Corrections to "Laver sequences for extendible and super-almost-huge cardinals"

(This appears as the Appendix in [1] and refers to results in that paper.)

The purpose of this section is to correct several errors that were published in [2]. These errors became apparent during the period that the present paper was being reviewed and modified. The first error is an erroneous statement about the existence of $\mathcal{E}_\kappa^{sah}$-Laver sequences that was based on an earlier (incorrect) version of the present paper. The other two errors are incorrect proofs of correctly stated theorems about the existence of $\mathcal{E}_\kappa^{ext}$-Laver sequences. We present an outline of the results that are in error and follows these with corrections.

Error #1: [2, Theorem 5.1(5)]. This theorem refers to a result that had been stated in an earlier version of this paper which we formulated in [2] as an axiom $\mathrm{SAH}_4(\kappa)$. We formulate an alternative version of this axiom and show that it has all the features that the version in [2] was supposed to have.

Error #2: [2, Theorem 4.1]. The theorem states that, assuming only that κ is extendible, the construction $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{ext})$ produces an $\mathcal{E}_\kappa^{ext}$-Laver sequence. The result is true and, though the ideas for a correct proof do appear in [2], the proof given there is not correct; we give a correct proof here.

Error #3: [2, Theorems 4.4 and 4.5]. These theorems are used in [2] to demonstrate that the construction $\mathbf{CC}(t, \mathcal{E}_\kappa^{ext})$ also produces an $\mathcal{E}_\kappa^{\theta}$-Laver sequence assuming only that κ is extendible. This result is also basically true, as we show below, but the statements of the theorems mentioned are incorrect.

Correction to Error #1

Our original approach to the proof that the functions obtained from either $\mathbf{CC}(t, \mathcal{E}_\kappa^{sah})$ or $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ are $\mathcal{E}_\kappa^{sah}$-Laver, and our original proof of Theorem (3) in particular, were somewhat different from our current version. In [2], an axiom that we called $\mathrm{SAH}_4(\kappa)$ was extracted from the original paper, and we argued in [2, Theorem 5.3] that $\mathrm{SAH}_4(\kappa)$ is sufficient to prove that f^R obtained in $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ is $\mathcal{E}_\kappa^{sah}$-Laver; in [2, Theorem 5.1(5)] that "κ is superhuge" strongly implies $\mathrm{SAH}_4(\kappa)$; and in [2, Theorem 5.1(4)] that $\mathrm{SAH}_4(\kappa)$ is strictly consistency-wise stronger than another axiom $\mathrm{SAH}_2(\kappa)$ that is also concerned with super-almost-huge cardinals. The error occurs in [2, Theorem 5.1(5)]: our proof, which first appeared in the original version of this paper, is incorrect; we don't know at this time the consistency strength of $\mathrm{SAH}_4(\kappa)$, or whether it is consistent with any known large cardinal axiom. The other results are correct but not particularly meaningful in the absence of a reasonable upper bound on $\mathrm{SAH}_4(\kappa)$.

Our plan here is to replace $\mathrm{SAH}_4(\kappa)$ with a different axiom, and show that the results originally obtained for $\mathrm{SAH}_4(\kappa)$ go through for our new axiom. Since we have not given the details of $\mathrm{SAH}_4(\kappa)$

as it appears in [2], there should be no confusion if we use the same name for our new version of this axiom. We begin by giving the statements of the axioms $\mathrm{SAH}_0(\kappa)$-$\mathrm{SAH}_3(\kappa)$ from [2], and then stating our new $\mathrm{SAH}_4(\kappa)$. We then provide some other background material and conclude with the relevant proofs.

If κ is super-almost-huge, let $\Lambda = \{\lambda : \lambda \text{ is an a.h. target for } \kappa\}$, and, for any class $\mathbf{C}$, $\mathbf{C}' = \{\nu : \nu \text{ is a limit point of } \mathbf{C}\}$.

$\mathrm{SAH}_0(\kappa)$: κ is super-almost-huge.

$\mathrm{SAH}_1(\kappa)$: κ is super-almost-huge and the class $\Lambda \cap \Lambda'$ is bounded.

