relation, for every $x \in W$:*

$$\pi(x) = \{\pi(y) \mid y \sqsubseteq x\}. \tag{1}$$

Proof. Let $B \subseteq W$ be defined by putting $z \in B$ if and only if the formula $\psi(z)$ holds, where $\psi(z)$ is the formula $\exists! g\, \phi(z, g)$ and $\phi(z, g)$ is the following formula:

$$\text{dom } g = W_z \cup \{z\} \text{ and, for all } x \in W_z \cup \{z\},\ g(x) = \{g(y) \mid y \sqsubseteq x\}.$$

Whenever there exists a g such that $\phi(z, g)$, we say that g is a *witness for* $\psi(z)$. When such a g defined on $W_z \cup \{z\}$ exists, it will typically be denoted π_z.

We will show that B is j-inductive, and then, from B, obtain the Mostowski Collapsing map. We first observe that $a \in B$: Since there is no $y \in W$ for which $y \sqsubseteq a$, $\{\pi_a(y) \mid y \sqsubseteq a\}$ must be empty. Therefore, the empty function is the unique function π_a with domain $W_a \cup \{a\} = \{a\}$ that satisfies $\pi_a(a) = \{\pi_a(y) \mid y \sqsubseteq a\}$. We have shown that $\psi(a)$ holds, so $a \in B$.

Now assume $z \in B$ and let π_z be the unique map defined on $W_z \cup \{z\}$ that is a witness for $\psi(z)$. We prove $j(z) \in B$. We define $\pi_{j(z)}$ on $W_{j(z)} \cup \{j(z)\}$ by

$$\pi_{j(z)}(x) = \begin{cases} \pi_z(x) & \text{if } x \sqsubseteq j(z), \\ \{\pi_z(y) \mid y \sqsubseteq j(z)\} & \text{if } x = j(z). \end{cases}$$

Notice that if $y \sqsubseteq j(z)$, then either $y = z$ or $y \sqsubseteq z$ (Lemma 3.9(2)(A)). Therefore, defining $\pi_{j(z)}(x)$ to be $\{\pi_z(y) \mid y \sqsubseteq j(z)\}$ when $x = j(z)$ makes sense. We verify that $\pi_{j(z)}$ is a witness for $\psi(j(z))$:

If $x \sqsubseteq j(z)$, then

$$\begin{aligned} \pi_{j(z)}(x) &= \pi_z(x) \\ &= \{\pi_z(y) \mid y \sqsubseteq x\} \\ &= \{\pi_{j(z)}(y) \mid y \sqsubseteq x\}. \end{aligned}$$

The last line follows because, by definition of $\pi_{j(z)}$, $\pi_{j(z)}$ agrees with π_z on all y for which $y \sqsubseteq x$ (since $x \sqsubseteq j(z)$).

On the other hand, if $x = j(z)$, then

$$\begin{aligned} \pi_{j(z)}(x) &= \{\pi_z(y) \mid y \sqsubseteq j(z)\} \\ &= \{\pi_{j(z)}(y) \mid y \sqsubseteq j(z)\}. \end{aligned}$$

Once again, by definition of $\pi_{j(z)}$, $\pi_{j(z)}$ agrees with π_z on y for which $y \sqsubseteq j(z)$, so the second equality in the display follows from the first.

This shows that a witness $\pi_{j(z)}$ for $\psi(j(z))$ exists; we show it is unique. Assume f is defined on $W_{j(z)} \cup \{j(z)\}$ and is also a witness for $\psi(j(z))$; in particular, that f satisfies $f(x) = \{f(y) \mid y \sqsubseteq x\}$. It is not hard to check that $f \restriction W_z$ is a witness for $\psi(z)$, so by uniqueness of π_z as a witness for $\psi(z)$, $f \restriction W_z = \pi_z$. By this observation, we have

$$f(j(z)) = \{f(y) \mid y \sqsubseteq j(z)\} = \{\pi_z(y) \mid y \sqsubseteq j(z)\} = \{\pi_{j(z)}(y) \mid y \sqsubseteq j(z)\} = \pi_{j(z)}(j(z)).$$

Hence $f = \pi_{j(z)}$, and we have established uniqueness. It follows that $j(z) \in B$.

We have shown B is j-inductive, and so $W = B$. We now define the Mostowski Collapsing map π on W as follows: For each $x \in W$,

$$\pi(x) = \pi_x(x).$$

Claim. For all $y \in W$,

$$\text{if } x \sqsubseteq y, \text{ then } \pi_x(x) = \pi_y(x). \tag{2}$$

Proof of Claim. Let $B = \{y \in W \mid \text{if } x \sqsubseteq y \text{ then } \pi_x(x) = \pi_y(x)\}$. We show B is j-inductive. Vacuously, $a \in B$. Suppose $x \in B$; we show $j(x) \in B$. Let y be such that $j(x) \sqsubseteq y$. Then, using the fact (twice) that $x \in B$, we have

$$\pi_y x = \{\pi_y(u) \mid u \sqsubseteq x\} = \{\pi_x(u) \mid u \sqsubseteq x\} = \{\pi_{j(x)}(u) \mid u \sqsubseteq x\} = \pi_{j(x)}(x).$$

Therefore $j(x) \in B$. Since B is j-inductive, $B = W$ and the result follows. □

We show that π satisfies (1) for each $x \in W$. Using statement (2), we have:

$$\pi(x) = \pi_x(x) = \{\pi_x(y) \mid y \sqsubseteq x\} = \{\pi(y) \mid y \sqsubseteq x\},$$

as required.

Finally, we show that π is the unique f satisfying, for all $x \in W$, $f(x) = \{f(y) \mid y \sqsubseteq x\}$. Given any such f, we show $f = \pi$. Let

$$B = \{x \in W \mid \text{for all } y \text{ such that } y = x \text{ or } y \sqsubseteq x,\ f(y) = \pi(y)\}.$$

We show B is j-inductive. The fact that $a \in B$ is immediate; in particular, $\pi(a) = \emptyset = f(a)$. Assume $x \in B$, so that $f(y) = \pi(y)$ for all y for which $y = x$ or $y \sqsubseteq x$. Then since $y \sqsubseteq j(x)$ implies $y = x$ or $y \sqsubseteq x$ (as we observed earlier),

$$f(j(x)) = \{f(y) \mid y \sqsubseteq j(x)\} = \{\pi(y) \mid y \sqsubseteq j(x)\} = \pi(j(x)).$$

To show $j(x) \in B$, we must also show that for $u \sqsubseteq j(x)$, $f(u) = \pi(u)$, but this follows from the fact that $x \in B$. We have shown $j(x) \in B$. Therefore B is j-inductive and $B = W$. It follows that $f = \pi$, as required. □

The proof makes use of Replacement in essentially the same ways as Dedekind's proof (Dedekind, 1888, ¶126) of definition by recursion (on a simply infinite set). Since Replacement is included among our background axioms, we have avoided the circularity that crept into Dedekind's argument.[19]

We establish several properties of π; at this point in the discussion, we shall denote the range of π by N.

THEOREM 3.12 (Properties of π and N)

(1) *N is a transitive set.*
(2) *π is 1-1.*
(3) *For all $x, y \in W$, $x \sqsubseteq y$ if and only if $\pi(x) \in \pi(y)$. In other words, π is an order isomorphism from $(W, \sqsubseteq)$ to $(N, \in)$.*

[19] In our proof, Replacement was used in the inductive step in the proof that the set $B = \{z \in W \mid \psi(z)\}$ contains every element of W; in particular, in the definition $\pi_{j(z)}(x) = \{\pi_z(y) \mid y \sqsubseteq j(z)\}$ whenever $x = j(z)$. Also, in the definition of π as $\pi(x) = \pi_x(x)$ for $x \in W$, Replacement is used implicitly to guarantee that $\{\pi_x \mid x \in W\}$ is a set.

(4) $(N, \in)$ *is a well-order. In particular,* $0 = \emptyset$ *is the* $\in$*-least element of* N.
(5) *Each* $n \in N$ *is a transitive set. Moreover,* $n = \{m \in N \mid m \in n\}$.

Proof of (1). Suppose $\pi(x) \in N$ and $u \in \pi(x)$. We must show $u \in N$. Since $\pi(x) = \{\pi(y) \mid y \sqsubseteq x\}$, it follows that $u = \pi(y)$ for some $y \in W$. Thus $u \in N$.

Proof of (2). Suppose $\pi(x) = \pi(z)$ but $x \neq z$. Note that $W_x \neq W_z$. Without loss of generality, assume there is $u \in W_x - W_z$, so $u \sqsubseteq x$ and $u \not\sqsubseteq z$. Then $\pi(u) \in \pi(x)$. Since $\pi(z) = \pi(x)$, then $\pi(u) \in \pi(z)$, whence $u \sqsubseteq z$, and this is a contradiction. We have shown π is 1-1.

Proof of (3). This follows immediately from the definition of π.

Proof of (4). The first part follows immediately from (3). For the second clause, let $m \in N$. It is clear from the definition of π that $\pi(a) = 0$. Let $x \in W$ be such that $\pi(x) = m$. Since $a \sqsubseteq x$, then by (3), $0 = \pi(a) \in \pi(x) = m$.

Proof of (5). The fact that each $n \in N$ is a transitive set follows from the fact that $\in$ is transitive as an order relation. To show that $n = \{m \in N \mid m \in n\}$, we perform a computation: Let $x \in W$ be such that $n = \pi(x)$.

$$\begin{aligned} n &= \pi(x) \\ &= \{\pi(y) \mid y \sqsubseteq x\} \\ &= \{\pi(y) \mid \pi(y) \in \pi(x)\} \quad \text{(because } \pi \text{ is an order isomorphism)} \\ &= \{m \in N \mid m \in n\}. \ \square \end{aligned}$$

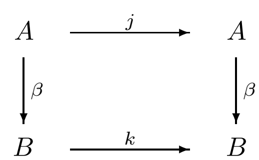
□

We will show that $N = \omega$. A few simple computations ($\pi(a) = 0$, $\pi(j(a)) = 1$, $\pi(j(j(a))) = 2$) seem to support the intuition that π "gives rise" to the natural numbers via its internal dynamics, though, in reality, the finite ordinals $0, 1, 2, 3, \ldots$ are already derivable from ZFC−Infinity. Nevertheless, what π does do is to produce the *set* ω together with the successor function.

It will be helpful at this point to represent a Dedekind self-map $j : A \to A$ with critical point a as a structure (A, j, a) having a single unary operation j and a distinguished element a.

Definition 3.13 (Category of Dedekind Algebras) A *Dedekind algebra* is a triple (A, j, a) where $j : A \to A$ is a Dedekind self-map with critical point a. If $\mathcal{A} = (A, j, a)$ and $\mathcal{B} = (B, k, b)$ are Dedekind algebras, $\beta : \mathcal{A} \to \mathcal{B}$ is a *Dedekind algebra arrow* if

(i) $\beta : A \to B$ is a function with $\beta(a) = b$, and
(ii) the following diagram is commutative:



The Dedekind algebras, together with the Dedekind algebra arrows, form a category, which we denote **DedAlg**. A **DedAlg**-arrow $\beta : \mathcal{A} \to \mathcal{B}$ may sometimes be written as $\beta : j \to k$.

Using the notation of the definition, a map $\beta : j \to k$ is an *iso* in **DedAlg** if there is a **DedAlg** arrow $\gamma : k \to j$ with $\beta \circ \gamma = 1_k$ and $\gamma \circ \beta = 1_j$. We make use of the following characterization, which has a straightforward proof.

PROPOSITION 3.14 *Suppose $\mathcal{A} = (A, j, a)$ and $\mathcal{B} = (B, k, b)$ are Dedekind algebras and $\beta : \mathcal{A} \to \mathcal{B}$ is a* **DedAlg**-*arrow. Then β is an iso if and only if $\beta : A \to B$, as a function on sets, is a bijection.*

We now show that the unique **DedAlg**-arrow $N \to N$ induced by π is the successor function $s : N \to N$.

THEOREM 3.15 (Derivation of the Successor Function) *Define $s = \pi \circ (j \restriction W) \circ \pi^{-1} : N \to N$. Then, for all $n \in N$, $s(n) = n \cup \{n\}$. Moreover, π is a* **DedAlg**-*iso and also the unique* **DedAlg**-*arrow from $j \restriction W$ to s. In particular, s is 1-1.*

$$\begin{array}{ccc} W & \xrightarrow{\;j \restriction W\;} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\;s\;} & N \end{array} \tag{3}$$

Proof. By the definition of s, diagram (3) must be commutative. Let $B = \{x \in W \mid s(\pi(x)) = \pi(x) \cup \{\pi(x)\}\}$. We show B is j-inductive. This is enough because every $n \in N$ is $\pi(x)$ for some $x \in W$, so, assuming $B = W$, we have $s(n) = s(\pi(x)) = \pi(x) \cup \{\pi(x)\} = n \cup \{n\}$.

To prove B is j-inductive, first we show $a \in B$: By commutativity,

$$s(\pi(a)) = \pi(j(a)) = \{\pi(u) \mid u \mathrel{\mathsf{E}} j(a)\} = \{\pi(a)\} = \{0\} = 0 \cup \{0\} = \pi(a) \cup \{\pi(a)\}.$$

Next, assume $x \in B$, so $s(\pi(x)) = \pi(x) \cup \{\pi(x)\}$. We show $j(x) \in B$, that is, $s(\pi(j(x))) = \pi(j(x)) \cup \{\pi(j(x))\}$. But

$$\begin{aligned} s(\pi(j(x))) &= \pi(j(j(x))) \\ &= \{\pi(y) \mid y \mathrel{\mathsf{E}} j(j(x))\} \\ &= \{\pi(y) \mid y \mathrel{\mathsf{E}} j(x) \text{ or } y = j(x)\} \\ &= \{\pi(y) \mid y \mathrel{\mathsf{E}} j(x)\} \cup \{\pi(y) \mid y = j(x)\} \\ &= \pi(j(x)) \cup \{\pi(j(x))\}. \end{aligned}$$

For the "moreover" clause, we have already established π is a **DedAlg**-isomorphism (by Theorem 3.12(2), (3)). Commutativity of diagram (3) allows us to conclude that s is 1-1. To complete the proof, we need to establish uniqueness of π. Suppose $h : j \restriction W \to s$ is a **DedAlg**-iso. Let $B \subseteq W$ be defined by

$$B = \{x \in W \mid h(x) = \pi(x)\}.$$

Since $h(a) = 0 = j(a)$, $a \in B$. Assume $x \in B$. Since h also makes diagram (3) commutative (replacing π with h), we have

$$\pi(j(x)) = j(s(x)) = h(j(x)),$$

and so $j(x) \in B$. We have shown B is j-inductive and hence that $B = W$, as required. $\square$

We can now verify that N is the set ω of finite ordinals. Recall that a set S is *inductive* if $\emptyset \in S$ and whenever $x \in S$, we have $x \cup \{x\}$ is in S, and that ω is defined to be the smallest inductive set.

THEOREM 3.16 *N is an inductive set. Indeed, $N = \bigcap\{I \mid I$ is inductive$\}$. Therefore, $N = \omega$.*

Proof. We have seen already that $\emptyset \in N$. Suppose $n \in N$. Then for some $x \in W$, $n = \pi(x)$. But now

$$n \cup \{n\} = s(n) = s(\pi(x)) = \pi(j(x)) \in N.$$

For the second clause, it is sufficient to show that $N \subseteq I$ for every inductive set I. Let I be any inductive set. Let $B = \{x \in W \mid \pi(x) \in I\}$. We show B is j-inductive. By definition $\pi(a) = \emptyset \in I$, so $a \in B$. If $x \in B$, then $n = \pi(x) \in I$. But now

$$j(x) \in B \Leftrightarrow \pi(j(x)) \in I \Leftrightarrow s(\pi(x)) \in I \Leftrightarrow s(n) \in I \Leftrightarrow n \cup \{n\} \in I,$$

and the last of these statements is true by definition of "inductive." Hence B is j-inductive, and so, for every $n \in N$, $n \in I$, as required. □

We have derived ω and $s : \omega \to \omega$ from the theory

$$\text{ZFC} - \text{Infinity} + \text{“there is a Dedekind self-map”}.$$

We are now in a position to give formal definitions of the concepts "finite" and "infinite."

DEFINITION 3.17 (Finite and Infinite Sets) A set X is *finite* if there is $n \in \omega$ for which there is a bijection from n to X. A set is *infinite* if it is not finite.

THEOREM 3.18 *(Dedekind).* (ZFC – Infinity) *The following are equivalent for a set A:*

(1) A is infinite.
(2) There is a Dedekind self-map $j : A \to A$.

Proof. We give an outline of the proof based on the development given here. For (2) ⇒ (1), we can simply review the work we have done so far. From a Dedekind self-map $j : A \to A$ with critical point a, we obtain a set W, defined as the smallest j-inductive set, which includes each $j^n(a)$. The Mostowski Collapse $\pi : W \to \omega$ is a bijection and produces the 1-1 successor function $s : \omega \to \omega$. Because s is 1-1, by the Pigeonhole Principle, ω is infinite. The composition $\pi^{-1} \circ s : \omega \to A$ must also be 1-1 and so A must also be infinite.

For (1) ⇒ (2), assume A is not empty and there is no bijection between A and any nonzero element $n \in \omega$. To obtain a Dedekind self-map with domain A, one can first establish the following two claims:

Claim 1. For each $n \in \omega$, there is a 1-1 map $n \to A$.

Using the Axiom of Choice (and Replacement), obtain a set $\{f_n \mid n \in \omega\}$ such that for each n, $f_n : n + 1 \to A$ is 1-1, and let $A_n = f_n[n + 1]$.

Claim 2. For each $n \in \mathbb{N}$, there is a least $m_n \in \omega$, with $n \le m_n$, such that A_{m_n} contains an element y_n of A that is not in $A_1 \cup A_{m_2} \cup \cdots \cup A_{m_{n-1}}$.

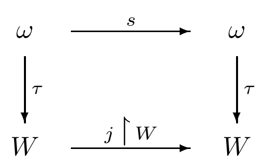
In the proof, to form the set of y_n, the Axiom of Choice is used. Using this set, we can define a function $f : \omega \to A$ by $f(n) = y_n$. Clearly, f is 1-1, as required. One may obtain from f a Dedekind self-map $j : A \to A$ with critical point $a = f(0)$ as follows:

$$j(x) = \begin{cases} f(n+1) & \text{if } x = f(n) \text{ for some } n \in \omega \\ x & \text{otherwise} \end{cases}$$

Clearly $a \notin \operatorname{ran} f$. Also, since $\operatorname{ran} f$ is closed under j, it follows j is 1-1. We have shown that j is a Dedekind self-map on A. □

On the way to establishing the definition by recursion theorem, we show that the successor function $s : \omega \to \omega$ is initial in the category **DedAlg**; indeed, that s is embedded isomorphically in all objects of **DedAlg**. We begin with a lemma that says that π^{-1} is the unique **DedAlg**-arrow from s to $j \upharpoonright W$.

LEMMA 3.19 *Let* $\tau : \omega \to W$ *be* π^{-1}*. Then* τ *is a* **DedAlg**-*iso and is the unique* **DedAlg** *arrow from* $(\omega, s, 0)$ *to* $(W, j \upharpoonright W, a)$.



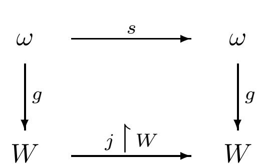
(4)

Proof. Since $\tau = \pi^{-1}$, τ is a bijection and $\tau(0) = a$. Also, the diagram (4) is commutative:

$$\tau \circ s = j \circ \tau \Leftrightarrow s = \pi \circ j \circ \tau \Leftrightarrow s \circ \pi = \pi \circ j.$$

Since π is a **DedAlg**-arrow, the last of these equations ($s \circ \pi = \pi \circ j$) holds true, and so the first one ($\tau \circ s = j \circ \tau$—see diagram (4)) does as well.

For uniqueness, we first observe that if g is a **DedAlg**-arrow from $(\omega, s, 0)$ to $(W, j \upharpoonright W, a)$, then, as a function on sets, g must be 1-1 and onto (see diagram (5)).



(5)

Let $A = \{n \in \omega \mid g(n) \notin \{g(0), g(1), \ldots, g(n-1)\}\}$. We show A is inductive; this will establish that g is 1-1. Clearly $0 \in A$. If $n \in A$ and $g(s(n)) = g(i)$ for some $i, 0 \leq i \leq n-1$, notice first that $i \neq 0$ since we would have in that case

$$a = g(0) = g(s(n)) = j(g(n)),$$

which is impossible since, for no $x \in W$ is it true that $x \sqsubseteq a$. Therefore, $i = s(k)$ for some $k, 0 \leq k < n-1$, and we have

$$j(g(n)) = g(s(n)) = g(s(k)) = j(g(k)).$$

Since j is 1-1, $g(n) = g(k)$ which contradicts the fact that $n \in A$. Therefore, $s(n) \in A$ as required.

To see that g is also onto, let $B \subseteq W$ be defined by $B = \{x \in W \mid \text{for some } n \in \omega, g(n) = x\}$. Clearly $a \in B$. If $x \in B$, let $n \in \omega$ with $g(n) = x$. We show $j(x) \in B$. But

$$j(x) = j(g(n)) = g(s(n)),$$

as required. Since B is j-inductive, $B = W$, and g is onto.

To complete the proof, we must show that $\tau = g$. But notice now that g^{-1} makes the following diagram commutative:

$$\begin{array}{ccc} W & \xrightarrow{j \restriction W} & W \\ \downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\ \omega & \xrightarrow{s} & \omega \end{array} \tag{6}$$

By uniqueness of π, $g^{-1} = \pi$, and so $g = (g^{-1})^{-1} = \pi^{-1} = \tau$. □

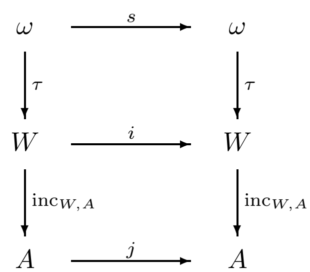
THEOREM 3.20 (Initiality of the Successor Function) *The successor function $s : \omega \to \omega$ is initial in* **DedAlg**. *In other words, for any Dedekind algebra (A, j, a), there is a unique* **DedAlg**-*arrow $\overline{\tau} : (\omega, s, 0) \to (A, j, a)$, as in diagram (7).*

$$\begin{array}{ccc} \omega & \xrightarrow{s} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ A & \xrightarrow{j} & A \end{array} \tag{7}$$

Moreover, $\overline{\tau}$ is 1-1. In addition, if $i : W \to W$ is any initial object in **DedAlg**, *then, for each Dedekind algebra (A, j, a), the unique* **DedAlg**-*arrow that maps i to j must also be 1-1.*

Proof. We have already completed some of the main steps of the proof: We obtained $W \subseteq A$ as the smallest j-inductive set. We showed the collapsing map $\pi : W \to \omega$, is the unique **DedAlg**-arrow from $j \restriction W$ to s, and that π is a bijection. Also, by the lemma, if $\tau = \pi^{-1}$, then τ is also a bijection on sets and is the unique **DedAlg**-arrow from s to $j \restriction W$.

The existence of $\overline{\tau}$ requires one additional step. Consider the following diagram, where $\mathrm{inc}_{W,A}$ is the inclusion map $W \hookrightarrow A$:



(8)

We now define $\overline{\tau} = \mathrm{inc}_{W,A} \circ \tau$. Clearly $\overline{\tau}(0) = a$ and, for all $n \in \omega$,

$$j \circ \mathrm{inc}_{W,A} \circ \tau = \mathrm{inc}_{W,A} \circ i \circ \tau = \mathrm{inc}_{W,A} \circ \tau \circ s, \tag{9}$$

so diagram (8) is commutative. It follows that

$$j \circ \overline{\tau} = \overline{\tau} \circ s,$$

as required.

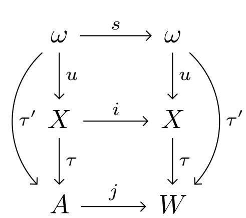
For uniqueness, suppose $h : \omega \to A$ satisfies the same conditions: $h(0) = a$ and $h \circ s = j \circ h$. We show $h = \overline{\tau}$ by proving by induction that $h(n) = \overline{\tau}(n)$ for all $n \in \omega$. Certainly $h(0) = a = \overline{\tau}(0)$ by assumption. Assuming $h(n) = \overline{\tau}(n)$ we show $h(s(n)) = \overline{\tau}(s(n))$. But

$$h(s(n)) = j(h(n)) = j(\overline{\tau}(n)) = \overline{\tau}(s(n)),$$

as required. This completes the induction and shows that $h = \overline{\tau}$.

The "moreover" clause follows because both τ and $\mathrm{inc}_{W,A}$ are 1-1. Finally, suppose (X, i, x) is initial in **DedAlg** and (A, j, a) is an object in **DedAlg**, and let $\tau : i \to j$ be the unique **DedAlg**-arrow guaranteed by initiality. Let $u : s \to i$ and $\tau' : s \to j$ be the unique **DedAlg**-isos guaranteed by the first part of the proof. Because the maps guaranteed by initiality are unique, the diagram below is commutative and we can show τ is 1-1. Given $x, y \in X$, let $x_0, y_0 \in W$ with $u(x_0) = x$ and $u(y_0) = y$. Then

$$\tau(x) = \tau(y) \Rightarrow \tau(u(x_0)) = \tau(u(y_0)) \Rightarrow \tau'(x_0) = \tau'(y_0) \Rightarrow x_0 = y_0 \Rightarrow x = y.$$



□

COROLLARY 3.21 *Suppose $j : A \to A$ is a Dedekind self-map and $W = \{a, j(a), j(j(a)), \ldots\}$. Then $(W, j \restriction W, a)$ is initial in* **DedAlg**.

Proof. Lemma 3.19 showed that $(W, j \restriction W, a)$ is isomorphic to $(\omega, s, 0)$ in **DedAlg**, and Theorem 3.20 showed that $(\omega, s, 0)$ is initial in **DedAlg**. The result follows. □

We apply Theorem 3.20 to show that the elements of W are precisely $a, j(a), j(j(a)), \dots$. As before, let $j : A \to A$ be a Dedekind self-map with critical point a and define $W \subseteq A$ as above. Let $i = j \restriction W$. We obtain by Theorem 3.20 the sequence $\langle i^0, i^1, \dots, i^n, \dots \rangle$ of iterates of i, where i^0 is by convention id_W:

Let $W^W = \{g \mid g : W \to W\}$ and let $J_i : W^W \to W^W$ be defined by

$$J_i(g) = i \circ g. \tag{10}$$

Using Theorem 3.20, let $\overline{\tau} : \omega \to W^W$ be the unique map for which $\overline{\tau}(0) = \mathrm{id}_W$ and diagram (11) is commutative:

$$\begin{array}{ccc} \omega & \xrightarrow{\;s\;} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ W^W & \xrightarrow{\;J_i\;} & W^W \end{array} \tag{11}$$

Define $i^n = \overline{\tau}(n)$ for each $n \in \omega$. We have the following result.

THEOREM 3.22

(1) $i^0 = \mathrm{id}_W$ *and, for each* $n \in \omega$, $i^{n+1} = i \circ i^n$, *so that* i^n *is the* n*th iterate of* i.

(2) $W = \{a, j(a), j^2(a), \dots\} = \{a, i(a), i^2(a), \dots\} = \{i^n(a) \mid n \in \omega\}$

Proof of (1). The case $n = 0$ follows by definition of $\overline{\tau}(0)$. Also, for $n \geq 0$, commutativity of diagram (11) gives us the following:

$$\overline{\tau}(n+1) = \overline{\tau}(s(n)) = J_i(\overline{\tau}(n)) = J_i(i^n) = i \circ i^n.$$

Proof of (2). By (1), for all $n \in \omega, i^n(a) \in \omega$. This shows that W contains all the terms $i^n(a)$. We show that these are the *only* elements of W. Let $B \subseteq W$ be defined by

$$B = \{x \in W \mid \text{for some } n \in \omega,\ x = i^n(a)\}.$$

Notice $a \in B$ since $a = i^0(a)$. Assume $x \in B$, so $x = i^n(a)$ for some $n \in \omega$. Then

$$i(x) = i(i^n(a)) = i^{n+1}(a) \in B.$$

We have shown that B is j-inductive, and so $B = W$. Therefore, every element of W is one of the terms $i^n(a)$. This completes the proof of (2). □

Theorem 3.20 may be strengthened in a number of ways. One of these strengthenings is achieved by taking the given $j : A \to A$ to be *arbitrary* rather than a Dedekind self-map; this step leads to a version of the Definition by Recursion Theorem for ω. The proof we have given of Theorem 3.20 does not generalize to this more general setting, but Dedekind does provide a proof that follows naturally in the present context.[20]

[20] As remarked before, the circularity in Dedekind's argument that was noted in Kanamori (2012) vanishes once we include Replacement in our background theory.

THEOREM 3.23 (Definition by Recursion Theorem for ω) *(*Dedekind*) Suppose $j : A \to A$ is a function and $a \in A$. There is a unique function $\overline{\tau} : \omega \to A$ such that $\overline{\tau}(0) = a$ and $\overline{\tau}(s(n)) = j(\overline{\tau}(n))$.*

$$\begin{array}{ccc} \omega & \xrightarrow{\;s\;} & \omega \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ A & \xrightarrow{\;j\;} & A \end{array} \tag{12}$$

Dedekind uses this Definition by Recursion Theorem to define the operations of addition, multiplication, and exponentiation on ω and to establish the familiar properties of these operations, sufficient to derive the second-order Peano axioms.

REMARK 3.24 A triple (A, j, a) is a *unary pointed algebra* if $a \in A$ and $j : A \to A$ is any self-map. It will be useful later to consider the category **UPA** of unary pointed algebras whose arrows are defined exactly as they are for **DedAlg**. It is easy to verify that for any unary pointed algebra (A, j, a), (A, j, a) is initial in **UPA** if and only if it is initial in **DedAlg**. In particular, $(\omega, s, 0)$ is initial in **UPA**, as is any triple that is **DedAlg**-isomorphic to $(\omega, s, 0)$. An initial object in **UPA** is called a *natural numbers object*.

Another way to strengthen Theorem 3.20 is to provide a *class* version of the result. To state the result properly in ZFC − Infinity, we need to represent ω as a (possibly proper) class. We denote the collection of finite ordinals, when viewed as a class, by $\overline{\omega}$. Define the *global successor function* $\overline{s}$ by $\overline{s}(x) = x \cup \{x\}$. We define $\overline{\omega}$ as follows:

$$\overline{\omega} = \begin{cases} \{0\} \cup \{\overline{s}(\alpha) \mid \alpha \in \text{ON and } \alpha < \gamma\} & \text{if } \gamma \text{ is the least nonzero limit ordinal,} \\ \{0\} \cup \{\overline{s}(\alpha) \mid \alpha \in \text{ON}\} & \text{if ON has no nonzero limit ordinal.} \end{cases}$$

Whether or not any form of the Axiom of Infinity holds, $\overline{\omega}$ still satisfies a form of the Principle of Induction since it is an initial segment of ON and therefore well-ordered; the proof is straightforward.

THEOREM 3.25 (Class Induction Over $\overline{\omega}$) (ZFC − Infinity) *Suppose* $\mathbf{C}$ *is a subclass of* $\overline{\omega}$ *with the following properties:*

(1) $0 \in \mathbf{C}$;
(2) whenever $n \in \mathbf{C}$, $\overline{s}(n) \in \mathbf{C}$.

Then $\mathbf{C} = \overline{\omega}$. □

A special case of definition by recursion on the ordinals allows us to recursively define sequences with indices in $\overline{\omega}$.

THEOREM 3.26 (Class Recursion Over $\overline{\omega}$) (ZFC − Infinity) *A class sequence* $\langle x_0, x_1, x_2, \ldots, x_n, \ldots\rangle$ *indexed by the elements of* $\overline{\omega}$ *can be specified by providing the following:*

(1) (Basis) *The value of* x_0.
(2) (Induction Step) *A formula for obtaining the value* x_{n+1} *from* x_n *for each* $n \in \overline{\omega}$.

We now state a class version of the initiality theorem.

THEOREM 3.27 (Strong Initiality of the Successor) (ZFC − Infinity) *Suppose* $\mathbf{C}$ *is a class,* $c \in \mathbf{C}$*, and* $j : \mathbf{C} \to \mathbf{C}$ *is 1-1 and has critical point* c*, and is itself a class function. Then there is a unique class map* $\overline{\tau} : \overline{\omega} \to \mathbf{C}$ *such that* $\overline{\tau}(0) = c$ *and the following is commutative:*

$$\begin{array}{ccc} \overline{\omega} & \xrightarrow{s} & \overline{\omega} \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \overline{\tau}} \\ \mathbf{C} & \xrightarrow{j} & \mathbf{C} \end{array} \tag{13}$$

Formally, we need to replace the expression "there is a unique class map" by more careful language stating that, from the formulas that define the classes $\mathbf{C}$ and j, one may obtain (working in the metatheory, using finitistic logic) a formula that defines $\overline{\tau}$ so that the properties ascribed to $\overline{\tau}$ hold true.

In this section we have given mathematical expression to the intuition that the discrete collection of natural numbers arises as a precipitation of the "dynamics of an unbounded field," realized as a Dedekind self-map $j : A \to A$. The "dynamics" that lead to this "precipitation" are the repeated application of j to its critical point and subsequent values, followed by the action of the Mostowski collapsing isomorphism. In the next section we identify principles that are at work in these dynamics, which are amenable to generalization.

§4 Properties of Dedekind Self-Maps and the Concept of a Blueprint

In this section we identify and name some of the characteristics of a Dedekind self-map $j : A \to A$ that are amenable to generalization to larger scale self-maps. At the end of this section, we organize these characteristics in the form of a conjecture concerning the kind of Dedekind self-map we expect as a candidate solution to the Problem of Large Cardinals. For this discussion, we fix a Dedekind self-map $j : A \to A$ with critical point a.

A first observation is that a blueprint $W = \{a, j(a), j(j(a)), \ldots\}$ of ω arises from repeated application of j to its critical point and subsequent values. As we mentioned earlier, this is a natural realization of the Plotinian principle *Multiplicity As Epiphenomenon*. Viewing existence of a Dedekind self-map as a fundamental axiom suggests that we are adopting the view that part of the "dynamics of the infinite" is the interaction between j and its critical point, and that "precipitates" from j emerge from this interplay between j and its critical point.

> **Critical Point Dynamics**. A key sequence of values emerges from j and its interaction with its critical point.

Another point about these dynamics that is implicit in the work we have done so far is that one may view the "critical sequence" $a, j(a), j(j(a)), \ldots$ as emerging from a sequence of successive *restrictions* of j to nested subsets of A:

$$\begin{aligned} A_0 &= A; \\ j_0 &= j : A \to A; \\ \mathrm{crit}(j_0) &= a; \\ A_{n+1} &= j[A_n]; \\ j_{n+1} &= j \restriction A_{n+1}; \\ \mathrm{crit}(j_{n+1}) &= j^{n+1}(a). \end{aligned}$$

Therefore $j^n(a)$ is a critical point of j_n, which is the restriction of j to A_n.

> **Restrictions of j and Critical Points**. Restrictions of j to subsets of its domain are directly related to the emergence of the critical sequence $a, j(a), j(j(a)), \ldots$.

Another characteristic of Dedekind self-maps that is apparent from our work in the previous section is a striking *preservation* property that it has: $j : A \to A$ preserves essential properties of its domain (A is Dedekind-infinite, and so likewise is the image $j[A]$ of j) and of itself (the property of being a Dedekind self-map propagates to the restriction $j \restriction j[A]$). One could say that this kind of preservation is a realization of the Plotinian principle *Preservation* since the fundamental character of j is preserved even as its critical sequence emerges from successive restrictions of j to the nested sequence of ranges $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$.

> **Preservation**. j exhibits strong preserves properties.

As observed before, we think of the set $W = \{a, j(a), j(j(a)), \ldots\}$ as a *blueprint* for the set of finite ordinals. A characteristic we would like to make as precise as possible is that, from j's interaction with its critical point, a blueprint of some central set or collection is produced. Before stating this intuition as a principle, we consider what the elements of a blueprint really are.

4.1 The Concept of a Blueprint In most of the early philosophies that hold that the natural numbers arise from a source, one finds also the view that the process of emergence of diversity from the source entails emergence of a *blueprint* for the multiplicity that will eventually come into being.

For both Plato and Plotinus, the world of forms plays this role of blueprint. In the *Timaeus*, Plato describes how the sensible world is crafted by the Demiurge by shaping unformed matter (the *receptacle*) using the eternal forms as templates.

A similar idea is found in Chinese philosophy. For instance, the blueprint of the universe from the perspective of *I Ching* is a higher world of images or forms. These images originate from the Creative principle (yang) and are nurtured into being by the Receptive principle (yin) (Anthony, 1998):

> In the Cosmic Mind, the image arises. The arising of the image was seen by the Chinese as the action of Yang; therefore, in the *I Ching*, Yang is called the Creative. Still, it is only half of the complementary whole. Its other half is Yin, its opposite and complementary force, that in the *I Ching* is called the Receptive. The image offered by Yang is received and nurtured by Yin, bringing it

> into being. The spin-off of this interaction was seen as an ongoing Creation, and the ever-moving Wheel of Change (p. 15).

Likewise, there is a fundamental blueprint—the Veda itself—in the unfoldment of the universe according to the ancient Vedic philosophers as well. For instance, the following account is typical (Oates, 2011):

> The totality of all the laws is the Veda; or, expressed from another perspective, Veda is the "root of all laws." Veda is referred to as a blueprint of creation, but Veda is not merely a description of the mechanics of intelligence in motion within itself; the self-interacting dynamics of consciousness generate Veda and therefore may be seen as the essence—the source of the laws which give rise to the infinite diversity of creation (p. 122).

In quantum field theory, it is reasonable to view the assortment of quantum fields as collectively forming a kind of "blueprint"; every observable in the universe arises from the process of *field collapse* (Brooks, 2016, p. 52ff).

There is an analogue to these philosophical notions of blueprint in the context of Dedekind self-maps. We have already mentioned this idea in a simple form: From a Dedekind self-map $j : A \to A$ with critical point a, one obtains the set $W = \{a, j(a), j(j(a)), \ldots\}$; in an intuitively natural sense, we view W as a "blueprint" for the set of finite ordinals.

This simple intuition can be elaborated a bit further. Again, we start with a Dedekind self-map $j : A \to A$ with critical point a. Let $W = \{a, j(a), j(j(a)), \ldots\}$. Recall $j \upharpoonright W : W \to W$. Let $\pi : W \to \omega$ be the Mostowski collapsing map. For each $n \in \omega$, let $i_n : W^W \to \omega^W$ be defined by $i_n(g) = \pi \circ g^n$, where g^n is the nth iterate of g for $n \geq 1$ and $g^0 = \mathrm{id}_W$. Let $\mathcal{E} = \{i_n \mid n \in \omega\}$. We wish to view the triple $(\mathcal{E}, a, j \upharpoonright W)$ as a "blueprint" in light of the following proposition.

PROPOSITION 4.28 *(Blueprints) For every $n \in \omega$ there exists $i \in \mathcal{E}$ such that $i(j \upharpoonright W)(a) = n$.*

The truth of the proposition is easy to verify: Given $n \in \omega$, note that $n = \pi(j^n(a))$. Let $i \in \mathcal{E}$ be i_n. Then

$$i(j \upharpoonright W)(a) = (\pi \circ (j \upharpoonright W)^n)(a) = \pi(j^n(a)) = n.$$

We think of the function $f = j \upharpoonright W$ as a way of representing ω in coded form. The encoding is initially obtained by the interaction between j and its critical point (producing the set W). Decoding is achieved by applying different elements of $\mathcal{E}$ to f and evaluating further at the critical point.

Before developing the formalism for blueprints further, we introduce an additional element that is often associated with blueprints.

4.2 Returning a Multitude to Its Source and the Role of Co-Dedekind Self-Maps Not only should a blueprint *generate* a multitude but it should facilitate the return of that multitude to its source. For instance, Plotinus writes: "By a natural necessity does everything proceed from, and return to unity" *Enneads III.3.1*.[21]

The idea that in the flow of life, diversity, once expressed, naturally returns to its source, to unity, is pervasive in the *Tao Te Ching*. One reads for example, "Returning is the motion of the *Tao*" (v. 40). Also: "It [*Tao*] flows far away. Having gone far, it returns" (v. 25). Taoist scholar Z.G. Sha and string theorist R. Xiu explain (Sha and Xiu, 2014, pp. 50–53) that the return from the "10,000 things" to Three to Two to One to *Tao* (called by them *reverse creation*) is the counterbalance to the "normal creation" expressed in verse 42, according to which from *Tao* emerges One, then Two, then Three, and ultimately the 10,000 things.

In the Vedic approach, we have already mentioned the fact that the Veda itself is seen to be the blueprint for material existence; but Veda is also understood to contain all the dynamics of *return* from expressed diversity to the starting point in the field of pure consciousness. Indeed, as explained in Nader (1995), p. 25, there is a "part" of the Veda, represented in the Vedic Literature, that is responsible for *expansion* and another that is responsible for *return*.

Finally, in quantum field theory, we find this same theme of return as a dynamic element of the blueprint, since quantum fields are responsible not only for creation of particles but also for their destruction (Brooks, 2016).

We have just now found considerable support in the ancient philosophies, and in QFT to some extent as well, for this concept of return. In a rather natural way, these dynamics of return are expressed mathematically using the dual notion of a *co-Dedekind self-map*.

Let us say that an onto self-map $h : A \to A$ is *co-Dedekind* if the preimage $h^{-1}(a)$, for some $a \in A$, has two or more elements. Whenever $a \in A$ is such that $|h^{-1}(a)| \geq 2$, a will be called a *co-critical point* of h.

Recall that, whenever $h : A \to A$ is onto, one can always obtain a function $s : A \to A$ (using the Axiom of Choice) whose range contains exactly one element from each preimage $h^{-1}(a)$, for $a \in A$; we will call such an s a *section of* h. Clearly, any section of an onto map must be 1-1. Moreover, we have the following:

THEOREM 4.29 (ZFC – Infinity) *Suppose A is a set. Then the following are equivalent:*

[21] Translation by Guthrie (1918); see p. 1077.

(1) *There is a Dedekind self-map on A.*
(2) *There is a co-Dedekind self-map on A.*

Proof. Suppose there is a Dedekind self-map $j : A \to A$ with critical point a. Let $a_0 \in \operatorname{ran} j$. Define $h : A \to A$ by

$$h(x) = \begin{cases} a_0 & \text{if } x \notin \operatorname{ran} j, \\ y & \text{otherwise, where } y \in A \text{ is unique such that } j(y) = x. \end{cases}$$

Certainly h is onto since even $h \restriction \operatorname{ran} j$ is onto. Therefore, some $b = j(x) \in \operatorname{ran} j$ is mapped by h to a_0. But because $a \notin \operatorname{ran} j$, $b \neq a$, and at the same time, by definition of h, $h(a) = a_0$. Therefore, $|h^{-1}(a_0)| \geq 2$, and so h is co-Dedekind.

Conversely, suppose $h : A \to A$ is co-Dedekind, and suppose $s : A \to A$ is a section of h. We show s itself is Dedekind. We have already observed that s is 1-1. Let $x \in A$ be such that $|h^{-1}(x)| \geq 2$, and let $u \neq v \in A$ be elements of $h^{-1}(x)$. Then one of u, v does not belong to the range of s and so is a critical point of s. □

The argument shows that any section s of a co-Dedekind self-map $h : A \to A$ is itself a Dedekind self-map; moreover, for any co-critical point x of h, some element of $h^{-1}(x)$ is a critical point of s.

We give an example to illustrate the "collapsing" or "returning" effect that co-Dedekind self-maps often have. A set A is *closed under pairs* if, whenever $x, y \in A$, $\{x, y\} \in A$. For the example, we first show that, in studying co-Dedekind self-maps $A \to A$, there is nothing lost if we assume A is a transitive set closed under pairs:

Proposition 4.30 (ZFC − Infinity) *There is a co-Dedekind self-map on a set if and only if there is a co-Dedekind self-map on a transitive set that is closed under pairs.*

Proof. Suppose $h : A \to A$ is a co-Dedekind self-map on A. It follows that A is infinite, so we may as well work in ZFC. Let $t : A \to A$ be a section of h that is a Dedekind self-map with critical point a. We will lift t to a Dedekind self-map $\hat{t} : B \to B$, where B is a transitive set that is closed under pairs and that includes A.

Claim. There is a Dedekind self-map $\hat{t} : B \to B$, where B is a transitive set that is closed under pairs and that includes A, and for which $\hat{t} \restriction A = t$.

Proof of Claim. We first observe that, for any set C, by forming the union $C \cup [C]^2 \cup \left[[C]^2\right]^2 \cup \cdots$, we obtain a set $U(C) \supseteq C$ that is closed under pairs. We build the set B as the union of the following chain:

$$A = A_0 \subseteq B_0 \subseteq A_1 \subseteq B_1 \subseteq \cdots,$$

where, for each $i \in \omega$, $B_i = U(A_i)$ and A_{i+1} is a transitive set that contains B_i. It is straightforward to verify that B has the desired properties.

We obtain a Dedekind self-map $\hat{t} : B \to B$ as follows:

$$\hat{t}(b) = \begin{cases} b & \text{if } b \notin A, \\ t(b) & \text{if } b \in A. \end{cases}$$

It is easy to see that $\hat{t}$ is a Dedekind self-map that extends t. □

To complete the proof of Proposition 4.30, we simply recall that, by Theorem 4.29, whenever there is a Dedekind self-map $B \to B$, there is also a co-Dedekind self-map $B \to B$. □

EXAMPLE 4.31 (Generate/Return Duality) Let A be a transitive nonempty set that is closed under pairs. Consider the self-map $F : A \to A$, defined by:

$$F(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ y & \text{where } y \text{ is any } \in\text{-minimal element of } x. \end{cases}$$

Because A is transitive, $\operatorname{ran} F \subseteq A$. We also observe that F is onto: Suppose $y \in A$. Then $\{y\} \in A$, and clearly $F(\{y\}) = y$. Finally, suppose $z \in A$ and consider the sets $x = \{z\}$ and $y = \{z, \{z\}\}$. The fact that no set is an element of itself ensures that $z, \{z\}$ are disjoint, and so $F(y) = z = F(x)$. Thus, $|F^{-1}(z)| > 1$. We have shown F is a co-Dedekind self-map.

Let $S_A = A \to A$ be defined by $S_A(x) = \{x\}$ for all $x \in A$. We observe that S_A is a section of F: For every $x \in A$,

$$F(S_A(x)) = F(\{x\}) = x.$$

The dual notions of Dedekind self-map and co-Dedekind self-map are expressed in the self-maps S_A and F. The map S_A plays the role of *generating* a blueprint for the finite ordinals: Given a, $S_A \restriction \{a, S_A(a), S_A^2(a), \ldots\}$ is an initial Dedekind self-map. We wish to show that, conversely, F plays the role of *collapsing* or *returning* the values of A to their point of origin.

We show that for every $x \in A$, there is $n \in \omega$ such that $F^n(x) = \emptyset$. Suppose not. Then for each $n \in \omega$, $F^n(x) \neq \emptyset$. It follows that the following is an infinite descending $\in$-chain:

$$\cdots \in F^n(x) \in F^{n-1}(x) \in \cdots \in F(x) \in x.$$

Such chains cannot exist in the presence of the Axiom of Foundation. The result follows.

Let $\mathcal{E} = \{i_n \mid n \in \omega\}$, where, for each n, $i_n : A^A \to A^A$ is defined by

$$i_n(f) = \begin{cases} f^n & \text{if } n > 0 \\ \mathrm{id}_A & \text{if } n = 0 \end{cases}. \tag{14}$$

Here, f^n denotes the nth iterate of f. We have the following:

PROPOSITION 4.32 *Suppose A is a transitive set that is closed under pairs. Let $W = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\} \subseteq A$.*

(1) For every $x \in A$—in particular, for every $x \in W$—there is $i \in \mathcal{E}$ such that $i(F)(x) = \emptyset$.
(2) For every $x \in W$, there is $i \in \mathcal{E}$ such that $i(S_A)(\emptyset) = x$.

The proposition indicates how every element of A is "returned to its source" via the interplay of F and a naturally occurring set $\mathcal{E}$ of functionals defined over A. Likewise, through the interplay of S_A and $\mathcal{E}$, we also see once again in the present context how a blueprint for the set of finite ordinals is generated. In summary, the dual self-maps S_A and F play the roles, respectively, of "generating a blueprint" and "returning elements to their source." □

We observe next that the functionals belonging to the class $\mathcal{E}$ have a useful preservation property, which we can generalize to a broader context.

Definition 4.33 Suppose C, D are sets, each equipped with binary relations E, R, respectively. A function $f : (C, E) \to (D, R)$ will be called Σ_0-*elementary* if, for every Σ_0^{ZFC} $\in$-formula $\phi(x_1, \dots, x_m)$ and all $c_1, \dots, c_m \in C$, $\phi^{(C,E)}(c_1, \dots, c_m) \Leftrightarrow \phi^{(D,R)}(f(c_1), \dots, f(c_m))$ (as usual, it is understood that E interprets $\in$ in C and R interprets $\in$ in D). Let S be a nonempty collection of functions $C \to D$. Suppose $i : S \to T$ is a functional defined so that for each $f \in S$, $i(f) : C' \to D'$. Suppose also that C', D' are equipped with binary relations E', R', respectively. We shall say that i is Σ_0-*preserving* if, whenever $f \in S$ is Σ_0-elementary, then $i(f)$ is also Σ_0-elementary.

To see the connection to the previous example, let $S_W = S_A \restriction W : W \to W$. We can define a binary relation $\sqsubseteq$ on W as we did earlier (p. 20) that satisfies $x \sqsubseteq y$ if and only if one can obtain y from x by applying S_W at most finitely many times to x: $y = S_W(S_W \dots (S_W(x)) \dots)$. Then it is easy to verify that each of the functionals $i_n \in \mathcal{E}$ is Σ_0-preserving.

As we shall see, the concept of a "blueprint" arises naturally in the context of large cardinals, and Σ_0-preserving functionals play an important role in that context. We now make the notion of a blueprint, suggested by our results here, more precise.

4.3 A Formal Treatment of Blueprints We turn now to a mathematical account of blueprints that arises from a careful analysis of Dedekind self-maps. Linking the characteristics discovered by ancient texts to those we find connected with the dynamics of a Dedekind self-map will provide us with material for a conjecture about the sort of behaviors and dynamics we should expect to find as we start to examine generalized Dedekind self-maps. In this subsection, then, we give a detailed account of blueprints, and how the behavior of a Dedekind self-map produces a blueprint, in the formal sense, of the set ω of natural numbers.

Starting with a Dedekind self-map $j : A \to A$ with critical point a and a set $X \subseteq A$, our goal is to give precise expression to the notion of a *blueprint for* X. The intention is that, first of all, we have a class $\mathcal{E}$ of Σ_0-preserving functionals on B^B, for some $B \subseteq A$, and, through the interaction of j, a, and $\mathcal{E}$, we obtain a dual pair of self-maps f, g (one of which is Dedekind, the other, co-Dedekind), each defined on B. The map f will *encode* the set X. Moreover, f will provide a way of *generating* the elements of X. Dually, g will provide a way to *return* elements of X to their source, a. We think of f as containing all the information about the elements of X in its "seed" (or encoded) form; in this sense, f may be thought of as a *substrate* for X. We consider the "blueprint" for X to consist not only of f, but also of the mechanism by which elements of X are obtained from f; this mechanism includes $\mathcal{E}$ and the critical point a.

One other aspect of our definition of blueprint is that we require that $f, g, \mathcal{E}$, and a all "come from" the underlying self-map j. The reason for this requirement is that we wish to think of $f, g, \mathcal{E}, a$ as arising from the dynamics of j, just as, for example, Plato's forms arise from the dynamics of the One.

In the formal definition, we first consider a simpler case in which elements of X are generated, but in which we do not necessarily have a mechanism for returning elements to their source. We will call the machinery by which elements of X are generated a *blueprint*. Then we consider the "ideal" case, in which we have both generation and return of elements of X from the blueprint; the resulting strengthened form of a blueprint will be called a *strong* blueprint.

We need the following definition: Suppose B is a set and $\mathcal{E}$ is a collection of functionals such that, for each $i \in \mathcal{E}$, dom $i \supseteq B^B$. Let $\mathcal{E}_0$ be defined by $\mathcal{E}_0 = \{i \restriction B^B \mid i \in \mathcal{E}\}$. Then $\mathcal{E}_0$ is called the *restriction of* $\mathcal{E}$ *to* B^B and we specify this relationship by writing $\mathcal{E}_0 \sqsubseteq_{r,B} \mathcal{E}$, or simply $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ when the meaning is clear from the context. In the definition below, point (2) refers to the concept of a class $\mathcal{E}$ of Σ_0-preserving functionals being "compatible" with a Dedekind self-map $j : A \to A$; this concept will be made precise in Remark 4.35(2), immediately following the definition.

Definition 4.34 (Blueprints) Suppose $j : A \to A$ is a Dedekind self-map with critical point a, and suppose $X \subseteq A$. A *j-blueprint* (or simply a *blueprint*) for X is a triple $(f, a, \mathcal{E})$, having the following properties:

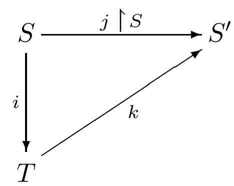
(1) For some set B, $f : B \to B$ is either a Dedekind self-map or a co-Dedekind self-map. (Note: The critical (co-critical) point for the self-map may or may not be equal to a.)
(2) The class $\mathcal{E}$ consists of functions each of whose domains includes B^B and $\mathcal{E}$ is compatible with j. Moreover, if $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ is the restriction of $\mathcal{E}$ to B^B, then, each $i \in \mathcal{E}_0$ has the following properties:
    (a) there exist C_i, D_i so that $i : B^B \to {D_i}^{C_i}$;
    (b) $C_i \supseteq B$;
    (c) if $C_i \neq D_i$, then there is a bijection $\pi_i : D_i \to C_i$ that is definable from j and a;
    (d) $a \in$ dom $i(f)$
    (e) there are partial orders E, E_i, R_i on B, C_i, D_i, respectively, so that for all Σ_0-elementary $f \in B^B$ (relative to E), $i(f)$ is Σ_0-elementary (relative to C_i, D_i); in other words, i is Σ_0-preserving.
(3) (*Encoding*) The self-map f is *definable from* $\mathcal{E}, j, a$.
(4) (*Decoding*) f generates X in the following sense:

$$\text{For every } x \in X, \text{ there is } i \in \mathcal{E}_0 \text{ such that } i(f)(a) = x.$$

Remark 4.35

(1) In the example given earlier, the generating function $x \mapsto \{x\}$ turned out to be a Dedekind self-map, but in some contexts, the generating function will be a co-Dedekind self-map. Condition (1) leaves room for either possibility.
(2) One point in the definition that remains vague is the requirement that the elements of $\mathcal{E}$ should be "compatible with j." For the example that we know about so far, and others we will see that belong to a relatively simple context, to say that the elements of $\mathcal{E} = \{i_0, i_1, i_2, \ldots, i_n, \ldots\}$ are "compatible with j" simply means they are definable from j and its critical point. In Example 4.31, no ambient self-map $j : A \to A$ was specified; however, since any set A that is closed under pairs is infinite, we can certainly find a Dedekind self-map on A, and the elements of $\mathcal{E}$, which are functionals that specify various iterations, can be shown to be definable from A itself (which is in turn defined from j by $A =$ dom j). When we expand to a more general context, the requirement that each $i \in \mathcal{E}$

is definable from j and its critical point will be too strong. In that context, we will make use of the notion of local compatibility described in Section §2 in a more general form. We outline the idea here, which will be applicable in contexts in which dom $i \subseteq$ dom j for each $i \in \mathcal{E}$; under this condition, we will say that $\mathcal{E}$ *is compatible with* j if some $i : S \to T \in \mathcal{E}$ (where $i \upharpoonright B^B : B^B \to C_i^{C_i}$) is a *right factor* of $j \upharpoonright S : S \to S'$; more precisely, for some $i \in \mathcal{E}$, $i : S \to T$, there is $k : T \to S'$ so that $k \upharpoonright C_i^{C_i}$ is Σ_0-preserving and $j \upharpoonright S = k \circ i$.



This requirement, together with the requirement that elements of $\mathcal{E}_0$ are Σ_0-preserving, is an attempt to capture the intuition that elements of $\mathcal{E}$ "arise from" j.

(3) In some contexts, the set $\mathcal{E}_0$ of maps $i : B^B \to {D_i}^{C_i}$ arise as restrictions of maps from a broader, naturally defined class $\mathcal{E}$. In such cases, for "decoding" purposes, $\mathcal{E}_0$ suffices, but for "encoding" purposes, the broader class $\mathcal{E}$ is needed.

The requirements on elements of $\mathcal{E}$ may appear needlessly general. Based on the example we have considered so far, it would be reasonable to expect that each $i \in \mathcal{E}_0$ would have type $B^B \to B^B$. Later, however, we will see examples in which it is natural for functions in B^B to be taken to functions in C^C, where $C \supseteq B$, or even D^C, where D is a bijective image of C, under a bijection π that is definable from j. This latter situation can arise when $D = X$, where X is the set that is being generated, but will not arise when $B^B \subseteq$ dom j. It will also happen sometimes that the codomain C^C of the a functional i on B^B may vary depending on i, as condition (2)(a) indicates. Nevertheless, for any such C, we always have $B \subseteq C$.

As we move toward a definition for *strong blueprint*, in which elements of X are also returned to their "source" element a, an obstacle needs to be addressed in the case that functionals $i \in \mathcal{E}_0$ are of the form $i : B^B \to D^C$, where $D \neq C$. In that case it is not clear how to meet the requirement of obtaining a dyad (f, g) for which f generates X and g returns elements of X to a.

For concreteness, we consider an example. Suppose we have a class $\mathcal{E}_0$ of weakly elementary functionals of the form $i : B^B \to D^C$, and $\pi : C \to D$ is a bijection definable from j and a. Suppose also we have obtained a generating Dedekind self-map f so that for every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(f)(a) = x$. Now the type of $i(f)$ must be $i(f) : C \to D$. The type presents no problem since $a \in B \subseteq C$. Now, to return elements of X back to a, we will need a co-Dedekind self-map $g : B \to B$ with the property that, for each $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(g)(x) = a$. Here again, the type of $i(g)$ must be $i(g) : C \to D$. This means that a must belong to D, but since D is only an image of C under π, it will not generally be possible for this requirement to be met.

To overcome this obstacle, we introduce the concept of a *conjugate class* $\mathcal{E}_0^*$. Given $\mathcal{E}_0$ containing functionals of type $B^B \to D^C$, as we have been discussing, and given $i \in \mathcal{E}_0$, we define $i^* : B^B \to C^D$ by $i^*(h) = \pi^{-1} \circ i(h) \circ \pi^{-1} : D \to C$.

Then we let $\mathcal{E}_0^* = \{i^* \mid i \in \mathcal{E}\}$. Now $i^*(h)$ has the right type. So now it does make sense to require that for each $x \in X$, there is $i \in \mathcal{E}_0$ so that $i^*(g)(x) = a$.

In the more typical context in which $\mathcal{E}_0$ contains functionals of type $B^B \to C^C$ (or $B^B \to C_i^{C_i}$ with $i \in \mathcal{E}_0$), the map π is taken to be id_C (or id_{C_i}, respectively), so that $i^* = i$ in such cases.

Using this device, we can give a satisfactory definition of strong blueprints:

Definition 4.36 (Strong Blueprints) Suppose $j : A \to A$ is a Dedekind self-map with critical point a, and suppose $X \subseteq A$. A *strong j-blueprint* (or simply a *strong blueprint*) for X is a quadruple $(f, g, a, \mathcal{E})$ having the following properties:

(1) For some set B, f and g are functions $B \to B$, and one of these is a Dedekind self-map, the other, a co-Dedekind self-map. (The values of the critical and co-critical points of the self-maps may or may not be equal to a). The pair (f, g) is called the *blueprint dyad* and satisfies one of the following: $g \circ f = \mathrm{id}_B$ or $f \circ g = \mathrm{id}_B$.

(2) The class $\mathcal{E}$ is compatible with j. Moreover, if $\mathcal{E}_0 \sqsubseteq_r \mathcal{E}$ is the restriction of $\mathcal{E}$ to B^B, then, for each $i \in \mathcal{E}_0$:

   (a) there exist C_i, D_i so that $i : B^B \to {D_i}^{C_i}$;
   (b) $C_i \supseteq B$;
   (c) if $C_i \neq D_i$, then there is a bijection $\pi : D_i \to C_i$ that is definable from j and a;
   (d) $a \in \mathrm{dom}\ i(f)$;
   (e) $\mathrm{dom}\ i(f) = \mathrm{dom}\ i(g)$
   (f) there are partial orders E, E_i, R_i on B, C_i, D_i, respectively, so that for all Σ_0-elementary $f \in B^B$ (relative to E), $i(f)$ is Σ_0-elementary (relative to C_i, D_i); in other words, i is Σ_0-preserving.

(3) (*Encoding*) The self-maps f and g are *definable from* $\mathcal{E}, j, a$.

(4) (*Decoding*)

   (a) The self-map f generates X in the following sense:

   For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(f)(a) = x$.

   (b) The self-map g collapses elements of X in the following sense:

   For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i^*(g)(x) = a$.

Note that in condition (4)(b), we have used the conjugate of i so that the functional type is correct. In most respects, the definition of strong blueprint is the same as the definition of blueprint, except that we have also required the existence of a dual to the generating function, which returns values in X to a.

In the sequel, we will make use of both concepts—blueprint and strong blueprint—as we consider more examples. One situation that arises is that, for a particular set X we may have a blueprint $(f, a, \mathcal{E})$, but not a strong blueprint, but, for an important subset Y of X, we are able to obtain a dual g of f so that $(f, g, a, \mathcal{E})$ is a strong blueprint for Y.

Remark 4.37 We now rework Example 4.31 to indicate how the maps defined in the example give rise to a formal blueprint, and also a formal strong blueprint. In that setting, A was a transitive set, closed under pairs, and we discussed a blueprint for the set $W = \{\emptyset, \{\emptyset\}, \ldots\} \subseteq A$. We did not specify a Dedekind self-map on A in Example 4.31, but *any* Dedekind self-map $j : A \to A$ can be used here. In the example, we defined a function F by

$$F(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ y & \text{where } y \text{ is any } \in\text{-minimal element of } x. \end{cases}$$

Formally, the function $F \restriction W$ corresponds to g in the strong blueprint definition since it "returns" elements of W to their source. Likewise, S_A was defined by $x \mapsto \{x\}$, and so the function $S_A \restriction W$ corresponds to f since it serves to generate elements. Also $\mathcal{E} = \mathcal{E}_0$, the collection $\{i_0, i_1, i_2, \ldots\}$ of iteration maps, is indeed a collection of Σ_0-preserving functionals. Therefore, a blueprint for W is given by $(S_A \restriction W, a, \mathcal{E})$, and a strong blueprint, by $(S_A \restriction W, F \restriction W, a, \mathcal{E})$. Notice that since each $i \in \mathcal{E}$ is, in this formal context, of type $W^W \to W^W$, each element of $\mathcal{E}$ is definable from W, which, being defined from $S_A : A \to A$, is definable from j (since $A = \text{dom } j$). Likewise, $F \restriction W$ can be shown to be definable from j. □

With our formal definition, we can now substantiate the claim, made near the beginning of the paper, that if $j : A \to A$ is a Dedekind self-map with critical point a, the set $W = \{a, j(a), j(j(a)), \ldots\}$ forms a "blueprint" for ω. For this purpose, we recall the main diagram of Theorem 3.15, which shows how $s : \omega \to \omega$ arises as the Mostowski collapse of $j \restriction W : W \to W$, where $W = \{a, j(a), j(j(a)), \ldots\}$ and $j : A \to A$ is a Dedekind self-map with critical point a. Diagram (15) summarizes the results of Theorem 3.15.

$$\begin{array}{ccc} W & \xrightarrow{j_W} & W \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \omega & \xrightarrow{s} & \omega \end{array} \tag{15}$$

Here, $j_W = j \restriction W$, π is the Mostowski collapsing isomorphism, and $s : \omega \to \omega$ is the usual successor function $n \mapsto n \cup \{n\}$.

Theorem 4.38 (Blueprint for ω Theorem) *Suppose $j : A \to A$ is a Dedekind self-map with critical point a. Let $W = \{a, j(a), j^2(a), \ldots\}$. Then there are a set $\mathcal{E} = \mathcal{E}_0$ of Σ_0-preserving functionals compatible with j and a co-Dedekind self-map $h : W \to W$ with co-critical point a such that $(j \restriction W, h, a, \mathcal{E})$ is a strong blueprint for ω.*

Proof. Let $j_W = j \restriction W$. First, we want to generate ω using j_W. From the commutative diagram, we see that to arrive at elements of ω, we will need to compose with π. We can derive the following facts from diagram (15):

$$\begin{aligned} \pi(a) &= 0; \\ \pi(j_W(a)) &= 1; \\ \pi(j_W^2(a)) &= 2. \end{aligned}$$

Thus, for each $n \in \omega$, it follows that $(\pi \circ j^n)(a) = n$. Now this formula suggests how to define elements of $\mathcal{E} = \mathcal{E}_0 = \{i_0, i_1, i_2, \ldots\}$. Instead of requiring i_n to be the functional that produces nth iterations, we will require i_n to produce nth iterations *composed with* π. Therefore, we define i_n as follows.

$$\text{For each } n \in \mathbb{N} \text{ and each } g : W \to W,\ i_n(g) = \pi \circ g^n. \tag{16}$$

Notice that i_n takes elements of W^W to elements of ω^W. With this definition, we have obtained a blueprint for ω: For each $n \in \omega$, there is $i \in \mathcal{E}_0$ such that $i(j_W)(a) = n$, since

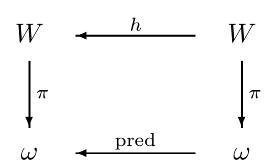
$$i_n(j_W)(a) = \pi(j_W^n(a)) = n.$$

Next, we obtain the dual h for j_W, which is also a map from W to W. Moreover, it must be the case that j_W is a section of h; that is, for each $j_W^n(a) \in W$, we should have $(h \circ j_W)(j_W^n(a)) = j_W^n(a)$. Since $j_W(j_W^n(a)) = j_W^{n+1}(a)$, we make use of the usual predecessor function pred : $\omega \to \omega$ to help in our definition of h. The function pred is defined as follows:

$$\text{pred}(n) = \begin{cases} n-1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

Clearly pred is co-Dedekind and we have $\text{pred}(s(n)) = n$ for all $n \in \omega$.

Consider the following diagram.



(17)

Diagram (17) makes it clear how h must be defined: $h = \pi^{-1} \circ \text{pred} \circ \pi$. We check that h, with this definition, is indeed a dual for j_W: For any $j_W^n(a) \in W$, we have:

$$\begin{aligned} (h \circ j_W)(j_W^n(a)) &= h(j_W(j_W^n(a))) \\ &= h(j_W^{n+1}(a)) \\ &= \pi^{-1}(\text{pred}(\pi(j_W^{n+1}(a)))) \\ &= \pi^{-1}(\text{pred}(n+1)) \\ &= \pi^{-1}(n) \\ &= j_W^n(a) \\ &= \text{id}_W(j_W^n(a)). \end{aligned}$$

We now verify that h is a co-Dedekind self-map. First we show h is onto: For each $j_W^n(a) \in W$, we have $h(\pi^{-1}(s(\pi(j_W^n(a))))) = j_W^n(a)$, since

$$\begin{aligned} h(\pi^{-1}(s(\pi(j_W^n(a))))) &= \pi^{-1}(\text{pred}(\pi(\pi^{-1}(s(\pi(j_W^n(a)))))) \\ &= \pi^{-1}(\text{pred}(s(\pi(j_W^n(a)))) \\ &= \pi^{-1}(\pi(j_W^n(a))) \\ &= j_W^n(a). \end{aligned}$$

Moreover, h has co-critical point a since $h(a) = a = h(j_W(a))$. We have shown h is a co-Dedekind self-map.

Next, we verify that, for every $n \in \omega$, there is $i \in \mathcal{E}_0$ such that $i^*(h)(n) = a$. Recall that, because our functionals i are of type $i : W^W \to \omega^W$, to get the collapsing step to work out, we need to use a conjugate $i^* : W^W \to W^\omega$ of i; in that case, the type of $i^*(h)$ is $i^*(h) : \omega \to W$, exactly as needed. Here is the required verification:

$$\begin{aligned} i_n^*(h)(n) &= i_n^*(\pi^{-1} \circ \text{pred} \circ \pi)(n) \\ &= \big(\pi^{-1} \circ i_n(\pi^{-1} \circ \text{pred} \circ \pi) \circ \pi^{-1}\big)(n) \\ &= \big(\pi^{-1} \circ \pi \circ \pi^{-1} \circ \text{pred}^n \circ \pi \circ \pi^{-1}\big)(n) \\ &= (\pi^{-1} \circ \text{pred}^n)(n) \\ &= \pi^{-1}(0) \\ &= a. \end{aligned}$$

To complete the proof, a few details need to be checked. We refer to Definition 4.36 where strong blueprints are defined. For (2)(c), we need to verify that π is defined from j and a; a review of the definition of π as the Mostowski collapsing map given in equation 1 (Theorem 3.11) shows that this is indeed the case. It follows that $\mathcal{E}$ is compatible with j, since each $i \in \mathcal{E}_0$ is a composition of an iteration function with π. For (3), we need to verify that j_W and h are also defined from $j, a, \mathcal{E}$. Certainly $j_W = j \restriction W$ is defined from j and W, but $W = \bigcap \mathcal{I}$ where $\mathcal{I} = \{B \subseteq A \mid B \text{ is } j\text{-inductive}\}$, and the j-inductive property is defined in terms of a and j. So W is defined from j and a. Also, h is defined from the successor $s : \omega \to \omega$, and, in our treatment, s is defined by $s = \pi \circ (j \restriction W) \circ \pi^{-1}$, the factors of which, as has already been indicated, are defined from j and a.

Next, we check that each $i_n \in \mathcal{E}$ is Σ_0-preserving. The partial order on ω will be the membership relation $\in$ and the partial order for W is the relation $\sqsubseteq$, as described on p. 20. Since we have already verified that ordinary iteration functionals $f \mapsto f^n$ are Σ_0-preserving for $\sqsubseteq$, it suffices to check that the Mostowski collapsing map $\pi : W \to \omega$ is also Σ_0-preserving, relative to $\sqsubseteq, \in$. Suppose $\phi(x_1, \dots, x_m)$ is a Σ_0 $\in$-formula. We show

$$\text{for all } a_1, \dots, a_m,\ \phi^{(W,\sqsubseteq)}(a_1, \dots, a_m) \Leftrightarrow \phi^{(\omega,\in)}(\pi(a_1), \dots, \pi(a_m)). \tag{18}$$

By the definition of π, (18) holds for atomic formulas. It suffices to show, using the Tarski-Vaught Criterion, that if there is $y \in \omega$ such that $\phi^{(\omega,\in)}(y, \pi(a_1), \dots, \pi(a_m))$, then there is $x \in W$ for which $\phi^{(\omega,\in)}(\pi(x), \pi(a_1), \dots, \pi(a_m))$. But this follows from the fact that π is a bijection.

The remaining verifications are straightforward. □

To close this section, we review the salient characteristics of Dedekind self-maps that we have identified so far.

4.4 Summary of Characteristics of a Dedekind Self-Map and a Conjecture We summarize the four characteristics of Dedekind self-maps that we have identified in our work so far. We will use these to formulate a conjecture on the right way to extend and generalize the notion of a Dedekind self-map to provide an account of large cardinals.

Properties of a Dedekind Self-Map

(1) *Critical Point Dynamics.* A key sequence of values emerges from j and its interaction with its critical point.

(2) *Restrictions of j and Critical Points*. Restrictions of j to subsets of its domain are directly related to the emergence of the critical sequence $a, j(a), j(j(a)), \ldots$.
(3) *Preservation*. j exhibits strong preserves properties.
(4) *Blueprint*. The interaction between j and its critical point produces a blueprint—even a *strong* blueprint—from which a set of central importance is generated.

We formulate a conjecture about the sort of Dedekind self-map $j : V \to V$ that would be a candidate for deriving large cardinals. We specify characteristics that appear to us to be natural generalizations of the characteristics of a *set* Dedekind self-map, and which, at the same time, are strong enough to obtain large cardinals. The fact that the sort of candidate mapping we have in mind exhibits characteristics that, in a straightforward way, generalize properties of a self-map whose existence we postulate as a foundational axiom makes our selection especially appropriate as an "intrinsic" justification in the sense of Gödel.

Dedekind Self-Map Conjecture

(1) Dedekind self-maps of the universe V, with rich preservation properties, account for the presence of large cardinals in the universe.
(2) The mechanism by which large cardinals and other mathematical objects arise from a Dedekind self-map $j : V \to V$ involves the interaction of j with its critical points.
(3) Emergence of a critical sequence for such a Dedekind self-map j is closely related to successive transformations of j obtained by restrictions of j to sets in V.
(4) The dynamics of such a Dedekind self-map $j : V \to V$ will result in emergence of a blueprint or strong blueprint for some significant class of sets—possibly the entire universe V.
(5) The dynamics of j are in some way present everywhere in V.
(6) Every mathematical object arises from the dynamics present in j.

Part (1) of the conjecture expresses our belief in the *Preservation* characteristic that we identified in the dynamics of a Dedekind self-map (reflecting the Plotinian *Preservation* principle as well). We observed how a set Dedekind self-map preserves essential characteristics of its domain (being Dedekind infinite) and of itself (restricting j to its range produces another Dedekind self-map). We use this observation as evidence for the conclusion that the $j : V \to V$ that we are seeking should also have strong preservation properties.

Part (2) arises from our belief, embodied in the *Critical Point Dynamics* characteristic (and reflecting the Plotinian principle *Multiplicity As Epiphenomenon*), that what is essential about a notion of infinity is the underlying dynamics that give rise to an infinite multitude, rather than the infinite multitude itself. These dynamics appear, in the context of *set* Dedekind self-maps, as the interaction of the Dedekind self-map with its critical point. We conjecture that large cardinals will also arise as "precipitations" of the dynamics of a Dedekind self-map $j : V \to V$ with one of its critical points.

Part (3) records the intuition, embodied in the *Restrictions of j and Critical Points* characteristic, obtained from our study of set Dedekind self-maps j, that successive restrictions of j are closely related to emergence of the critical sequence for j. We expect to find a similar phenomenon in properly chosen Dedekind self-maps $V \to V$.

Part (4) expresses our belief, reflecting the *Blueprint* characteristic, that the dynamics of the right sort of Dedekind self-map $j : V \to V$ will produce a *blueprint* for a class of central importance—possibly V itself—paralleling the fact that in the realm of infinite sets, a Dedekind self-map gives rise to perhaps the most important of infinite sets (the set of finite ordinals) by way of a blueprint.

Parts (5) and (6) are parts of our conjecture that are not based on characteristics that we have discovered in the dynamics of a Dedekind self-map, but are inspired by the last two Plotinian principles, *Everywhere Present* and *Everything from the Dynamics of the Source*. While the first two Plotinian principles found a fairly natural realization in the dynamics of a set Dedekind self-map, we conjecture that the remaining two Plotinian principles provide us with a *prediction* about the kind of Dedekind self-map $j : V \to V$ we can expect to find as an ultimate realization of the intuition that everything arises from the dynamics of the source. Because of the emphasis in (5) and (6) on the "ultimate" structure of the universe, we would not necessarily expect to find these principles at work in the more localized context of a *set* Dedekind self-map, particularly in the generation of the (very small) collection of finite ordinals.

We acknowledge that there is some overlap in the concepts presented in (4) and (6). The emphasis in (4) is the emergence of a blueprint from which important (possibly *all*) sets emerge, whereas the emphasis in (6) is on the idea that nothing in the universe arises by any means other than the dynamics of j. This leaves open the possibility that a blueprint arises from j, giving rise to some key proper subclass of V, and yet all sets in V still are seen to arise from the dynamics of j, but possibly not directly from the blueprint.

We now begin a search for a Dedekind self-map of the universe that satisfies our stated criteria and test the points of the conjecture.

§5 Obtaining Infinite Sets from a Dedekind Self-Map $j : V \to V$. We begin our study of Dedekind self-maps defined on V, working in ZFC$-$Infinity, with the observation that, unlike a *set* Dedekind self-map, Dedekind self-maps defined on V need not produce an infinite set; the global successor function $\overline{s}$, defined by $\overline{s}(x) = x \cup \{x\}$ is an obvious counterexample. In this section, we discuss methods of strengthening such Dedekind self-maps so that some form of the Axiom of Infinity is derivable. We will make use of the principles discussed in the previous section to guide our selection of stronger properties that we will require of such a $j : V \to V$.

We describe two methods for introducing stronger properties for Dedekind self-maps $j : V \to V$:

(1) require j to satisfy certain *preservation properties*;
(2) seek a version of j for which an infinite set arises through the interaction of j with its critical point.

Remark 5.39 Our intention in approach (1) is that we expect a map $j : V \to V$ to preserve the structure of its domain, in accordance with part (1) of the Dedekind Self-Map Conjecture; little steps in this direction are captured by the idea that j preserves particular *properties* of its domain.

On the other hand, approach (2) reflects part (2) of the Dedekind Self-Map Conjecture according to which important objects in the mathematical universe are expected to arise from the interaction between the self-map and its critical point.

We begin with (1); we introduce several ways of adding extra preservation properties to j. We introduce some terminology. (Some of these terms were discussed at greater length in Section §2.)

Recall (Section §2) that a *terminal object* in V is any set that has just one element. Also, suppose $j : V \to V$ is a 1-1 class function. A set $X \in V$ is said to be a *strong critical point* if $|X| \neq |j(X)|$. It is possible that, even if j has a strong critical point, j may not be a Dedekind self-map. The relationship between strong critical points and critical points is addressed in Proposition 7.63 and Theorem 7.68, below.

DEFINITION 5.40 *Suppose $j : V \to V$ is any function.*

(1) j is said to preserve disjoint unions *if, whenever $X, Y \in V$ are disjoint, $j(X), j(Y)$ are also disjoint and $j(X \cup Y) = j(X) \cup j(Y)$.*
(2) j is said to preserve coproducts *if, whenever X, Y are disjoint, $|j(X \cup Y)| = |j(X) \cup j(Y)|$. Moreover, j* preserves finite coproducts *if, whenever $n \in \overline{\omega}$ and $X_1, X_2, \ldots, X_n$ are disjoint, $|j(X_1 \cup X_2 \cup \cdots \cup X_n)| = |j(X_1) \cup j(X_2) \cup \cdots \cup j(X_n)|$.*
(3) j preserves subsets *if, whenever $X \subseteq Y$, we have $j(X) \subseteq j(Y)$.*
(4) j preserves intersections *if, whenever $X, Y \in V$, $j(X \cap Y) = j(X) \cap j(Y)$.*
(5) j preserves singletons *if, for any X, $j(\{X\}) = \{j(X)\}$.*
(6) j preserves terminal objects *if, whenever T is terminal, $j(T)$ is terminal. In particular, j maps singleton sets to singleton sets since, in* **Set***, the terminal objects are precisely the singleton sets.*
(7) j preserves initial objects *if, whenever I is initial, $j(I)$ is also initial. In particular, $j(\emptyset) = \emptyset$, since $\emptyset$ is the unique initial object in* **Set***. When j preserves initial objects, we shall often say instead that j preserves the empty set.*
(8) j preserves transitive sets *if $j(X)$ is a transitive set whenever X is.*
(9) j is cofinal *if, for every set a, there is a set A such that $a \in j(A)$.*

REMARK 5.41 We point out that our definition in (2) of preservation of coproducts is a weakening of the usual definition given in category theory (see Awodey (2011)) and is strictly weaker than preservation of disjoint unions, as can be seen in Example 5.51.

We also observe that, among the properties listed in Definition 5.40, the property of being *cofinal* is the only one that is not a preservation property. Our emphasis in this section will be to strengthen a bare $j : V \to V$ with various combinations of preservation properties, in accord with our intuition that the j we are looking for must exhibit strong preservation properties. At the same time, the intuition we are guided by also tells us (point (5) of the Dedekind Self-Map Conjecture, p. 45) that "everything" should arise from the dynamics of j. This principle is realized to some extent at least by the view that, relative to j, every set is *internal* (borrowing terminology from the nonstandard approach to set theory (Kanovei & Reeken, 1997a, 1997b)); in other words, the sets that are of importance in mathematics should be the ones that belong to the various images of j. Later in the paper, as we examine additional properties that would be natural for j to have, we will be able to give a stronger justification for the cofinal property; see Remark 9.108.

Finally, we note that the definition does not require $j : V \to V$ to be definable in V. Indeed, we will have occasion to consider self-maps j that are not definable in V. The mechanism for studying such maps is to work in the expanded language $\{\in, \mathbf{j}\}$, in the theory $\mathrm{ZFC}_{\mathbf{j}}$; see Section §2. Hereafter, we will be careful to indicate theorems and definitions that require j to be definable.

We make some observations about relationships between these preservation properties.

PROPOSITION 5.42 *Let $j : V \to V$ be a function.*

(1) Whenever j preserves disjoint unions, j preserves subsets.
(2) Whenever j preserves disjoint unions, j preserves the empty set. However, there is an example of a 1-1 $j : V \to V$ that preserves both subsets and intersections but not the empty set; there is also an example of a 1-1 $j : V \to V$ that preserves finite coproducts but not the empty set.
(3) Whenever j preserves disjoint unions and intersections, j preserves unions; that is, for all sets X, Y, $j(X \cup Y) = j(X) \cup j(Y)$.

Proof. For (1), given sets X, Y with $X \subseteq Y$, note that $Y = X \cup (Y - X)$ is a disjoint union, and so we may write $j(Y) = j(X) \cup j(Y - X)$. Clearly, $j(X) \subseteq j(Y)$.

For (2), assume $j(\emptyset) \neq \emptyset$. Then, although $\phi \cup \phi$ is a disjoint union, $j(\phi) \cup j(\phi)$ is not. For the second part, first note that the function $j(X) = \mathcal{P}(X)$ is 1-1 and preserves subsets but not the empty set. Then, observe that the function $j(X) = \omega \times \{X\}$ has the property that $|j(X)| = \omega$ for all X; it follows that j preserves finite coproducts but not the empty set.

For (3), suppose j preserves disjoint unions and intersections, and suppose X, Z are sets with $X \subseteq Z$. We first prove the following claim:

$$j(Z - X) = j(Z) - j(X).$$

Suppose $u \in j(Z) - j(X)$. Since $X, Z - X$ are disjoint, $j(Z) = j(X) \cup j(Z - X)$ is a disjoint union and, since j preserves intersections and the empty set, $j(X) \cap j(Z - X) = \emptyset$. It follows that $u \in j(Z - X)$. Conversely, suppose $u \in j(Z - X)$. Since j preserves subsets, $u \in j(Z)$. We show $u \notin j(X)$. If $u \in j(X)$, then u belongs to the empty intersection $j(X) \cap j(Z - X)$, yielding a contradiction. Therefore, $u \notin j(X)$. We have proven the claim.

To prove (3), let X, Y be sets. We may write $X \cup Y$ as a disjoint union: $X \cup Y = X \cup (Y - (X \cap Y))$. Applying j's preservation properties and the claim, we have:

$$\begin{aligned} j(X \cup Y) &= j\Big(X \cup (Y - (X \cap Y))\Big) \\ &= j(X) \cup j\Big(Y - (X \cap Y)\Big) \\ &= j(X) \cup (j(Y) - (j(X) \cap j(Y))) \\ &= j(X) \cup j(Y). \end{aligned}$$

□

As we show in Theorem 5.50, the requirement in (2), that j preserves disjoint unions, can be weakened to preservation of coproducts if we also assume that j is 1-1 and preserves terminal objects.

THEOREM 5.43 (ZFC − Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map with critical point a. Suppose also that j preserves disjoint unions and singletons. Then there is an infinite set.*

REMARK 5.44 It is easy enough to prove the theorem directly but we take a more roundabout approach, which will introduce two other useful properties that are not strictly speaking "preservation properties." □

Suppose $j : V \to V$ is any class self-map in a model of ZFC − Infinity. Let **HF** denote the class of hereditarily finite sets. We will say that j *is bijective on* **HF**-*terminals* if $j \upharpoonright S : S \to S$ is a bijection where $S = \{z \in \mathbf{HF} \mid z \text{ is a singleton}\}$. Note that the property of being bijective on **HF**-terminals is a slight strengthening of the property of preserving all terminal objects belonging to **HF**.

LEMMA 5.45 (ZFC − Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map with critical point a. Suppose also that j preserves disjoint unions and singletons. Then*

(1) The singleton $\{a\}$ is also a critical point.
(2) The self-map j is bijective on **HF**-*terminals.*

Proof of (1). Suppose, for a contradiction, that $\{a\} = j(z)$. If z is not a singleton, let $z_0 \in z$ and, because $\{z_0\}, z - \{z_0\}$ is a partition of z, by preservation of singletons and disjoint unions, $j(z) = \{j(z_0)\} \cup j(z - \{z_0\})$, and so $|j(z)| > 1$, which is impossible. Therefore, z is a singleton set $\{y\}$. We have $\{a\} = j(z) = j(\{y\}) = \{j(y)\}$, and so $a = j(y)$, contradicting the fact that a is a critical point of j. □

Proof of (2). Let $S = \{z \in \mathbf{HF} \mid z \text{ is a singleton}\}$. We show j is the identity on S. We make use of the fact, which follows by an application of the axiom Trans (see Section 2.2), that $\mathbf{HF} = \bigcup_{n \in \overline{\omega}} V_n$. We proceed by induction on rank(x) for $x \in S \subseteq \mathbf{HF}$. For the base case, we only need to verify $j(\{\emptyset\}) = \{\emptyset\}$, which follows by preservation of the empty set (Proposition 5.42(1)) and of singletons:

$$j(\{\emptyset\}) = \{j(\emptyset)\} = \{\emptyset\}.$$

For the induction step, assume $j(x) = x$ for all $x \in V_n$ for which x is a singleton. Let $x \in V_{n+1}$ with $x = \{y\}$. Then y is a finite set belonging to V_n; say $y = \{y_1, \ldots, y_k\}$. Then for $1 \le i \le k$, $\{y_i\} \in V_n$ and so, by the induction hypothesis, $j\big(\{y_i\}\big) = \{y_i\}$, and we have:

$$j(y) = j\big(\{y_1, \ldots, y_k\}\big) = j\left(\bigcup_{i=1}^{k} \{y_i\}\right) = \bigcup_{i=1}^{k} j\big(\{y_i\}\big) = \bigcup_{i=1}^{k} \{y_i\} = y.$$

This completes the induction and the proof. □

REMARK 5.46 The qualification in the hypothesis of the theorem that j is a class map is necessary because of the induction that is done in the proof of (2): The formula used for the induction involves occurrences of j; if j is being treated as a realization of a function symbol **j**, it is not automatically the case that induction can be performed; see Section 2.3.

Theorem 5.43 now follows from the following lemma:

LEMMA 5.47 (ZFC − Infinity) *Suppose $j : V \to V$ is a Dedekind self-map with critical point a such that $\{a\}$ is a second critical point of j and j is bijective on* **HF***-terminals. Then there is an infinite set.*

Proof. It suffices to show $a \notin \mathbf{HF}$, since in that case any transitive set that includes a must be infinite (and there must be at least one such set by the Trans axiom). For a contradiction, assume $a \in \mathbf{HF}$. Then $\{a\} \in \mathbf{HF}$. Since j is bijective on **HF**-terminals, there is $\{x\} \in \mathbf{HF}$ such that $j(\{x\}) = \{a\}$, and this is impossible because $\{a\}$ is a critical point of j. □

We now give a ZFC example to show that the hypotheses of Lemma 5.47 can be realized in the presence of an infinite set in the universe.

EXAMPLE 5.48 Define $j : V \to V$ by

$$j(x) = \begin{cases} x & \text{if } x \text{ is finite} \\ \mathcal{P}(x) & \text{otherwise} \end{cases}$$

It follows immediately that j is a Dedekind self-map with critical point ω. It is also clear that $\{\omega\}$ is a second critical point for j and that j is bijective (indeed, the identity) on **HF**-terminals. □

We do not have a similar ZFC example for the hypotheses of Theorem 5.43, but we can at least show that they are consistent with ZFC:

EXAMPLE 5.49 We review the well-known fact from model theory that any model of ZFC has a submodel M that admits a nontrivial elementary embedding $j : M \to M$ (see Chang and Keisler (1973), Theorems 3.3.10, 3.3.11(d), for details). One begins by extending the language of set theory with countably many constants corresponding to some infinite well-ordered set $I = \{i_0 < i_1 < \cdots\}$ and extending ZFC with axioms that assert that these constants are indiscernibles. Using Ramsey's Theorem and the Compactness Theorem, and the fact that ZFC has a model, one shows that the extended theory is consistent. Let $\mathcal{N} = \langle N, E\rangle$ be the reduct of a model of this theory. Now $I \subset N$ is a set of indiscernibles for $\mathcal{N}$. Assuming, without loss of generality, that $\mathcal{N}$ has built-in Skolem functions, we let $M = \mathfrak{H}^{\mathcal{N}}(I)$. One may then extend any order-preserving $f : I \to I$ to an elementary embedding $j : M \to M$ by defining $j(t[i_{m_1}, \ldots, i_{m_k}]) = t[f(i_{m_1}), \ldots, f(i_{m_k})]$. Defining f by $f(i_n) = i_{n+1}$ ensures that the resulting j has i_0 as a critical point (cf. Bell and Machover (1977), Problem 7.12(i))). Since j is elementary, it follows that, in the model $\langle M, E, j\rangle$, j preserves singletons and disjoint unions.

We note that, as distinct from the hypotheses in Theorem 5.43, the Dedekind self-map $j : M \to M$ described here is not definable in M. □

We turn to a second set of preservation properties on a Dedekind self-map $j : V \to V$ that produce an infinite set.

THEOREM 5.50 (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 self-map with a strong critical point. Suppose j preserves finite coproducts and terminal objects. Then there is an infinite set.*

Proof. We first show that

$$\text{for every finite set } X,\ |j(X)| = |X|. \tag{19}$$

We start with the case $X = \emptyset$. We show that $|j(\emptyset)| = 0$. Let y be such that $j(\{\emptyset\}) = \{y\}$. Assume for a contradiction that $|j(\emptyset)| \geq 1$. There are two cases to consider, both of which lead to the conclusion that $|j(\emptyset) \cup j(\{\emptyset\})| \geq 2$: If $j(\emptyset) \cap j(\{\emptyset\}) = \emptyset$, then we have

$$|j(\emptyset) \cup j(\{\emptyset\})| = |j(\emptyset)| + |j(\{\emptyset\})| = |j(\emptyset)| + |\{y\}| \geq 2.$$

If $j(\emptyset) \cap j(\{\emptyset\}) \neq \emptyset$, it follows that $y \in j(\emptyset)$. Since j is 1-1, it is not possible that $j(\emptyset) = \{y\}$. It follows that $|j(\emptyset)| \geq 2$, from which we conclude again that $|j(\emptyset) \cup j(\{\emptyset\})| \geq 2$.

However, since j preserves coproducts, $|j(\emptyset) \cup j(\{\emptyset\})| = |j(\emptyset \cup \{\emptyset\})| = |j(\{\emptyset\})| = |\{y\}| = 1$, and we have a contradiction. We have shown that $j(\emptyset) = \emptyset$.

We consider next the case in which $X \neq \emptyset$, Write $X = \{x_1, \ldots, x_n\}$. Because j is 1-1 and preserves finite coproducts and terminal objects,

$$|j(X)| = |j(\{x_1\}) \cup \cdots \cup j(\{x_n\})| = |\{y_1, \ldots, y_n\}| = |X|,$$

where, for each i, $\{y_i\} = j(\{x_i\})$.

Finally, we show there is an infinite set: Let Z be a strong critical point of j, so that $|j(Z)| \neq |Z|$. The statement (19) implies that Z is not finite. Therefore, Z is infinite. □

Example 5.51 (Simple Model of Theorem 5.50) Assuming that ω does exist, we construct a Dedekind self-map $j : V \to V$ having the properties mentioned in the hypothesis of Theorem 5.50.

$$j(x) = \begin{cases} x & \text{if } x \text{ is finite,} \\ \mathcal{P}(x) & \text{if } x \text{ is infinite.} \end{cases}$$

The first clause of the definition ensures that j preserves terminal objects, since such objects must always be finite. Since the powerset operator is 1-1, so is j. Notice ω is a strong critical point by Cantor's Theorem; it is easy to see that ω is a critical point as well. We show that j preserves finite coproducts. Suppose $X_1, \ldots, X_n$ are disjoint sets. If each X_i is finite, the fact that $|j(X_1 \cup \cdots \cup X_n)| = |j(X_1) \cup \cdots \cup j(X_n)|$ is obvious. Assume at least one X_i is infinite. Without loss of generality, assume $X_1, \ldots, X_m$ are finite and $X_{m+1}, \ldots, X_n$ are infinite, and that $m < n$. We have

$$\begin{aligned} |j(X_1 \cup \cdots \cup X_n)| &= |\mathcal{P}(X_1 \cup \cdots \cup X_n)| \\ &= 2^{\max\{|X_1|, \ldots, |X_n|\}} \\ &= \max\{2^{|X_1|} \ldots 2^{|X_n|}\} \\ &= |X_1 \cup \cdots \cup X_m \cup \mathcal{P}(X_{m+1}) \cup \cdots \cup \mathcal{P}(X_n)| \\ &= |j(X_1) \cup \cdots \cup j(X_n)|. \square \end{aligned}$$

The requirement in Theorem 5.50 that j preserves finite coproducts can be weakened to preservation of coproducts under additional assumptions. We catalog two such assumptions here, but in the sequel, we will attempt to generalize Theorem 5.50 rather than the variants we consider now.

THEOREM 5.52 *Suppose $j : V \to V$ is a 1-1 class map with a strong critical point. Suppose j preserves coproducts and terminal objects. Suppose also that j reflects $\in$, that is, whenever $j(x) \in j(y)$ we have $x \in y$.*[22] *Then there is an infinite set.*

Proof. We show that for each finite set X, $|j(X)| = |X|$. The rest of the proof is the same as for Theorem 5.50. We proceed by induction on $n = |X|$. The base case $n = 1$ follows because j preserves terminals. For the induction step, write $X = \{x_1, \ldots, x_n, x_{n+1}\}$. Let $X_n = \{x_1, \ldots, x_n\}$. Recalling that $|j(\{x_{n+1}\})| = 1$ and that by the induction hypothesis $|j(X_n)| = n$, we observe that the only way $|j(X_n) \cup j(\{x_{n+1}\})|$ could fail to equal $n+1$ is if $j(\{x_{n+1}\}) \in j(X_n)$, but this is impossible because j reflects $\in$. Therefore, using the fact that j preserves coproducts, we have:

$$\begin{aligned} |j(\{x_1, \ldots, x_{n+1}\})| &= |j(X_n \cup \{x_{n+1}\})| \\ &= |j(X_n) \cup j(\{x_{n+1}\})| \\ &= n+1, \end{aligned}$$

as required. □

THEOREM 5.53 *Suppose $j : V \to V$ is a 1-1 class map with a strong critical point Z. Suppose j preserves coproducts and terminal objects. Suppose also that $|j(Z)| > |Z|$. Then there is an infinite set.*

Proof. As in the proof of Theorem 5.50, $j(\emptyset) = \emptyset$. We prove the following claim:

Claim. For each nonempty finite set X, $|j(X)| \leq |X|$.

Proof. We proceed by induction on $n = |X|$. The base case $n = 1$ is obviously true since j preserves terminals. For the induction step, write $X = \{x_1, \ldots, x_{n+1}\}$ and let $X_n = \{x_1, \ldots, x_n\}$. Since, by the induction hypothesis, $|j(X_n)| \leq n$ and since $|j(x_{n+1})| = 1$, the largest value $|j(X_n) \cup j(x_{n+1})|$ could have is $n+1$. Therefore $|j(X)| = |j(X_n) \cup j(x_{n+1})| \leq n+1$. This completes the induction and the proof of the claim.

To complete the proof of the theorem, we observe by the claim that if Z has the property that $|j(Z)| > |Z|$, then Z is not finite. □

We note here that both Theorems 5.52 and 5.53 require j to be definable in V. The proofs of these theorems depend on this assumption since, in each case, an induction is performed that involves j. No such dependency occurs in the proof of Theorem 5.50.

§6 Emergence of a Nonprincipal Ultrafilter from a $j : V \to V$. Our techniques so far for strengthening a bare Dedekind self-map $j : V \to V$ so that it is strong enough to yield an infinite set have been of the first variety mentioned in Remark 5.39: We have required j to satisfy certain preservation properties. In this section we make use of the second approach also mentioned there: An infinite set is produced by enhancing the interaction between j and its critical point. In particular, we show that if a is a critical point of j and for some set A, $a \in j(A)$,

[22] This preservation property is formally introduced in Definition 7.64.

then under certain conditions (most of which are preservation properties of j) the set $\{X \subseteq A \mid a \in j(X)\}$ is a nonprincipal ultrafilter. Note that such a set, intuitively speaking, arises from the interaction between j and its critical point.

THEOREM 6.54 *Suppose $j : V \to V$ is a class Dedekind self-map with critical point a and there is a set A such that $a \in j(A)$. Let $D = \{X \subseteq A \mid a \in j(X)\}$.*

(1) *If j preserves subsets, intersections, and the empty set, then D is a filter.*
(2) *If, in addition to the conditions in (1), j preserves terminals and $\{a\}$ is a second critical point of j, then D is a nontrivial filter.*
(3) *If, in addition to the conditions in (1), j preserves disjoint unions, then D is an ultrafilter. Moreover, if the conditions in (2) also hold, D is a nonprincipal ultrafilter (and A is infinite).*

In (1), preservation of the empty set does not follow from the other properties in the hypothesis and (1): Proposition 5.42(2) provides an example of a 1-1 $j : V \to V$ that preserves subsets and intersections but not the empty set, namely, $j(X) = \mathcal{P}(X)$. Moreover, this map j has a critical point $\emptyset$ for which there is a set A such that $\emptyset \in j(A)$, namely, $A = \emptyset$.

Motivation for the condition in part (2) of the theorem that asserts that $\{a\}$ is a second critical point can be found in Lemmas 5.45 and 5.47. We note here that this requirement is not trivial: There are Dedekind self-maps $j : V \to V$ with the property that for *no* critical point a of j is it the case that $\{a\}$ is a second critical point. For instance, consider $S : V \to V$ defined by $S(x) = \{x\}$. The map S is a Dedekind self-map with a proper class of critical points. Moreover, for any critical point a of S, $\{a\} \in \operatorname{ran} S$ and is therefore not a critical point of S.

REMARK 6.55 In the absence of (3), properties (1) and (2) are rather weak. We give an example to illustrate this point, but also to motivate a more complex example for which (3) does hold.

Suppose A is a set with two or more elements, and define $j : V \to V$ by $j(X) = X^A = \{f \mid f : A \to X\}$. Let $\mathrm{id}_A : A \to A$ denote the identity map. Then $\mathrm{id}_A \in j(A)$. Clearly id_A and $\{\mathrm{id}_A\}$ are critical points of j. Also, for any sets X, Y, $X \neq Y \Rightarrow X^A \neq Y^A$, and so j is 1-1; indeed, j is a Dedekind self-map. Since $\emptyset^A = \emptyset$, j preserves the empty set. For any set X, $j(\{X\}) = \{X\}^A = \{t\}$, where $t : A \to \{X\}$ is the unique function from A to $\{X\}$; therefore, j preserves terminals. Finally, one checks that$j$ preserves intersections by observing that $(X \cap Y)^A = X^A \cap Y^A$.

Since this particular $j : V \to V$ satisfies properties (1) and (2) of Theorem 6.54, it follows that D is a nontrivial filter. However, j does not satisfy property (3): Given two nonempty disjoint subsets B and C of A with $B \cup C = A$, while it is true that B^A and C^A are disjoint, it is not the case that $B^A \cup C^A = (B \cup C)^A$; indeed, id_A belongs to the set on the right-hand side, but not to the set on the left-hand side.

In fact, D is rather uninteresting: Although it is true that $A \in D$ (since $\mathrm{id}_A \in j(A)$), D contains no other set: If $X \subsetneqq A$, then it is not the case that $\mathrm{id}_A \in X^A$. To make D more interesting, in Example 6.57 we weaken the requirement for membership in D using an equivalence relation. □

Proof of (1). $A \in D$ by the definition of D and $\emptyset \notin D$ since $j(\emptyset) = \emptyset$. D is closed under intersections since j preserves intersections. D is also closed under supersets since j preserves subsets. We have shown that D is a filter.

Proof of (2). We verify D is nontrivial. For this, it suffices to show that D has no element of the form $\{z\}$. Assuming as in the hypothesis that j preserves terminals and $\{a\}$ is a second critical point, we assume also that D has an element of the form $\{z\}$. It follows that $a \in j(\{z\})$ and since j preserves terminals, $j(\{z\}) = \{y\}$ for some $y \in j(A)$. It follows that $a = y$, and so $\{a\} = j(\{z\})$; in other words, $\{a\} \in \operatorname{ran} j$ contradicting the assumption that $\{a\}$ is a second critical point for j. We have shown that D is a nontrivial filter.

Proof of (3). Assuming j preserves disjoint unions, we show that D is an ultrafilter: Suppose $X \subseteq A$ and $X \notin D$. Let $Y = A - X$. We show $a \in j(Y)$. Since j preserves disjoint unions, $j(A) = j(X) \cup j(Y)$; since $a \notin j(X)$, then $a \in j(Y)$, as required. If the conditions in (2) also hold, then, by the argument establishing (2), D is a nonprincipal ultrafilter. It follows that A itself is infinite. □

Remark 6.56 We remark that the definition of D in the first line of the theorem statement requires j to be definable in V; if it were not, there would be no guarantee that D is a *set*.

Example 6.57 Using an appropriately defined equivalence relation, we modify the example in Remark 6.55 so that the filter D derived from j is a nonprincipal ultrafilter.

We begin with a fixed (proper) filter U on A. We do not require U to be a nonprincipal ultrafilter, so it is possible for A to be finite at this stage. If $f : A \to Z$ is a partial function, we call f *U-good* if $\{x \in A \mid f(x) \text{ is defined}\} \in U$. For this example only, we re-define the sets of the form Z^A in the following way:

$$Z^A = \{f \mid f : A \to Z \text{ is a } U\text{-good partial function}\}.$$

For $f, g : A \to Z$ that are U-good, we define $\sim$ (more formally, $\sim_{A,U,Z}$) by

$$f \sim g \text{ iff } \{x \in A \mid f(x) = g(x)\} \in U.$$

We note that, for any $f \in Z^A$ for which $|Z| \geq 1$, there is a total function $f' : A \to Z$ such that $f \sim f'$: Let $z \in Z$. Define f' by

$$f'(x) = \begin{cases} f(x) \text{ if } x \in \text{dom } f, \\ z \text{ otherwise.} \end{cases}$$

Clearly $f' \sim f$.

Denote the equivalence class that contains f by $[f]_U$, or simply $[f]$. For each set Z, we define

$$Z^A/U = \{[f]_U \mid f : A \to Z \text{ is } U\text{-good}\}.$$

Finally, define $j : V \to V$ by $j(Z) = Z^A/U$. We first show, in Claims (1)–(5) below, that j has property (1), mentioned in Theorem 6.54:

Claim 1. $j(\emptyset) = \emptyset$.

Proof. Immediate.

Claim 2. j is 1-1.

Proof. Suppose $Y \neq Z$ are sets; without loss of generality, let $y \in Y - Z$. Consider the constant function $f^y : A \to Y$ defined by $f^y(x) = y$. Clearly f^y agrees nowhere with any partial function $A \to Z$. Therefore $f^y \in Y^\omega - Z^\omega$, and so $j(Y) \neq j(Z)$.

Claim 3. j preserves terminal objects.

Proof. We show j takes singleton sets to singleton sets. Given a singleton set $\{z\}$, let f_z be the unique function $A \to \{z\}$. Notice that if $g : A \to \{z\}$ is U-good, then $g \sim f_z$. Therefore, we have

$$\begin{aligned} j(\{z\}) &= \{z\}^A/U \\ &= \{[g] \mid g \text{ is a } U\text{-good partial function } A \to \{z\}\} \\ &= \{[f_z]\}. \end{aligned}$$

Claim 4. $[\mathrm{id}_A]$ is a critical point of j. Therefore, j is a Dedekind self-map. Moreover, $[\mathrm{id}_A] \in j(A)$.

Proof. Certainly, $[\mathrm{id}_A]$ is not itself of the form Z^A/U, so $[\mathrm{id}_A]$ is a critical point. Since $\mathrm{id}_A \in A^A$, certainly $[\mathrm{id}_A] \in A^A/U = j(A)$.

Claim 5. j preserves intersections.

Proof. Suppose X, Y are sets and $Z = X \cap Y$. Suppose $[f] \in X^A/U \cap Y^A/U$. Let $B, C \in U$ be such that for all $x \in B$, $f(x) \in X$ and for all $x \in C$, $f(x) \in Y$. Since U is closed under instersections, $B \cap C \in U$, and we have that for all $x \in B \cap C$, $f(x) \in X \cap Y$, so $[f] \in (X \cap Y)^A/U$. For the converse, if $E \in U$ is such that for all $x \in E$, $f(x) \in X \cap Y$, then it follows easily, using the fact that U is closed under supersets, that $[f] \in X^A/U$ and $[f] \in Y^A/U$.

We have shown property (1) of Theorem 6.54 holds for j. Therefore, if $D = \{X \subseteq A \mid [\mathrm{id}_A] \in j(X)\}$, then D is a filter. In the present context, this is not surprising in light of the next claim.

Claim 6. $D = U$.

Proof. Suppose $X \subseteq A$. We show $X \in D$ if and only if $X \in U$. For one direction, we have:

$$\begin{aligned} X \in D &\Rightarrow [\mathrm{id}_A] \in j(X) = X^A \\ &\Rightarrow \exists f \in X^A\,(f \sim \mathrm{id}_A) \\ &\Rightarrow \exists B \in U\,(f \restriction B = \mathrm{id}_B \text{ and } B \subseteq X). \end{aligned}$$

Since $B \in U$ and $B \subseteq X$, it follows that $X \in U$. For the other direction, suppose $X \in U$. Define a U-good partial function $f \in X^A$ in the following way: Let dom $f = X$ and define f on elements by $f(x) = x$ for all $x \in X$. In other words, $f = \mathrm{id}_X$. Clearly, $\mathrm{id}_A \sim f$ and so $[\mathrm{id}_A] \in X^A = j(X)$. It follows that $X \in D$.

We next show that j preserves disjoint unions only if our starting filter U was already an ultrafilter:

Claim 7. The following are equivalent:

(A) j preserves disjoint unions.
(B) U is an ultrafilter (equivalently, D is an ultrafilter).

Proof. First, observe that $(A) \Rightarrow (B)$ follows from Theorem 6.54, part (3), using the fact that $D = U$. For the converse, suppose X, Y are disjoint and let $Z = X \cup Y$; we show $j(X)$ and $j(Y)$ are disjoint, and that $j(X) \cup j(Y) = j(X \cup Y)$.

Disjointness of $j(X)$ and $j(Y)$ follows from the fact that $X^A \cap Y^A = \emptyset$. It is obvious that $j(X) \cup j(Y) \subseteq j(Z)$. To prove $j(Z) \subseteq j(X) \cup j(Y)$, let $[f] \in Z^A/D$. Since f is U-good, $S \in U$, where $S = \{x \in A \mid f(x) \in Z\}$. Let $S_X = \{x \in A \mid f(x) \in X\}$ and $S_Y = \{x \in A \mid f(x) \in Y\}$. Since $S = S_X \cup S_Y \in U$ and U is an ultrafilter, one of S_X, S_Y belongs to U, say S_X. Then $[f] = [f \restriction S_X] \in X^A/U$. We have shown that each $[f]$ in Z^A/U belongs to $\big(X^A/U\big) \cup \big(Y^A/U\big)$.

Assuming from the beginning that U is an ultrafilter, we have established that j preserves disjoint unions and that D is therefore, by Theorem 6.54, also an ultrafilter. None of the arguments so far requires A to be infinite or D to be nonprincipal. In the present setting, the only way D could turn out to be a nonprincipal ultrafilter, given the assumptions we have made so far on U, is if $\{[\mathrm{id}_A]\}$ is a critical point of j, which could happen only if U itself was initially assumed to be nonprincipal. Therefore, our example shows that existence of a nonprincipal ultrafilter is equivalent to existence of a Dedekind self-map having properties (1)–(3) in Theorem 6.54. The next claim establishes the remaining details.

Claim 8. Assume U is an ultrafilter. Then the following are equivalent:

(A) $\{[\mathrm{id}_A]\}$ is a critical point of j.
(B) U (equivalently, D) is a nonprincipal ultrafilter on A, whence A is infinite.

Proof. The fact that $(A) \Rightarrow (B)$ follows from Theorem 6.54(2), using the fact that $D = U$. For the converse, assume U is nonprincipal but $\{[\mathrm{id}_A]\}$ is in the range of j, so that $\{[\mathrm{id}_A]\} = Z^A/U$, for some set Z; we will arrive at a contradiction.

First, we show that Z itself must be a singleton set: If Z has at least two distinct elements y, z, the constant functions $f^y : A \to Z : x \mapsto y$ and $f^z : A \to Z : x \mapsto z$ agree nowhere, and so $[f^y] \neq [f^z]$; hence, $|j(Z)| = |Z^A/U| > 1$, contradicting our assumption that $\{[\mathrm{id}_A]\} = Z^A/U$. Therefore, $Z = \{z\}$ for some z. Let f be the unique function from A to $\{z\}$. Then $Z^A/U = \{[f]\} = \{[\mathrm{id}_A]\}$; in other words, $f \sim \mathrm{id}_A$. It follows that $\{x \in A \mid f(x) = \mathrm{id}_A(x)\} \in U$; that is, $\{z\} = \{x \in A \mid z = x\} \in U$. Since U is nonprincipal, this is impossible. We have shown therefore that $\{[\mathrm{id}_A]\}$ is a critical point of j.

Claims 7 and 8 could have been presented and proven in reverse order, with slight modifications; the only change in the proofs is that "nonprincipal" must be replaced with "nontrivial" in the new version of Claim 8. We leave the details to the reader. □

The condition in Theorem 6.54 that the critical point a must belong to a set of the form $j(A)$ for some set A is necessary in order to be able to conclude that an infinite set exists; indeed, Dedekind self-maps can be built in the theory ZFC − Infinity that satisfy parts (1)–(3) of the theorem; for such Dedekind self-maps, the theorem tells us that, in a universe without infinite sets, for *no* critical point a of j (for which $\{a\}$ is a second critical point) is it possible to find a set A for which $a \in j(A)$. We give such an example next.

Example 6.58 This example shows that in every ZFC − Infinity universe, one may obtain a Dedekind self-map $j : V \to V$ satisfying parts (1)–(3) of Theorem 6.54, and yet for no critical point a of j for which $\{a\}$ is also a critical point is it the case that $a \in j(A)$ for any A. Working in such a universe V, define $j : V \to V$ by

$$j(X) = \overline{s}[X] = \{\overline{s}(x) \mid x \in X\},$$

where $\overline{s}$ is the global successor function. Since $\overline{s}$ is 1-1, so is j. It is straightforward to verify that j preserves disjoint unions, intersections, and the empty set, and that both $\{\{1\}\}$ and $\{\{\{1\}\}\}$ are critical points of j (note though that $\{1\} \in \operatorname{ran} j$ since $j(1) = \{1\}$). We verify that j preserves terminal objects: We compute $j(\{x\})$ for any x:

$$j(\{x\}) = \overline{s}[\{x\}] = \{\overline{s}(y) \mid y \in \{x\}\} = \{\overline{s}(x)\},$$

which is also a singleton.

Finally, we show that if a is any critical point of j for which $\{a\}$ is also a critical point, there is no set A for which $a \in j(A)$: Suppose for a contradiction that there are a, A so that both a and $\{a\}$ are critical points of j and $a \in j(A) = \overline{s}[A]$. It follows from $a \in j(A)$ that

$$a \in \operatorname{ran} \overline{s}. \tag{20}$$

We complete the proof by showing that $\{a\}$ is a critical point of j if and only if $a \notin \operatorname{ran} \overline{s}$, contradicting (20). This final part of the proof follows from the following chain of equivalences:

$$\begin{aligned} \{a\} \in \operatorname{ran} f &\Leftrightarrow \exists X\,(\{a\} = \overline{s}[X]) \\ &\Leftrightarrow \exists x, X\,(x \in X \wedge \{a\} = \{\overline{s}(x)\}) \\ &\Leftrightarrow \exists x, X\,(x \in X \wedge a = \overline{s}(x)) \\ &\Leftrightarrow a \in \operatorname{ran} \overline{s}. \end{aligned}$$

Notice that the Dedekind self-map $j : V \to V$ in this example can be defined in the theory ZFC − Infinity; its properties—namely, (1)–(3) of Theorem 6.54—are not strong enough to imply the existence of an infinite set. □

We list several sufficient conditions here for the critical point a to belong to a set of the form $j(A)$.

(a) j is *cofinal*; that is, for every $a \in V$, there is $A \in V$ with $a \in j(A)$.
(b) j strongly preserves $\in$, preserves rank, and both preserves and reflects ordinals (definitions on p. 61) (in that case, for at least one critical point a of j, $a \in j(a)$; note that we must assume in this case that j is definable in V; see p. 63).
(c) a is a *(weakly) universal element* for j (defined on p. 13 in Section 2.3).

The fact (a) that cofinality of j suffices is obvious; we mention this condition here because, as was mentioned in Remark 5.41, we think of the cofinal property as being a natural one for the sort of j we are seeking. A proof that (b) suffices can be found on p. 63. The consequence of the conditions in (b)—that for some critical point a, $a \in j(a)$—is a clear example of "interaction between j and its critical point." The stronger versions of j that we will consider later always have this property. The construction given in Example 6.57 provides an example of (c), as we prove in Remark 6.61.

Combining Theorem 6.54 with condition (a) leads to an easily stated sufficient condition for the existence of a nonprincipal ultrafilter:

PROPOSITION 6.59 (ZFC−Infinity) *Suppose $j : V \to V$ is a cofinal class Dedekind self-map with critical points a and $\{a\}$. Suppose also that j preserves disjoint unions, intersections, and terminal objects. Then there is a nonprincipal ultrafilter over some (infinite) set A.*[23]

Note that the hypotheses of Proposition 6.59 imply that j preserves the empty set (Proposition 5.42(2)).

Proof. By cofinality of j, there is a set A such that $a \in j(A)$. Define D by $D = \{X \subseteq A \mid a \in j(X)\}$. Then by the proof of Theorem 6.54, D is a nonprincipal ultrafilter on A. □

EXAMPLE 6.60 (Reduced Product Construction) We revisit the example given in Example 6.57, specializing to the case in which $A = \omega$. This example will provide us with a class Dedekind self-map with the properties listed in Theorem 6.54(2) and Theorem 5.50. It will also set the stage for a generalization in which $A = \kappa$, where κ is some uncountable cardinal.[24]

We begin by fixing a nonprincipal ultrafilter D on ω. We recall several definitions from Example 6.57. Given any set X, if f, g are both D-good partial functions from ω to X, we declare f, g are equivalent, and write $f \sim g$, if the set of $n \in \omega$ at which f, g are both defined and equal belongs to D. Let $[f]$ denote the $\sim$-equivalence class containing f and let

$$X^\omega/D = \{[f] \mid f \text{ is a } D\text{-good partial function from } \omega \text{ to } X\}.$$

Define $j_D : V \to V$ by

$$j_D(X) = X^\omega/D.$$

We note as before that, for any $f \in X^\omega$ for which $|X| \geq 1$, there is a total function $f' : \omega \to X$ such that $f \sim f'$. Let $x_0 \in X$. Then f' is defined by:

$$f'(n) = \begin{cases} f(n) \text{ if } n \in \text{dom } f, \\ x_0 \text{ otherwise.} \end{cases}$$

Clearly $f' \sim f$.

Claims 1–6 below are proved in Example 6.57 (setting $A = \omega$).

Claim 1. $j_D(\emptyset) = \emptyset$.

Claim 2. j_D is 1-1.

Claim 3. j_D preserves terminal objects.

Claim 4. Both $[\text{id}_\omega]$ and $\{[\text{id}_\omega]\}$ are critical points of j_D. Also,

$$D = \{X \subseteq A \mid [\text{id}_\omega] \in j_D(X)\}.$$

Claim 5. j_D preserves disjoint unions. Consequently, j_D preserves all finite coproducts.

[23] We show in Section 11 that these conditions on a Dedekind self-map $j : V \to V$ are satisfied when j is a WA_0-embedding.

[24] This generalization is given in Example 9.109, starting on page 92.

Claim 6. j_D preserves intersections.

The following claim shows that our example also illustrates Theorem 5.50:

Claim 7. ω is a strong critical point of j_D.

Proof. Define, for each $n \in \omega$, the function $f_n : \omega \to \omega$ by $f_n(i) = n + i$. Then, whenever $m \neq n$, f_m, f_n disagree everywhere. Therefore $\{[f_n] \mid n \in \omega\}$ is an infinite subset of $j_D(\omega) = \omega^\omega/D$. We show that $j_D(\omega)$ is in fact uncountable. Suppose $\{[g_n] \mid n \in \omega\}$ is an infinite subset of ω^ω/D. Define $h : \omega \to \omega$ by

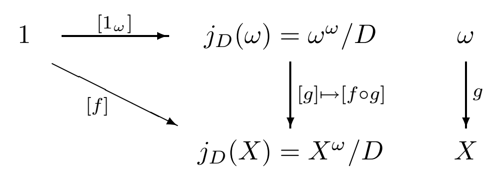
$$h(n) = \text{least element of } \omega - \{g_i(n) \mid i < n\}.$$

Then for all $i \in \omega$ and all $n > i$, $h(n) \neq g_i(n)$; in particular $[h] \neq [g_i]$ for all $i \in \omega$. We have shown $j_D(\omega)$ is uncountable, so $\omega < |j_D(\omega)|$. Therefore, ω is a strong critical point. □

REMARK 6.61 The reduced product construction obtained in Example 6.60 provides an example of a universal element. As in that example, given a nonprincipal ultrafilter D on ω, $j_D : V \to V$ is defined by $j_D(X) = X^\omega/D$. The self-map j_D can be turned into a functor **Set** $\to$ **Set** by defining its values on **Set**-arrows:

$$j_D(f) : X^\omega/D \to Y^\omega/D : [g] \mapsto [f \circ g], \tag{21}$$

for any $f : X \to Y$. Then $[1_\omega] \in j_D(\omega)$ is a weakly universal element for j_D:



(22)

Given $[f] \in j_D(X) = X^\omega/D$, with $f : \omega \to X$, f itself is the required function. We show that $j_D(f)([1_\omega]) = [f]$:

$$j_D(f)([1_\omega]) = [f \circ 1_\omega] = [f]. \ \square \tag{23}$$

The results of this section and the last provide a characterization of the Axiom of Infinity in terms of Dedekind self-maps $j : V \to V$, summarized in the following theorem.

THEOREM 6.62 (ZFC – Infinity) *The following statements are equivalent.*

(1) *There is an infinite set.*
(2) *There is a Dedekind self-map $j : V \to V$ with a strong critical point that preserves finite coproducts and terminal objects.*
(3) *There is a class Dedekind self-map $j : V \to V$ with critical point a such that*
   (i) *the map j preserves disjoint unions, intersections, and terminals;*
   (ii) *there is a set A such that $a \in j(A)$;*
   (iii) *the set $\{a\}$ is a second critical point of j.*[25]

[25] Parts (2) and (3) of Theorem 6.62 appear to be asserting the existence of proper class functions. To state the result in a formally correct way, we could re-state our “theorem” as a *schema* of theorems, one for each functional formula of set theory.

Proof. The implication (2) $\Rightarrow$ (1) is established by Theorem 5.50, and the implication (1) $\Rightarrow$ (2) follows from Example 5.51. The implication (3) $\Rightarrow$ (1) is established by Theorem 6.54. The implication (1) $\Rightarrow$ (3) is proved in the following way: Given an infinite set A, obtain in the usual way a nonprincipal ultrafilter D on A, and define $j : V \to V$ by $j(X) = X^A/D$. Example 6.60 shows that j satisfies the properties listed in (3). □

In this section we have shown how an infinite set, in the form of a nonprincipal ultrafilter, is obtained from a Dedekind self-map $j : V \to V$ that has been supplemented with a combination of simple preservation properties and a property obtained from interaction between j and its critical point. In this case we have used both methods (1), (2) mentioned on p. 46 for introducing stronger properties of Dedekind self-maps.

Our next objective is to make use primarily of method (2) to produce infinite sets from Dedekind self-maps $j : V \to V$, using methods from category theory. For this next step, we will need to clarify the relationship between the different notions of critical point that we will be using.

§7 The Relationship Between the Different Notions of Critical Point

In this section, we clarify the relationship between critical points and strong critical points. We also introduce a related concept: Suppose $j : V \to V$ is a 1-1 class function. A set X in the domain of j is a *weak critical point* if $j(X) \neq X$. The relationships between these three notions of critical point are summarized in the following proposition:

Proposition 7.63 (ZFC − Infinity)

(1) *Suppose $j : V \to V$ is a 1-1 function. If X is a strong critical point or a critical point, then X is also a weak critical point.*
(2) *There is a 1-1 class map $j : V \to V$ having a weak critical point that has no strong critical point.*
(3) *There is a 1-1 class map $j : V \to V$ having a weak critical point but no critical point.*
(4) *There is a 1-1 class map $j : V \to V$ that has a strong critical point but no critical point.*
(5) *There is a 1-1 class map $j : V \to V$ that has a critical point but not a strong critical point.*

Proof of (1). If X is a strong critical point, certainly $j(X) \neq X$, so X is also a weak critical point. If X is a critical point, then since X is not in the range of j, $j(X) \neq X$, so X is a weak critical point.

Proof of (2). Obtain $j : V \to V$ as follows.

$$j(x) = \begin{cases} x & \text{if } x \neq 1 \text{ and } x \neq \{1\}, \\ \{1\} & \text{if } x = 1, \\ 1 & \text{if } x = \{1\}. \end{cases}$$

Here, $j(1) \neq 1$, so 1 is a weak critical point. However, $|j(x)| = |x|$ for all $x \in V$, so j has no strong critical point.

Proof of (3) and (4). Obtain $j : V \to V$ as follows.

$$j(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } x \neq 1, \\ 0 & \text{if } x = 1, \\ 1 & \text{if } x = 0. \end{cases}$$

Now $j(0) \neq 0$ so 0 is a weak critical point. However, $\operatorname{ran} j = \operatorname{dom} j$, so j has no critical point. Also, notice that $|j(0)| \neq 0$, so 0 is a strong critical point.

Proof of (5). Define $j : V \to V$ by

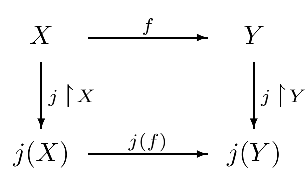
$$j(x) = \begin{cases} \{n+1\} & \text{if } x = \{n\} \text{ for some } n \in \overline{\omega}, \\ x & \text{otherwise.} \end{cases}$$

Certainly j is 1-1 and has critical point $\{0\}$. However, for each $n \in \overline{\omega}$, $|\{n+1\}| = |j(\{n\})| = |\{n\}| = 1$, so j has no strong critical point. □

Despite these differences, when j satisfies certain additional preservation properties, these three notions of critical point coincide. Moreover, we will be especially interested in maps that satisfy these properties; we give the definitions below.

DEFINITION 7.64 Suppose $j : V \to V$ is a 1-1 function.

(1) *j preserves* $\in$ if, whenever $x, y \in \operatorname{dom} j$ and $x \in y$, then $j(x) \in j(y)$. *j reflects* $\in$ if, whenever $j(x) \in j(y)$, we have $x \in y$. Finally, *j strongly preserves* $\in$ if j preserves and reflects $\in$.
(2) *j preserves cardinals* if, whenever $\gamma \in \mathrm{ON}$ is a cardinal, $j(\gamma)$ is also a cardinal.
(3) *j preserves functions* if, for any $f : X \to Y$ and $X, Y \in \operatorname{dom} j$, $j(f)$ is a function from $j(X)$ to $j(Y)$
(4) *j preserves images* if, whenever $f : X \to Y$ is a function, $j(f)$ is a function and $j(\operatorname{ran} f) = \operatorname{ran} j(f)$.
(5) *j preserves functional application* (or often *preserves application*) if j preserves functions and, whenever $f : X \to Y$ is a function, then, for all $x \in X$, $j(f(x)) = j(f)(j(x))$. This property of j can be represented by a commutative diagram:



(6) *j preserves ordinals* if, whenever α is an ordinal, $j(\alpha)$ is also an ordinal.
(7) *j reflects ordinals* if, whenever $j(x)$ is an ordinal, x is also an ordinal.[26]
(8) *j preserves rank* if, for every set x, $j(\operatorname{rank}(x)) = \operatorname{rank}(j(x))$.

Any 1-1 function $j : V \to V$ that satisfies both parts of (1), as well as (2)–(4), will be called *basic structure-preserving*, or *BSP*.

[26] Reflecting ordinals is also a *preservation property* since j reflects ordinals if and only if j preserves *non*-ordinals.

Theorem 7.65 (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 class function.*

(1) If j preserves $\in$, then, for all ordinals α, β, if $j(\alpha) \in j(\beta)$, then $\alpha \in \beta$.
(2) If j preserves ordinals and $\in$, then, for all ordinals α, $j(\alpha) \geq \alpha$.
(3) Suppose j preserves ordinals and strongly preserves $\in$. Suppose $x \in V$ has the following two properties:

   (a) *x and $j(x)$ have the same rank*
   (b) *for all y for which $\mathrm{rank}(y) < \mathrm{rank}(x)$, we have $j(y) = y$.*

   Then $j(x) = x$.
(4) If j preserves ordinals and rank, strongly preserves $\in$, and has a weak critical point, then there is an ordinal α such that $j(\alpha) \neq \alpha$. In particular, $j \restriction ON : ON \to ON$ has a weak critical point.
(5) Suppose j is BSP, preserves functional application, ordinals, and rank, and has a weak critical point. Then j has a weak critical point that is a cardinal.

Proof of (1). Suppose $j(\alpha) \in j(\beta)$. If $\alpha \notin \beta$, then either $\alpha = \beta$ or $\beta \in \alpha$ (since $\in$ is a total ordering on ON). If $\alpha = \beta$, then we have $j(\beta) = j(\alpha) \in j(\beta)$ which is impossible by irreflexivity of $\in$. If $\beta \in \alpha$, then $j(\beta) \in j(\alpha) \in j(\beta)$, and this contradicts the fact that $\in$ is both irreflexive and transitive. The result follows.

Proof of (2). Suppose not; let α be least such that $j(\alpha) < \alpha$. Since j preserves ordinals and $\in$, $j(j(\alpha)) < j(\alpha)$. But now $j(\alpha)$ is a smaller ordinal β with the property that $j(\beta) < \beta$, contradicting leastness of α.

Proof of (3). We first show $x \subseteq j(x)$: Let $y \in x$. Since j preserves $\in$, $j(y) \in j(x)$. Since $\mathrm{rank}(y) < \mathrm{rank}(x)$, $y = j(y)$. It follows $y \in j(x)$. Conversely, if $y \in j(x)$, then y is of lower rank (since $\mathrm{rank}(j(x)) = \mathrm{rank}(x)$), so $y = j(y)$. Since $j(y) = y \in j(x)$ and j strongly preserves $\in$, it follows that $y \in x$, as required.

Proof of (4). Suppose $j(\alpha) = \alpha$ for all ordinals α. Let x be a weak critical point for j; that is, $j(x) \neq x$. Let $\alpha = \mathrm{rank}(x) + 1$, and let $X = V_\alpha$. Let $M = \{x \in X \mid j(x) \neq x\}$. The fact that M is a set follows from an application of Separation. Let $B = \{\mathrm{rank}(x) \mid x \in M\}$. B is a set by Replacement. Also, $B \neq \emptyset$ since $M \neq \emptyset$. Let $\gamma = \inf B$ and let $y \in M$ be such that $\mathrm{rank}(y) = \gamma$. Since $y \in M$, $j(y) \neq y$, but, using (3), we can also show that $j(y) = y$, yielding the needed contradiction. To apply (3), and complete the proof of (4), it suffices to establish condition (3)(a); note that (3)(b) already holds by the leastness of $\mathrm{rank}\, y$. But (3)(a) holds because j preserves rank and because of the assumption that j is the identity on ON:

$$\mathrm{rank}(j(y)) = j(\mathrm{rank}(y)) = \mathrm{rank}(y).$$

We have shown, therefore, that $j \restriction \mathrm{ON} : \mathrm{ON} \to \mathrm{ON}$ has a weak critical point.

Proof of (5). Let κ be the least ordinal moved by j (which must exist by (4)). By (2), $j(\kappa) > \kappa$. Suppose $\alpha < \kappa$ and $f : \alpha \to \kappa$ is an onto function. By leastness of κ, $j(\alpha) = \alpha$. Since j preserves functions, $j(f) : j(\alpha) \to j(\kappa)$ is also a function, and since j preserves images,

$$\mathrm{ran}\, j(f) = j(\mathrm{ran}\, f) = j(\kappa). \tag{24}$$

Since $j(\alpha) = \alpha$, $j(f) : \alpha \to j(\kappa)$. We show $f = j(f)$: For any $\beta \in \alpha$, because j preserves functional application and $j(\beta) = \beta$, we have

$$j(f)(\beta) = j(f)(j(\beta)) = j(f(\beta)) = f(\beta).$$

The last step follows because $f(\beta) \in \kappa$ and κ is the least ordinal moved by j. Therefore $j(f) = f$ and so $\operatorname{ran} j(f) \subseteq \kappa$, which contradicts (24). Therefore, no such onto function exists, and κ is a cardinal. □

REMARK 7.66 The proofs of (2) and (4) given above rely on the fact that j is a class map (and since (5) makes use of (4), it implicitly relies on definability of j as well). For (2), if j were not definable, finding the least α for which $j(\alpha) < \alpha$ would require the use of the Least Ordinal Principle, which does not automatically hold in $\mathrm{ZFC}_{\mathbf{j}}$ (Section 2.3). In (4), without definability of j, the class M could not be shown to be a set without an application of a Separation principle for **j**-formulas (Section 2.3).

COROLLARY 7.67 (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 class function having a weak critical point. If j strongly preserves $\in$ and preserves rank and ordinals, then there is a least ordinal α moved by j; moreover, $\alpha < j(\alpha)$.*

Proof. By Theorem 7.65(4), there is an ordinal β with $j(\beta) \neq \beta$; it follows that there is a least ordinal α with this property.[27] By Theorem 7.65(2), it follows that $\alpha < j(\alpha)$, as required. □

With these preservation properties in mind, we can return to the task of verifying a point mentioned earlier, regarding Theorem 6.54. In that theorem, one of the hypotheses concerning $j : V \to V$ was that one of its critical points a belongs to a set of the form $j(A)$. We described earlier (p. 57) several sufficient conditions for this hypothesis to hold true. One of the conditions described there is the following:

j is a class map and strongly preserves $\in$,
preserves rank, and also preserves and reflects ordinals.

We explain why this condition suffices: Theorem 7.63 shows that if $j : V \to V$ is a Dedekind self-map, j has a weak critical point; moreover, since we are assuming j strongly preserves $\in$ and j preserves ordinals and rank, there is, by Corollary 7.67, a least ordinal α_0 such that $\alpha_0 < j(\alpha_0)$. This ordinal α_0 must be a critical point of j because (a) for all ordinals α, $\alpha \leq j(\alpha)$ (Theorem 7.65(2)), and (b) by the ordinal reflecting property, no non-ordinal is mapped to α_0. It follows therefore that α_0 itself is the required set A; that is, $\alpha_0 \in j(\alpha_0)$. As we consider Dedekind self-maps that give rise to the bigger large cardinals, it will be typical for j to have such a critical point.

We show now that when a self-map $j : V \to V$ satisfies a modest subset of the preservation properties described so far, the three notions of critical point coincide.

THEOREM 7.68 (ZFC − Infinity) *Suppose $j : V \to V$ is a 1-1 class function that is BSP and preserves and reflects ordinals, preserves rank, and preserves functional application. Then the following are equivalent.*

(1) j has a weak critical point.
(2) j has a critical point.
(3) j has a strong critical point.

[27] As observed earlier, the existence of such an ordinal α requires j to be definable in V.

In particular, the least ordinal κ moved by j is a weak critical point, a critical point, and a strong critical point; $\kappa < j(\kappa)$; and κ is a cardinal.

Proof. We have already seen that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). It suffices to prove (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

Assume j has a weak critical point. Using Corollary 7.67 and Theorem 7.65(5), there is a cardinal κ that is moved by j and that is the least ordinal moved by j. Assume for a contradiction that $\kappa \in \operatorname{ran} j$, and let $j(x) = \kappa$. Because j reflects ordinals, x must also be an ordinal. Because $\alpha \leq j(\alpha)$ for all ordinals α, it follows that $x < \kappa$. However, existence of such an ordinal x contradicts the leastness of κ. It follows that κ is a critical point of j. Also, since κ is a cardinal and j preserves cardinals, $j(\kappa)$ is also a cardinal. Since $|\kappa| = \kappa < j(\kappa) = |j(\kappa)|$, we conclude that κ is also a strong critical point of j.

To prove the last clause of the theorem, note that the argument given in the last paragraph shows that, whenever j has a weak critical point, there is a least ordinal κ moved by j which is both a critical point and a strong critical point; which is a cardinal; and which satisfies $\kappa < j(\kappa)$. the least ordinal κ moved by j is □

In the sequel, it will at times be useful to know whether a $j : V \to V$ that has a strong critical point also has a critical point. Theorem 7.68 provides one set of criteria for this implication, but we offer here an alternative set, also consisting of preservation properties. We first introduce one additional notion of preservation.

Definition 7.69 Suppose $j : V \to V$ is a 1-1 class function. Then j *preserves cardinality* if for every set X,

$$j\big(|X|\big) = |j(X)|.$$

We observe here that preservation of cardinality is a stronger form of preservation than preservation of *cardinals*: If j preserves cardinality and λ is a cardinal, then $j(\lambda) = j\big(|\lambda|\big) = |j(\lambda)|$, which is clearly a cardinal. On the other hand, the following Dedekind self-map shows that preservation of cardinals does not in general imply preservation of cardinality:

$$j(x) = \begin{cases} x & \text{if } x \text{ is a cardinal} \\ \mathcal{P}(x) & \text{otherwise} \end{cases}$$

Lemma 7.70 *Suppose $j : V \to V$ is a 1-1 class function and preserves subsets and transitive sets. Then for all ordinals α, if $j(\alpha)$ is an ordinal, then $j(\alpha) \geq \alpha$.*

Proof. Assume j is 1-1, preserves subsets and transitive sets, but for some ordinal α, $j(\alpha) < \alpha$; let α be the least such. Then $j(\alpha) \subsetneqq \alpha$ and so by preservation of subsets and 1-1-ness, $j(j(\alpha)) \subsetneqq j(\alpha)$. It follows that both $j(\alpha)$ and $j(j(\alpha))$ are sets of ordinals. Since j preserves transitive sets, $j(j(\alpha))$ itself must be an ordinal. Now $j(\alpha)$ is a smaller ordinal than α and yet $j(j(\alpha)) < j(\alpha)$, contradicting the leastness of α. □

Lemma 7.71 *Suppose $j : V \to V$ is a 1-1 (not necessarily definable) map. Suppose also that j preserves subsets, transitive sets, and cardinality. In addition, suppose that the following two properties hold:*

(a) *There is an ordinal κ such that κ is a strong critical point of j and is the least such.*

(b) *Whenever both* α *and* $j(\alpha)$ *are ordinals,* $j(\alpha) \geq \alpha$.

Then κ *is the least cardinal moved by* j *and is a critical point of* j.

Proof. First observe that since j preserves cardinality, $j(\kappa) \neq \kappa$:

$$j(\kappa) = j\big(|X|\big) = |j(X)| \neq \kappa. \tag{25}$$

It follows from (b) that $j(\kappa) > \kappa$.

We claim that $\kappa \notin \operatorname{ran} j$: If $\kappa \in \operatorname{ran} j$, let Y be such that $j(Y) = \kappa$; let $\lambda = |Y|$. Since j preserves cardinality, we have

$$j(\lambda) = j\big(|Y|\big) = |j(Y)| = \kappa.$$

By (b), $j(\lambda) \geq \lambda$; since $\lambda \neq \kappa$, we have $\lambda < \kappa$ and $\lambda < j(\lambda)$. But then λ is a strong critical point that is smaller than κ, contradicting (a). We have shown that κ is a critical point of j.

Finally, if $\lambda < \kappa$ is a cardinal moved by j, since j preserves cardinals, it would follow that λ is a strong critical point $< \kappa$, which is impossible. It follows, therefore, that κ is the least cardinal moved by j. □

Theorem 7.72 *Suppose* $j : V \to V$ *is a 1-1 class map and* j *has a strong critical point. Suppose also that* j *preserves subsets, transitive sets, and cardinality. Then* j *has a critical point. Indeed, some cardinal is moved by* j *and the least such cardinal is a critical point for* j.

Proof. Since j has a strong critical point, using definability of j, we can find a set X of least cardinality for which $|j(X)| \neq |X|$; let $\kappa = |X|$. We will use the previous two lemmas to show that κ is a critical point for j and the least cardinal moved by j.

First notice that condition (a) of Lemma 7.71 holds, since κ is the least strong critical point of j. Also, condition (b) of Lemma 7.71 holds by Lemma 7.70 and (25). Now the result follows from Lemma 7.71. □

Assuming that the j of Theorem 7.72 also strongly preserves $\in$ and preserves rank and ordinals, we may, by Theorem 7.67, add to the conclusion of Theorem 7.72 the statement that the least cardinal moved is also the least ordinal moved.

Notice that Lemma 7.71 provides criteria for a 1-1, possibly undefinable self-map $j : V \to V$ according to which a strong critical point (in particular, the *least* strong critical point) is a critical point, whereas, to arrive at the same conclusion, Theorem 7.72 requires that j be definable. We will make use of the weaker hypothesis provided in Lemma 7.71 in our proof of Theorem 9.105(1).

Remark 7.73 So far, we have used the notation $\operatorname{crit}(j)$ to denote any specified critical point of j. In the sequel, we may also use this notation to denote the *least ordinal moved* by j, in cases where this least ordinal κ actually is a critical point of j and has the property $\kappa < j(\kappa)$. In this usage, $\operatorname{crit}(j)$ will not be referring to a pre-specified critical point but will be an ordinal that is definable (as the least ordinal moved) from self-maps meeting these requirements. By Theorem 7.67, these requirements will be met whenever j is a class map that strongly preserves $\in$ and preserves rank and ordinals.

§8 Dedekind Self-Maps $j : V \to V$ from Adjoint Functors and Dedekind Monads. In earlier sections, we have shown how equipping a Dedekind self-map $j : V \to V$ with certain preservation properties guarantees the existence of an infinite set, and also how requiring $j : V \to V$ to exhibit a combination of preservation properties and particular dynamics in its interaction with its critical point produces a nonprinicipal ultrafilter, and hence also an infinite set. In this section, we show how a *set* Dedekind self-map is naturally derived from a Dedekind self-map $j : V \to V$ when j arises as a certain kind of *monad*—that is, as a composition $G \circ F$ of adjoint functors of a certain kind. This work will lead to a new equivalent form of the Axiom of Infinity in terms of *Dedekind monads*.

Summarizing some of W. Lawvere's early work, we recalled in Corazza (2010) the fact that if $G : V^{\circlearrowleft} \to V$ is the *forgetful functor*—taking a self-map $g : X \to X$ to its domain X—and $F : V \to V^{\circlearrowleft}$ is left adjoint to G, then if $j = G \circ F$, it follows that 1 is a strong critical point for j and $j(1)$ is infinite. Moreover, as we show below, 1 is also a critical point of j and, although j itself may not be 1-1 on objects, j is isomorphic to another $j' = G \circ F'$ with $F' \dashv G$ for which j' *is* 1-1 on objects. We will say that j is *essentially Dedekind*.

Extending Proposition 2.3 of Corazza (2010) a bit further, we show below (p. 68) that if $\eta : 1_V \to G \circ F$ is the unit of the adjunction, then $\eta_1(0) \in GF(1)$ is a universal element of G and hence (see Section 2.3),

$$V = \{G(f)(\eta_1(0)) \mid f \text{ is an arrow in } V^{\circlearrowleft}\}.$$

The observation that an infinite set arises as $j(\mathrm{crit}(j))$ is a realization of the second method (described on p. 46) for strengthening the properties of a Dedekind self-map $V \to V$ so that it gives rise to an infinite set (at its critical point). We improve this result in this section by showing that in fact the j of the Lawvere Construction, in its interaction with its critical point, produces a (set) Dedekind self-map $j(1) \to j(1)$. While it is true (as we show below) that $F(1) : j(1) \to j(1)$ is already a Dedekind self-map (in fact, an *initial* Dedekind self-map), it is not derived from the interaction between j and $\mathrm{crit}(j)$ (but rather from the interaction of F and $\mathrm{crit}(j)$). With a somewhat deeper look at the structure of j, one does obtain a Dedekind self-map $j(1) \to j(1)$ "derived from" the interaction of j and $\mathrm{crit}(j)$, in a sense that we will make precise.

These observations lead to a natural question. Is there some criterion for emergence of a set Dedekind self-map that is "internal" to j, and not dependent upon externally defined categories and functors like $V^{\circlearrowleft}, F$, and G? We offer in this section one such criterion: If j is a *Dedekind monad*, whose properties are simple generalizations of the j described above, then it produces a Dedekind self-map in a way that is analogous to the way such a self-map arises from the $j : V \to V$ of the Lawvere Construction.

The rest of this section is devoted to working out the details for the results mentioned in the outline above. We will close the section with questions about the extent to which the results given here might generalize to a large cardinal context.

We begin with a precise statement of Lawvere's results, mentioned above, as formulated in Corazza (2010).

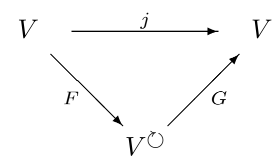
THEOREM 8.74 (Lawvere's Theorem). *Suppose V is a model of* ZFC − Infinity. *Then the following are equivalent:*

(1) *V satisfies the Axiom of Infinity*

(2) *There is a functor* $j : V \to V$ *that factors as a composition* $G \circ F$ *of functors satisfying:*

 (A) $F \dashv G$ *(*F *is left adjoint to* G*)*
 (B) $F : V \to V^{\circlearrowright}$
 (C) $G : V^{\circlearrowright} \to V$ *is the forgetful functor, defined by* $G(A \to A) = A$.

In particular, F *preserves all colimits and* G *preserves all limits. Moreover,* 1 *is a strong critical point of* j*; indeed,* $j(0) = 0$ *and* $|j(1)| = \omega$.



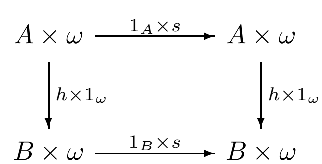
Whenever $j : V \to V$ arises as $j = G \circ F$ as in Lawvere's Theorem, namely, when $G : V^{\circlearrowright} \to V$ is the forgetful functor, defined by $G(A \to A) = A$, and $F \dashv G$, we shall say that j *is obtained from the Lawvere construction*.

Whenever an infinite set exists, the proof of (1) ⇒ (2) (see for example Corazza (2010)) shows that, in the presence of ω, the required left adjoint $F : V \to V^{\circlearrowright}$ may be defined as follows: Define F on objects by

$$F(A) = 1_A \times s : A \times \omega \to A \times \omega, \tag{26}$$

where the map $1_A \times s$ is defined by $(1_A \times s)(a, n) = (a, n+1)$. Also, if $h : A \to B$ is a V-arrow, then $F(h) : F(A) \to F(B)$ is defined by

$$F(h) = h \times 1_\omega. \tag{27}$$



For these particular functors F, G, the composite functor $j = G \circ F$ will be called the *Lawvere functor*.

The next result shows that when j is obtained from the Lawvere construction, it is *essentially Dedekind* (that is, j is naturally isomorphic to a functor $j' : V \to V$ that is 1-1, but not onto, on objects), and the set $1 = \{0\}$ is not only a strong critical point but a critical point as well.

THEOREM 8.75 *Suppose* $j = G \circ F$ *is obtained from the Lawvere construction. Then the following statements hold:*

(1) j *is essentially Dedekind.*
(2) *For any nonempty set* A, $j(A)$ *is infinite.*
(3) *1 is a critical point of* j.

Proof of (1). Since we have already shown that existence of a left adjoint of G produces a Dedekind self-map, existence of the left adjoint guarantees that ω exists. We can then define a functor F' as in equations (26) and (27), giving us $F' \dashv G$. To see $j' = G \circ F'$ has a critical point, note that $j'(0) = 0$, and also that, for all nonempty A, $j'(A)$ is infinite (since $j'(A) = A \times \omega$); thus, 1, for example, is a critical point. To see that j' is 1-1 on objects, suppose $A \neq B$. If $A = \emptyset$, then $j'(B)$ is infinite while $j'(A) = \emptyset$. If $A \neq \emptyset$, say $a \in A - B$. Then $(a, 0) \in A \times \omega - B \times \omega$. In each case, $j'(A) \neq j'(B)$. We have shown j' is 1-1, but not onto, on objects.

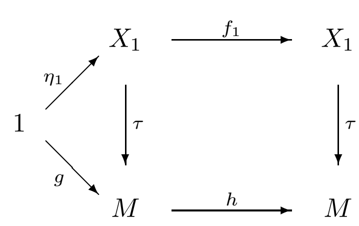
A fact from category theory (cf. Awodey (2011) or Mac Lane (1978)) is that left adjoints of the same functor must be naturally isomorphic. In this case, this means that there is, for each set A, a $V^{\circlearrowleft}$-iso $\sigma_A : F(A) \to F'(A)$ that is natural in A. Applying G, we obtain another iso $G(\sigma_A) : G(F(A)) \to G(F'(A))$ (note that every functor preserves isos). It is straightforward to show that $G(\sigma_A)$ is natural in A. Therefore, j and j' are naturally isomorphic and j' is a Dedekind self-map. It follows that j is an essentially Dedekind self-map. □

Proof of (2). Suppose A is nonempty. Now $j(A)$ must be infinite because$j \cong j'$:

$$|j(A)| = |j'(A)| = |A \times \omega| \geq \omega. \ \square$$

Proof of (3). We show 1 is not one of the objects in the range of j. Note that, because $j \cong j'$, $j(0) = 0$. Therefore, by (2), the only finite object in the range of j is $j(0) = 0$. Therefore, 1 is not an object in the range of j. □.

Next we show that if $j = G \circ F$ is obtained from the Lawvere construction, $F(1)$ is an initial Dedekind self-map. Write $F(1) = f_1 : X_1 \to X_1$. It follows from Theorem 8.75(3) that X_1 is infinite. Using the properties of the unit η of the adjunction $F \dashv G$, it is shown in Corazza (2010), p. 69, that f_1 has the following property: Suppose $h : M \to M$ is an object in $V^{\circlearrowleft}$ and $g : 1 \to G(h) = M$ is a V-arrow. Then there is a unique $V^{\circlearrowleft}$-arrow $\tau : f_1 \to h$ that makes the following diagram commutative:



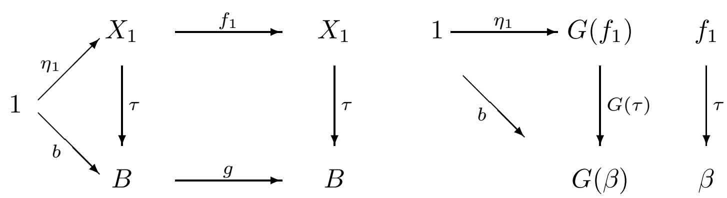
(28)

In other words, $(X_1, f_1, \eta_1(0))$ is initial in the category **UPA** of unary pointed algebras (and is thus a natural numbers object), and therefore, by Remark 3.24, is initial in **DedAlg**.

THEOREM 8.76 *Suppose $j = G \circ F$ is obtained from the Lawvere construction. Let $\eta : 1_V \to G \circ F$ denote the unit of the adjunction. Write $F(1) = f_1 : X_1 \to X_1$. Then f_1 is an initial Dedekind self-map with critical point $\eta_1(0)$.* □

We conclude this review with the fact that, in the present context, $\eta_1(0) \in G(f_1)$ is a universal element for G: Suppose $g : B \to B \in V^{\circlearrowleft}$ and $b \in G(\beta) = B$. For purposes of display, we identify b with the arrow $\bar{b} : 1 \to B : 0 \to b$. Since $(X_1, f_1, \eta_1(0))$ is a natural numbers object, there is a unique arrow $\tau : f_1 \to \beta$ in

$V^{\circlearrowleft}$ so that the triangle below is commutative, that is, $b = \tau(\eta_1(0)) = G(\tau)(\eta_1(0))$. Therefore, $\eta_1(0)$ is a universal element for G.



Since G is a cofinal functor, we also have (Section 2.3):

$$V = \{G(g)(\eta_1(0)) \mid g \text{ is a } V^{\circlearrowleft}\text{-arrow}\}.$$

We have shown the following:

THEOREM 8.77 *Suppose $j = G \circ F$ is obtained from the Lawvere construction. Let $\eta : 1_V \to G \circ F$ denote the unit of the adjunction. Write $F(1) = f_1 : X_1 \to X_1$. Then $\eta_1(0) \in G(f_1)$ is a universal element for G and we may write*

$$V = \{G(g)(\eta_1(0)) \mid g \textit{ is a } V^{\circlearrowleft}\textit{-arrow}\}.\square$$

As we remarked at the beginning of this section, while discovery of an infinite set arising from the interaction between a j obtained from the Lawvere construction and its critical point meets expectations suggested by our Dedekind Self-Map Conjecture, the fact that a (set) Dedekind self-map arises from interaction between F and this critical point, rather than between j and its critical point, falls short of our expectation. Moreover, it is reasonable to seek properties internal to such a functor j, independent of the fact that j is a composition of two other functors G, F that are tied to another category different from V (in our case, $V^{\circlearrowleft}$), which would guarantee that $j(\text{crit}(j))$ is infinite and that some (set) Dedekind self-map arises from interaction between j and $\text{crit}(j)$. We provide a way to achieve this objective in the following subsection.

8.1 Dedekind Monads In this section, we show how the functor $j : V \to V$ obtained in the Lawvere construction produces, in a natural way, a *set* Dedekind self-map. We then develop sufficient conditions on a functor $j : V \to V$, based on properties internal to the structure of j, for producing a set Dedekind self-map. We begin with some definitions and background results.

DEFINITION 8.78 (Dedekind Maps) Let us say that a function $f : A \to B$ is a *Dedekind map* if

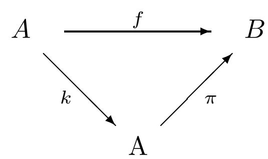
(1) $|A| = |B|$
(2) f is 1-1 but not onto.

We will call any element $b \in B$ not in the range of f a *critical point of f*.

The concept of a Dedekind map is a generalization of the concept of a Dedekind self-map. As we now show, Dedekind maps always factor as a composition of a bijection and a Dedekind self-map.

PROPOSITION 8.79 *Suppose $f : A \to B$ and $b \in B$. Then f is a Dedekind map with critical point b if and only if there exist functions π, k so that $f = \pi \circ k$, where*

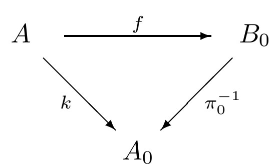
$\pi : A \to B$ is a bijection and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$.



Proof. For one direction, suppose A, B are sets, $b \in B$, and $f : A \to B$ can be factored as $f = \pi \circ k$, where $\pi : A \to B$ is a bijection and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$. We show f is a Dedekind map. Certainly, f is 1-1. We show b is a critical point of f. Since π is onto, there is $a \in A$ with $\pi(a) = b$. Note that $a = \pi^{-1}(b)$ is, by assumption, a critical point of k. If $b \in \operatorname{ran} f$, then let $x \in A$ be such that $f(x) = b$. Then by commutativity of the diagram, $\pi(k(x)) = b$, and so $k(x) = \pi^{-1}(b) = a$, which is impossible since $a \notin \operatorname{ran} k$. We have shown b is a critical point of f, f is 1-1, and $|A| = |B|$ (by way of π), as required.

For the other direction, suppose $f : A \to B$ is a Dedekind map with critical point b. Since $|A| = |B|$, there is a bijection $\pi : A \to B$. Let $B_0 = f[A] \subseteq B$ and let $A_0 = \pi^{-1}[B_0] \subseteq A$. Let $\pi_0 = \pi \restriction A_0$. We show $\operatorname{ran} \pi_0 = B_0$: Suppose $y \in A_0$. Then $y = \pi^{-1}(x)$ for some $x \in B_0$; that is, for some $x \in B_0$, $\pi(y) = x$. This shows $\operatorname{ran} \pi_0 \subseteq B_0$. If $x \in B_0$, then let $y \in A_0$ with $\pi^{-1}(x) = y$. Then $x = \pi(y) \in \operatorname{ran} \pi$. We have shown therefore that π_0 is onto. Since π_0 is a restriction of the bijection π, π_0 is also 1-1. Therefore, $\pi_0 : A_0 \to B_0$ is a bijection.

Note that $f : A \to B_0$ is a bijection. Let $k = \pi_0^{-1} \circ f : A \to A_0$. Then k is a bijection, and so, viewed as a map $k : A \to A$, k is a Dedekind self-map. The following diagram is commutative:



We claim that $\pi^{-1}(b)$ is a critical point for $k : A \to A$: Suppose $j(x) = \pi^{-1}(b)$ for some $x \in A$. Then

$$b = \pi(k(x)) = \pi(\pi^{-1}(f(x))) = f(x),$$

which is impossible because b is a critical point of f.

Now notice that, for any $x \in A$, since $k(x) = \pi_0^{-1}(f(x))$, then $\pi(k(x)) = \pi_0(k(x)) = f(x)$. We have shown that $f = \pi \circ k$, π is a bijection, and $k : A \to A$ is a Dedekind self-map with critical point $\pi^{-1}(b)$, as required. $\square$

To make further progress in locating within the dynamics of the functor $j : V \to V$ obtained from Lawvere's construction a naturally defined Dedekind self-map, we will view j as a *monad*—a concept that we now define.

As we have already observed in the case of the categories $\mathbf{Set}^{\circlearrowleft}$ and $\mathcal{C} = \mathbf{Set}$, any adjunction $(F, G, \eta, \varepsilon)$ determines another functor $T : \mathcal{C} \to \mathcal{C}$ by composition: $T = G \circ F$. By virtue of the properties of the adjunction, T becomes the functor part of another structure, called a monad. We introduce some of the basic results about monads here as a preliminary to our discussion below about Dedekind monads. See Mac Lane (1978) and Awodey (2011) for more on monads.

DEFINITION 8.80 (Monads) Given a category $\mathcal{C}$, a *monad* is a triple (T, η, μ) for which $T : \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta : 1 \to T$ and $\mu : T^2 \to T$ are natural transformations, so that, as in the commutative diagrams (29) and (30) below,

(i) for any object A in $\mathcal{C}$ and $x \in T^3(A)$, $\mu_A(\mu_{T(A)}(x)) = \mu_A\left(T(\mu_A)(x)\right)$.
(ii) for any object A in $\mathcal{C}$ and $x \in T(A)$, $\mu_A(\eta_{T(A)}(x)) = x = \mu_A(T(\eta_A)(x))$.

Note that the maps $T(\eta_A) : T(A) \to T^2(A)$, for $A \in \mathcal{C}$, are themselves the components of a natural transformation.

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu_T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \tag{29}$$

$$\begin{array}{ccccc} T & \xrightarrow{\eta_T} & T^2 & \xleftarrow{T\eta} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array} \tag{30}$$

We shall often refer to a functor T as a "monad" when we mean that T is the functor part of a monad (T, η, μ). The transformation η is called the *unit* of the monad, and μ is called the *multiplication* operation for the monad.

Any adjunction $(F, G, \eta, \varepsilon)$ gives rise to a monad (T, η, μ) by way of the following definitions:

(i) $T = G \circ F$
(ii) η is the same in both structures
(iii) for all objects A in $\mathcal{C}$ and $x \in T^2(A)$, $\mu_A(x) = \left(G(\varepsilon_{F(A)})\right)(x)$.

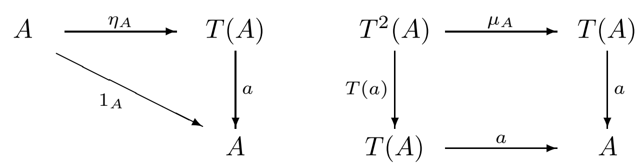
If $j = G \circ F$ is obtained from the Lawvere construction, the monad (j, η, μ) obtained from j as in (i)–(iii) above will be called a *Lawvere monad*.

A monad T can be used to define a new category, entirely within $\mathcal{C}$, called the category of T-*algebras*, denoted $\mathcal{C}^T$. This category is defined as follows:

$$\mathcal{C}^T = \{(A, a) \mid A \text{ is a } \mathcal{C}\text{-object and } a : T(A) \to A\},$$

where each (A, a) in $\mathcal{C}^T$ satisfies the equations:

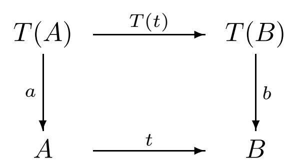
$$a \circ \eta_A = 1_A \qquad a \circ T(a) = a \circ \mu_A. \tag{31}$$



(32)

Arrows $t : (A, a) \to (B, b)$ in $\mathcal{C}^T$ are $\mathcal{C}$-arrows $t : A \to B$ making Diagram (34) commutative:

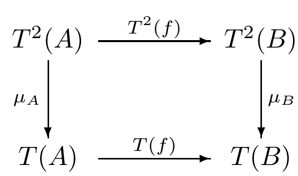
$$b \circ j(t) = t \circ a. \tag{33}$$



(34)

Functors $F^T : \mathcal{C} \to \mathcal{C}^T$ and $G^T : \mathcal{C}^T \to \mathcal{C}$ (the forgetful functor) can be defined as follows:

$$\begin{aligned} G^T((A, a)) &= A \\ F^T(A) &= (T(A), \mu_A) \\ G^T(t) &= t : A \to B \text{ where } t : (A, a) \to (B, b) \\ F^T(f) &= T(f) \text{ where } f : A \to B \text{ and} \\ &\quad\ \text{the diagram below commutes} \end{aligned}$$



(36)

It can be shown (Awodey, 2011) that $F^T \dashv G^T$ and $T = G^T \circ F^T$. The adjunction $F^T \dashv G^T$ is called the *induced T-algebra adjunction.*

When a monad T arises from an adjunction $F \dashv G$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ (and this is the situation we have in our case of interest), there is a way to measure the degree to which $\mathcal{C}^T$ is an isomorphic copy of $\mathcal{D}$: The *Eilenberg-Moore comparison functor* $\Phi : \mathcal{D} \to \mathcal{C}^T$ is defined as follows for any $\mathcal{D}$-object D and arrow $f : D \to D'$:

$$\begin{aligned} \Phi(D) &= (G(D), G(\varepsilon_D)) \\ \Phi(f) &= G(f) : (G(D), G(\varepsilon_D)) \to (G(D'), G(\varepsilon_{D'})) \end{aligned}$$

where $\varepsilon : F \circ G \to 1_{\mathcal{D}}$ is the co-unit of the adjunction $F \dashv G$. One can show (Mac Lane, 1978) that Φ is the unique functor satisfying

$$G^T \circ \Phi = G \text{ and } \Phi \circ F = F^T. \tag{37}$$

In many important cases, Φ is an iso. An adjunction $F \dashv G$ is said to be *monadic* if Φ is an iso. Moreover, for any functor G, if G has a left adjoint F so that the corresponding Eilenberg-Moore comparison functor Φ is an iso, then G is said to be a *monadic functor.*

REMARK 8.81 We note that whenever (T, μ, η) is a monad, the induced T-algebra adjunction $F^T \dashv G^T$ is monadic. This follows immediately from the fact that $T = G^T \circ F^T$. □

The following lemma explains our interest in monadic functors, which we state without proof.[28]

LEMMA 8.82 *Let $G : V^{\circlearrowleft} \to V$ be the forgetful functor. Then for every left adjoint F of G, the adjunction $F \dashv G$ is monadic. In particular, the functor G is monadic.*

REMARK 8.83 (T-Algebras) It is almost always the case that, for any monad (T, η, μ) with $T : V \to V$, the category V^T contains at least one T-algebra that has more than one element. The only two exceptions[29] are the monads induced by the following functors S_1, S_2:

(1) for all $A \in V$, $S_1(A) = 1$
(2) for all nonempty $A \in V$, $S_2(A) = 1$, but $S_2(0) = 0$.

Neither of these functors is obtainable as a composition $G \circ F$, as in the Lawvere construction, since, as we have already shown, for any such functor, we have that $G(F(1))$ is infinite. We will make use of this observation in the proof of Lemma 8.89 □

The next two lemmas are folklore results that we will need in the proof that Dedekind monads (to be defined shortly) produce Dedekind self-maps.

LEMMA 8.84[30] *Suppose $G : \mathcal{D} \to \mathcal{C}$ is a monadic functor. Then*

(1) *G is faithful; that is, $G \restriction \mathcal{D}(D, D')$ is 1-1, for every pair of $\mathcal{D}$-objects D, D';*
(2) *G reflects isos; that is, whenever $f : A \to B$ is a $\mathcal{D}$-arrow and $G(f)$ is an iso in $\mathcal{C}$, then f itself is also an iso (in $\mathcal{D}$).*

Proof. Let F be left adjoint to G so that the adjunction $F \dashv G$ is a monadic adjunction and let $T = G \circ F : \mathcal{C} \to \mathcal{C}$. Define, as described above, the functor $G^T : \mathcal{C}^T \to \mathcal{C}$.

For part (1), we first prove that G^T is faithful. Suppose $f, g : (D, d) \to (D', d')$ in $\mathcal{C}^T$, and suppose $G^T(f) = G^T(g)$. This means that $f : D \to D'$ and $g : D \to D'$ are equal as $\mathcal{C}$-arrows, as required.

To finish the proof of (1), suppose $f, g : D \to D'$ are $\mathcal{D}$-arrows and $G(f) = G(g)$. By Equation (37), $G^T(\Phi(f)) = G^T(\Phi(g))$. By the first part of the proof of (1), $\Phi(f) = \Phi(g)$. Applying Φ^{-1} to both sides yields $f = g$, as required.

[28] A proof follows from Exercise PPTT, p. 116, in Barr and Wells (1985), with further elaboration in Campion (2017).
[29] A proof of this observation, due to T. Leinster, can be found in Leinster (2017a). See also the footnote following the definition of **T**-algebras in Corazza (2017).
[30] See Leinster (2017b) for more information about these results.

For part (2), we first prove that G^T reflects isos. Suppose $f : (A, a) \to (B, b)$ is a $\mathcal{C}^T$-arrow, so that $f : A \to B$ is a $\mathcal{C}$-arrow for which the following is commutative:

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(B) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array} \tag{38}$$

Suppose also that $f : A \to B$ is a $\mathcal{C}$-iso. Let g be the inverse of f in $\mathcal{C}$. We show that $g : B \to A$ is in fact a $\mathcal{C}^T$-arrow $g : (B, b) \to (A, a)$ by showing that the following diagram is commutative:

$$\begin{array}{ccc} T(B) & \xrightarrow{T(g)} & T(A) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{g} & A \end{array} \tag{39}$$

From Diagram (38) we have the following in $\mathcal{C}$:

$$b \circ T(f) = f \circ a.$$

Since $T(f) \circ T(g) = T(f \circ g) = 1_B$, we can compose with $T(g)$ on the right and compose with g on the left to obtain:

$$g \circ b = a \circ T(g),$$

which demonstrates commutativity of Diagram (39). Now $g : (B, b) \to (A, a)$ is the inverse of $f : (A, a) \to (B, b)$ in $\mathcal{C}^T$ since composition of $\mathcal{C}^T$-arrows is done by composing the corresponding $\mathcal{C}$-arrows and corresponding diagrams.

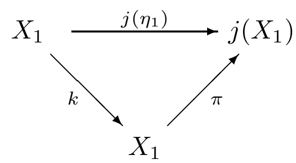
Next, we show that $G : \mathcal{D} \to \mathcal{C}$ itself reflects isos. Suppose $f : D_1 \to D_2$ is a $\mathcal{D}$-arrow and $G(f) : G(D_1) \to G(D_2)$ is a $\mathcal{C}$-iso. By Equation (37), $G(f) = G^T(\Phi(f))$. Now $\Phi(f) : \Phi(D_1) \to \Phi(D_2)$ is a $\mathcal{C}^T$-arrow. Since $G^T(\Phi(f))$ is a $\mathcal{C}$-iso, by the first half of the proof of (2), it follows that $\Phi(f)$ is a $\mathcal{C}^T$-iso. Since Φ^{-1} is also an iso and since functors that are isos preserve isos, it follows that $\Phi^{-1}(\Phi(f)) = f$ is an iso in $\mathcal{D}$, and the result follows. □

Lemma 8.85 (Faithful Functors Reflect Isos) *Suppose $F : \mathbf{Set} \to \mathcal{D}$ is a faithful functor. Then F reflects isos.*

Proof. See Goldblatt (1984), Exercise 4, p. 460. □

We can now show that, whenever (j, η, μ) is a Lawvere monad, there is a naturally defined *set* Dedekind self-map $k : X_1 \to X_1$.

THEOREM 8.86 *Let (j, η, μ) be a Lawvere monad. Let $X_1 = j(1)$. Then $j(\eta_1) : X_1 \to j(X_1)$ is a Dedekind map with factorization $j(\eta_1) = \pi \circ k$, as described above.*



In particular, $k : X_1 \to X_1$ is a Dedekind self-map.

This theorem will follow as a corollary to a more general result, formulated in Theorem 8.91 below.

Theorem 8.86 tells us that if a functor $j : V \to V$ happens to admit the special factorization $j = G \circ F$ as in the Lawvere construction, then a *set* Dedekind self-map is derivable. But is there some criterion for emergence of a set Dedekind self-map that is "internal" to j, and not dependent upon externally defined categories and functors (namely, $V^{\circlearrowleft}, F$, and G)? We suggest one such criterion, based on the following definition.

DEFINITION 8.87 (Dedekind Monad) Suppose (j, η, μ) is a monad, where $j : V \to V$. Then (j, η, μ) is a *Dedekind monad* if the following properties hold:

(1) On objects, j is 1-1 but not onto.
(2) $j(0) = 0$.
(3) For some V-object $c \notin \operatorname{ran} j$,

$$|c| < |j(c)| = |j(j(c))|.$$

The set c mentioned in (3) will be called a *canonical critical point of j*. If $j : V \to V$, (j, η, μ) is a monad, and j is naturally isomorphic to the functor part of a Dedekind monad, then we shall say that (j, η, μ) is an *essentially Dedekind monad*.[31]

We will now show from the theory ZFC−Infinity that, whenever we have a functor $j : V \to V$ that is the functor part of an essentially Dedekind monad (j, η, μ), then there is a set Dedekind self-map k that naturally arises from j. We begin with a lemma that tells us that whenever the components of the unit of a monad are 1-1, the monad reflects isos; this property of a monad is one of the keys to ensure that it gives rise to a Dedekind self-map.

LEMMA 8.88 (ZFC − Infinity) *Suppose $j : V \to V$ and (j, η, μ) is a monad. Suppose also that for each set B, $\eta_B : B \to j(B)$ is 1-1. Then*

(A) *j is a faithful functor, and*
(B) *j reflects isos.*

[31] Note that an essentially Dedekind monad is not necessarily isomorphic to a Dedekind monad in the sense of *monad isomorphism*; having the functor parts of two monads be naturally isomorphic is a weaker condition than requiring the two monads to admit a monad isomorphism between them. This stronger requirement, though perhaps more natural, is not necessary for our purposes.

Proof of (A). Let $F^j : V \to V^j, G^j : V^j \to V$ be the adjoint functors induced by j, as described above, and let Θ be the natural bijection for the adjunction. By Remark 8.81, G^j is monadic. It follows from Lemma 8.84 that G^j is faithful. Recalling that $j = G^j \circ F^j$, it therefore suffices to show that F^j is faithful.

Let $f, g : A \to B$ be functions and assume $F^j(f) = F^j(g)$; we show $f = g$. We let $\overline{F^j(f)} = \Theta_{A,F^j(B)}(F^j(f))$ and $\overline{F^j(g)} = \Theta_{A,F^j(B)}(F^j(g))$. Applying Lemma 2.1, we have

$$\eta_B \circ f = \overline{F^j(f)} = \overline{F^j(g)} = \eta_B \circ g.$$

Since η_B is 1-1, $f = g$, as required.

Proof of (B). We show $j = G^j \circ F^j$ reflects isos. Since G^j is monadic (by Remark 8.81), G^j reflects isos (by Lemma 8.84), so it suffices to show that F^j reflects isos. But this follows from the fact that F^j is faithful (by Lemma 8.85). $\square$

LEMMA 8.89 (ZFC – Infinity) *Suppose $j : V \to V$ and (j, η, μ) is an essentially Dedekind monad. Then for each set B, $\eta_B : B \to j(B)$ is 1-1. In particular, j reflects isos.*

Proof. The last clause follows from the first together with Lemma 8.88. Suppose $j : V \to V$ and (j, η, μ) is an essentially Dedekind monad. Let $j' : V \to V$ be such that (j', η', μ') is a Dedekind monad and j' is naturally isomorphic to j.

Suppose x, y are distinct elements of B. We show $\eta_B(x) \neq \eta_B(y)$. Let $G^j : V^j \to V$ and $F^j : V \to V^j$ be the adjoint functors defined from j, as described above.

We wish to obtain a j-algebra (X, a) for which X has two or more elements. Recall from Remark 8.83 that among monads whose functor part T is defined on V, there are only two for which all T-algebras have at most one element. One of these takes every set to 1; the other takes every nonempty set to 1 and takes 0 to 0.

We first show why j cannot be a functor of the first type. Since j' is a Dedekind monad, $j'(0) = 0$. Since $j \cong j'$, $|j(0)| = |j'(0)| = 0$, and so $j(0) = 0$ also. Therefore, j could not be a functor of the first type.

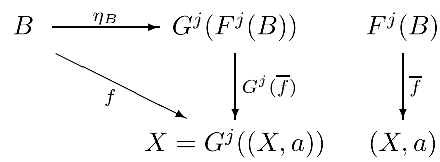
Suppose j is a functor of the second type. Let c be a canonical critical point for j'. Notice $c \neq 0$ since $j'(0) = 0$ but $|c| < |j'(c)|$. Therefore, we have $1 \leq |c| < |j'(c)| = |j(c)|$. It follows that $|j(c)| > 1$, which contradicts the assumption that j is of the second type (which would require that $j(c) = 1$).

We have shown that there is a j-algebra (X, a) for which $|X| > 1$. Pick two distinct elements u, v in X.

Next, we define $f : B \to G^j((X, a)) = X$ by

$$f(z) = \begin{cases} u & \text{if } z = x \\ v & \text{if } z = y \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

By the universal property of η, there is a (unique) $\overline{f} : F^j(B) \to (X, a)$ so that the following diagram is commutative:



Now if $\eta_B(x) = \eta_B(y)$, it follows by commutativity of the diagram that $f(x) = f(y)$, which is impossible. We have shown $\eta_B(x) \neq \eta_B(y)$. Therefore η_B is 1-1. $\square$

Our final lemma shows that, although an essentially Dedekind monad may not be a Dedekind self-map (as a functor), it does have the other two properties of Dedekind monads.

LEMMA 8.90 *Suppose (j, η, μ) is an essentially Dedekind monad and (j', η', μ') is a Dedekind monad with $j' : V \to V$ and $j \cong j'$. Let c be a canonical critical point of j'. Then $j(0) = 0$ and*

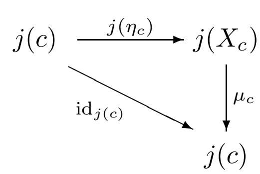
$$|c| < |j(c)| = |j(j(c))|.$$

Proof. Let $(j, \eta, \mu), (j', \eta', \mu'), c$ be as in the hypothesis. The fact that $j(0) = 0$ was shown in the proof of Lemma 8.89. Because j' is a Dedekind monad and $j \cong j'$, we have $|c| < |j'(c)| = |j(c)|$.

Let $\sigma : j \to j'$ be a natural iso. Since each of $j(\sigma_c) : j(j(c)) \to j(j'(c))$ and $\sigma_{j'(c)} : j(j'(c)) \to j'(j'(c))$ is a bijection, the composition $\sigma_{j'(c)} \circ j(\sigma_c) : j(j(c)) \to j'(j'(c))$ is a bijection as well. It follows that $|j(c)| = |j'(c)| = |j'(j'(c))| = |j(j(c))|$. $\square$

THEOREM 8.91 (ZFC$-$Infinity) *Suppose $j : V \to V$ and (j, η, μ) is an essentially Dedekind monad. Let (j', η', μ') be a Dedekind monad with $j \cong j'$ and such that c is a canonical critical point of j'. Let $X_c = j(c)$. Then there is a Dedekind self-map $k : X_c \to X_c$.*

Proof. By diagram (30), since (j, η, μ) is a monad, we have



Commutativity of the diagram implies that $j(\eta_c)$ is 1-1. Note that if $j(\eta_c)$ were a bijection, then, since j reflects isos (by Lemma 8.89(C)), it would follow that $\eta_c : c \to j(c) = X_c$ is also a bijection, which is impossible since, by Lemma 8.90, $|c| < |j(c)| = |X_c|$. Let $b \in j(X_c) - \operatorname{ran} j(\eta_c)$.

Next we observe that $j(\eta_c)$ is a Dedekind map (Definition 8.78). We have just shown $j(\eta_c)$ is 1-1 and has a critical point b. The fact that $|X_c| = |j(X_c)|$ follows from Lemma 8.90. We may therefore apply Proposition 8.79 to conclude that there is a Dedekind self-map $k : j(c) \to j(c)$. $\square$

We give an example to show that the concept of a Dedekind monad is not trivial; in other words, there do exist monads $V \to V$ that are not Dedekind. Verification of details for this example can be found in Awodey (2011).

Consider the mapping $\mathcal{P} : V \to V : X \mapsto \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the powerset of X. For any $f : X \to Y$, define $\mathcal{P}(f)$ by

$$\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) : x \mapsto f[x].$$

Clearly, $\mathcal{P}$ is a functor. One may easily verify that $\mathcal{P}$ is 1-1 on objects and has a critical point (also a strong critical point) $\emptyset$.

Define the unit $\eta : 1 \to \mathcal{P}$ by

$$\eta_A : A \to \mathcal{P}(A) : x \mapsto \{x\}.$$

Finally, the multiplication operation $\mu : \mathcal{P}^2 \to \mathcal{P}$ is defined by

$$\mu_A : \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A) : Z \mapsto \bigcup Z.$$

It can be shown (Awodey, 2011) that $(\mathcal{P}, \eta, \mu)$ is a monad on V—the *power set monad*—that satisfies part (1) of the definition of a Dedekind monad. However, $(\mathcal{P}, \eta, \mu)$ is not a Dedekind monad since $\mathcal{P}(0) \neq 0$ and, for any set c, $|\mathcal{P}(c)| < |\mathcal{P}(\mathcal{P}(c))|$, violating parts (2) and (3) of the definition. Note that existence of the power set monad is provable in ZFC $-$ Infinity; it does not imply the existence of an infinite set.

As promised earlier, we can now show that any monad (j, η, μ) obtained from the Lawvere construction is an essentially Dedekind monad, and so, by Theorem 8.91, gives rise to a set Dedekind self-map $X_1 \to X_1$ (recalling that in this case, 1 is always a critical point of j and $X_1 = j(1)$). Theorem 8.86 will then follow as a corollary.

THEOREM 8.92 *Suppose $j : V \to V$ is obtained as $j = G \circ F$ as in the Lawvere construction and let (j, η, μ) be the monad induced by j. Then (j, η, μ) is an essentially Dedekind monad.*

Proof. We observed in Theorem 8.75 that j is naturally isomorphic to the Lawvere functor $j' : V \to V$, defined on objects by $j'(A) = A \times \omega$, which is the functor part of a Dedekind monad. It follows that (j, η, μ) is an essentially Dedekind monad. □

The problem that originally motivated our discussion about Dedekind monads was to show that if $j = G \circ F$ is obtained from the Lawvere construction, not only is it true that an infinite set arises from the interaction between j and its critical point 1, but it is also true that a (set) Dedekind self-map is directly derivable from this interaction. Returning to this original context, our proof of Theorem 8.86 shows that any such j determines a Lawvere monad (j, η, μ) with the property that $j(\eta_1)$ is a Dedekind map and factors as $j(\eta_1) = \pi \circ k$, where π is a bijection and $k : X_1 \to X_1$ is a Dedekind self-map.

$$\begin{array}{ccc} X_1 & \xrightarrow{j(\eta_1)} & j(X_1) \\ & {}_{k}\searrow \quad \nearrow_{\pi} & \\ & X_1 & \end{array}$$

Whether k is truly derivable just from the interaction between j and its critical point depends on how the bijection π is defined. We show in the next example how a definable (without parameters) bijection π can be obtained in the special case of the Lawvere functor j (definition on p. 67). In that case, the derived Dedekind self-map $k : X_1 \to X_1$ is obtained as $k = \pi^{-1} \circ j(\eta_1)$, which certainly meets the intuitive requirement of being "derived from" the interaction between j and its critical point.

Example 8.93 (Dedekind Self-Map Definable from the Lawvere Functor). We outline here the computations for obtaining a Dedekind self-map definable from the Lawvere functor. Verifications are left to the reader.[32]

We recall the definitions of F and G for the Lawvere functor $j = G \circ F$. F is defined on objects by $F(A) = 1_A \times s : A \times \omega \to A \times \omega$, where $(1 \times s)(a, n) = (a, s(n)) = (a, n+1)$. The definition of F on $V^{\circlearrowright}$-arrows is given by the following: Given $f : A \to B$ in $V^{\circlearrowright}$, $F(f) : F(A) \to F(B)$ is the $V^{\circlearrowright}$-arrow $\phi = \phi_f$ defined by $\phi(a, n) = (f(a), n)$; one verifies that this definition of ϕ makes the following diagram commutative:

$$\begin{array}{ccc} A \times \omega & \xrightarrow{1_A \times s} & A \times \omega \\ \downarrow \phi & & \downarrow \phi \\ B \times \omega & \xrightarrow{1_B \times s} & B \times \omega \end{array} \tag{40}$$

Given a set A and an $V^{\circlearrowright}$ object $\beta : B \to B$, the natural bijection $\Theta_{A,\beta}$: $V^{\circlearrowright}(F(A), \beta) \to V(A, G(\beta))$ is defined, for any $\rho \in V^{\circlearrowright}(F(A), \beta)$, by

$$\Theta_{A,\beta}(\rho)(a) = \rho(a, 0).$$

The inverse of Θ, $\Theta^{-1}_{A,\beta} : V(A, G(\beta)) \to V^{\circlearrowright}(F(A), \beta)$, is defined as follows: For any $f : A \to B = G(\beta)$,

$$\Theta^{-1}_{A,\beta}(f) : A \times \omega \to B : (a, n) \mapsto \beta^n(f(a)). \tag{41}$$

The unit $\eta : 1 \to G \circ F$ of the adjunction is defined as follows. For each A, $\eta_A : A \to G(F(A)) = j(A) = A \times \omega$ is defined by

$$\eta_A = \Theta_{A,F(A)}(1_{F(A)}),$$

so that

$$\eta_A(a) = \Theta_{A,F(A)}(1_{F(A)})(a) = 1_{F(A)}(a, 0) = (a, 0).$$

[32] More details can be found in Corazza (2016), pp. 141–143.

Dually, we define the co-unit $\varepsilon : F \circ G \to 1$. For each $\beta : B \to B$, $\varepsilon_\beta : F(G(\beta)) \to \beta$ (that is, $\varepsilon_\beta : (1_B \times s : B \times \omega \to B \times \omega) \to (\beta : B \to B)$) is defined by

$$\varepsilon_\beta = \Theta^{-1}_{G(\beta),\beta}(1_{G(\beta)}).$$

Applying (41), we have

$$\varepsilon_\beta = \Theta^{-1}_{A,\beta}(1_{G(\beta)}) : (b, n) \mapsto \beta^n(b). \tag{42}$$

Since $(F, G, \eta, \varepsilon)$ is an adjunction, it determines a monad (j, η, μ), where $j = G \circ F$ and $\mu : j^2 \to j$ is defined by

$$\mu_A(x) = \left(G(\varepsilon_{F(A)})\right)(x),$$

for all sets A in V and $x \in j^2(A)$.

We unwind the definition of μ. First, we observe that the domain of $\varepsilon_{F(A)}$ is

$$(F \circ G \circ F)(A) = F(G(1_A \times s)) = F(A \times \omega) = 1_{A \times \omega} \times s : A \times \omega \times \omega \to A \times \omega \times \omega,$$

and the codomain is $1_A \times s$. We have, for every $((a, m), n) = (a, m, n) \in A \times \omega \times \omega$,

$$\varepsilon_{F(A)}((a, m), n) = (1_A \times s)^n(a, m) = (a, s^n(m)) = (a, m + n)$$

(recalling that this map also makes the appropriate diagram commutative).

Now

$$\mu_A = G(\varepsilon_{F(A)}) : (G \circ F \circ G \circ F)(A) \to (G \circ F)(A),$$

—in other words

$$G(\varepsilon_{F(A)}) : A \times \omega \times \omega \to A \times \omega$$

—is computed as follows:

$$\mu_A(a, m, n) = G(\varepsilon_{F(A)})(a, m, n) = (a, m + n).$$

This monad (j, η, μ) generates the category V^j of j-algebras, where $V^j = \{(A, a) \mid a : A \times \omega \to A\}$, where each j-algebra (A, a) satisfies

$$a \circ \eta_A = 1_A \quad \text{and} \quad a \circ j(a) = a \circ \mu_A.$$

Note that $j(a) : A \times \omega \times \omega \to A \times \omega$.

Next, we look more closely at the Dedekind characteristics of this monad. We first recall that, since $j(1) = 1 \times \omega$ and $j(j(1)) = 1 \times \omega \times \omega$, we have the Dedekind properties $|\mathrm{crit}(j)| < |j(\mathrm{crit}(j))| = |j(j(\mathrm{crit}(j)))|$. Next, we compute η_1 and $j(\eta_1)$.

$$\eta_1 : 1 \to j(1) = 1 \times \omega : 0 \mapsto (0, 0). \tag{43}$$

Referring to the definition of F on $V^{\circlearrowleft}$-arrows, $F(\eta_1) : 1 \times \omega \to 1 \times \omega \times \omega$ is defined to be the map $\phi = \phi_{\eta_1}$—defined by $\phi(0, n) = (\eta_1(0), n) = (0, 0, n)$—making the following commutative:

$$\begin{array}{ccc} 1 \times \omega & \xrightarrow{1_1 \times s} & 1 \times \omega \\ \Big\downarrow \phi & & \Big\downarrow \phi \\ 1 \times \omega \times \omega & \xrightarrow{1_{1 \times \omega} \times s} & 1 \times \omega \times \omega \end{array} \tag{44}$$

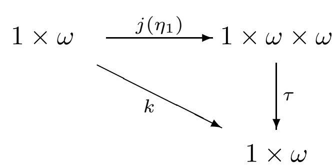
Therefore,

$$j(\eta_1) = G(F(\eta_1)) : 1 \times \omega \to 1 \times \omega \times \omega : (0, n) \mapsto (0, 0, n).$$

We obtain a Dedekind self-map $k : 1 \times \omega \to 1 \times \omega$ following the techniques of the proof of Proposition 8.79, in a definable way. For this purpose, we propose to use the following bijection: $\tau : 1 \times \omega \times \omega \to 1 \times \omega$:

$$\tau(0, m, n) = (0, \langle m, n \rangle),$$

where $\langle \text{-,-} \rangle$ is a definable (bijective) pairing function $(m, n) \mapsto n + \frac{(n+m)(n+m+1)}{2}$. The Dedekind self-map k is defined, as in the proof of Proposition 8.79, to be $\tau \circ j(\eta_1)$ (note that setting $\pi = \tau^{-1}$, we can also write $k = \pi^{-1} \circ j(\eta_1)$):



A straightforward computation yields

$$k(0, n) = \tau(j(\eta_1)(0, n)) = (0, \langle 0, n \rangle).$$

For instance,

$$k(0, 0) = (0, 0); \quad k(0, 1) = (0, 2); \quad k(0, 2) = (0, 5).$$

Since k is strictly increasing in the second component, the lexicographically least critical point of k is $(0, 1)$. □

The concept of a Dedekind monad leads to yet another equivalent of the Axiom of Infinity:

THEOREM 8.94 (ZFC − Infinity) *The following are equivalent.*

(1) *There is an infinite set.*
(2) *There is a Dedekind monad.*[33]

33 Some care is needed in the formulation of this theorem since we appear to be quantifying over a class. To re-state (1) ⇒ (2) properly, we would specify a formula for the functor $F : V \to V^{\circlearrowleft}$ (defined above) that takes a set A to $1_A \times s : A \times \omega \to A \times \omega$ and assert that it is left adjoint to the forgetful functor $G : V^{\circlearrowleft} \to V : (h : A \to A) \mapsto A$, and that the monad induced by $G \circ F$ is a Dedekind monad. To re-state (2) ⇒ (1) properly requires a schema of statements, one for each formula ψ that defines a Dedekind monad (i, η, μ); let ρ be a subformula of ψ that defines i. For each such ψ, we would have a formula $\sigma(a, k)$ that asserts (using ρ) that a is a critical point of j satisfying the Dedekind conditions and $k : j(a) \to j(a)$ is a Dedekind self-map. Then, for each such ψ, we would include the following statement in the schema:

$$\psi \to \exists a, k\, \sigma(a, k).$$

Proof. For (1) $\Rightarrow$ (2), use the fact that existence of an infinite set implies existence of ω. Then the Lawvere functor is the functor part of a Dedekind monad (Example 8.93). Conversely, (2) $\Rightarrow$ (1) follows from Theorem 8.86 and the fact that existence of a Dedekind self-map on a set implies existence of an infinite set. □

8.2 Possibilities for Generalizing to a Large Cardinal Context In this section we have encountered characteristics of interesting Dedekind self-maps $j : V \to V$ that make themselves known in a category-theoretic context. In this subsection, we ask whether some of these characteristics can be reasonably expected to hold for an "ultimate" Dedekind self-map—one that satisfies the properties of the Dedekind Self-Map Conjecture.

For this discussion, let us call a Dedekind self-map $j : V \to V$ *adequate* if it has the properties listed in the Dedekind Self-Map Conjecture.

Questions. Suppose $j : V \to V$ is an adequate Dedekind self-map.

(1) Must it be the case that there are sets a, A for which $a \in j(A)$ is a universal element for j?
(2) Is j the functor part of a monad?

Question (1) is motivated by the fact that a universal element played a key role in the emergence of an infinite set given in Example 6.60—see the proof in Remark 6.61. Also, as we will see, weakly universal elements reappear when we generalize Example 6.60 to obtain measurable cardinals (Example 9.109). In that case (p. 93), the critical point $\kappa \in j(\kappa)$ is weakly universal for j; this relationship gives convincing expression to the principle of Critical Point Dynamics (see the discussion in Remark 9.108).

Question (2) is motivated by the fact that our category-theoretic way of obtaining a $j : V \to V$ that produces an infinite set has been to compose adjoint functors; the result in every case has been a monad. Do monads play a role in generating large cardinals as we scale to stronger preservation properties? A Dedekind monad $j : V \to V$ has the special characteristic that its interaction with its critical point gives rise to a set Dedekind self-map—another clear expression of Critical Point Dynamics—and it would seem natural to find relationships of this kind in an adequate Dedekind self-map $j : V \to V$.

We will address these questions in Section 11.3 (see Theorems 11.130 and 11.132).

§9 Deriving Large Cardinals from Dedekind Self-Maps $j : V \to V$.

In this section we strengthen the preservation properties, and other properties, discussed in Sections §5–§6 with the aim of producing large cardinals from Dedekind self-maps $V \to V$. We provide strengthened versions of Theorems 5.43, 5.50 and then enrich our generalization of Theorem 5.43 further to produce an ineffable cardinal. We also provide a stronger version of Theorem 6.54 that yields a measurable cardinal. In addition, we show that, without assuming too much more than the target large cardinal, it is possible to construct examples of Dedekind self-maps that have the properties mentioned in these theorems.

9.1 Deriving Inaccessible Cardinals We begin by defining a few new preservation properties.

DEFINITION 9.95 Suppose $j : V \to V$ is a Dedekind self-map.

(1) *j preserves countable disjoint unions* if, whenever $\langle X_n \mid n \in N\rangle$ (where either $N \in \omega$ or $N = \omega$) is a sequence of disjoint sets, $j\left(\bigcup_{n\in N} X_n\right) = \bigcup_{n\in N} j(X_n)$.
(2) Suppose α is an ordinal. Then *j preserves unboundedness at α* if $j(\alpha)$ is an ordinal and, whenever $A \subseteq \alpha$ is unbounded in α and $j(A) \subseteq j(\alpha)$, we have that $j(A)$ is unbounded in $j(\alpha)$. In general, j is said to *preserve unboundedness* if j preserves ordinals and for each ordinal α, j preserves unboundedness at α.
(3) *j preserves power sets* if, for all X, $j(\mathcal{P}(X)) = \mathcal{P}(j(X))$.

THEOREM 9.96 (Generalization of Theorem 5.43 to Inaccessibles) (ZFC−Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map that strongly preserves $\in$ and preserves ordinals and rank. Let $\kappa = \mathrm{crit}(j)$.*

(A) *Suppose j preserves countable disjoint unions and singletons. Then $\kappa > \omega$.*
(B) *Suppose that, in addition to (A), j is BSP and preserves functional application, images, and unboundedness. Then κ is an uncountable regular cardinal.*
(C) *Suppose that, in addition to (B), j preserves power sets. Then κ is an inaccessible cardinal.*

Proof of (A). Recalling Remark 7.73, by Theorem 5.43, ω exists; the proof of that theorem shows that, under these hypotheses, $j \restriction \mathbf{HF} = \mathrm{id}_{V_\omega}$; it follows that $\kappa \geq \omega$; note also that $j(0) = 0$. We have, by Theorem 7.65 and the properties that j preserves:

$$\begin{aligned}
\omega &= \{0\} \cup \{1\} \cup \{2\} \cup \cdots \\
&= \{j(0)\} \cup \{j(1)\} \cup \{j(2)\} \cup \cdots \\
&= j(\{0\}) \cup j(\{1\}) \cup j(\{2\}) \cup \cdots \\
&= j\left(\{0\} \cup \{1\} \cup \{2\} \cup \cdots\right) \\
&= j(\omega).
\end{aligned}$$

It follows that $\kappa > \omega$.

Proof of (B). By Theorem 7.65(5), κ is a cardinal. By (A), κ is an uncountable cardinal. Suppose $f : \alpha \to \kappa$ and $\operatorname{ran} f$ is unbounded in κ. Using the fact that j is BSP and preserves functional application, we can reason as in the proof of Theorem 7.65(5) to show that $j(f) : j(\alpha) \to j(\kappa)$ has domain α and $j(f)(\beta) = f(\beta)$ for all $\beta < \alpha$. Because j preserves images, we have

$$\operatorname{ran} j(f) = j(\operatorname{ran} f) \subseteq \kappa, \tag{45}$$

but, because j also preserves ordinals and unboundedness and because $\operatorname{ran} f$ is unbounded in κ,

$$j(\operatorname{ran} f) \text{ is unbounded in } j(\kappa). \tag{46}$$

Clearly, since $\kappa < j(\kappa)$, (45) and (46) contradict each other. Therefore, all functions from α to κ have bounded range. It follows, therefore, that κ is regular. We have shown κ is an uncountable regular cardinal.

Proof of (C). We begin by showing that, for every $A \subseteq \alpha$, where $\alpha < \kappa$, we have $j(A) = A$. Because $\alpha < \kappa$ and $\in$ is preserved, $A \subseteq j(A)$. To show $j(A) \subseteq A$, we first observe that $j(A) \subseteq \alpha$. Using the fact that j preserves $\in$ and power sets, we have

$$A \in \mathcal{P}(\alpha) \Rightarrow j(A) \in j(\mathcal{P}(\alpha)) = \mathcal{P}(j(\alpha)) = \mathcal{P}(\alpha).$$

Now, suppose $\gamma \in j(A) - A$. Since α is the disjoint union of A and $\alpha - A$ and $\gamma \notin A$, it follows that $\gamma \in \alpha - A$. Since j preserves disjoint unions, $\alpha = j(\alpha) = j(A) \cup j(\alpha - A)$ and $j(A), j(\alpha - A)$ are disjoint. Since $\gamma \in \alpha - A$, $\gamma = j(\gamma) \in j(\alpha - A)$, and so $\gamma \notin j(A)$, which is a contradiction. We have shown $A = j(A)$.

Continuing with the proof that κ is a strong limit, suppose, for a contradiction, that there is a surjective function $g : \mathcal{P}(\alpha) \to \kappa$, where $\alpha < \kappa$. Since j preserves functions and power sets and $j(\alpha) = \alpha$, we have that $j(g) : \mathcal{P}(\alpha) \to j(\kappa)$. Note that, for each $A \in \mathcal{P}(\alpha)$, $g(A) \in \kappa$, whence $j(g(A)) = g(A)$. Therefore, since j preserves functional application, for each $A \in \mathcal{P}(\alpha)$,

$$j(g)(A) = j(g)(j(A)) = j(g(A)) = g(A).$$

Therefore, we have

$$\operatorname{ran} j(g) = \operatorname{ran} g = \kappa. \tag{47}$$

We also have, by preservation of images,

$$\operatorname{ran} j(g) = j(\operatorname{ran} g) = j(\kappa) > \kappa. \tag{48}$$

Clearly, (47) and (48) contradict each other, and so no such function g exists. We have shown κ is a strong limit, and hence, inaccessible. □

The next result generalizes the approach in Theorem 5.50. One new property that we will require of j in this theorem is that the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is cofinal in ON, where κ is the least cardinal moved by j (and is itself a critical point of j). This property is not a preservation property; however it is a strengthening of the *cofinal* property (Definition 5.40), which we take to be a well-motivated property for j to have. To see the connection to the cofinal property, assume the critical sequence is cofinal and let a be a set, with $a \in V_\alpha$ for some α. Let $n \in \omega$ be such that $\alpha < j^{n+1}(\kappa)$. Then $a \in j(A)$ where $A = V_{j^n(\kappa)}$.

We observe that, by Replacement, it is not possible for the critical sequence of j to be cofinal in ON if j is definable in V, so for the proof of the theorem, we will not be able to rely on earlier results that assumed definability of j.

Definition 9.97 *Suppose $j : V \to V$ is a map.*

(1) j preserves countability *if whenever A is countable, $j(A)$ is countable.*
(2) *Suppose j has a critical point that is a cardinal κ. Then the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ of j is cofinal in ON if, for each ordinal α, there is $n \in \omega$ with $\alpha < j^n(\kappa)$.*
(3) *Suppose j has a critical point that is a cardinal κ. Then j* preserves small sequences relative to κ *if for every γ-sequence $s = \langle x_\alpha \mid \alpha < \gamma \rangle$, where $\gamma < \kappa$ and $x_\alpha \in \kappa$ for all $\alpha < \gamma$, we have $j(s) = s$. Moreover, such a sequence s will be called a* small sequence relative to κ.

Remark 9.98 We observe that if j preserves small sequences relative to κ, then κ is the least ordinal moved by j: If not, let $\alpha < \kappa$ be an ordinal for which $j(\alpha) \neq \alpha$ and let β be such that $\alpha < \beta < \kappa$. The sequence $s = \langle x_\delta \mid \delta < \beta \rangle$, where $x_\delta = \delta$ for each δ, is a small sequence relative to κ. Since $j(s) = s$, it follows that $j(x_\alpha) = x_\alpha$, contradicting the fact that $j(\alpha) \neq \alpha$.

THEOREM 9.99 (Generalization of Theorem 5.50 to Inaccessibles) *Suppose $j : V \to V$ is a Dedekind self-map (not necessarily definable). Suppose there is a cardinal κ that is both a critical point and a strong critical point for j and that is the least cardinal moved by j.*

(1) *If j preserves finite coproducts and terminal objects, then κ is infinite.*
(2) *If, in addition to (1), j preserves $\in$, ordinals, unboundedness, images, countability, cardinality, subsets, and small sequences relative to κ, then κ is an uncountable regular cardinal.*
(3) *If, in addition to (2), j preserves power sets and has a cofinal critical sequence, then κ is inaccessible.*

REMARK 9.100 We do not consider that Theorem 9.99 provides compelling evidence for the existence of an inaccessible cardinal, since the requirement that j's critical sequence be cofinal in ON lacks sufficient intuitive motivation (especially since it violates Replacement), despite its connection to the cofinal property. We include this theorem in our discussion because, when we start to consider Dedekind self-maps $j : V \to V$ that are elementary embeddings—a step that we feel *is* well-motivated—we will be forced to deal with the fact that j must be undefinable in V, and some examples of such a j will indeed have a cofinal critical sequence. In anticipation of such results, we illustrate in this theorem, in a rather simple context, some of the issues that one must address in working with such self-maps. We shall see that, in the context of the stronger axioms and properties that we will study in the remaining sections of the paper, all the properties listed in Theorem 9.99 will turn out to be derivable consequences, despite the unintuitive flavor that some of these seem to have at first blush.

There are several ways to formulate the theorem precisely. To avoid conflicting with Replacement in part (3), any such formulation will take place in the theory $\mathrm{ZFC}_{\mathbf{j}}$ (Section 2.3). We would then consider parts (1)–(3) to be expressed as formal axioms added to $\mathrm{ZFC}_{\mathbf{j}}$. Since it is possible, even in ZFC+BTEE, that $\mathbf{j}^N(\kappa)$ does not exist for some finite ordinal N within the theory and that the critical sequence is not strictly increasing, some care is needed in the formulation of (3). One approach is to consider the critial sequence $\langle j^n(\kappa) \mid n \in \omega\rangle$ to be enumerated in the metatheory. Another way is to assume that we are working in a transitive model of $\mathrm{ZFC}_{\mathbf{j}}$, since in such models, all finite ordinals are standard. The formulation that gives the strongest conclusion requires us to include as an additional axiom the Least Ordinal Principle (Section 2.3), which guarantees that the critical sequence has the expected properties.

Proof of (1). By the proof of Theorem 5.50, any strong critical point of j must be infinite and by hypothesis κ itself is a strong critical point. Note that the hypothesis of Theorem 5.50 does not require j to be definable in V.

Proof of (2). Since j preserves cardinality, $j(\kappa)$ is a cardinal. We observe that $\kappa < j(\kappa)$: By assumption, $\kappa \neq j(\kappa)$. If $j(\kappa) < \kappa$, then since j preserves $\in$, we would have $j(j(\kappa)) < j(\kappa)$, contradicting the fact that κ is the least cardinal moved.

Since $\kappa < j(\kappa)$ and both are infinite cardinals, it follows that $j(\kappa)$ is uncountable. Since j preserves countability, κ itself must be uncountable.

We show that κ is regular by showing the all unbounded subsets of κ have size κ. For a contradiction, assume A is unbounded in κ and $|A| = \gamma < \kappa$. Since j preserves subsets, $j(A) \subseteq j(\kappa)$. Let $s = \langle x_\alpha \mid \alpha < \gamma\rangle$ be the increasing enumeration of A.

Clearly, s is a small sequence relative to κ, and so $j(s) = s$. Then, because j preserves images, we have

$$\kappa = \sup(A) = \sup(\operatorname{ran} s) = \sup(\operatorname{ran} j(s)) = \sup(j(\operatorname{ran} s)) = \sup(j(A)),$$

and this contradicts the fact that $j(A)$ is unbounded in $j(\kappa)$ (recall that j preserves unboundedness at κ). It follows that no such set A exists. We have shown that κ is regular.

Proof of (3). We show that κ is a strong limit cardinal. Assume not. Let $\alpha < \kappa$ be such that $|\mathcal{P}(\alpha)| \geq \kappa$. Let n be such that $|\mathcal{P}(\alpha)| < j^n(\kappa)$ (using the fact that the critical sequence is cofinal in ON). Notice that, by induction (either in the metatheory, within an ambient transtive model, or within the theory strengthened by adding the Least Ordinal Principle, as described in Remark 9.100), for each $m \in \omega$ and each set X, we have:

(a) j^m preserves $\in$,
(b) $j^m(\kappa) < j^{m+1}(\kappa)$,
(c) $j^m(\mathcal{P}(\alpha)) = \mathcal{P}(j^m(\alpha))$, and
(d) $j^m(|X|) = |j^m(X)|$.

Applying these observations yields the following:

$$j^n(\kappa) \leq j^n(|\mathcal{P}(\alpha)|) = |\mathcal{P}(j^n(\alpha))| = |\mathcal{P}(\alpha)| < j^n(\kappa).$$

This is a contradiction, and establishes that κ is a strong limit, and hence, inaccessible. □

9.2 Deriving an Ineffable Cardinal We introduce more preservation properties to enrich the hypotheses of Theorem 9.96 still further in order to produce an *ineffable* cardinal. We begin with two specialized preservation properties that are relevant only when $j : V \to V$ is a Dedekind self-map having a least ordinal moved, denoted κ, which is also a critical point for j. A function $f : \kappa \to \mathcal{P}(\kappa)$ will be called *locally bounded on* κ if, for all $\alpha < \kappa$, $f(\alpha) \subseteq \alpha$.

Definition 9.101 (Two Specialized Preservation Properties) Suppose $j : V \to V$ is a Dedekind self-map and κ is both a critical point for j and the least ordinal moved by j.

(1) *j preserves closed subsets of κ* if, j preserves cardinals and for each closed set $C \subseteq \kappa$, $j(C)$ is a closed subset of $j(\kappa)$.
(2) *j preserves locally bounded functions at κ* if the following conditions hold:
   (a) j preserves functions;
   (b) for each locally bounded function $f : \kappa \to \mathcal{P}(\kappa)$, $\kappa \in \operatorname{dom} j(f)$ and $j(f)(\kappa) \subseteq \kappa$.

In typical applications of (2), whenever f is locally bounded at κ, we will have $j(f) \restriction \kappa = f$.[34] This means that j has the effect of extending f to a larger domain;

[34] Reasoning as in Theorem 9.96(C), one shows that whenever $f : \kappa \to \mathcal{P}(\kappa)$ is locally bounded at κ and j preserves functions, functional application, $\in$, disjoint unions, and power sets, then $j(f) \restriction \kappa = f$.

in so doing, (2) asserts that the extension $j(f)$ preserves, at κ, the property that f enjoyed at each α; namely, $j(f)(\kappa) \subseteq \kappa$.

We introduce two other more general preservation properties that will also be relevant later when we produce a measurable cardinal. Suppose $f : A \to B$ and $g : A \to B$ are functions. The *equalizer* $E_{f,g}$ *of* f *and* g is the set $\{x \in A \mid f(x) = g(x)\}$. Generalizing somewhat, suppose A, B are sets, $A \subseteq B$ and B is transitive, and suppose $f : B \to \mathcal{P}(B)$ is a function. Let $*$ denote one of the set operations $\cap, \cup, -$ (intersection, union, set difference). Then $f^*_A : B \to \mathcal{P}(B)$ is the *function of* $*$*-type derived from* f if f^*_A is defined by $f^*_A(x) = f(x) * A$ for all $x \in B$. Note that the equalizer $E^{A,C}_{f,g}$ of two such derived functions is defined in the same way as for ordinary functions, where $C \subseteq B$ and $g : B \to \mathcal{P}(B)$ is another given function:

$$E^{A,C}_{f,g} = \{x \in B \mid f^*_A(x) = g^*_C(x)\} = \{x \in B \mid f(x) * A = g(x) * C\}.$$

DEFINITION 9.102 (Two General Preservation Properties) Suppose $j : V \to V$ is a Dedekind self-map that preserves functions. Let $*$ denote one of the set operations $\cap, \cup, -$

(1) j *preserves equalizers* if for any $f, g : A \to B$ as above, $j(E_{f,g}) = E_{j(f),j(g)}$; that is, if $E_{f,g}$ is the equalizer of f and g, then $j(E_{f,g})$ is the equalizer of $j(f) : j(A) \to j(B)$ and $j(g) : j(A) \to j(B)$.
(2) Suppose j preserves power sets. Then j *preserves equalizers of type* $*$ if whenever $f^*_A, g^*_C : B \to \mathcal{P}(B)$ are functions of $*$-type derived from given functions $f, g : B \to \mathcal{P}(B)$, where $A, C \subseteq B$ and B is transitive, then

$$\begin{aligned} j(E^{A,C}_{f,g}) = E^{j(A),j(C)}_{j(f),j(g)} &= \{x \in j(B) \mid j(f)^*_{j(A)}(x) = j(g)^*_{j(C)}(x)\} \\ &= \{x \in j(B) \mid j(f)(x) * j(A) = j(g)(x) * j(C)\}. \end{aligned}$$

Recall that a cardinal κ is *ineffable* if for each sequence $\langle A_\alpha : \alpha < \kappa\rangle$ satisfying $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, there is a set $A \subseteq \kappa$ such that $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary in κ.

THEOREM 9.103 (Generalization of Theorem 9.96 to Ineffables) (ZFC − Infinity) *Suppose $j : V \to V$ is a class Dedekind self-map that strongly preserves $\in$ and preserves ordinals and rank. Let $\kappa = \mathrm{crit}(j)$. Suppose j satisfies the properties listed in (A)–(C) of Theorem 9.96. Suppose also that j preserves intersections, equalizers of type $\cap$, closed subsets of κ, and locally bounded functions at κ. Then κ is ineffable.*

Proof. Let $f = \langle A_\alpha \mid \alpha < \kappa\rangle$ be a sequence satisfying $A_\alpha \subseteq \alpha$ for each $\alpha < \kappa$. Since j preserves functions and power sets, $j(f) : j(\kappa) \to \mathcal{P}(j(\kappa))$ is a function. Since $\kappa < j(\kappa)$ and f is locally bounded at κ and j preserves locally bounded functions, it follows that $j(f)(\kappa) \subseteq \kappa$; let $A = j(f)(\kappa)$. Let $g : \kappa \to \mathcal{P}(\kappa)$ denote the inclusion map $\alpha \mapsto \alpha$. Consider the functions $f^\cap_\kappa, g^\cap_A$ of type $\cap$ derived from f, g respectively; in particular, $f^\cap_\kappa(\alpha) = A_\alpha \cap \kappa = A_\alpha$ and $g^\cap_A(\alpha) = A \cap \alpha$, for all $\alpha < \kappa$. Let S be the equalizer of $f^\cap_\kappa, g^\cap_A$; that is,

$$S = \{\alpha < \kappa \mid f^\cap_\kappa(\alpha) = g^\cap_A(\alpha)\} = \{\alpha \mid A_\alpha = A \cap \alpha\}.$$

To complete the proof, we show that S is stationary in κ; as a first step, we show $\kappa \in j(S)$. Since j preserves equalizers of type $\cap$,

$$j(S) = \{\alpha < j(\kappa) \mid j(f)(\alpha) = j(A) \cap \alpha\}.$$

Since $A \subseteq \kappa$, we may show that $j(A) \cap \kappa = A$: If $\alpha \in A$, by preservation of $\in$, $\alpha = j(\alpha) \in j(A)$. Conversely, if $\alpha \in j(A) \cap \kappa$, then $j(\alpha) \in j(A)$, so by strong preservation of $\in$, $\alpha \in A$. It now follows that $\kappa \in j(S)$.

We now prove S is stationary. Suppose C is closed and unbounded in κ. Since j preserves unboundedness and closed sets, $j(C)$ is closed and unbounded in $j(\kappa)$. As in the argument in the previous paragraph, $C = j(C) \cap \kappa$; therefore, κ is a limit point of $j(C)$ and hence $\kappa \in j(C)$. Since we also have $\kappa \in j(S)$, we conclude that $j(C) \cap j(S) \neq \emptyset$. Since j preserves intersections, $j(C) \cap j(S) = j(C \cap S)$. But now if $C \cap S = \emptyset$, then, since $j(0) = 0$, we would have $j(C \cap S) = \emptyset$, which is a contradiction. Hence $C \cap S \neq \emptyset$. We have shown S is stationary. Therefore, κ is ineffable. □

Remark 9.104 Theorems 9.96, 9.99, and 9.103 provide evidence that combining ever more of the right preservation properties (and possibly other well-motivated properties) leads to ever stronger large cardinals. But as this list of properties grows to the point of being unwieldy, it is reasonable to consider studying Dedekind self-maps $j : V \to V$ that preserve *all* first-order properties—namely, elementary embeddings $j : V \to V$—rather than cataloging each property that we intend to use, as we have so far. In order to adopt this hypothesis as we climb the hierarchy of large cardinals, we must abandon the requirement that j is a *class* map, since Kunen's theorem implies that definable elementary embeddings $j : V \to V$ are inconsistent. But, as described earlier, we may continue to make progress in this direction by working in the theory $\mathrm{ZFC}_{\mathbf{j}}$ in the language $\{\in, \mathbf{j}\}$.

It so happens that, having arrived at an ineffable cardinal, we have also nearly arrived at the level in the large cardinal hierarchy in which elementary embeddings from a transitive model of ZFC to itself first make their appearance. While a cardinal κ is ineffable if and only if,[35] for each partition $f : [\kappa]^2 \to 2$, there is a stationary set $H \subseteq \kappa$ such that f is constant on $[H]^2$, a related but slightly stronger type of partition relation leads to elementary embeddings $N \to N$ for transitive models N of ZFC: Recall that λ is *ω-Erdös* if

$$\lambda \text{ is least such that } \lambda \to (\omega)^{<\omega}.$$

If there is an ω-Erdös cardinal λ and $\mathfrak{A}$ is a model whose language has less than λ symbols and whose domain contains every element of λ, then $\mathfrak{A}$ has a set of indiscernibles of ordertype ω. The presence of such indiscernibles makes it possible to define an elementary embedding. Given a transitive $M \models \mathrm{ZFC}$ with built-in or definable Skolem functions and $I \subseteq \mathrm{ON}^M$ of indiscernibles for M having ordertype ω, define $B = \mathfrak{H}^M(I) \prec M$, the Skolem hull of I in M. Let $\pi : B \to N$ be the transitive collapsing map, and let $e : N \to M$ denote the induced elementary embedding ($e = \pi^{-1}$). Define $i_0 : I \to I$ so that i_0 takes each element α of I to the next element $s_I(\alpha)$ of I above α. Let $i : B \to B$ be the canonical elementary embedding defined from e on Skolem terms. Letting $j = \pi \circ i \circ \pi^{-1}$, it follows that j is a nontrivial elementary embedding with cofinal critical sequence $J = \pi'' I$; in particular, $\langle N, \in, j\rangle \models \mathrm{ZFC} + \mathrm{BTEE}$. In Corazza (2006), we called the model $\langle N, \in, j\rangle$ the *canonical transitive model of ZFC + BTEE derived from M, I.*

[35] A proof that this formulation is equivalent to the one used in the proof of Theorem 9.103 above can be found in Kanamori and Magidor (1978).

Considering that the move from assuming as hypotheses the long list of preservation properties needed to derive an ineffable cardinal to assuming existence of a nontrivial elementary embedding involves a relatively small jump in large cardinal strength, we would argue that adopting this new requirement on the Dedekind self-maps $j : V \to V$ that we are studying is a natural step to take at this point.

Theorem 9.105 below shows that such models provide examples of Dedekind self-maps $j : V \to V$ having the properties listed in Theorems 9.99 and 9.103. □

As we seek examples of Dedekind self-maps $j : V \to V$ with the properties listed in Theorems 9.96 and 9.99, we note that the optimal large cardinal assumption for such examples would be existence of an inaccessible, but our best examples require a stronger assumption, namely, the existence of a transitive model of ZFC + BTEE. On the other hand, as we just observed in Remark 9.104, such a model is much closer to optimal as an example of the properties of Theorem 9.103.

The following theorem provides an example for the properties of all three theorems: 9.96, 9.99, and 9.103.

THEOREM 9.105 *Assume there is an ω-Erdös cardinal. Then there is a transitive model $\mathcal{M} = \langle M, \in, j\rangle$ of ZFC + BTEE in which the following properties hold:*

(1) *The embedding j is a Dedekind self-map. The least ordinal κ moved by j (which must exist) is both a critical point and a strong critical point for j, and is infinite.*
(2) *The embedding j is BSP, strongly preserves $\in$, and preserves ordinals, rank, countable disjoint unions, singletons, functional application, images, unboundedness, and power sets.*
(3) *The embedding j preserves finite coproducts, terminal objects, countability, cardinality, and small sequences relative to κ, and has a cofinal critical sequence.*
(4) *The embedding j preserves intersections, equalizers of type $\cap$, closed subsets of κ, and locally bounded functions at κ.*

Proof. The transitive model $\mathcal{M} = \langle M, \in, j\rangle$ of ZFC + BTEE that we use is the canonical transitive model obtained from a set of indiscernibles of order type ω, given to us by the ω-Erdös cardinal, as described in Remark 9.104.

As remarked earlier, in the canonical transitive model we are using, the embedding j has a critical sequence that is cofinal. The rest of the properties in parts (2)–(4) follow immediately from the fact that j is a BTEE-embedding; some of these details are spelled out in Corazza (2006), pp. 338–342.

We note that the reliance on definability of j that was observed in the proof of Theorem 9.96 is not needed in the proof of (2) since elementarity alone suffices for the proof. Note however that definability *was* needed in the proof of Theorem 9.96 to show that some critical point of j is a cardinal κ, that $\kappa < j(\kappa)$, and that κ is infinite. Therefore, we take a different path to establish (1), which makes use of Lemma 7.71.

First, by definability of 0 and ω, it follows that $j(0) = 0$ and $j(\omega) = \omega$. Since $\text{Induction}_{\mathbf{j}}$ holds in M (since M is transitive—see Section 2.3), one may show that $j(n) = n$ for all $n \in \omega$. Transitivity of M also implies that the Least Ordinal $\text{Principle}_{\mathbf{j}}$ holds in M (Section 2.3) and so, for each ordinal α, $j(\alpha) \geq \alpha$. Our plan therefore is to establish that the least ordinal κ moved by j (which must exist by the

Least Ordinal Principle$_{\mathbf{j}}$) is a strong critical point—and therefore the *least* strong critical point—and make use of Lemma 7.71 and its proof to conclude that κ is also a critical point of j. Note that since $j(n) = n$ for all $n \in \omega$, it will also follow that κ is infinite.

We show that κ is a cardinal. Suppose $\gamma < \kappa$ and there is an onto $f : \gamma \to \kappa$. By elementarity, $j(f) : \gamma \to j(\kappa)$ is also onto, which contradicts the fact that $\kappa < j(\kappa)$. Therefore, there is no such function and κ is a cardinal.

Therefore both κ and $j(\kappa)$ are cardinals and $\kappa < j(\kappa)$. Therefore, κ is a strong critical point for j; indeed κ is the least strong critical point of j. Now κ satisfies (a) of Lemma 7.71, and we have already established that (b) holds as well. By elementarity of j, the rest of the hypotheses of Lemma 7.71 also hold. We conclude that κ is a critical point of j. □

Remark 9.106 Part (2) of the theorem lists the properties of the hypothesis of Theorem 9.96; Part (3) lists additional properties from the hypothesis of Theorem 9.99; and Part (4) lists remaining properties from the hypothesis of Theorem 9.103.

Open Question. Is it possible, starting from a universe whose only large cardinal is an inaccessible, to build a class Dedekind self-map $j : V \to V$ that satisfies the properties of Theorem 9.96(1)–(3)?

9.3 Deriving a Measurable Cardinal Our third and final generalization of the work in Sections §5–§6 to large cardinals strengthens Theorem 6.54 so that the generated ultrafilter yields a measurable cardinal. Following Eklof and Mekler (1990), p. 26, we shall say that an infinite cardinal κ is *ω-measurable* if there is a nonprincipal ω_1-complete ultrafilter on κ. It is well-known (Eklof and Mekler, 1990, Corollary 2.12) that there is an ω-measurable cardinal if and only if there is a measurable cardinal; in particular, if κ is ω-measurable, there is a measurable cardinal λ such that $\lambda \leq \kappa$. The next theorem and the techniques of proof, along with the methods used in Example 9.109, are small modifications of the work of Blass (1976).

Theorem 9.107 (Generalization of Theorem 6.54 to Measurables) *Suppose $j : V \to V$ is a functor, definable in V, having a strong critical point, which preserves countable disjoint unions, intersections, equalizers, and terminal objects, and which has a weakly universal element $a \in j(A)$ for j, for some set A. Then $|A|$ is ω-measurable. In particular, there exists a measurable cardinal.*

Remark 9.108 We have not required j to be a Dedekind self-map; we do not have a proof, under the given hypotheses, that it must have this property. As we show now, however, j must be *essentially* Dedekind (see p. 66 for the definition): In the proof given below, Claim 1 shows that j and j_D are naturally isomorphic. As shown in Example 9.109 (below), j_D itself is 1-1, and, as in Example 6.57 j_D has a critical point (this is obvious, but $[\mathrm{id}]_\kappa$ is an example).

The hypothesis requires j to be definable in V. This condition is used in the proof when the filter D is defined (and assumed to be a *set*).

Although the hypotheses require j to have a strong critical point, they do not require the weakly universal element $a \in j(A)$ itself to be a strong critical point or a critical point. However, under somewhat stronger hypotheses, the natural choice of a weakly universal element is both a critical point and a strong critical point—see the remarks following Example 9.109.

We observe that most of the properties that we have required j to have in this theorem are preservation properties, providing further realization of our Dedekind Self-Map Conjecture that large cardinals are expected to emerge when j is endowed with the right preservation properties. However, as in Theorem 6.54, one of the properties of j is that there are a, A such that $a \in j(A)$; in this case, a is required to be a weakly universal element of j. We have offered several kinds of justification for existence of such a relationship, but the stronger requirement that $a \in j(A)$ is weakly universal for j accords well with our Conjecture for the following reason: Whenever $a \in j(A)$ is weakly universal for j and j also happens to be cofinal, we will have:

$$\forall x \, \exists f \, j(f)(a) = x \tag{49}$$

In other words, every set in the universe is expressible in the form $j(f)(a)$ for some function f. The function f may be thought of as an *approximation* to a blueprint for sets in V. This situation accords with the principle that *all* sets should arise from the dynamics of j (point (6) of the Conjecture).

The fact that the cofinal property together with the existence of a weakly universal element leads to (49) adds credibility to the cofinal property. Indeed, in a typical context, wherein j preserves functions, the property (49) on its own *implies* that j is cofinal: Assuming (49) holds, let b be a set; we show $b \in j(B)$ for some B. Let $f : A \to B$ be such that $j(f)(a) = b$. Since $j(f) : j(A) \to j(B)$, it follows that $b \in j(B)$. □

Proof of Theorem 9.107. Let $D = \{X \subseteq A \mid a \in j(X)\}$. As in the first part of Theorem 6.54, D is an ultrafilter (we do not claim yet that it is nonprincipal). Note that by Proposition 5.42, j preserves the empty set. Define $j_D : V \to V$ by $j_D(X) = X^A/D$.

Claim 1. j is naturally isomorphic to j_D. That is, there are, for all sets B, bijections $\overline{\phi}_B : j_D(B) \to j(B)$, natural in B. e

Proof of Claim 1. We first define an onto map $\phi_B : B^A \to j(B)$, and then show that ϕ_B induces a bijection $\overline{\phi}_B : B^A/D \to j(B)$. Define ϕ_B by

$$\phi_B(f) = j(f)(a).$$

We use the fact that a is a weakly universal element to show that ϕ_B is onto: Suppose $y \in j(B)$. By weak universality, there is $f : A \to B$ so that $y = j(f)(a) = \phi_B(f)$, as required.

Now define $\overline{\phi}_B : B^A/D \to j(B)$ by

$$\overline{\phi}_B([f]) = j(f)(a).$$

Like ϕ_B, $\overline{\phi}_B$ is onto. To see it is well-defined and 1-1, it suffices to show that, for all partial functions $f, g : A \to B$, $f \sim g$ if and only if $j(f)(a) = j(g)(a)$.

$$\begin{aligned} f \sim g \quad &\Leftrightarrow \quad E = E_{f,g} \in D \\ &\Leftrightarrow \quad a \in j(E) \\ &\Leftrightarrow \quad a \in \{z \mid j(f)(z) = j(g)(z)\} = E_{j(f),j(g)} \\ &\Leftrightarrow \quad j(f)(a) = j(g)(a). \end{aligned}$$

The proof that the $\overline{\phi}_B$ are components of a natural transformation is straightforward.

Claim 2. D is nonprincipal.

Proof of Claim 2. Let Z be a strong critical point for j. By Claim 1, Z is also a strong critical point for j_D. Assume D is principal. Then there is $u \in A$ such that $\{u\} \in D$; it follows that $D = \{X \subseteq A \mid u \in X\}$. For a contradiction, it suffices to exhibit a bijection $j_D(Z) \to Z$.

For each $z \in Z$, let $c_z : A \to Z$ be the constant function defined by $c_z(x) = z$ for all $x \in A$. Suppose $g : A \to Z$ is total and let $z = z_g = g(u)$. Let $E = E_{c_z,g} = \{x \in A \mid c_z(x) = g(x)\}$. Since $u \in E$, it follows that $E \in D$, and so $[c_z] = [g]$. Since every $[f] \in Z^A/D$ has a representative g that is total, and observing that for any such g, $[g] = [c_{z_g}]$, we have that $Z^A/D = \{[z] \mid z \in Z\}$. The map $[c_z] \mapsto z$ is therefore a 1-1 correspondence between Z^A/D and Z.

Claim 3. D is ω_1-complete.

Proof of Claim 3. It suffices to show that if $\{X_n \mid n \in \omega\}$ are disjoint subsets of A with $\bigcup_{n\in\omega} X_n \in D$, then for some $n \in \omega$, $X_n \in D$. Let $X = \bigcup_{n\in\omega} X_n$. Since $X \in D$ and since j preserves countable disjoint unions,

$$a \in j(X) \iff a \in \bigcup_{n\in\omega} j(X_n) \iff \exists n \in \omega\, a \in j(X_n) \iff \exists n \in \omega\, X_n \in D.$$

We have shown that $|A|$ is ω-measurable. □

We consider next an example that corresponds to Theorem 9.107. The large cardinal hypothesis required here is precisely the large cardinal that is derived from the hypotheses of Theorem 9.107—namely, existence of a measurable cardinal.

Example 9.109 Suppose κ is a measurable cardinal and D is a nonprincipal, κ-complete ultrafilter on κ. Define $j_D : V \to V$ as was done in the proof of Theorem 9.107: $j_D(X) = X^\kappa/D$. Defining j_D on functions $f : X \to Y$ by $j_D(f)(g) = [f \circ g]$, as before, turns j_D into a functor. The proof of the fact that j_D is 1-1 and preserves disjoint unions, intersections, and terminal objects is essentially the same (replacing ω with κ) as the corresponding verifications given in Example 6.60.

Claim 1. $[\mathrm{id}_\kappa] \in j_D(\kappa)$. Moreover,

$$D = \{X \subseteq \kappa \mid [\mathrm{id}_\kappa] \in j_D(X)\}. \tag{50}$$

In addition, $[\mathrm{id}_\kappa] \in j_D(\kappa)$ is a universal element for j_D.

Proof of Claim 1. The proof of (50) is like the corresponding proof for Theorem 6.54. The proof that $[\mathrm{id}_\kappa] \in j_D(\kappa)$ is a universal element for j_D follows the logic given in Example 6.61.

Claim 2. j_D preserves countable disjoint unions.

Proof of Claim 2. Suppose $X = \bigcup_{n\in\omega} X_n$ is a countable disjoint union. It is clear that the sets in $\{X_n^\kappa/D \mid n \in \omega\}$ are also disjoint and that $\bigcup_{n\in\omega}(X_n^\kappa/D) \subseteq X^\kappa/D$. To show the converse, let $f : \kappa \to X$. For each $n \in \omega$, let $S_n = \{\alpha < \kappa \mid f(\alpha) \in X_n\}$. By κ-completeness, some S_n belongs to D. It follows that $[f] \in X_n^\kappa/D \subseteq \bigcup_{n\in\omega}(X_n^\kappa/D)$.

Claim 3. j_D preserves equalizers.

Proof of Claim 3. Let $f, g : X \to Y$ and let $E = E_{f,g} = \{x \in X \mid f(x) = g(x)\}$. We show that $j_D(E) = E^\kappa/D$ is the equalizer $E_{j_D(f),j_D(g)}$ of $j_D(f), j_D(g)$:

$$\begin{aligned}[h] \in j_D(E) &\iff \{\alpha < \kappa \mid h(\alpha) \in E\} \in D \\ &\iff \{\alpha < \kappa \mid f(h(\alpha)) = g(h(\alpha))\} \in D \\ &\iff [h] \in E_{j_D(f),j_D(g)}.\end{aligned}$$

Claim 4. κ is a strong critical point for j_D.

Proof of Claim 4. By κ-completeness and the fact that D is nonprincipal, all members of D have size κ; in particular, all final segments $[\alpha, \kappa)$ belong to D. We show $\kappa < |j_D(\kappa)| = |\kappa^\kappa/D|$. Let $\langle f_\alpha \mid \alpha < \kappa\rangle$ be a sequence of κ functions $\kappa \to \kappa$. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = \sup\{f_\beta(\alpha) \mid \beta < \alpha\} + 1.$$

Then for each β, $\{\alpha \mid f_\beta(\alpha) < g(\alpha)\} \supseteq (\beta, \kappa)$ and $(\beta, \kappa) \in D$. Therefore, $|\kappa^\kappa/D| > \kappa$, as required.

We have shown that, assuming κ is measurable, the self-map $j_D : V \to V$ defined by $X \to X^\kappa/D$ is a Dedekind self-map having the properties listed in the hypotheses of Theorem 9.107.

It can also be shown (Corazza, 2010) that κ is a critical point of j_D and the least ordinal moved by j_D. Furthermore, since D is (at least) ω_1-complete, it can be shown that, if we identify κ^κ/D with its transitive collapse, $[\mathrm{id}_\kappa]$ is mapped to κ by the collapsing map; in this sense, $\kappa \in j_D(\kappa)$ itself is a weakly universal element[36] of j_D. □

[36] We note, however, that $j_D : V \to V$ is not a cofinal functor. The usual construction of an ultrapower embedding, however, provides an example in which κ is weakly universal element for the embedding, and the embedding *is* cofinal. Suppose we are given a nontrivial elementary embedding $i_U : V \to V^\kappa/U \cong N$, with U a normal measure on κ and N an inner model of ZFC. We treat V and N as categories, whose objects are the sets they contain and whose arrows are the functions between sets. It follows that i_U is a functor and also that $\kappa = [\mathrm{id}_\kappa]_U$. We first observe that $\kappa \in i_U(\kappa)$ is a weakly universal element for i_U: Suppose $x \in i_U(A)$ for some $A \in V$. Then for some $f : \kappa \to V$, $x = [f]$, and $i_U(A) = [c_A]$. Since $\{\alpha < \kappa \mid f(\alpha) \in c_A(\alpha)\} \in U$, there is $g : \kappa \to A$ (in V) so that $f \sim_U g$, and so $x = [g]$. It follows that $i_U(g)(\kappa) = x$:

$$\begin{aligned} i_U(g)(\kappa) = x &\Leftrightarrow \Big([c_g]([\mathrm{id}_\kappa]) = [g]\Big)^N \\ &\Leftrightarrow \{\alpha < \kappa \mid c_g(\alpha)(\mathrm{id}_\kappa(\alpha)) = g(\alpha)\} \in U \\ &\Leftrightarrow \{\alpha < \kappa \mid g(\alpha) = g(\alpha)\} \in U,\end{aligned}$$

and the last of these statements is true. We have shown $\kappa \in i_U(\kappa)$ is a weakly universal element of i_U. We wish to show that i_U is cofinal (using the natural definition of cofinal for two categories: A functor $G : \mathcal{C} \to \mathbf{Set}$ is *cofinal* if, for every set y, there is $c \in \mathcal{C}$

Example 9.109 together with Theorem 9.107 provides the following characterization: There is a measurable cardinal if and only if there is a class Dedekind self-map $j : V \to V$ that is a functor, preserves unions, intersections, equalizers, and terminal objects, and has a weakly universal element. In Blass (1976), Blass strengthens this characterization by using the concept of an *exact functor*: A functor is exact if it preserves all finite limits and colimits (see Mac Lane (1978) for definitions). The main result of (Blass, 1976) is the following:[37]

THEOREM 9.110 (Trňkova-Blass Theorem) *There exists a measurable cardinal if and only if there is an exact functor from V to V, definable in V, having a strong critical point.* □

We summarize our results related to Dedekind self-maps and measurable cardinals in the following corollary.

COROLLARY 9.111 (Measurable Cardinals and Dedekind Self-Maps) (ZFC−Infinity) *The following statements are equivalent.*

(1) *There is a measurable cardinal.*
(2) *There is a class Dedekind self-map $j : V \to V$ having the following properties:*
   (i) *j is a functor;*
   (ii) *j has a strong critical point;*
   (iii) *j preserves countable disjoint unions, intersections, equalizers, and terminal objects;*
   (iv) *there is a weakly universal element $a \in j(A)$ for j, for some set A.*
(3) *There is an exact functor $j : V \to V$, definable in V, having a strong critical point.*

In this section we have strengthened the properties of Dedekind self-maps $j : V \to V$ so that various large cardinals can be derived. Requiring j to satisfy a number of elementary preservation properties, in accord with *Preservation* and Conjecture point (1) (p. 45), had as a consequence the fact that the canonical critical point κ of j is inaccessible (Theorem 9.96); supplementing with a few more specialized preservation properties led to the conclusion that κ is ineffable (Theorem 9.103).

such that $y \in G(c)$). Suppose $x \in N$. We find $A \in V$ so that, in N, $x \in i_U(A)$. Let α be such that $x \in V_\alpha^N$. But now $x \in i_U(V_\alpha)$ since

$$i_U(V_\alpha) = V^N_{i_U(\alpha)} = V_{i_U(\alpha)} \cap N \supseteq V_\alpha \cap N = V_\alpha^N.$$

Now it is easy to check that

$$N = \{i_U(f)(\kappa) \mid f : \kappa \to V \text{ and } \kappa \in \text{dom } i_U(f)\}.$$

A related fact is that one cannot carry out a similar argument for any kind of elementary embedding $j : V \to V$; the argument breaks down because V cannot be represented as the transitive collapse of an ultrapower (for example, see Kanamori (1994), Proposition 5.7(e)). In fact, as is shown in Theorem 11.130, if j is a WA_0-embedding, then for *no* sets a, A for which $a \in j(A)$ is it the case that a is a universal element for j.

[37] The result proved in Blass (1976) is formulated differently, but is shown in Corazza (2010), Theorem 2.12, to be equivalent to the statement given here.

We then showed that Dedekind self-maps with these properties are realized in the canonically constructed model of ZFC+BTEE, obtained from an ω-Erdös cardinal, whose consistency strength is only slightly greater than that of ineffability.

Based on these results, we provisionally declare that existence of ineffable cardinals in V is justified by the fact that they are derivable from a Dedekind self-map on V equipped with preservation properties that are naturally motivated by our basic intuition concerning the characteristics of "the infinite."

We also strengthened the notion of a Dedekind self-map $j : V \to V$ in a different direction, guided by a combination of *Preservation* (Conjecture point (1)) and *Critical Point Dynamics* (Conjecture point (2)), by introducing a somewhat different set of preservation properties and also requiring existence of a weakly universal element $a \in j(A)$ for some set A. This approach led to the sharper result that existence of a measurable cardinal is equivalent to existence of a class Dedekind self-map equipped with these properties. The self-map j, which was built in Example 9.109 starting from a measurable cardinal κ, and which was shown to have all the properties listed in Theorem 9.107, has, in its cleanest form (see the comments following Example 9.109), the characteristic that $\kappa \in j(\kappa)$ is a critical point, a strong critical point, and a universal element for j. The requirement that there is $a \in j(A)$, which takes the form "$\kappa \in j(\kappa)$" in this case, is a convincing application of Critical Point Dynamics, as is the definition of the ultrafilter D by $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Again, based on these results, we provisionally declare the naturalness of measurable cardinals as they likewise arise from a strengthening of the concept of a Dedekind self-map $j : V \to V$ in accordance with intuitively compelling first principles.

In both cases (regarding existence of ineffable and measurable cardinals), our declarations are only provisional: The first principles that we identified earlier have only been partially realized in the work done so far. Our work so far shows that application of *Preservation* and *Critical Point Dynamics* brings to light several small types of large cardinals. We have yet to see how Conjecture points (3)–(6) might play a role in generating stronger large cardinal notions.

9.4 Emergence of a Blueprint in a Transitive Model of ZFC + BTEE

To take the next step, we demonstrate how even a canonical transitive model of ZFC + BTEE shows the first signs of the emergence of a blueprint, arising from the dynamics of the embedding. Our initial example will not fully satisfy all the requirements specified in Definition 4.34 but will clearly exhibit key features of a formal blueprint. We will then show, in the next section, how the theory ZFC + BTEE can be strengthened slightly to yield a blueprint—indeed a *strong* blueprint—for an interesting set that lives in the vicinity of the critical point.

The next theorem tells us that if we start with a canonical transitive model $(N, \in, j)$ of ZFC+BTEE with critical point κ, built from (L_ρ, I), where ρ is ω-Erdös and I is a set of indiscernibles of ordertype ω, obtained from the ω-Erdös property, then, working in the model $L_{j(\kappa)}$, there must exist a self-map $\ell : V_\kappa \to V_\kappa$ that exhibits important characteristics of a formal blueprint. The function ℓ will have the property that, for each $\mathrm{Coll}(\omega, <\kappa)$-generic G over $L_{j(\kappa)}$ (where $\mathrm{Coll}(\omega, <\kappa)$ is the Levy collapse) and each $x \in L_{j(\kappa)}$, there is an elementary embedding h, defined in $L_{j(\kappa)}[G]$, such that $h(\ell \upharpoonright \mu + 1)(\kappa) = x$, where $\mu < \kappa$ is such that $\ell \upharpoonright \mu + 1 \in$ dom h. This follows from the work in Cheng and Gitman (2015) in conjunction with observations in Corazza (2006) concerning models of remarkable cardinals derived from canonical models of ZFC + BTEE. We begin with several definitions from Cheng and Gitman (2015).

Definition 9.112 (Remarkable cardinals) A cardinal κ is *remarkable* if, in the $\mathrm{Coll}(\omega, <\kappa)$ forcing extension $V[G]$, for every regular cardinal $\lambda > \kappa$, there is a V-regular cardinal $\bar{\lambda} < \kappa$ and an elementary embedding $j : H^V_{\bar{\lambda}} \to H^V_\lambda$ with critical point γ such that $j(\gamma) = \kappa$.

Here, and for the rest of this section, "critical point" has its usual meaning, in the context of elementary embeddings, as the least ordinal moved by the embedding.

Definition 9.113 (Remarkable embeddings) In a $\mathrm{Coll}(\omega, <\kappa)$ forcing extension $V[G]$, an elementary embedding $h : H^V_{\bar{\lambda}} \to H^V_\lambda$ is $(\mu, \bar{\lambda}, \kappa, \lambda)$-*remarkable* if $\lambda > \kappa$ and $\bar{\lambda} < \kappa$ are V-regular, $\mathrm{crit}(h) = \mu$, and $h(\mu) = \kappa$.

Definition 9.114 (Laver property) Suppose κ is remarkable and $f : \kappa \to V_\kappa$ is a partial function.

(1) Suppose $\lambda > \kappa$ is regular, and $x \in H_\lambda$. The function $f : \kappa \to V_\kappa$ λ-*anticipates* x if, whenever G is $\mathrm{Coll}(\omega, <\kappa)$-generic over V, there is, in $V[G]$, a $(\mu, \bar{\lambda}, \kappa, \lambda)$-remarkable embedding $h : H_{\bar{\lambda}} \to H_\lambda$ with the following properties:

   (i) $f \restriction (\mu + 1) \in H_{\bar{\lambda}}$.
   (ii) $\mu \in \mathrm{dom}\ \ell$.
   (iii) $h\Big(f \restriction (\mu + 1)\Big)(\kappa) = x$.

(2) The function f has the *remarkable Laver property* if, for each regular $\lambda > \kappa$, each G that is $\mathrm{Coll}(\omega, <\kappa)$-generic over V, and each $x \in H_\lambda$, we have in $V[G]$ that f λ-anticipates x.

It is shown in Cheng and Gitman (2015) that, assuming that κ is remarkable, the partial function ℓ_W defined below has the remarkable Laver property: Let W be a well-ordering of V_κ of ordertype κ. Define (the partial function) $\ell_W : \kappa \to V_\kappa$ inductively as follows. Suppose $\ell_W \restriction \xi$ has been defined. If there is λ such that

$$1 \Vdash \text{"there is a set that } \ell_W \restriction \xi \text{ does not } \lambda\text{-anticipate"},$$

then $\ell_W(\xi)$ is the W-least $\mathbf{a}$ such that

$$1 \Vdash \ \lambda \text{ is least for which there is a set that } \ell_W \restriction \xi \text{ does not } \lambda\text{-anticipate, and } \ell_W \restriction \xi \text{ does not } \lambda\text{-anticipate } \mathbf{a}$$

Theorem 9.115 (Cheng and Gitman, 2015, Theorem 3.7) *If κ is a remarkable cardinal, then ℓ_W has the remarkable Laver property.*

Theorem 9.116 *Let $(N, \in, j)$ be the canonical transitive model of ZFC + BTEE with critical point κ, derived from M, I, where $M = L_\rho$, ρ is ω-Erdös, and $I \subseteq L_\rho$ is a set of indiscernibles of ordertype ω for L_ρ. Inside $L_{j(\kappa)}$ there is a self-map $\ell : V_\kappa \to V_\kappa$ with the property that, for each $x \in L_{j(\kappa)}$ and each G that is $\mathrm{Coll}(\omega, <\kappa)$-generic over $L_{j(\kappa)}$, there exist, in $L_{j(\kappa)}[G]$, $\mu, \bar{\lambda}, \lambda, h : H_{\bar{\lambda}} \to H_\lambda$ so that, in $L_{j(\kappa)}[G]$, h is elementary, $h(\mu) = \kappa$, and $h\Big(\ell \restriction (\mu + 1)\Big)(\kappa) = x$.*

Proof. It is shown in Corazza (2006), Theorem 3.10, that, under these hypotheses, $L_{j(\kappa)} \models$ "κ is remarkable".[38] Let W be the restriction of the canonical well-ordering of L to $L_\kappa \times L_\kappa$; certainly $W \in L_{j(\kappa)}$. Since $L_{j(\kappa)} \models$ "κ is inaccessible", $L_{j(\kappa)} \models L_\kappa = V_\kappa$. The proof of Theorem 3.7 in Cheng and Gitman (2015) shows that ℓ_W has the Laver property in $L_{j(\kappa)}$. Working in $L_{j(\kappa)}$, we define $\ell : V_\kappa \to V_\kappa$ by[39]

$$\ell(x) = \begin{cases} \ell_W(x) & \text{if } x \in \text{dom } \ell_W \\ x & \text{otherwise} \end{cases}$$

Let $x \in L_{j(\kappa)}$ and let G be $\text{Coll}(\omega, {<}\kappa)$-generic over $L_{j(\kappa)}$. In $L_{j(\kappa)}$, let $\lambda > \kappa$ be regular so that $x \in H_\lambda$. By Theorem 9.115, there is, in $L_{j(\kappa)}[G]$, a $(\mu, \bar{\lambda}, \kappa, \lambda)$-remarkable embedding $h : H_{\bar{\lambda}} \to H_\lambda$ having properties (i)–(iii) as above; in particular,

$$L_{j(\kappa)}[G] \models h\Big(\ell_W \restriction (\mu + 1)\Big)(\kappa) = x.$$

Since (in $L_{j(\kappa)}[G]$) $\ell \restriction (\mu + 1) = \ell_W \restriction (\mu + 1)$, the result follows. □

A crude summary of the result is that the canonical transitive model $(N, \in, j)$ of ZFC + BTEE with critical point κ, derived from L_ρ, I yields, within the model $L_{j(\kappa)}$, a kind of approximation $\ell : L_\kappa \to L_\kappa$ to a blueprint for $L_{j(\kappa)}$. It is only an approximation for a number of reasons. First of all, the decoding that allows us to realize a given x as $h\Big(\ell \restriction (\mu+1)\Big)(\kappa)$ occurs only in forcing extensions (via Levy collapse); therefore, the *Decoding* requirement of Definition 4.34 is only approximately satisfied. In addition, most of the other requirements listed in Definition 4.34 are not quite satisfied in the present context. For instance, although the various $(\mu, \bar{\lambda}, \kappa, \lambda)$-remarkable embeddings have the desired preservation properties, there is no way to collect them into a class $\mathcal{E}$ meeting the requirements of Definition 4.34; nor is it clear that this class of embeddings is compatible—in any of the senses that were discussed in Remark 4.35—with the ambient BTEE-embedding $j : N \to N$.

Historically, of course, the version of the Laver property that we see here is a deliberate weakening of the concept of a Laver function for supercompact cardinals to the weaker context of remarkable cardinals, but in our treatment, we propose a different perspective. We are ascending the hierarchy of large cardinals by imposing ever stronger requirements on a Dedekind self-map $j : V \to V$, in accordance with a set of adopted intuitive principles. The result of Theorem 9.116 can be viewed as a first sign that the concept of *Blueprint* is realizable when j realizes enough of these principles, with the recognition that the realization of *Blueprint* in this case is far from ideal. We conjecture that if the properties of j are strengthened further, the *Blueprint* principle will be realized more fully.

To take the next step, which will lead to a much more satisfactory realization of *Blueprint*, we observe an important difference between the j obtained in the

[38] We note here that the definition of remarkability used in Corazza (2006) differs from the one we are using in this paper. However, Theorem 2.8 of Cheng and Gitman (2015) shows that the two definitions are equivalent.

[39] The definition given here fails to satisfy property (1) in the formal definition of blueprint (Definition 4.34). It is easy enough in this case to artificially define ℓ in such a way that it becomes a co-Dedekind self-map, but, as the remarks following the proof indicate, ℓ fails to be a blueprint in the formal sense for more substantial reasons.

BTEE model described in Theorem 9.105 and the j of Theorem 9.111(3): The map in Theorem 9.111(3) is required to be definable in V, while the map described in Theorem 9.105 *cannot* be definable in V. Note that a BTEE-embedding $j : V \to V$ is easily seen to be an exact functor, but, in apparent contrast to Theorem 9.111, in a ZFC + BTEE universe V, it is not generally the case that a measurable cardinal exists, since only an ω-Erdös cardinal is needed to produce a model of this theory. The reason for this apparent discrepancy is that the ultrafilter D derived from an exact functor with a strong critical point—and which is a witness to existence of a measurable cardinal—has the form $\{X \subseteq A \mid \phi(X)\}$, where ϕ is a formula that depends on j—see Corazza (2010), pp. 72–74; in the absence of definability, D cannot be shown to be a set.[40]

Since the requirement that a Dedekind self-map $j : V \to V$ be an elementary embedding is the fullest possible realization of *Preservation*, and since our application of intuitive principles has already led to a justification of measurable cardinals, we have sufficient motivation to accept as true the theory ZFC + BTEE plus one additional axiom that asserts that the ultrafilter derived from $\mathbf{j}$ and κ *exists* (where $\mathbf{j}$ is the BTEE-embedding and κ is its critical point). We call this additional axiom the *Measurable Ultrafilter Axiom*, or *MUA*.

> **Measurable Ultrafilter Axiom (MUA)**. The class $\{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$ is a set.

As described before, we understand here that κ, which denotes the critical point of $\mathbf{j}$, is a constant that has been added by definitional extension.

As described in Section §2, the theory ZFC + BTEE + MUA is strong enough to produce many measurable cardinals, and a transitive model of the theory is obtainable from a 2^κ-supercompact (where κ is the critical point of the MUA-embedding). We state these facts more precisely in the following theorem.

Theorem 9.117 (Corazza, 2006, Propositions 9.9, 9.10)

(1) *Suppose $\langle M, \in, j\rangle$ is a model of* ZFC + BTEE + MUA *and κ is the critical point of j. Then κ is a measurable cardinal having Mitchell order $> \kappa$.*
(2) *If there is a cardinal κ that is 2^κ-supercompact, there is a transitive model of* ZFC + BTEE + MUA.

Reasoning as before, since an MUA-embedding is an elementary embedding, it gives fullest possible expression to the *Preservation* principle. Also, the derived measurable ultrafilter is an expression of *Critical Point Dynamics*. However, unlike earlier formulations of Dedekind self-maps $j : V \to V$ that we have seen, any MUA-embedding produces a blueprint for an important set living in the vicinity of its critical point. At the same time, we will see that MUA-embeddings do not in general embody the principles (3), (5), or (6) in the Dedekind Self-Map Conjecture; these limitations will suggest further enhancements that will lead toward a satisfactory

[40] Existence of the ultrafilter D defined in Theorem 9.107 likewise depends on definability of j. However, in this case we cannot claim that a BTEE-embedding has all the properties listed in the theorem: As shown in Theorem 2.17 of Corazza (2010) (see also the remarks at the end of the footnote beginning on p. 93), for a BTEE-embedding $j : V \to V$ with critical point κ, it is not generally true that $\kappa \in j(\kappa)$ is weakly universal for j.

realization of even these points from the Conjecture and toward an extremely strong kind of Dedekind self-map. We develop these points in the next section.

§10 The theory ZFC + BTEE + MUA If $\langle M, \in, j\rangle$ is a model of ZFC + BTEE + MUA, where $\kappa = \mathrm{crit}(j)$, the familiar argument shows that the derived ultrafilter $U_j = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ is a normal measure on κ; moreover, Theorem 9.117(1) tells us that U_j has high Mitchell order. We can use this fact to obtain from j and κ a blueprint $(\ell, \kappa, \mathcal{E})$ for the stage $V_{\kappa+1}$ of the universe, where $\ell : V_\kappa \to V_\kappa$ is the blueprint map and $\mathcal{E}$ consists of restrictions of certain ultrapower elementary embeddings. In particular, we will show that for each $X \subseteq V_\kappa$, there is a normal measure U on κ such that if i_U is the canonical embedding derived from U, $i_U(\ell)(\kappa) = X$. In this way, every set in the stage $V_{\kappa+1}$ can be seen as arising from or being generated by the interplay of κ, j, and ℓ, where ℓ is encoded and decoded by $\mathcal{E}$.

The essence of the construction of ℓ is a $V_{\kappa+1}$-*Laver function*. Recall from Section 2.4 that, for any set X, a function $f : \kappa \to V_\kappa$ is X-*Laver at* κ if, for each $x \in X$, there is a normal measure U on κ such that $i_U(f)(\kappa) = x$.

In the present setting, the function f will be defined using a variation Laver's original construction 1968, encoding information about all possible normal measures over κ.[41] We define f and then explain how ℓ is obtained from f. We first define a formula $\psi(g, x, \lambda)$ that is needed both in the definition of f and in the proof that f has the desired properties:

$$\psi(g, x, \lambda) : \quad g : \lambda \to V_\lambda \wedge x \subseteq V_\lambda \wedge \text{“for all normal measures } U \text{ on } \lambda,\ i_U(g)(\lambda) \neq x\text{”}.$$

When $\psi(g, x, \lambda)$ holds true, it means that g is *not* a $V_{\lambda+1}$-Laver sequence at λ: Some subset x of V_λ cannot be computed as $i_U(g)(\lambda)$ for any choice of U. We can now define f:

$$f(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } f \restriction \alpha \text{ is } V_{\alpha+1}\text{-Laver at } \alpha, \\ x & \text{otherwise, where } x \text{ satisfies } \psi(f \restriction \alpha, x, \alpha). \end{cases} \tag{51}$$

The definition tells us that $f(\alpha)$ has nonempty value just when the restriction $f \restriction \alpha$ is *not* $V_{\alpha+1}$-Laver at α, and in that case, its value is a witness to non-Laverness.

Theorem 10.118 ($V_{\kappa+1}$-Laver Functions Under MUA) *The function f defined in (51) is a $V_{\kappa+1}$-Laver function at κ.*

Proof. Let $j : V \to V$ be the Dedekind self-map, with critical point κ, given to us in a model of ZFC + BTEE + MUA. Suppose f is not $V_{\kappa+1}$-Laver at κ, so, in particular, for some y, $\psi(f, y, \kappa)$ holds. We consider $j(f) : j(\kappa) \to V_{j(\kappa)}$.

First we show that $j(f) \restriction \kappa = f$. For each $\alpha < \kappa$, we have $j(f)(\alpha) = j(f)(j(\alpha)) = j\,(f(\alpha))) = f(\alpha)$ (since $f(\alpha) \in V_\kappa$). By elementarity, $j(f)$ has the same definition as f. In particular, we have that, for each $\alpha < j(\kappa)$,

$$j(f)(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \text{ is not a cardinal or } j(f) \restriction \alpha \text{ is } V_{\alpha+1}\text{-Laver at } \alpha, \\ x & \text{otherwise, where } x \text{ satisfies } \psi(j(f) \restriction \alpha, x, \alpha). \end{cases}$$

[41] The construction and proof given here are essentially the same as those found in Hamkins (2002) and in Corazza (1998), Proposition 1.8.

In particular, since $f = j(f) \restriction \kappa$ is not $V_{\kappa+1}$-Laver, computation of $j(f)(\kappa)$ uses the second clause of the definition for $j(f)$ and $\psi(j(f) \restriction \kappa, j(f)(\kappa), \kappa)$ is true. Therefore, $x = j(f)(\kappa)$ is a witness to the fact that $f = j(f) \restriction \kappa$ is not $V_{\kappa+1}$-Laver. Recall from the definition of ψ that any such witness x must be a subset of V_κ, so we have that $j(f)(\kappa) \subseteq V_\kappa$.

Let $D = U_j$ be the normal measure derived from j; that is, $D = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. By MUA, D is a *set*. Let $i = i_D : V \to V^\kappa/D \cong N$ be the canonical embedding, and define $k : N \to V$ by $k([h]) = j(h)(\kappa)$. One can show (Jech, 1978) that k is an elementary embedding with critical point $> \kappa$ and makes the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{\ \ j\ \ } & V \\ & \searrow^{i_D} & \uparrow k \\ & & N \end{array} \tag{52}$$

By diagram (52), we have

$$\begin{aligned} j(f)(\kappa) &= (k \circ i)(f)(\kappa) \\ &= \big(k(i(f))\big)(k(\kappa)) \\ &= k\big(i(f)(\kappa)\big). \end{aligned}$$

Now since $j(f)(\kappa) \subseteq V_\kappa$ and $\mathrm{crit}(k) > \kappa$, $k\big(j(f)(\kappa)\big) = j(f)(\kappa)$. Since k is 1-1 and

$$k\big(j(f)(\kappa)\big) = j(f)(\kappa) = k\big(\big(i(f)(\kappa)\big),$$

it follows that

$$j(f)(\kappa) = i(f)(\kappa).$$

The import of this last equation is that, while it is claimed that $\psi(j(f) \restriction \kappa, j(f)(\kappa), \kappa)$ holds true, we have just exhibited a normal measure D on κ such that $i_D(f)(\kappa) = j(f)(\kappa)$, and we have a contradiction. We conclude, therefore, that f is $V_{\kappa+1}$-Laver after all. □

We turn to the construction of ℓ:

$$\ell(x) = \begin{cases} f(x) & \text{if } x \text{ is an ordinal} < \kappa, \\ x & \text{otherwise.} \end{cases} \tag{53}$$

We now show that ℓ is the blueprint map for a blueprint $(\ell, \kappa, \mathcal{E})$ for $V_{\kappa+1}$, assuming MUA. We will describe the members of $\mathcal{E}$ in a more precise way in the discussion in Remark 10.120, below.

THEOREM 10.119 (Existence of Blueprint Self-Maps under MUA) (ZFC+BTEE+MUA). *Suppose $j : V \to V$ is a Dedekind self-map given by a model of* ZFC + BTEE + MUA, *with critical point κ. Then the function $\ell : V_\kappa \to V_\kappa$ defined in (53) has the following property:*

$$\forall X \subseteq V_\kappa\ \exists U\ (U \text{ is a normal measure on } \kappa \text{ and } i_U(\ell)(\kappa) = X). \tag{54}$$

Moreover, ℓ is a co-Dedekind self-map.

For the rest of this section, we shall say that a self-map having the property (54) has the *Laver property at* κ.

Proof. Note that for all $\alpha < \kappa$, $f(\alpha) = \ell(\alpha)$. In particular, if $T = \{\alpha < \kappa \mid f(\alpha) = \ell(\alpha)\}$ and U is a normal measure on κ, then $T \in U$. Therefore, if $i = i_U$ is the canonical embedding,

$$\kappa \in i(T) = i\left(\{\alpha < \kappa \mid f(\alpha) = \ell(\alpha)\}\right) = \{\alpha < i(\kappa) \mid i(f)(\alpha) = i(\ell)(\alpha)\}.$$

It follows that, for every normal measure U on κ, $i_U(f)(\kappa) = i_U(\ell)(\kappa)$. It follows that ℓ has the Laver property at κ since f does.

To see that ℓ is a co-Dedekind self-map, it is sufficient to show that, whenever $f : \kappa \to V_\kappa$ is $V_{\kappa+1}$-Laver for subsets of V_κ, for each $x \in V_\kappa$, $|f^{-1}(x)| = \kappa$. Given $x \in V_\kappa$, let $T_x = \{\alpha < \kappa \mid f(\alpha) = x\}$. Let U be a normal measure on κ such that $i_U(f)(\kappa) = x$. Note that $i_U(x) = x$ since $x \in V_\kappa$. Then since $\kappa \in \{\alpha < i(\kappa) \mid i(f)(\alpha) = x\} = i(T_x)$, it follows that $T_x \in U$. Therefore $|f^{-1}(x)| = |T_x| = \kappa$. □

Remark 10.120 (The Blueprint Coder $\mathcal{E}$) We describe the blueprint coder for the blueprint of $V_{\kappa+1}$ in more detail. Suppose $j : V \to V$ is a Dedekind self-map given by a model of ZFC + BTEE + MUA, with critical point κ. For each normal measure U on κ, let $i_U : V \to V^\kappa/U \cong M$ be the canonical embedding and let $\overline{i_U} = i_U \restriction V_{\kappa+1} : V_{\kappa+1} \to N = (V_{i_U(\kappa)+1})^M$. We define $\mathcal{E}$ by

$$\mathcal{E} = \{\overline{i_U} \mid U \text{ is a normal measure on } \kappa\}.$$

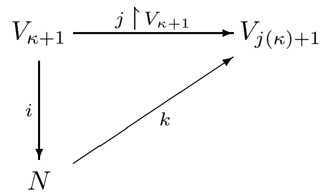
Taking this step allows us to formally define $\mathcal{E}$ as a class; this step is necessary since, formally speaking, we cannot collect all embeddings of the form $i : V \to M$ into a single class. Nothing is lost in restricting the embeddings in this way: Tracing through the proofs above, it is straightforward to verify that a $V_{\kappa+1}$-Laver function can be obtained by making use of $\mathcal{E}$ in place of the full elementary embeddings $i : V \to M$ that were used previously. Moreover, one verifies that ℓ is defined in the same way as before. Note that $\mathcal{E}$ is the same as the class $\mathcal{E}_\kappa^{\theta_m}$ defined in Section 2.4, which was shown there to precisely capture the notion of a measurable cardinal.

We show that $(\ell, \kappa, \mathcal{E})$ is a blueprint for $V_{\kappa+1}$; we use the criteria specified in Definition 4.34 (p. 39). The map ℓ satisfies (1) because it is a co-Dedekind self-map. For (2), note that $V_\kappa^{V_\kappa} \subseteq V_{\kappa+1} = \text{dom } i$, for each $i \in \mathcal{E}$. For (2)(a)–(e), let us define $\mathcal{E}_0$ by

$$\mathcal{E}_0 = \{\bar{i} \restriction V_\kappa^{V_\kappa} \mid \bar{i} \in \mathcal{E}\}.$$

Let $i_0 \in \mathcal{E}_0$ and let $i : V \to M$ be such that $\bar{i} = i \restriction V_{\kappa+1}$ and $i_0 = \bar{i} \restriction V_\kappa^{V_\kappa}$. For (2)(a),(b),(d), notice $i_0 : V_\kappa^{V_\kappa} \to T_i^{T_i}$, where $T_i = V_{i(\kappa)}^M$, and we have $V_\kappa \subseteq T_i$ and $\kappa \in \text{dom } i_0(\ell)$. For (2)(e), note that since each map in $\mathcal{E}_0$ is the restriction of an elementary embedding $V_{\kappa+1} \to N$, each is Σ_0-preserving.

Verification of the compatibility requirement makes use of the criteria in Remark 4.35. Applying those criteria to the present context, we must find $i : V_{\kappa+1} \to N \in \mathcal{E}$ and $k : N \to V_{j(\kappa)+1}$ so that $k \restriction \left(V_{i(\kappa)}^N\right)^{V_{i(\kappa)}^N}$ is Σ_0-preserving and $j \restriction V_{\kappa+1} = k \circ i$.



However, as was shown in Section 2.4 (p. 19), the class $\mathcal{E}_\kappa^{\theta_m}$ is *locally compatible* with j, and this property ensures existence of the necessary embeddings i and k described above.

For (3), we must argue that ℓ is definable from $\mathcal{E}, j, \kappa$. Certainly, the $V_{\kappa+1}$-Laver function that we defined is derived from $\mathcal{E}, j, \kappa$, and ℓ is definable from this function.

Finally, to prove (4), suppose $X \subseteq V_\kappa$. Since, from the previous theorem, ℓ has the Laver property at κ, we can find U such that $X = i_U(\ell)(\kappa) = \overline{i_U}(\ell)(\kappa)$. Since $\ell \in V_\kappa^{V_\kappa}$, it follows that $\overline{i_U} \restriction V_\kappa^{V_\kappa}(\ell)(\kappa) = X$ as well, and $\overline{i_U} \restriction V_\kappa^{V_\kappa} \in \mathcal{E}_0$. □

Next, we show that there is a strong blueprint for $V_{\kappa+1} - V_\kappa$, the collection of subsets of V_κ of size κ. This strong blueprint shows the generating and collapsing effects of ℓ and its dual, ℓ^{op}, which will be defined to be a certain section of ℓ.

We remark here that we cannot do much better than $V_{\kappa+1} - V_\kappa$. In fact, as we now show, there is no way to define a section s of ℓ with the property that, for each $x \in V_\kappa$, there is $i \in \mathcal{E}_0$ for which $i(s)(x) = \kappa$. Let $s : V_\kappa \to V_\kappa$ be a section of ℓ and let $x \in V_\kappa$. Let $i = i_U$ be a canonical embedding, as usual. Then

$$i(s)(x) = i(s)\,(i(x)) = i(s(x)) = s(x) \neq \kappa.$$

The fact that elements of V_κ are "left out" of the dynamics of the strong blueprint accords with our expectation: For any MUA-embedding $j : V \to V$ with critical point κ, we are *given* κ, and thereby V_κ as well, so the only sets that need to "return" to κ are those that lie outside of V_κ.

EXAMPLE 10.121 (Strong Blueprint for $V_{\kappa+1} - V_\kappa$) Let $j : V \to V$ be a Dedekind self-map given by a model of ZFC+BTEE+MUA, with critical point κ. Let $(\ell, \kappa, \mathcal{E})$ be a blueprint for $V_{\kappa+1}$. We define the dual map ℓ^{op} so that $(\ell, \ell^{\text{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V_{\kappa+1} - V_\kappa$.

We define $\ell^{\text{op}} : V_\kappa \to V_\kappa$ as follows:

$$\ell^{\text{op}}(x) = \begin{cases} \alpha & \text{if } \alpha \text{ is the least ordinal in } \ell^{-1}(x), \text{ if there is one,} \\ y & \text{otherwise, where } y \text{ is an arbitrary element of } \ell^{-1}(x). \end{cases}$$

As $\ell^{\text{op}}(x) \in \ell^{-1}(x)$ for each $x \in V_\kappa$, it is obvious that ℓ^{op} is a Dedekind self-map and a section of ℓ. We point out here that, while the definition of $\ell^{\text{op}}(x)$ requires finding the least ordinal α satisfying a certain formula—namely that α belongs to $\ell^{-1}(x)$—this formula is *not* a **j**-formula. This is important because, from the theory $\text{ZFC} + \text{BTEE} + \text{MUA}$, it is not in general possible to compute the least ordinal for which a **j**-formula holds; see Corazza (2006).

Claim. Suppose $X \in V_{\kappa+1} - V_\kappa$, U is a normal measure on κ, and $i = i_U$ is the canonical embedding with critical point κ for which $i(\ell)(\gamma) = X$. Then $\gamma \geq \kappa$.

Note that the Claim (once proven) continues to hold true if i_U is replaced with $\overline{i_U}$.

Proof of Claim. Suppose $\alpha < \kappa$. We compute $i(\ell)(\alpha)$, using the fact that $i(\alpha) = \alpha$:

$$i(\ell)(\alpha) = i(\ell)\,(i(\alpha)) = i(\ell(\alpha)) = \ell(\alpha) \in V_\kappa.$$

The fact that $i(\ell(\alpha)) = \ell(\alpha)$ follows because $\ell(\alpha) \in V_\kappa$ and i is the identity on V_κ. We have shown that if $X \in V_{\kappa+1} - V_\kappa$ and $i(\ell)(\gamma) = X$, then $\gamma \geq \kappa$. □

We verify the main property of ℓ^{op}: Let $X \in V_{\kappa+1} - V_\kappa$. Let U be a normal measure on κ and $i = i_U$ the canonical embedding so that $i(\ell)(\kappa) = X$. Note that, by elementarity, $i\left(\ell^{\mathrm{op}}\right)$ is defined, for each $x \in V_{i(\kappa)}$, by

$$i\left(\ell^{\mathrm{op}}\right)(x) = \begin{cases} \alpha & \text{if } \alpha \text{ is the least ordinal in } i(\ell)^{-1}(x), \text{ if there is one,} \\ y & \text{otherwise, where } y \text{ is an arbitrary element of } i(\ell)^{-1}(x). \end{cases}$$

By the claim, κ is the least ordinal in $i(\ell)^{-1}(X)$. By definition of $i(\ell)^{\mathrm{op}}$, it follows that $i(\ell)^{\mathrm{op}}(X) = \kappa$. Again, note that the same argument goes through if i_U is replaced by $\overline{i_U}$.

We can now formally establish that $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V_{\kappa+1} - V_\kappa$ by verifying the properties in Definition 4.36. We have already shown that $(\ell, \kappa, \mathcal{E})$ is a *blueprint* for X, where $X = V_{\kappa+1} - V_\kappa$ and where $\mathcal{E}_0$ is the set of restrictions of elements of $\mathcal{E}$ to $V_\kappa^{V_\kappa}$. What remains is to establish the following points, and these were demonstrated in the paragraphs above:

(A) dom $\ell^{\mathrm{op}} =$ dom ℓ.
(B) ℓ^{op} is a section of ℓ.
(C) For each $i \in \mathcal{E}_0$, dom $i(\ell^{\mathrm{op}}) =$ dom $i(\ell)$
(D) For every $x \in X$, there is $i \in \mathcal{E}_0$ such that $i(\ell^{\mathrm{op}})(x) = \kappa$. □

We arrived at the theory ZFC+BTEE by noticing (1) the needed strengthening of a Dedekind self-map $j : V \to V$ to produce an infinite set was obtained by requiring j to have fairly natural preservation properties; (2) by strengthening these preservation properties further, certain large cardinals could be derived; (3) the strongest kind of preservation possible is obtained when j is an elementary embedding, and the theory ZFC+BTEE is the formal assertion of the existence of such an embedding from V to V.

In our initial study of Dedekind self-maps, we found they exihibited not only interesting preservation properties, but led naturally to the concept of a nonprincipal ultrafilter. Generalizing these ideas led to an example and corresponding theorems in which a Dedekind self-map $j : V \to V$ exhibits strong preservation properties and a nonprincipal ultrafilter plays a key role. The results in this case provided motivation for the existence of a measurable cardinal. The construction of the ultrafilter in this case could not be carried out directly in the theory ZFC + BTEE because of definability restrictions, and so we were led to postulate a supplementary axiom to ZFC + BTEE, namely, MUA, which asserts that the ultrafilter naturally derived from a BTEE-embedding exists as a set. The strengthened theory ZFC + BTEE + MUA implies existence of many measurable cardinals and also realizes more fully the points of our Dedekind Self-Map Conjecture (p. 45)—in particular, we showed how a blueprint for $V_{\kappa+1}$ and a strong blueprint for $V_{\kappa+1} - V_\kappa$ emerge from the interaction between an MUA-embedding with its critical point.

However, one point from the Conjecture (point (3)) that we have not yet encountered in our formulations of properties of Dedekind self-maps is the role of restrictions of j in generating the critical sequence for j. Although it is true that the "generating" effect of j, in the theory ZFC + BTEE + MUA, was captured nicely by the blueprint and strong blueprint that are derived from j, it is still natural to ask about the properties of the sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ and the role of restrictions of j.

This topic reveals limitations in the theory ZFC + BTEE + MUA. We mention some known results (Corazza, 2006) and introduce some new refinements.

(1) *Critical sequence may not exist.* In the theory ZFC + BTEE + MUA, the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ cannot be shown to "exist," even as a j-class, without supplementing the theory with additional axioms. (One cannot even guarantee that whenever n is a nonstandard integer in the theory, $j^n(\kappa)$ exists as a set.) It can be shown that, for each particular (metatheoretic) natural number n, the theory proves that the sequence $\langle \kappa, j(\kappa), \ldots, j^n(\kappa)\rangle$ exists as a set. On the other hand, whenever we work in a *transitive model* of ZFC + BTEE + MUA, this problem is corrected, and the critical sequence can indeed be shown to be a j-class in the model.

(2) *Stages $V_{j^n(\kappa)}$ may not form an elementary chain.* The theory ZFC + BTEE + MUA shows, for each particular (metatheoretic) natural number n, that $V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)}$, but the sequence of models $V_\kappa, V_{j(\kappa)}, V_{j^2(\kappa)}, \ldots$ cannot be shown to be a j-class. Once again, this sequence *can* be shown to be a j-class inside any transitive model of the theory. However, even inside such a transitive model, without additional axioms, the reasonable conjecture $V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)} \prec \ldots \prec V$ cannot be proven.

(3) *Boundedness of the critical sequence is undecidable.* A natural question, which the theory ZFC + BTEE + MUA cannot answer, is whether the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is bounded. As was mentioned before, assuming existence of a 2^κ-supercompact cardinal, there is a transitive model of ZFC + BTEE + MUA; in that model, the critical sequence is bounded. On the other hand, letting σ denote the sentence "the critical sequence is unbounded," any I_3 embedding $i : V_\lambda \to V_\lambda$ (with critical point κ and with λ a limit greater than κ) gives rise to a transitive model $(V_\lambda, \in, i)$ of ZFC + BTEE + MUA + σ.

(4) *Restrictions $j \upharpoonright V_{j^n(\kappa)}$ $(n \geq 1)$ may not exist.* The restriction $j \upharpoonright V_\kappa$ can be shown to exist as a set in ZFC + BTEE + MUA (it is equal to id_{V_κ}); using elementarity of j, one can show that $V_\kappa \prec V_{j(\kappa)}$, and hence that $j \upharpoonright V_\kappa : V_\kappa \to V_{j(\kappa)}$ is an elementary embedding. However, it is not possible to show that restrictions of j to $V_{j^n(\kappa)}$, for $n \geq 1$, exist as sets in ZFC + BTEE + MUA. In fact, the theory ZFC + BTEE + $\exists g\,(g = \mathbf{j} \upharpoonright \mathbf{j}(\kappa))$ is sufficient to prove the consistency of ZFC + BTEE + MUA: Let $j : V \to V$ be the embedding in a model of ZFC + BTEE + MUA and let $\kappa = \mathrm{crit}(j)$, and assume $j \upharpoonright V_{j(\kappa)}$ is a set. Let $g = j \upharpoonright V_{j(\kappa)}$. Since $\mathcal{P}(\mathcal{P}(\mathcal{P}(\kappa))) \in V_{j(\kappa)}$, one can define a 2^κ-supercompactness measure U_g by

$$U_g = \{X \subseteq P_\kappa 2^\kappa \mid g[2^\kappa] \in g(X)\},$$

which ensures that κ is 2^κ-supercompact. But this degree of supercompactness has been shown (Corazza, 2006, Proposition 9.10) to be sufficient to build a transitive model of the theory ZFC + BTEE + MUA. By Gödel Incompleteness, therefore, one cannot prove from ZFC + BTEE + MUA the existence even of $\mathbf{j} \restriction V_{\mathbf{j}(\kappa)}$ (see Lemma 10.122(ii) and the remarks following).

The limitation described in (4) holds the key to pushing beyond ZFC + BTEE + MUA toward a theory in which more points of the Dedekind Self-Map Conjecture are realized and the limitations described in (1)–(4) above can be removed. It can be shown that, as we consider extensions of ZFC + BTEE in which axioms asserting existence of restrictions of j to ever larger sets, we arrive at theories having ever stronger large cardinal consequences. The limit of this direction of generalization is the assertion that the restriction of j to *every* set exists as a set. We consider this very strong extension of ZFC + BTEE in the next section.

To close this section, we review what is known about various strengthenings of ZFC + BTEE, obtained by adding axioms that assert existence of a restriction of $\mathbf{j}$ to some set. Each of the theories mentioned below consists of ZFC + BTEE plus some statement of the form "$\mathbf{j} \restriction A$ is a set," for some set A. Most of these theories are stronger than ZFC + BTEE + MUA (cf. Corazza (2006)).

(A) The theory ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction \kappa^+)$ has consistency strength at least that of a strong cardinal.
(B) The theory ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction \mathcal{P}(\kappa))$ has consistency strength at least that of a Woodin cardinal. Moreover, from this theory, it is possible to derive the axiom MUA: Let $g = j \restriction \mathcal{P}(\kappa)$ and let $U = \{X \subseteq \kappa \mid \kappa \in g(X)\}$.
(C) The theory ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction \mathbf{j}(\kappa))$ is strong enough to prove the consistency of the theory ZFC + BTEE + MUA, as mentioned above.
(D) The theory ZFC + BTEE + $\exists z\,\big(z = \mathbf{j} \restriction \mathcal{P}(\mathcal{P}(\mathbf{j}(\kappa)))\big)$ directly proves κ is huge with κ huge cardinals below it.[42]
(E) The theory ZFC + BTEE + $\exists\lambda\exists z\,(\lambda$ is an upper bound for the critical sequence and $z = \mathbf{j} \restriction \lambda)$ is inconsistent.

For any set X for which $|X| \leq \kappa$, the restriction $j \restriction X$ does exist as a set in ZFC+BTEE+MUA (however, note that restrictions of j to sets of larger cardinality require stronger axioms, as (A) demonstrates). This follows from two observations, which we prove below.

LEMMA 10.122 *Suppose T is an extension of the theory* ZFC + BTEE *and $j : V \to V$ is the embedding with critical point κ. Suppose $j \restriction X$ can be proven to exist (as a set) in the theory T. Then:*

(i) *For any $Y \subseteq X$, $j \restriction Y$ is also a set.*
(ii) *For any Y for which there is a bijection $X \to Y$, $j \restriction Y$ is also a set.*

Proof of (i). Suppose $Y \subseteq X$ and let $i = j \restriction X$. Note that

$$j \restriction Y = \{(u, v) \mid u \in Y \text{ and } i(u) = v\},$$

which is a set by ordinary Replacement. □

[42] We note here that the statement $\exists z\ (z = \mathbf{j} \restriction \mathbf{j}(\kappa))$ is already enough to obtain a blueprint for $V_{\kappa+2}$. A proof can be found in Theorem 86 of Corazza (2016).

Proof of (ii). Suppose $f : X \to Y$ is a bijection. By elementarity, $j(f) : j(X) \to j(Y)$ is also a bijection. Let $i = j(f)$ and $k = j \restriction X$, both of which are sets in T. We have

$$\begin{aligned} j \restriction Y &= \{(y, z) \mid z = j(y)\} \\ &= \{(y, z) \mid \exists x \in X\, y = f(x) \text{ and } z = i(k(x))\}. \end{aligned}$$

and the last expression is a set by ordinary Replacement. □

A consequence of (ii) is that the following theories are equivalent: ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction V_{\kappa+1})$ and ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction P(\kappa))$. Using the fact that $j(\kappa)$ is inaccessible, the following theories are also equivalent: ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction V_{\mathbf{j}(\kappa)})$ and ZFC + BTEE + $\exists z\,(z = \mathbf{j} \restriction \mathbf{j}(\kappa))$.

As our sampling of results suggests, the large cardinal consequences of asserting the existence of restrictions of j to sets X increase in strength as those sets X increase in size. Part (E) shows one limitative result in this direction. When the critical sequence is unbounded, however, we are free to require restrictions of j to sets of any size without introducing inconsistency. The next section explores this possibility.

§11 Wholeness Axiom Embeddings $j : V \to V$. Our analysis of the embedding $j : V \to V$ that we get from a model of ZFC + BTEE + MUA shows that our strategy for strengthening the notion of a Dedekind self-map $V \to V$ based on observations we have made about set Dedekind self-maps, in Properties of a Dedekind Self-Map (p. 118), has been reasonably successful, as far as it goes: We have arrived at a formulation of a Dedekind self-map of the universe whose properties ensure the existence of many measurable cardinals. However, we have yet to tap the full potential of these properties. For example, our blueprint for generating sets (point (4) of Properties) has taken us only up to $V_{\kappa+1}$. Also, point (2) in Properties suggests that the critical sequence derived from j plays a special role (for set Dedekind self-maps, it forms a blueprint for ω) and that the critical sequence "emerges" from a sequence of restrictions of the Dedekind self-map under consideration. However, under MUA, the critical sequence for j is not even formally defined. Worse, it is not possible to define restrictions $j \restriction V_{j^n(\kappa)}$ for $n > 0$; doing so entails much stronger large cardinal consequences than those available in the theory ZFC + BTEE + MUA. While it is true that a *set* Dedekind self-map $j : A \to A$ has the property that restrictions of j to subsets of A play an important role in unfolding the dynamics of j, one could say that the theory ZFC + BTEE + MUA *masks* the analogous dynamics of an MUA-embedding $j : V \to V$ because restrictions of j to sets of size $> \kappa$ cannot be proven to exist in the universe.

The last paragraph of the previous section suggested a way to proceed further and to address the limitations we have just outlined: A path to stronger large cardinals is to postulate that restrictions of an embedding $j : V \to V$ to ever larger sets be themselves sets. While working in the theory ZFC + BTEE serves to maximize the preservation properties a Dedekind self-map from V to V could have (point (3) of Properties), insisting also that restrictions of j to various sets are also sets in the universe not only leads to significant strengthenings of the theory in the direction of stronger large cardinals, but also accords with our original intuition about which properties such a j *should* have (Conjecture point (5), p. 45).

These considerations suggest the following stronger axiom as a natural strengthening of MUA:

> **Axiom of Amenability$_{\mathbf{j}}$**. For every set X, $\mathbf{j} \upharpoonright X : X \to \mathbf{j}(X)$ is a set.

Intuitively, the Axiom of Amenability$_{\mathbf{j}}$ is a way of ensuring that the "dynamics" of an elementary embedding $j : V \to V$ are present "everywhere" within the universe. The inconsistency result (point (E) on p. 105) tells us that the only way the theory ZFC + BTEE + Amenability$_{\mathbf{j}}$ could be consistent is if the critical sequence has no upper bound in the universe,[43] and, in particular, the embedding $j : V \to V$ must be cofinal.

In the literature, the set of axioms BTEE + Amenability is given the name the *(weak) Wholeness Axiom* or WA$_0$.[44] We have the following result:

Theorem 11.123 (Consequences of the Wholeness Axiom) (Corazza, 2000) *Let* $(V, \in, j)$ *be a model of* ZFC + WA$_0$ *and let* κ *denote the least ordinal moved by* j*. Then* j *is a Dedekind self-map (neither a set nor a proper class) that is BSP and has critical point* κ*. Moreover,* κ *is super-*n*-huge for every* n *(and is in fact the* κ*th such cardinal).* □

Using WA$_0$, we are in a position to see more clearly the extent to which the concept of a Dedekind self-map points the way to a deeper understanding of the origin of large cardinals. Referring once again to Properties of a Dedekind Self-Map (p. 118), we observe that the fact that any WA$_0$-embedding is an elementary embedding gives expression to point (3) in Properties (*Preservation*) and realizes point (1) in the Conjecture. Also, the theorem highlights the role of critical point dynamics in the emergence of large cardinals (point (1) in Properties and Conjecture point (2)). But now, in the context of ZFC+WA$_0$, there are also natural realizations of points (2) and (4) in Properties, as well as the remaining points of the Conjecture; we discuss these next.

We begin with point (4) in Properties—the emergence of a blueprint. In the WA$_0$ setting, we replace the use of elementary embeddings derived from a normal measure that were used with MUA with *extendible embeddings* (see Section 2.3 for a definition). Intuitively, extendible cardinals arise from a WA$_0$-embedding (and its iterates) by restriction: Suppose $j : V \to V$ is a WA$_0$-embedding and $\kappa = \mathrm{crit}(j)$. Suppose η is an ordinal $> \kappa$. If $\kappa < \eta < j(\kappa)$, it is easy to see that $j \upharpoonright V_\eta : V_\eta \to V_{j(\eta)}$ is an extendible embedding with critical point κ. Likewise, if $j(\kappa) \leq \eta < j(j(\kappa))$,

[43] Moreover, ZFC+BTEE+Amenability$_{\mathbf{j}}$ *is consistent,* relative to large cardinals that are even stronger than those implied by this theory; for instance if there is an I$_3$-cardinal κ, with corresponding I$_3$-embedding $i : V_\lambda \to V_\lambda$ having critical point κ, $(V_\lambda, \in, i)$ is a model of ZFC + BTEE + Amenability.

[44] It is called "weak" because a slightly stronger version of the Wholeness Axiom is also known. The (full) Wholeness Axiom (WA) is BTEE + Separation$_{\mathbf{j}}$, where Separation$_{\mathbf{j}}$ is the usual Separation axiom, applied to $\mathbf{j}$-formulas. Amenability$_{\mathbf{j}}$ is a consequence (Corazza, 2006) of Separation$_{\mathbf{j}}$. In the literature, BTEE+Amenability$_{\mathbf{j}}$ is denoted WA$_0$ to indicate it is slightly weaker than WA. Nevertheless, it has been shown (Corazza, 2006) that all the known large cardinal consequences of WA can also be shown to be consequences of WA$_0$. Therefore, in this paper, as we introduce the Wholeness Axiom, we have emphasized the more intuitively appealing Amenability$_{\mathbf{j}}$ axiom as a starting point.

$(j \circ j) \upharpoonright V_\eta : V_\eta \to V_{j(j(\eta))}$ is another extendible embedding with critical point κ. Proceeding in this way demonstrates that κ itself is an extendible cardinal and that this fact is witnessed by the j-class of restrictions of iterates of j to various V_η, $\eta > \kappa$.

11.1 Blueprints Arising from WA_0 Using the notion of extendible elementary embeddings, one can define from j and its critical point κ a blueprint $\ell : V_\kappa \to V_\kappa$ for *all sets in the universe*; the function ℓ will be, as in our analysis of ZFC + BTEE + MUA, a co-Dedekind self-map. We will show that, for every set $x \in V$, there is an extendible embedding i such that $i(\ell)(\kappa) = x$. In this way, every set in the universe can be seen as arising from or being generated by the interplay of κ, j and ℓ.

The essence of the construction of ℓ is a *Laver function* $f : \kappa \to V_\kappa$. A Laver function[45] is a function $f : \kappa \to V_\kappa$ with the property that for any set x, there is an extendible embedding $i : V_\eta \to V_\xi$ such that:

(1) $\kappa = \mathrm{crit}(i)$
(2) $\mathrm{rank}(x) < \eta < i(\kappa) < \xi$
(3) $i(f)(\kappa) = x$.

This definition is in contrast with that for an X-Laver function (in particular, a $V_{\kappa+1}$-Laver function, as discussed in connection with ZFC + BTEE + MUA) which restricts the possible values of x to the set X.

The weak Wholeness Axiom guarantees the existence of a Laver function. The proof given here is nearly identical to that in Corazza (2010), Theorem 2.19, except we use here the theory $\mathrm{ZFC} + \mathrm{WA}_0$ rather than ZFC + WA.

Theorem 11.124 (Corazza, 2000) (WA_0) *Suppose $j : V \to V$ is a WA_0-embedding with critical point κ. Then there is a Laver function $f : \kappa \to V_\kappa$.*

Proof. We build a formula $\phi(g, x)$ that asserts that g is *not* Laver, with witness x, as follows: Let $\psi(\eta, \zeta, i, \alpha)$ be a formula that states formally "$i : V_\eta \to V_\zeta$ is an elementary embedding with critical point α." We let $\phi(g, x)$ be the following formula:

$$\exists\alpha \Big[g : \alpha \to V_\alpha \wedge \forall\eta\forall\zeta\,\forall i\,[(\psi(\eta,\zeta,i,\alpha) \wedge \mathrm{rank}(x) < \eta < i(\alpha) < \zeta) \to i(g)(\alpha) \neq x]\Big].$$

Define $f : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} \emptyset & \text{if } f \upharpoonright \alpha \text{ is Laver at } \alpha \text{ or } \alpha \text{ is not a cardinal,} \\ x & \text{otherwise, where } x \text{ satisfies } \phi(f \upharpoonright \alpha, x). \end{cases}$$

[45] In the literature, such a function is called a *weakly extendible* Laver function (Corazza, 2010, Definition 2.18). *Extendible* Laver functions were introduced in Corazza (2000), where the existence of such a function was proved to follow from (and to be equivalent to) the existence of an extendible cardinal; the definition of extendible Laver functions parallels more closely the definition originally developed by Laver for supercompact cardinals. *Weakly* extendible Laver functions were introduced in Corazza (2010) to simplify some of the proofs in the context of ZFC + WA.

Let $D = U_j$ be the normal ultrafilter over κ that is derived from j; that is:

$$D = \{X \subseteq \kappa \,|\, \kappa \in j(X)\}.$$

By Amenability, D is a set, since $D = \{X \subseteq \kappa \mid \kappa \in g(X)\}$, where $g = j \restriction \mathcal{P}(\kappa)$.

Define sets S_1 and S_2 by

$$\begin{array}{rcl} S_1 & = & \{\alpha < \kappa \,|\, f \restriction \alpha \text{ is Laver at } \alpha\} \\ S_2 & = & \{\alpha < \kappa \,|\, \phi(f \restriction \alpha, f(\alpha))\}. \end{array}$$

Clearly, $S_1 \cup S_2 \in D$. To complete the proof, it suffices to prove that $S_1 \in D$, and for this, it suffices to show $S_2 \notin D$.

Toward a contradiction, suppose $S_2 \in D$. Reasoning as in the ZFC + BTEE + MUA case (p. 99), we have $f = j(f) \restriction \kappa$ and so $\phi(f, j(f)(\kappa))$ holds in V. Let $x = j(f)(\kappa)$. Since $j(f) : j(\kappa) \to V_{j(\kappa)}$, $\text{rank}(x) < j(\kappa)$, so we can pick $\eta > \kappa$ so that $\text{rank}(x) < \eta < j(\kappa)$. Let $i = j \restriction V_\eta : V_\eta \to V_\zeta$, where $\zeta = j(\eta)$. By Amenability, i is a set, and is an elementary embedding with critical point κ. Clearly, in V, $\text{rank}(x) < \eta < i(\kappa) < \zeta$ and $i(f)(\kappa) = x$, contradicting the fact that $\phi(f, j(f)(\kappa))$ holds in V. Therefore $S_2 \notin D$, as required. □

We now define the blueprint map ℓ exactly as we did in the context of ZFC + BTEE + MUA:

$$\ell(x) = \begin{cases} f(x) & \text{if } x \text{ is an ordinal} < \kappa, \\ x & \text{otherwise.} \end{cases} \tag{55}$$

We have the following:

THEOREM 11.125 (Existence of Blueprint Self-Maps) (WA_0) *Suppose $j : V \to V$ is a WA_0-embedding with critical point κ. Then $\ell : V_\kappa \to V_\kappa$ defined as in equation (55) is a blueprint self-map; that is, for any set x, there is an extendible elementary embedding $i : V_\eta \to V_\xi$ with critical point κ and $\eta \geq \kappa + 1$ such that $i(\ell)(\kappa) = x$. Moreover, ℓ is a co-Dedekind self-map.*

The proof is exactly the same as the one given for the MUA case in Theorem 10.119, replacing canonical embeddings i_U with extendible embeddings i.

We show next that there is, just as in the MUA case, a natural dual to ℓ, which we will once again denote ℓ^{op}, which sends every sufficiently large set back to the "point" κ. The precise statement is given in the following theorem:

THEOREM 11.126 (WA_0) *Let $j : V \to V$ be a WA_0-embedding with critical point κ. For each blueprint self-map $\ell : V_\kappa \to V_\kappa$, there is a Dedekind self-map $\ell^{\text{op}} : V_\kappa \to V_\kappa$ with the following properties:*

(1) For every $x \notin V_\kappa$, there is an extendible elementary embedding $i : V_\eta \to V_\xi$ with critical point κ and $\eta \geq \kappa + 1$ such that

$$i(\ell^{\text{op}})(x) = \kappa$$

(2) ℓ^{op} is a section of ℓ; in particular, $\ell \circ \ell^{\text{op}} = \text{id}_{V_\kappa}$.

Again, the proofs are essentially identical to those given in the MUA case (see p. 103), replacing embeddings i_U obtained from a normal measure with extendible embeddings. In this case, we are not restricted to subsets of V_κ, as we were in the MUA case, but the reasoning is the same since now we have a (full) Laver function that gives access to sets of arbitrarily large rank.

As before, we do not claim that, for $x \in V_\kappa$, there is an i for which $i(\ell^{\mathrm{op}})(x) = \kappa$. Indeed, as before, there is no way to define a section s of ℓ so that this is true, and the proof of this fact is identical to the one given in the previous section.

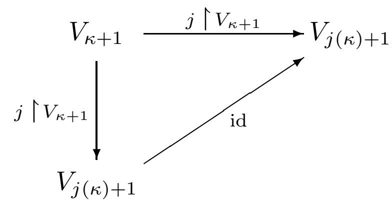
We turn now to a more detailed examination of our blueprint for the universe under WA.

Remark 11.127 (The Blueprint Coder $\mathcal{E}$ for V) We describe now in more detail the blueprint coder for the blueprint of V, given to us by WA_0. Suppose $j : V \to V$ is a WA_0-embedding, given to us by a model of $\mathrm{ZFC} + \mathrm{WA}_0$, with critical point κ. The collection $\mathcal{E}$ is the class of extendible embeddings $V_\beta \to V_\eta$ with critical point κ. The set $\mathcal{E}_0$ is defined to be the restriction of $\mathcal{E}$ to $V_\kappa^{V_\kappa}$, that is, $\mathcal{E}_0 = \{i \restriction V_\kappa^{V_\kappa} \mid i \in \mathcal{E}\}$. Note that each element of $\mathcal{E}_0$ is of the form $i : V_\kappa^{V_\kappa} \to V_{i(\kappa)}^{V_{i(\kappa)}}$.

We indicate why the triple $(\ell, \kappa, \mathcal{E})$ is a blueprint for V, and also why $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V - V_\kappa$. We refer to Definitions 4.34 and 4.36.

We start by verifying the properties mentioned in Definition 4.34. For (1), we have already seen that ℓ is a co-Dedekind self-map with co-critical point κ. We note that verification of properties (2)(a)–(d) is straightforward, and is like the MUA case. For (2)(e), for each $i \in \mathcal{E}_0$, we let $B = V_\kappa$ and $C = D = V_{i(\kappa)}$, and take the membership relation $\in$ on these sets to be the required partial order in each case. Since i is the restriction of an elementary embedding, it is clear that it is Σ_0-preserving.

To complete verification of (2), we must establish the compatibility requirement. We must find $i : V_{\kappa+1} \to V_{i(\kappa)+1} \in \mathcal{E}$ that is a right factor of $j \restriction V_{\kappa+1} : V_{\kappa+1} \to V_{j(\kappa)+1}$. However, by Amenability$_j$, this restriction of j is itself a set and belongs to $\mathcal{E}$. We can therefore let $i = j \restriction V_{\kappa+1}$ and let the required mapping k simply be the identity $V_{j(\kappa)+1} \to V_{j(\kappa)+1}$.



For (3), we must argue that ℓ is definable from $\mathcal{E}, j, \kappa$. A review of the definition of ℓ and the Laver function on which it is based makes this point clear. Finally, for (4), the fact that, for each $x \in V$, there is $i \in \mathcal{E}$ such that $i(\ell)(\kappa) = x$ guarantees that $\left(i \restriction V_\kappa^{V_\kappa}\right)(\ell)(\kappa) = x$, and note that $i \restriction V_\kappa^{V_\kappa} \in \mathcal{E}_0$.

Verification of the remaining points in Definition 4.36 to show that $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ is a strong blueprint for $V - V_\kappa$ is now straightforward in light of Theorem 11.126. □

We consider next how Conjecture point (3)—which anticipates that the critical sequence of j will arise in connection with restrictions of j—is realized in the theory $\mathrm{ZFC} + \mathrm{WA}_0$.

11.2 Restrictions of a WA_0-Embedding and Its Critical Sequence We begin by recalling the role of successive restrictions of a *set* Dedekind self-map in the emergence of the critical sequence $a, j(a), j(j(a)), \ldots$:

$$\begin{aligned} A_0 &= A; \\ j_0 &= j : A \to A; \\ \mathrm{crit}(j_0) &= a; \\ A_1 &= j[A_0]; \\ j_1 &= j \restriction A_1; \\ \mathrm{crit}(j_1) &= j(a); \\ A_{n+1} &= j[A_n]; \\ j_{n+1} &= j \restriction A_{n+1}; \\ \mathrm{crit}(j_{n+1}) &= j^{n+1}(a). \end{aligned}$$

Something similar occurs when we consider a certain sequence of restrictions of a given WA_0-embedding $j : V \to V$ with critical point κ. We observed in our study of ZFC + BTEE + MUA that $j \restriction V_\kappa : V_\kappa \to V_{j(\kappa)}$ is an elementary embedding, but we were unable to consider restrictions like $j \restriction V_{j(\kappa)}, j \restriction V_{j(j(\kappa))}, \ldots$ because the theory was not strong enough to admit such restrictions as sets in the universe. In the theory $\mathrm{ZFC} + \mathrm{WA}_0$, we no longer have this limitation; indeed, the sequence $\langle \kappa, j(\kappa), j(j(\kappa)), \ldots \rangle$ is a j-class in any model of the theory, and this j-class is, by observations made at the end of the last section, necessarily unbounded in the ordinals. Working in the theory $\mathrm{ZFC} + \mathrm{WA}_0$, we are now able to observe that each of these restricted embeddings serves to bring to light the next term in the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$, by analogy with set Dedekind self-maps. In particular, the observation that $j \restriction V_\kappa : V_\kappa \to j(V_\kappa) = V_{j(\kappa)}$ is an elementary embedding brings to light the image $j(\kappa)$ of κ. This is analogous to the discovery of the critical point $j(a)$ for $j \restriction B$ obtained by restricting $j : A \to A$ to its range: $j \restriction B : B \to B$, with $B = j[A]$.

If we now restrict j to the codomain of $j \restriction V_\kappa$, we obtain (the set) $j \restriction V_{j(\kappa)}$. Elementarity tells us that, for any $x \in V_{j(\kappa)}$, $j(x) \in j(V_{j(\kappa)}) = V_{j(j(\kappa))}$. In this way, the next term of j's critical sequence appears, namely, $j(j(\kappa))$. In general, the critical sequence for j can be seen to arise as the sequence of successive ranks of the codomains obtained by considering restrictions $j \restriction V_\kappa, j \restriction V_{j(\kappa)}, j \restriction V_{j(j(\kappa))}$, and so forth, all of which are sets because of $\mathrm{Amenability}_j$.

We consider next another sequence of restrictions of j that also leads to the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$. We begin with the following well-known definition of *application* of embeddings:

DEFINITION 11.128 Suppose $i, k : V \to V$ are WA_0-embeddings. Then

$$i \cdot k = \bigcup_{\alpha \in \mathrm{ON}} i(k \restriction V_\alpha).$$

We shall say in this case that i *is applied to* k*.*

It can be shown[46] that, if i, k are WA_0-embeddings $V \to V$, then $i \cdot k$ is also a WA_0-embedding; likewise for WA-embeddings. Moreover, for any such embeddings i, k, one shows (Dehornoy, 2000) that $\mathrm{crit}(i \cdot k) = i(\mathrm{crit}(k))$.

46 A proof can be found in Corazza (2017), p. 188ff.

We may now view the critical sequence of j as emerging from repeated self-application of j.

$$\begin{array}{rcl} \kappa & = & \mathrm{crit}(j) \\ j(\kappa) & = & \mathrm{crit}(j \cdot j) = \mathrm{crit}(j(j \restriction V_\kappa)) \\ j(j(\kappa)) & = & \mathrm{crit}(j \cdot (j \cdot j)) = \mathrm{crit}(j(j \restriction V_{j(\kappa)}))) \\ \ldots & & \ldots \end{array}$$

Here, the analogy with set Dedekind self-maps is even stronger (recall p. 111): In the case of set Dedekind self-maps $j : A \to A$, if $a \notin \mathrm{ran}\, j$ then $j(a) \notin \mathrm{ran}\,(j \restriction j[A])$. In the setting of repeated self-application of a WA-embedding $j : V \to V$, if $\kappa \notin \mathrm{ran}\, j$ (likewise, $\kappa \notin \mathrm{ran}\, j \restriction V_\kappa$), then $j(\kappa) \notin \mathrm{ran}\, j \cdot j$ (likewise, $j(\kappa) \notin \mathrm{ran}\, j(j \restriction V_\kappa)$). And likewise, if $\kappa = \mathrm{crit} j$ then $j(\kappa) = \mathrm{crit} j \cdot j$.

We note here that, in order for the sequence $j, j \cdot j, j \cdot (j \cdot j), \ldots$ to be well-defined, it appears to be necessary to use the stronger version WA of the Wholeness Axiom, in place of WA_0.[47] In models of $\mathrm{ZFC} + \mathrm{WA}_0$, even $j \cdot (j \cdot j)$ may not be defined; see Corazza (2006), p. 398, for a discussion of this point.

The intuitive justification for accepting this stronger version WA of the Wholeness Axiom is the same as for the justification of $\mathrm{Amenability}_{\mathbf{j}}$: The schema $\mathrm{Separation}_{\mathbf{j}}$ asserts in an even more complete way that the dynamics of the embedding j are present locally, as sets in the universe (realizing Conjecture point (5)).

We have exhibited two realizations of point (3) of the Dedekind Self-Map Conjecture. Moreover, we have indicated how WA-embeddings provide a full realization of the points of our original Dedekind Self-Map Conjecture, and, at the same time, provide an axiomatic account of virtually all large cardinals. We summarize in a table these Conjecture points and how they have been realized in the theory ZFC + WA. We also cross-reference with points from the Properties list (p. 118), obtained from reflecting upon the properties of a *set* Dedekind self-map, and the list of Plotinian Principles (p. 6), extracted from Plotinus's account of the dynamics of the ultimate nature of things; these lists originally guided the formulation of the Conjecture.

[47] See the footnote on p. 107 for more details about WA.

Conjecture Point	**Realization in ZFC + WA**
(1) *Preservation.* Dedekind self-maps of the universe V, with rich preservation properties, account for the presence of large cardinals in the universe. (Property (3), Plotinian Principle (2).)	The critical point κ of a WA-embedding has all large cardinal properties up to super-n-huge for every n.
(2) *Critical point dynamics.* The mechanism by which large cardinals and other mathematical objects arise from a Dedekind self-map $j : V \to V$ involves the interaction of j with its critical points. (Property (1).)	With respect to a WA-embedding j, the critical point of j is where the strongest large cardinal properties first arise. Moreover, a blueprint $(\ell, \kappa, \mathcal{E})$ for all sets in V arises from interaction between j and its critical point.
(3) *Critical sequence and restrictions of j.* Emergence of a critical sequence for such a Dedekind self-map j is closely related to successive transformations of j obtained by restrictions of j to sets in V. (Property (2).)	Using the application operation $\cdot$, successive critical points arise from application of j to its own restrictions to larger and larger stages of the universe: $\kappa = \mathrm{crit}(j)$ $j(\kappa) = \mathrm{crit}(j \cdot j) = \mathrm{crit}(j(j \upharpoonright V_\kappa))$ $j(j(\kappa)) = \mathrm{crit}(j \cdot (j \cdot j))$ $= \mathrm{crit}(j(j \upharpoonright V_{j(\kappa)})))$ $\ldots \quad \ldots$
(4) *Emergence of a blueprint.* The dynamics of such a Dedekind self-map $j : V \to V$ will result in emergence of a blueprint or strong blueprint for some significant class of sets—possibly the entire universe V. (Property (4), Plotinian Principle (1).)	With respect to a WA-embedding j, a blueprint $(\ell, \kappa, \mathcal{E})$ for all sets in V and a strong blueprint $(\ell, \ell^{\mathrm{op}}, \kappa, \mathcal{E})$ for $V - V_\kappa$ arise from interaction between j and its critical point.

Conjecture Point	**Realization in ZFC + WA**
(5) *Everywhere present.* The dynamics of j are in some way present everywhere in V. (Plotinian Principle (3).)	A WA_0-embedding j has the property that for every set X, $j \upharpoonright X$ is also a set. A WA-embedding j has the property that, for any set X, any subcollection Y of X that is definable from j is a set.
(6) *Nothing but dynamics of j.* Every mathematical object arises from the dynamics present in j. (Plotinian Principle (4).)	(Virtually) every mathematical object is represented as a set in V. Every set x in V is realized as $i(\ell)(\kappa)$ for some $i \in \mathcal{E}$, where $\mathcal{E}$ is a class that arises from the dynamics of j, and ℓ is defined from $j, \kappa, \mathcal{E}$.

11.3 Dedekind Self-Map Properties That Do Not Scale In Section 8.2, we remarked upon two properties exhibited by Dedekind self-maps $j : V \to V$ that often arise when j is obtained as a functor: existence of a weakly universal element for j and the realization of j as a monad. We argued that these properties give convincing expression to some of our guiding principles and realize certain points in our Dedekind Self-Map Conjecture. We asked in that section whether these properties hold for the "ultimate" Dedekind self-map—a version of $j : V \to V$ that satisfies all the points of the Conjecture. We address these questions here.

In Example 9.109, we showed that if κ is a measurable cardinal and D is a nonprincipal, κ-complete ultrafilter on κ, if $j_D : V \to V$ is defined by $j_D(X) = X^\kappa/D$ on objects and $j_D(f)(g) = [f \circ g]$ on functions $f : X \to Y$, then j_D is a Dedekind self-map with many nice preservation properties and with the additional property that $\kappa \in j_D(\kappa)$ is weakly universal for j_D (identifying κ^κ/D with its transitive collapse). We noted though that in this case j_D is not cofinal. We also observed (footnote on p. 93) that if instead we consider the ultrapower embedding $i_U : V \to V^\kappa/U \cong N$ with critical point κ (where U is a normal measure on a measurable cardinal κ), then not only is it true that $\kappa \in i_U(\kappa)$ is a weakly universal element for i_U, but also i_U is cofinal, yielding the intuitively desirable conclusion that $N = \{i_U(f)(\kappa) \mid f : \kappa \to V \text{ and } \kappa \in \mathrm{dom}\ i_U(f)\}$. A natural question is whether a Dedekind self-map $V \to V$ that is strong enough to give rise to all large cardinals will have both of these properties. We have already seen that any WA_0-embedding is cofinal; does such an embedding admit a universal element?

We show now that this is not the case; that, indeed, whenever $j : V \to V$ is a WA_0-embedding, there do not exist a, A for which $a \in j(A)$ is weakly universal for j. In Corazza (2010), Theorem 2.17, we established this result in the special case in which $a = \kappa$ and $j(A) = j(\kappa)$. We improve this result here, showing that for no choices of a, A is $a \in j(A)$ weakly universal for j. We need the following lemma:

Lemma 11.129 (Corazza, 2006, Corollary 8.7) *Working in the theory* $\mathrm{ZFC}+\mathrm{WA}_0$, *if $j : V \to V$ is a* WA_0*-embedding with critical point κ, then, whenever $\lambda \geq \kappa$ is a cardinal, $j(\lambda) > \lambda$.*

THEOREM 11.130 (No Weakly Universal Element for WA_0-Embeddings) *Work in the theory* $\mathrm{ZFC}+\mathrm{WA}_0$. *Let* $j : V \to V$ *be a* WA_0*-embedding with critical point* κ. *Then there is no weakly universal element for* j*; that is, for all sets* a, A *with* $a \in j(A)$*,* a *is not a weakly universal element for* j.

Proof. We show that

$$\forall a\, \exists z\, \forall f\, (a \in \mathrm{dom}\ \ j(f) \Rightarrow j(f)(a) \neq z). \tag{56}$$

We argue that this is sufficient to prove the theorem: Observe first that $j : V \to V$ is cofinal: Suppose $x \in V$. Then $x \in V_\alpha$ for some $\alpha \geq \kappa$; since, by the lemma, $j(\alpha) > \alpha$, we have

$$x \in V_\alpha \subseteq V_{j(\alpha)} = j(V_\alpha).$$

Now, if $a \in j(A)$ were a weakly universal element for j, it would follow (see Remark 9.108) that $V = \{j(f)(a) \mid a \in \mathrm{dom}\ (j(f))\}$. Thus, to prove the theorem, it is enough to establish (56). Let a be a set.

Case I: $a \notin \mathrm{ran}\, j$.

By cofinality of j, we can find an infinite set A for which $a \in j(A)$. Let $D = \{X \subseteq A \mid a \in j(X)\}$. By Theorem 6.54, D is a nonprincipal ultrafilter on A. Let $\lambda \geq \kappa$ be a beth fixed point for which $A \in V_\lambda$. Pick some $y \in A$.

Suppose $g : B \to V$ is such that $a \in \mathrm{dom}\ \ j(g)$. We show that, for some $f : A \to V$, $j(f)(a) = j(g)(a)$. Since j preserves intersections, we have $a \in j(A) \cap j(B) = j(A \cap B)$, and so $A \cap B \in D$. Define $f : A \to V$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in A \cap B \\ y & \text{otherwise} \end{cases}$$

Since $A \cap B \subseteq \{x \in A \mid f(x) = g(x)\}$, $\{x \in A \mid f(x) = g(x)\} \in D$. It follows that $j(f)(a) = j(g)(a)$.

Let $R = \{j(f)(a) \mid f : A \to V\}$ and let $T = \{j(h)(a) \mid h : A \to V_\lambda\}$. We show that whenever $f : A \to V$ is such that $j(f)(a) \in V_{j(\lambda)}$, there is $h : A \to V_\lambda$ such that $j(f)(a) = j(h)(a)$. Since $j(f)(a) \in V_{j(\lambda)}$, we have $S \in D$, where $S = \{x \in A \mid f(x) \in V_\lambda\}$ (note that, by elementarity of j, $j(V_\lambda) = V_{j(\lambda)}$). Define $h : A \to V_\lambda$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in S \\ y & \text{otherwise} \end{cases}$$

Since $S \subseteq \{x \in A \mid f(x) = h(x)\}$, it follows that $\{x \in A \mid f(x) = h(x)\} \in D$, whence $j(f)(a) = h(f)(a)$.

We have shown that whenever $j(f)(a) \in R$ and $j(f)(a) \in V_{j(\lambda)}$, then $j(f)(a) \in T$; that is, $R \cap V_{j(\lambda)} \subseteq T$. We now show that there must exist $z \in V_{j(\lambda)}$ such that, for all $g : B \to V$ for which $a \in \mathrm{dom}\ \ j(g)$, $j(g)(a) \neq z$. Notice by elementarity that $j(\lambda)$ is a beth fixed point, and that, by Lemma 11.129, $j(\lambda) > \lambda$. Then

$$|T| \leq |V_\lambda|^{|A|} = \lambda^{|A|} < j(\lambda) = |V_{j(\lambda)}|.$$

It follows that for some $z \in V_{j(\lambda)}$, there is no $f : A \to V_\lambda$ such that $j(f)(a) = z$, hence no $g : B \to V$ with $j(g)(a) = z$.

Case II: $a \in \operatorname{ran} j$.

For this case, we show that if $a \in \operatorname{ran} j$ and $f : A \to V$ with $a \in \operatorname{dom}\ j(f)$, then $j(f)(a) \in \operatorname{ran} j$; it will then follow that there must exist z (for instance, $z = \kappa$) for which $j(f)(a) \neq z$ for any choice of f.

Given a, f as above, let b be such that $a = j(b)$. Then

$$j(f)(a) = j(f)(j(b)) = j(f(b)) \in \operatorname{ran} j,$$

as required. □

We turn next to the other question raised in Section 8.2: Is an adequate Dedekind self-map a monad? We do not have a complete answer to this question, but the next theorem makes it seem unlikely that any WA-embedding could be the functor part of a monad. We begin with a lemma.

LEMMA 11.131 *Suppose $j : V \to V$ is a WA-embedding with critical point κ. Suppose $j' : V \to V$ is definable from j and has the following property: For each $x \in V$, $|j'(x)| = |j(x)|$. (We will say that j' and j are weakly isomorphic.) Then*

(1) *the sequence $\kappa, |j'(\kappa)|, |j'(j'(\kappa))|, \ldots$ is cofinal in ON, and*
(2) *j' is not definable in V.*

We note that the requirement in the hypothesis that j' be definable from j is necessary; otherwise, to formalize the use of j' would require adding a second unary function symbol to the language.

Proof. We first note that (2) follows from (1): If $j' : V \to V$ is any map that is definable in V, the sequence $\kappa, |j'(\kappa)|, |j'(j'(\kappa))|, \ldots$ has an upper bound, by Replacement. For definiteness, define $H : \omega \to V$ by recursion:

$$\begin{aligned} H(0) &= \kappa \\ H(n+1) &= j'(H(n)) \end{aligned}$$

Define $K : \omega \to V$ by $K(n) = |H(n)|$. Now, by Replacement, there is λ such that $K[\omega] \subseteq \lambda$.

To complete the proof, it suffices to prove (1). Since j' is definable from j, there is a j-formula $\gamma(x, y, z_1, \ldots, z_m)$ that defines j' in V. It is shown in Corazza (2006) there is a formula $\Phi(n, x, y)$ that asserts that, whenever $n \in \omega$, $y = j^n(x)$. A similar formula $\Phi'(n, x, y, z_1, \ldots, z_m)$ can be defined—replacing subformulas of the form $v = j(u)$ by formulas of the form $\gamma(u, v, z_1, \ldots, z_m)$—which asserts that, whenever $n \in \omega$, $y = (j')^n(x)$. A simple application of $\text{Induction}_{\mathbf{j}}$ allows us to show that for each $n \in \omega$ and for all $y, y' \in V$,

$$\Phi(n, \kappa, y) \wedge \Phi'(n, \kappa, y', z_1, \ldots, z_m) \Rightarrow |y| = |y'|. \tag{57}$$

In other words, for all $n \in \omega$, $|j^n(\kappa)| = |(j')^n(\kappa)|$.

Therefore, to see $\kappa, |j'(\kappa)|, |j'(j'(\kappa))|, \ldots$ is cofinal in ON, let γ be a cardinal and let n be such that $\gamma < j^n(\kappa)$. Then by equation (57), the result follows. □

We note that the argument does not go through for WA_0-embeddings because of our reliance on $\text{Induction}_{\mathbf{j}}$; the defining j-formula for H may be arbitrarily complex.

THEOREM 11.132 *Suppose $j : V \to V$ is a WA-embedding. Suppose $\mathcal{C}$ is a category that is closed under small copowers and that is definable in V. Then there do not exist functors $G : \mathcal{C} \to \mathbf{Set}$ and $F : \mathbf{Set} \to \mathcal{C}$ such that $F \dashv G$ and $j = G \circ F$.*

Proof. Suppose j and $\mathcal{C}$ are as in the hypothesis and there exist functors $G : \mathcal{C} \to \mathbf{Set}$ and $F : \mathbf{Set} \to \mathcal{C}$ such that $F \dashv G$ and $j = G \circ F$.

Since G has a left adjoint and $\mathcal{C}$ is closed under small copowers, it follows from Lemma 2.3 that there is a natural iso $\mathcal{C}(A, -) \to G$. It follows that $F \dashv \mathcal{C}(A, -)$. We define a functor $F' : \mathbf{Set} \to \mathcal{C}$ as follows, using again the fact that $\mathcal{C}$ is closed under small copowers. For any set I, $F'(I) = \amalg_I A$. For any function $f : I \to J$, define $F'(f) : \amalg_I A \to \amalg_J A$ to be the unique function guaranteed to exist by the universal mapping property for coproducts. It is easy to verify that $F' \dashv \mathcal{C}(A, -)$. By uniqueness of adjoints (Lemma 2.2), F' is naturally isomorphic to F. Let $j' = \mathcal{C}(A, -) \circ F'$. Notice that, as j' is the composition of functors that are explicitly defined in V,

$$j' \text{ is definable in } V. \tag{58}$$

On the other hand, we have

$$j' = \mathcal{C}(A, -) \circ F' \cong G \circ F = j.$$

It follows that j' and j are weakly isomorphic. Since j' is definable from j (by (58)), we have from Lemma 11.131 the following:

$$j' \text{ is not definable in } V. \tag{59}$$

Since (58) and (59) cannot both hold, we have a contradiction. □

§12 Conclusion This article has been an investigation into the question, "Why should large cardinals exist in the universe? What kind of instrinsic justification can be found for such exotic mathematical entities?" Our angle for approaching this question has been to try to find what it is about our concept of "the infinite" that would suggest that large cardinals should indeed exist. If we wish to examine the intuition about "the infinite" that underlies the use of infinite sets in mathematics, a natural starting point is to ask what intuition informs the Axiom of Infinity in set theory. However, the intuition that we find at the basis of the Axiom of Infinity is simply that it should be possible to view unbounded sequences like the natural numbers as completed sets, as actual mathematical objects. In other words, ω, as a completed set, *should* exist in the universe. This intuition, though revolutionary in its time, has not proven rich enough to point to the existence of extremely large infinities like large cardinals.

This limitation in intuitive richness led us to ask a question about the infinite that was familiar to many ancient cultures, though quite foreign to contemporary thought on the subject: What imparts to ω the characteristic of being infinite? In the ancient view, the natural numbers were seen to originate from some source. It was by virtue of the internal dynamics of this source that the natural numbers emerged. And it was because of a power or characteristic internal to this source that what emerged, as the natural numbers unfolded, was an *infinite* multitude. The emphasis concerning the infinite in this world view is on the nature and dynamics of the source rather than on the expression of those dynamics. In modern terminology, the natural numbers—indeed, the entire diversity of existence—were seen to be a side effect of the unseen dynamics of the source. We asked whether this view of the "origin" of ω could be axiomatized in a reasonable way.

We observed that a similar relationship between source and multiplicity was discovered as a solution to the problem in physics of finding the ultimate constituents of the material universe: the particles out of which everything is made are in reality an expression of the dynamics of underlying quantum fields. The "truth" about each class of particles is that they arise from the dynamics of an underlying quantum field.

Proceeding by analogy, we asked if the infinite multitude of natural numbers could also be seen as emerging from the "dynamics" of a source of some kind. As a realization of this intuitive perspective, we suggested that a Dedekind self-map $j : A \to A$, with critical point a, could play the role of "source"; the "dynamics" of j were seen to be the repeated application of j to its critical point and subsequent images: $a, j(a), j(j(a)), \ldots$. In this dynamic unfoldment, we see the emergence of a precursor to the familiar finite ordinals. In his own work, Dedekind declared that, starting from any such self-map j, the sequence $W = \{a, j(a), j(j(a)), \ldots\}$ that emerges *is* the set of natural numbers, up to isomorphism. We carried his analysis one step further and observed that, applying a form of the Mostowski collapsing function (whose existence can be established without reliance on the natural numbers), we find that ω arises as the collapse of W, and the usual successor map $s : \omega \to \omega$ arises as the collapse of $j \restriction W$.

In studying the details of the emergence of ω from a Dedekind self-map, we located principles that appeared to be at work in this emergence and that seemed to be amenable to generalization to a global context suitable for motivating large cardinals. We identified the following four principles:

Properties of a Dedekind Self-Map

(1) *Critical Point Dynamics*. A key sequence of values emerges from j and its interaction with its critical point.
(2) *Restrictions of j and Critical Points*. Restrictions of j to subsets of its domain are directly related to the emergence of the critical sequence $a, j(a), j(j(a)), \ldots$.
(3) *Preservation*. j exhibits strong preserves properties.
(4) *Blueprint*. The interaction between j and its critical point produces a blueprint—even a *strong* blueprint—from which a set of central importance is generated.

We observed that many of these properties that we discovered for Dedekind self-maps give expression to principles according to which the One or Source gives rise to multiplicity, as understood in many ancient philosophies, represented by the work of Plotinus. The Plotinian principles that we identified were the following.

Plotinian Principles of The One

(1) *Multiplicity As Epiphenomenon*. Multiplicity arises as a side effect of the internal dynamics of The One.
(2) *Preservation*. The transformations that lead from The One to multiplicity do not modify the nature of The One in any way.
(3) *Everywhere Present*. Though transcendent, The One is present in every grain of manifest existence.
(4) *Everything from the Dynamics of the Source*. Every existent thing arises from the dynamics of the source.

Viewing these extracted principles as input for a new intuition about "the infinite," we formulated a conjecture about the possibility of giving an account of large cardinals on the basis of versions of Dedekind self-maps—defined now on V rather than on arbitrary sets—whose properties are enriched in accordance with these principles. Our Dedekind Self-Map Conjecture is the following.

Dedekind Self-Map Conjecture

(1) Dedekind self-maps of the universe V, with rich preservation properties, account for the presence of large cardinals in the universe.
(2) The mechanism by which large cardinals and other mathematical objects arise from a Dedekind self-map $j : V \to V$ involves the interaction of j with its critical points.
(3) Emergence of a critical sequence for such a Dedekind self-map j is closely related to successive transformations of j obtained by restrictions of j to sets in V.
(4) The dynamics of such a Dedekind self-map $j : V \to V$ will result in emergence of a blueprint or strong blueprint for some significant class of sets—possibly the entire universe V.
(5) The dynamics of j are in some way present everywhere in V.
(6) Every mathematical object arises from the dynamics present in j.

With our principles and conjecture in hand, we began our quest for enrichments of a bare Dedekind self-map $j : V \to V$ that could provide a properly motivated account of large cardinals.

The first challenge in considering such self-maps was to strengthen j's properties sufficiently to ensure even the existence of ω, since a bare $j : V \to V$ could be defined even in the theory ZFC $-$ Infinity. We developed two ways to achieve this goal. One way involved strengthening j with preservation properties; one result in this direction was the following: If a class map $j : V \to V$ preserves disjoint unions and singletons, the universe must contain an infinite set.

Another approach was to obtain an infinite set directly from the action by a suitably defined Dedekind self-map $j : V \to V$ on its least critical point. An example of this second approach was the Lawvere Construction (Theorem 8.74) wherein $j = G \circ F$, $G : V^{\circlearrowright} \to V$ is the forgetful functor, F is a left adjoint of G, 1 is the least critical point of j, and $j(1)$ is infinite. Abstracting properties of the Lawvere Construction, we showed that whenever j is the functor part of a Dedekind monad, not only is it the case that $j(\mathrm{crit}(j))$ is infinite, but, in addition, a Dedekind self-map on a set is directly derivable from the interaction of j with its critical point, so that the Critical Point Dynamics property is even more fully realized.

We argued that each of these two ways of arriving at an infinite set is in keeping with our theme for finding the right generalization of Dedekind self-maps based on insights culled from ancient texts.

Further applying the first of these approaches, we added other naturally motivated preservation requirements to a Dedekind self-map $j : V \to V$. The resulting stronger versions of such a j led to the emergence of inaccessible and ineffable cardinals; see Theorems 9.96, 5.50, and 9.103. Applying a combination of both approaches led to a justification of measurable cardinals; see Theorem 9.107. Moreover, *any* measurable cardinal in V will give rise to a Dedekind self-map $V \to V$ having the conditions specified in Theorem 9.107; see Example 9.109.

In the discussion at the beginning of Section 9.4, we observed that, up to this point, our efforts to strengthen a bare Dedekind self-map $j : V \to V$ have made use of the principles of *Preservation* and *Critical Point Dynamics*, corresponding to Conjecture points (1) and (2). When we took the step to require j to preserve *all* first-order properties by working in a model $(V, \in, j)$ of ZFC + BTEE, we saw the first signs of the emergence of a blueprint for significant sets (Conjecture point (4)), combining the work of Cheng and Gitman (2015) with that of Corazza (2006). We showed that, working in a transitive model $(N, \in, j)$ of ZFC + BTEE with critical point κ, built from (L_ρ, I), where ρ is ω-Erdös and I is a set of indiscernibles of ordertype ω, obtained from the ω-Erdös property, there must exist a self-map $\ell : V_\kappa \to V_\kappa$ in $L_{j(\kappa)}$ that exhibits characteristics of a formal blueprint: The function ℓ will have the property that, for each $\mathrm{Coll}(\omega, < \kappa)$-generic G over $L_{j(\kappa)}$ and each $x \in L_{j(\kappa)}$, there is an elementary embedding h, defined in $L_{j(\kappa)}[G]$, such that $h(\ell \restriction \mu+1)(\kappa) = x$, where $\mu < \kappa$ is such that $\ell \restriction \mu+1 \in \mathrm{dom}\ h$. Since realization of elements in $L_{j(\kappa)}$ as $h(\ell \restriction \mu+1)(\kappa)$ is accomplished only via elementary embeddings h defined in a forcing extension, the *Blueprint* principle is only partially realized in this case.

Working in a transitive model $M = (V, \in, j)$ of the theory ZFC+BTEE obtainable from an ω-Erdös cardinal, a question arose: This model satisfies the criteria given in the Trňkova-Blass Theorem (Theorem 9.110) for existence of a measurable cardinal, in that j itself is an exact functor with a strong critical point. Yet, since ω-Erdös cardinals are much weaker than measurables, this model cannot contain a measurable cardinal. The reason for the apparent paradox is that, in order for the criteria of Theorem 9.107 to hold, the functor (j in this case) must be definable in the ambient universe, whereas BTEE-embeddings are never definable. As a result, though the collection $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ (where $\kappa = \mathrm{crit}(j)$) is provably a *set* in the context of Theorem 9.107, it is not generally a set when j is a BTEE-embedding (without additional hypotheses).

Having already established the naturalness of measurable cardinals based on the techniques developed so far, we postulated, by way of a new axiom MUA, the existence (as a set) of this ultrafilter U, derived from j; note that this postulate may also be viewed as an application of Critical Point Dynamics. The resulting theory ZFC + BTEE + MUA is strong enough to imply that the critical point κ is measurable of high Mitchell order. With the theory ZFC + BTEE + MUA we also witnessed the emergence of a blueprint (satisfying all the formal requirements) for a significant set: an MUA-embedding yields a blueprint for the set $V_{\kappa+1}$.

The theory ZFC + BTEE + MUA is limited, however, by the fact that it provides limited information about the critical sequence $\kappa, \mathbf{j}(\kappa), \ldots$, and most of the natural restrictions of $\mathbf{j}$ to sets of the form $V_{\mathbf{j}^n(\kappa)}$ that we would like to study, by analogy with the Dedekind self-maps defined on a set, cannot in general be proved to exist in an MUA universe.

Replacing MUA by the stronger Axiom of Amenability$_{\mathbf{j}}$ gave us a much stronger theory: WA_0 is the theory BTEE + Amenability$_{\mathbf{j}}$, and provides us with a realization of yet another point in the Conjecture: Since the restriction of the WA_0-embedding to every set is also a set, Conjecture point (5)—which asserts that the dynamics of j are in some way present everywhere in V—is realized in this theory. Moreover, we showed that from ZFC + WA_0, the strong properties of all the most widely studied large cardinals can be accounted for as properties of the critical point of the embedding. And, moreover, the embedding gives rise to a blueprint for the *entire universe of sets*. In particular, WA_0 embeddings provide a realization of Conjecture points (1), (2), (4)–(6).

Finally, taking the step from WA_0-embeddings to WA-embeddings made it possible to realize Conjecture point (3) as well: The critical sequence is seen in that context to arise in conjunction with natural restrictions of the embedding.

We arrived at an extension of ZFC that could account for large cardinals by adopting a revised version of the Axiom of Infinity (which asserts the existence of a Dedekind self-map), based on insights concerning the emergence of the natural numbers taken both from ancient texts drawn from several cultures around the world and on analogies from the approach of modern physics to solve the problem of finding the ultimate constituents of the physical universe. Studying how the natural numbers emerge on the basis of this new axiom of infinity revealed a number of principles that were amenable to generalization in the direction of large infinities. Applying these principles to strengthen a Dedekind self-map by introducing ever stronger preservation properties and by applying critical point dynamics, and then monitoring our resulting maps for emergence of a blueprint and for the role of restrictions of the map in the emergence of the critical sequence, we eventually arrived at a concept of Dedekind self-map that exhibited all the characteristics we conjectured should be there. Having found a self-map that realized these principles, we believe we have provided a candidate solution to the Problem of Large Cardinals: an intrinsically justified extension of ZFC that provides an account of the most widely used large cardinals.

To conclude the paper, we address two additional points.

(1) There are a number of large cardinal notions whose consistency strength exceeds that of ZFC + WA. Can these be accounted for using an extension of the approach that we have taken in this paper?
(2) For our intuition about the nature of the "infinite," as we sought a richer version of the Axiom of Infinity, we drew upon a breakthrough in modern physics—for the purpose of drawing an analogy—according to which particles in the universe are seen as precipitations of unbounded quantum fields. A WA-embedding provides an analogy for a single super quantum field that would in principle, through its own internal dynamics, account for everything in the material universe. Is there more than analogy going on here? Can the theory ZFC + WA, or some strengthening of it, provide a useful mathematical foundation for a physical "theory of everything"?

12.1 Beyond the Wholeness Axiom All of the theories ZFC + $\exists\kappa\, I_i(\kappa)$, for $i = 0, \ldots, 3$ (see Kanamori (1994) for definitions) are consistencywise strictly stronger than ZFC+WA (Corazza, 2000, Theorem 3.13). Each of the large cardinals defined by these theories is specified by an elementary embedding either from a rank to itself or from a proper inner model to itself. Since none of these elementary embeddings is a self-map $V \to V$, some of the intuitive motivation that we used to arrive at WA is no longer relevant for justifying existence of these stronger types of large cardinals, though the principles of *Preservation* and *Critical Point Dynamics* still play a major role. In particular, we do not, in any of these cases, find, or expect to find, a blueprint for all sets arising from the embedding's interaction with its critical point. To provide some kind of direct intrinsic justification for the large cardinals arising in these theories will require a different approach from the one we have studied here.

Nevertheless, it is possible to make use of our principles to give some justification for a very strong large cardinal notion due to Woodin, from which the axioms I_1–I_3 can be derived; in this indirect way, they can be considered to be "justified." Woodin's large cardinal is called by him a *weak Reinhardt cardinal*. We formulate the idea as an axiom; see Corazza (2010) for a fuller discussion:

> **Weak Reinhardt Axiom (WRA)**. There is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ such that $V_\kappa \prec V$.

Unlike the Dedekind self-maps that have been our primary focus in this paper, the axiom WRA does not assert the existence of a Dedekind self-map $V \to V$. Nevertheless, WRA does assert the existence of a very strong type of Dedekind self-map defined on a *set*. As we show now, though our intuitive principles are not a perfect fit, they still can be used to a reasonable extent to provide intrinsic justification for this large cardinal axiom.

In Corazza (2010) we prove the following:

Theorem 12.133 *Work in the theory ZFC+ WRA. Then*

(1) *There are arbitrarily large I_1 cardinals in the universe, and also arbitrarily large cardinals that are super-n-huge for every n.*
(2) *Let $\mathcal{E}^\lambda$ denote the set of all i such that*

$$V_\lambda \models \text{“}i \text{ is an extendible embedding with critical point } \kappa\text{”}.$$

Then there is a co-Dedekind self-map $\ell^\lambda : V_\kappa \to V_\kappa$ such that $(\ell^\lambda, \kappa, \mathcal{E}^\lambda)$ is a blueprint for V_λ.
(3) *There are α and $f : \alpha \to V_\alpha$ such that*

$$V = \{i(f)(\alpha) \mid i \text{ is an extendible embedding with critical point } \alpha\}. \quad (60)$$

In particular, there is a co-Dedekind self-map $\ell : V_\alpha \to V_\alpha$ such that $(\ell, \alpha, \mathcal{E})$ is a blueprint for V, where $\mathcal{E}$ is the class of all extendible embeddings with critical point α.

Part (1) shows that the theory is much stronger even than ZFC+$\exists\kappa\, I_1(\kappa)$. Part (2) arises from the observation that, if $i : V_{\lambda+1} \to V_{\lambda+1}$ is a WRA-embedding, then $(V_\lambda, \in, i \restriction V_\lambda)$ is a model of ZFC + WA.

Part (3) improves upon (2) by showing that the axiom realizes the *Blueprint* principle in the stronger sense that one can obtain a blueprint for *all of* V, though this occurs in a somewhat unexpected way: While ZFC+ WRA guarantees existence of extendible Laver sequences, the critical point κ of the embedding does not admit such a sequence (though there is such a sequence relative to V_λ); indeed, the least weak Reinhardt cardinal is neither extendible nor supercompact (Corazza, 2010, p. 60). Nevertheless, (3) is indeed an expression of a certain kind of critical point dynamics: Suppose $j : V_{\lambda+1} \to V_{\lambda+1}$ is a WRA-embedding with critical point κ, and let $i = j \restriction V_\lambda : V_\lambda \to V_\lambda$. Then the critical sequence $\kappa, j(\kappa), j(j(\kappa)), \ldots$ is unbounded in λ and it follows that, in V_λ, there is a proper class of extendible cardinals. Because $V_\kappa \prec V_\lambda$, V_κ has a proper class of extendibles, and because $V_\kappa \prec V$, so does V. Picking any extendible cardinal α, we can (by Corazza (2000), Theorem A.6) obtain an extendible Laver sequence at α, and equation (60) follows. The techniques of Section §11 can then be applied to obtain the final result.

Returning to strengthenings of ZFC + WA in which the underlying self-map has domain V, we recall that the strongest form of this type of elementary embedding was proposed by Reinhardt (1974). In this formulation, Separation and Replacement for j-formulas are accepted as axiom schema (as would happen automatically in any class theory like NBG or KM), but the Axiom of Choice is not included. The critical point of such an embedding is known as a *Reinhardt cardinal*. We shall denote the corresponding theory (formulated in the languagae $\text{ZFC}_{\mathbf{j}}$) ZF + R. Woodin has shown (cf. Goldberg (2018)) that, starting from a choiceless universe V that admits such an embedding with critical point κ, there is a forcing extension that satifies ZFC and in which there is a limit ordinal $\lambda > \kappa$ and an elementary embedding $i : V_\lambda \to V_\lambda$ having critical point κ; in other words, κ becomes an I_3 cardinal in the forcing extension. (And recall that for such λ and i, $(V_\lambda, \in, i)$ is a model of ZFC + WA.) Here, j is certainly a Dedekind self-map $V \to V$ and exhibits *Preservation* and *Critical Point Dynamics*. However, the *Blueprint* principle is not realizable in the ways we have described in this paper because, without AC, it is impossible to define the necessary Laver sequences.

Nevertheless, from an intuitive standpoint, whereas ZFC + WA may be seen as a well-motivated theory, based on the techniques of justification provided in this paper, a Reinhardt embedding could be seen as a kind of idealized theoretical precursor of a WA-embedding. Certainly, the presence of Replacement for j-formulas in the Reinhardt theory is more natural than a theory that excludes it. And omitting AC can be seen, as Koellner (2014)[48] describes it, as breaking through a barrier imposed by AC that is analogous to the way in which measurable cardinals break through the barrier imposed by $V = L$. Returning to our original motivation for introducing Dedekind self-maps in the first place, the move from ZFC + WA to ZF + R can be seen as being analogous to "breaking through" the particle-view of the material universe (and the view of the infinity of ω as primarily a collection of discrete quantities) to the view that particles are precipitations of quantum fields (and the view that natural numbers are precipitations of an underlying Dedekind self-map). From this perspective, just as particles are seen to arise from *field collapse*, and as ω and $s : \omega \to \omega$ can be seen as arising from a "collapse" of a Dedekind self-map's interaction with its critical point (via Mostowski collapse), so likewise the theory ZFC + WA can be seen, at a higher level, to arise as a "collapse" of the choiceless theory ZF + R, where in this case "collapse" refers to the production of a forcing extension.[49]

12.2 The Theory ZFC + WA As a Foundation for Physics In this subsection, we review some recent work that makes use of our theory ZFC + WA as a starting point for a mathematical foundation for the ontological interpretation of quantum mechanics, due to David Bohm.

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48 See slide 22.

49 Recall that a forcing extension is a quotient ("collapse") $M^B/G = M[G]$ of a universe $M^{\mathbf{B}}$ of names, where M is the ground model and B is a complete boolean algebra in M, by a generic ultrafilter G. Recall also that a model ZFC + WA is found in $V[G]$ in the form of a model $(V[G]_\lambda, \in, i)$ for some elementary embedding i and some ordinal $\lambda > \text{crit}(i)$, where V is a model of ZF + R.

David Bohm, a close and trusted colleague of Einstein in the 1940s and 1950s, after publishing a treatise *Quantum Theory* (Bohm, 1951) on the standard Copenhagen interpretation of quantum mechanics[50] —a work that was well-received in the physics community, particularly by Einstein[51]—became convinced of the limitations of the Cophenhagen interpretation and suspicious of the prevalent view that what occurs at the quantum level is unknowable.[52]

A major motivating concern was the physics community's commitment to a *fragmentary reductionist paradigm*, which Bohm often referred to as *atomism* (Bohm, 1981). The atomistic view is that, to understand the world, it is sufficient to identify and understand fundamental, separate parts and their relationships. On that view, the whole is nothing more than the sum of parts. To illustrate to the nonspecialist the undesirable limitation of this paradigm, Bohm offered the following analogy (Bohm and Peat, 2000): Imagine that a watch has been shattered into many small pieces. It is unlikely that a study of these separate fragments will be enough to reveal how the watch is supposed to work. On the other hand, knowing how the watch works as a first step makes the job of rebuilding the watch from parts a very different kind of activity. The example, in an elementary context, points to the need to begin the quest for understanding the universe by starting with wholeness rather than, as is currently done, with parts.

Bohm pointed out that many of the troubling unsettled questions in quantum theory as it is widely understood today arise from a failure to adopt the wholeness-first paradigm. He gave as one example the famous EPR[53] paradox. The EPR paradox is a thought experiment presented as part of a series of debates between Einstein and Bohr concerning quantum mechanics; the paradox was intended to demonstrate the incompleteness of quantum mechanics. The experiment showed that the result of a measurement on one particle of an entangled quantum system[54] can have an instantaneous effect on another particle, regardless of the distance between the two particles. The only two explanations considered possible (at least at the time of these debates) are that either the state of the second particle was already determined before the first was measured, or some communication occurred between the two particles. The first explanation violates Bell's inequality (Bricmont, 2016, p. 121) while the second conflicts with Einstein's theory of relativity since, if the particles are far enough apart, a message would have to travel between the particles faster than the speed of light.

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[50] In the Cophenhagen interpretation, physical systems do not have definite properties prior to being measured; quantum mechanics predicts only the probabilities of certain values being produced from measurement; the act of measurement causes a *collapse of the wave function* which causes the system to produce one of these possible values. (Wikipedia `https://en.wikipedia.org/wiki/Copenhagen_interpretation`).

[51] See the Wikipedia article on David Bohm, `http://en.wikipedia.org/wiki/David_Bohm`.

[52] In the Copenhagen interpretation, there is no "actuality" at the quantum level; something actually appears only as a result of a collapse of the wave function.

[53] Einstein-Podolsky-Rosen

[54] When two particles, like photons, are allowed to interact initially so that they will subsequently be defined by a single wave function, they are said to be *entangled*; when they are separated, they will still share a wave function so that measuring one will determine the state of the other. For more on quantum entanglement, see Cramer (2016).

Bohm (1981) offered a third possible explanation which also provides a segue to his wholeness-first approach to physics. The particles in the EPR experiment must not be viewed as separate entities but rather as projections of a higher-dimensional reality. To make the point accessible to the layman, Bohm used the following analogy (Bohm, 1981, pp. 186–87): Imagine a typical fish tank with transparent glass walls, filled with water and containing some fish, being viewed with two television cameras, one pointing from the east, the other from the north. Consider then the images of these cameras displayed on two televisions. Changes in the image on one screen will be highly correlated with those on the other and could lead one to conclude that some interaction is occurring between the images. However, this appearance arises because of the common source of the projections. In a sense, the television images are implicitly contained in the higher-dimensional reality of the fish tank, and the technology of the television cameras serves to *unfold* these images.

Likewise, Bohm (1981, p. 188) argued that the two-particle system of the EPR experiment should be viewed as a three-dimensional projection of a six-dimensional reality. The unseen higher-dimensional reality in this example is what Bohm calls an *implicate order* and the observed three-dimensional particles and relationships are the corresponding *explicate order*. The experiment has the effect of *unfolding* the observed particles and relationships, which represent only a limited viewpoint or aspect of the higher-dimensional implicate order.

Another analogy Bohm used was the hologram. An object and its representation in a hologram provide an analogy for explicate and implicate orders, respectively. The act of encoding the object as a hologram is called by Bohm *enfoldment*. The form and structure of the entire object is *enfolded* in each region of the hologram. When a light is shone on the hologram, the form and structure are *unfolded* to produce a detailed image of the original object.

Bohm applies this world view to develop his ontological interpretation of quantum mechanics (sometimes referred to as *Bohmian mechanics, pilot-wave theory,* or the *de Broglie-Bohm theory*).[55] His approach offers the same computational outcomes as the Copenhagen interpretation, but in Bohm's approach it becomes possible to give an account of each individual quantum process and also to give a coherent account of measurement. This latter account is achieved by representing the wave function as a quantum field (a Ψ-field) which serves, among other things, as a function from which actual probabilities can be derived (Piechocinska, 2005, p. 19). In this approach, each particle has a well-defined position and momentum and has

[55] The de Broglie-Bohm theory is not a favorite in the physics community, but quantum field theorist J. Bricmont (2016) offers a compelling defense of the theory and cogently argues that this theory is the only candidate quantum theory (p. 292) that gives a coherent account of all of the following:

(1) Trajectories associated with particles without running into contradiction.
(2) The measurement formalism, including the collapse rule.
(3) The no hidden variables theorems, which are explained by the contextuality of measurements and the active role of the measuring devices.
(4) The apparent randomness of quantum mechanics, which follows, in a fully deterministic theory, from rather natural assumptions about initial conditions.
(5) The unavoidable nonlocality fo any theory reproducing quantum predictions.

a quantum field (Ψ-field) associated with it. Adhering to the view of wholeness as primary, the particle and its quantum field are seen at a deeper level to be different aspects of the same process.

Bohm (1981) considers that the starting point for understanding the universe is a kind of universal implicate order, which is "one unbroken whole, including the entire universe, with all its 'fields' and 'particles' " (p. 189). He explains:

> ... matter as a whole can be understood in terms of the notion that the implicate order is the immediate and primary actuality... while the explicate order can be derived as a particular, distinguished case of the implicate order. (p. 197)

The dynamics by which enfoldment and unfoldment continuously occur is called by Bohm *holomovement*. Holomovement causes various aspects of the implicate order to appear in the manifest world and then later disappear. He explains (1981):

> [What is fundamental and primary in actuality] can perhaps best be called *Undivided Wholeness in Flowing Movement*. This view implies that flow is, in some sense, prior to that of the "things" that can be seen to form and dissolve in this flow.... That is, there is a universal flux that cannot be defined explicitly but which can be known only implicitly, as indicated by the explicitly definable forms and shapes... that can be abstracted from the universal flux. In this flow, mind and matter are not separate substances. Rather, they are different aspects of one whole and unbroken movement. (p. 11)

In Bohm's approach, we find a close parallel to the world view we have suggested in this article: the multitude of individual particles should be viewed as precipitations of the dynamics of an underlying source or wholeness. Moreover, we have argued that this principle is naturally realized in the mathematical context in the theory ZFC + WA, in which the fundamental dynamics are embodied in a WA-embedding $j : V \to V$, and all mathematical objects (i.e. *sets*) arise from its dynamics. The universe V is the wholeness of mathematics, transformed within itself by j, which, like Bohm's universal flux or holomovement is necessarily *undefinable*.

Seeing these parallels herself, physicist Barbara Piechocinska, in her Ph.D. thesis (2005), laid the groundwork for a systematic development of Bohm's theory using, as a mathematical starting point, the theory ZFC + WA.

To make use of the foundation ZFC + WA, Piechocinska begins with results from Laver (1995), and additional observations in Corazza (2017), concerning the monogenic left-distributive algebra $(\mathcal{A}_j, \cdot)$. For an I_3-embedding $j : V_\lambda \to V_\lambda$ with critical point κ, $\mathcal{A}_j$ is defined as the smallest collection satisfying the following:

$$\begin{array}{ll} - & j \in \mathcal{A}_j \\ - & \text{if } k, \ell \in \mathcal{A}_j, \text{ then } (k \cdot \ell) \in \mathcal{A}_j. \end{array} \tag{61}$$

We would like to think of $\mathcal{A}_j$ as a j-class in a model $(V, \in, j)$ of ZFC + WA, but at first glance this appears to be impossible since each of its elements is already a j-class. However, in Corazza (2017), we show how to abstract from the definition just given a set of *expressions*, defined with the same clauses as in (61), but independently of the notion of elementary embeddings. Then any particular

$i \in \mathcal{A}_j$ can be viewed as a realization of an expression, obtained by substituting suitable restrictions of j for each element in the expression. In this way, the encoding of $\mathcal{A}_j$ *is* a j-class and allows us to prove, now in the context of ZFC + WA, the main theorems found in Corazza (2017) about $\mathcal{A}_j$.

Proposition 12.134 (Laver, 1995) *$(\mathcal{A}_j, \cdot)$ is a left-distributive algebra with a single generator j. In particular, for all $i, k, \ell \in \mathcal{A}_j$, $i \cdot (k \cdot \ell) = (i \cdot k) \cdot (i \cdot \ell)$.*

We also wish to define the *applicative iterates of* j, denoted $j^{[n]}$, $n \in \omega$.

$$\begin{array}{rcl} j^{[1]} & = & j \\ j^{[2]} & = & j \cdot j \\ j^{[n+1]} & = & j \cdot j^{[n]} \end{array}$$

Let $\mathcal{S}_j = \{j^{[n]} \mid n \in \omega\}$. The next proposition is proved by induction on n.

Proposition 12.135 $\mathrm{crit}(j^{[n+1]}) = j^n(\kappa)$. □

Clearly $\mathcal{S}_j \subseteq \mathcal{A}_j$, so, by our coding technique, $\mathcal{S}_j$ may also be considered to be a j-class.

As observed earlier (p. 111), each element of $\mathcal{A}_j$ is a WA-embedding, so, intuitively speaking, each represents, like j, the dynamics of wholeness.

Define a "successor function" $s_j : \mathcal{S}_j \to \mathcal{S}_j$ by $s_j(i) = j \cdot i$. It is easy to see that $(\mathcal{S}_j, s_j, j)$ is an initial Dedekind algebra, isomorphic to $(\omega, s, 0)$. However, in this version of the "natural numbers," each element fully embodies the dynamics of wholeness. In this context $\mathcal{A}_j$ plays a role similar to that of the usual rational numbers; this point is made clear in Proposition 12.139 below.

We write

$$\mathrm{crit}(\mathcal{A}_j) = \{\gamma \mid \text{for some } k \in \mathcal{A}_j,\ \gamma = \mathrm{crit}(k)\}.$$

We need two results from Laver (1995).

Theorem 12.136 (Laver, 1995) *$\mathrm{crit}(\mathcal{A}_j)$ has order-type ω. In fact, for each $n \in \omega$, $\mathrm{crit}(\mathcal{A}_j) \cap [\kappa_n, \kappa_{n+1})$ is finite. Also, for each $n > 2$, $\mathrm{crit}(\mathcal{A}_j) \cap (\kappa_n, \kappa_{n+1})$ is nonempty.* □

Theorem 12.137 (Laver, 1995) *If $i, k \in \mathcal{A}_j$ and $i \restriction \mathrm{crit}(\mathcal{A}_j) = k \restriction \mathrm{crit}(\mathcal{A}_j)$, then $i = k$.* □

Using Theorem 12.136, we define $e : \omega \to \mathrm{crit}(\mathcal{A}_j)$ to be the unique increasing enumeration of $\mathrm{crit}(\mathcal{A}_j)$. For every pair (i, k) from $\mathcal{A}_j$ for which $i \neq k$, let $m_{i,k}$ be the least $n \in \omega$ such that $i(e(n)) \neq k(e(n))$. By Theorem 12.137, $m_{i,k}$ exists. Define $d : \mathcal{A}_j \times \mathcal{A}_j \to \mathbb{R}$ by

$$d(i,k) = \begin{cases} 0 & \text{if } i = k \\ 1/(m_{i,k}+1) & \text{otherwise} \end{cases}$$

The next proposition has a straightforward proof.

Proposition 12.138 *d is a metric on $\mathcal{A}_j$.*

Recall that a metric space (X, ρ) is *dense-in-itself* if for every $x \in X$, there is a sequence $\langle y_n \rangle_{n \in \omega}$, where each y_n is different from x, which converges to x.

The next proposition is proved in Corazza (2017).

PROPOSITION 12.139 *$(\mathcal{A}_j, d)$ is dense-in-itself.*

One direction for founding physics applications on the theory ZFC + WA would be to consider a *completion* of $(\mathcal{A}_j, d)$. A natural way to do this is to embed $(\mathcal{A}_j, d)$ in the space ω^ω of all functions from ω to ω. We let $\omega^{\uparrow\omega}$ denote the set of strictly increasing functions from ω to ω. Recall that defining $\rho : \omega^\omega \times \omega^\omega \to \mathbb{R}$ by

$$\rho(f,g) = \begin{cases} 0 & \text{if } f = g \\ 1/(m+1) & \text{if } f \neq g \text{ and } m \text{ is least for which } f(m) \neq g(m) \end{cases}$$

yields the result that (ω^ω, ρ) is a complete metric space.

We may embed (A_j, d) into (ω^ω, ρ) by mapping each $k \in \mathcal{A}_j$ to the function $f_k : \omega \to \omega$ defined by:

$$f_k(n) = e^{-1}(k(e(n))). \tag{62}$$

One shows easily (Corazza, 2017) that the map $F : k \mapsto f_k \in \omega^{\uparrow\omega}$ is an isometry. Therefore, the image B of $(\mathcal{A}_j, d)$ under F is a representation of $\mathcal{A}_j$ by increasing functions $\omega \to \omega$.[56]

Let $\bar{\mathcal{A}}_j$ be the closure of B in ω^ω. Let $\bar{d}$ be the restriction of ρ to $\bar{\mathcal{A}}_j \times \bar{\mathcal{A}}_j$. Then $(\bar{\mathcal{A}}_j, \bar{d})$ is the completion of $(\mathcal{A}_j, d)$, which is separable since $\mathcal{A}_j$ is countable and dense-in-itself. Indeed, $(\bar{\mathcal{A}}_j, \bar{d})$ is a perfect Polish space. Using this fact, it is possible to define a Hilbert space on $\bar{\mathcal{A}}_j$ and thereby provide a context for studying quantum mechanics; see Corazza (2017) for details.

Piechocinska makes use of the space $\mathcal{A}_j$ in a different way to find applications to physics. It is known that another realization of a monogenic left-distributive algebra can be found in the context of the *braid group B_∞ with infinitely many generators* $b_1, b_2, \ldots$ (Dehornoy, 2000). One can define an operation $\cdot$ on B_∞ that makes B_∞ a left-distributive algebra. It follows that if we form the left-distributive subalgebra $B_1 = \langle b_1 \rangle$ of B_∞, then B_1 is also an example of a *free* (monogenic) left-distributive algebra; an isomorphic copy of B_1 is the algebra B^{sp} of *special braids* (self-coloring braids).

Piechocinska (2005) uses this connection to explore several applications in physics. She first shows how to encode the structure of braids into the plane, which in turn yields a *Temperley-Lieb (planar) algebra*. She then summarizes the ways in which such algebras are used and interpreted in statistical and quantum physics, including the study of *anyons* (a generalization of bosons and fermions); lattice statistical physics (using the observation that links are definable from braids); the *Potts model* (in statistical mechanics, it is a model of interacting spins on a crystalline lattice); and von Neumann algebras (algebras of *observables in quantum physics*; Piechocinska mentions in her work the fact that every Temperley-Lieb algebra can be represented as a tower of von Neumann algebras).

Piechocinska's work shows one way to move in the direction of building physics—in particular, quantum mechanics—by "starting from wholeness." Starting from the theory ZFC + WA, she locates elements of the theory that have the potential for describing fundamental processes at the foundation of physics. In her work, she viewed the algebra $\mathcal{A}_j$ as a *process algebra* that could be represented in a mathematical form familiar to the world of physics (namely, via *special braids*) and

[56] Dougherty-Jech [24] refer to B as an *embedding algebra*.

reviewed results known to physicists about role of these algebras in understanding physical phenomena. The original goal (Piechocinska, 2003) of her program was to view $\mathcal{A}_j$, or some variant thereof, as a process algebra that could describe the move of wholeness as a ground for all physical phenomena—*holomovement*—as envisioned in Bohm's theories. This goal was an objective of a number of researchers at the time her thesis was being written, most prominently Bohm's associate Basil Hiley, who sought to advance Bohm's work. In more recent times, Hiley's work toward this objective has been quite successful, though it is not related in any obvious way to Piechocinska's. See Hiley (2011).

Though Piechocinska's work did not fully achieve the intended objective, it lays the groundwork for addressing an interesting foundational question. While Hiley has made progress at a technical level in mathematically representing the holomovement of Bohm, is it possible to unfold a mathematical representation of this kind in a canonical way, starting from a mathematical foundation for wholeness, in the spirit of Piechocinska's thesis?

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