$\mathrm{SAH}_2(\kappa)$: κ is super-almost-huge, and the class $\Lambda \cap \Lambda'$ is unbounded, and there is μ such that for all regular $\rho > \mu$ the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is nonstationary in ρ.

$\mathrm{SAH}_3(\kappa)$: κ is super-almost-huge and for arbitrarily large regular ρ, the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is stationary in ρ.

Our new version of $\mathrm{SAH}_4(\kappa)$ is the following:

$\mathrm{SAH}_4(\kappa)$: κ is super almost huge and there are unboundedly many λ such that λ is an a.h. target, and for some coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ that satisfies $\mathcal{B}(\kappa, \lambda)$, the set $\{\alpha < \lambda : \langle U_\eta : \kappa \leq \eta < \alpha\rangle \text{ satisfies } \mathcal{B}(\kappa, \alpha)\}$ is unbounded in λ.

To compare relative strengths of these hypotheses, we introduced in [2] the following notation: Given properties $A(\kappa), B(\kappa)$ that depend on an infinite cardinal κ, we write:

$A(\kappa) \xrightarrow{\text{ZFC}} B(\kappa)$	iff	"$A(\kappa)$ implies $B(\kappa)$"
	iff	$\text{ZFC} \vdash A \longrightarrow B$;
$A(\kappa) \xrightarrow{con} B(\kappa)$	iff	"$A(\kappa)$ is consistency-wise at least as strong as $B(\kappa)$"
	iff	$Con(\text{ZFC} + A(\kappa)) \xrightarrow{\text{ZFC}} Con(\text{ZFC} + B(\kappa))$;
$A(\kappa) \xrightarrow{s} B(\kappa)$	iff	"$A(\kappa)$ is strictly consistency-wise stronger than $B(\kappa)$"
	iff	$A(\kappa) \xrightarrow{con} B(\kappa)$ and $A(\kappa) \xrightarrow{\text{ZFC}} Con(\text{ZFC} + B(\kappa))$;
$A(\kappa) \xrightarrow{si} B(\kappa)$	iff	"$A(\kappa)$ strongly implies $B(\kappa)$"
	iff	$A(\kappa) \xrightarrow{\text{ZFC}} B(\kappa)$ and $\{\alpha < \kappa : B(\alpha)\}$ has normal measure 1.

The terminology *strongly implies* was introduced in [3].

1.1 Theorem. *$SAH_4(\kappa) \xrightarrow{con} SAH_2(\kappa)$.*

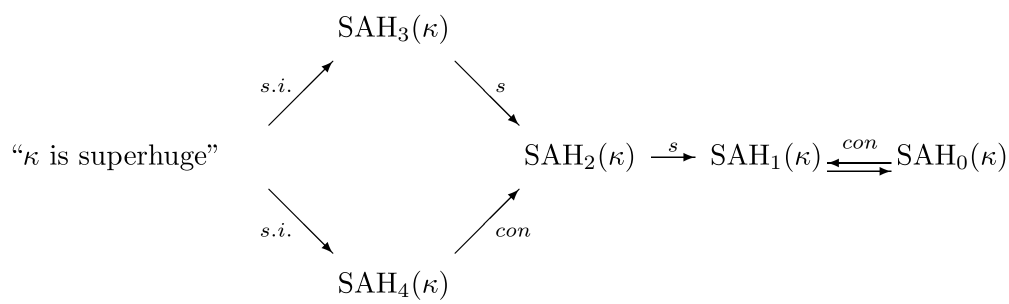
Proof. It is clear that $\mathrm{SAH}_4(\kappa)$ implies that $\Lambda \cap \Lambda'$ is unbounded. If there is a regular cardinal in which Λ is stationary, let ρ be the least such. Then $\Lambda \cap \Lambda' \cap \Lambda''$ is also stationary in ρ. Then if $\lambda = \min \Lambda \cap \Lambda' \cap \Lambda''$, it follows that $V_\lambda \models \mathrm{SAH}_2(\kappa)$. ∎

1.2 Theorem. *"κ is superhuge" $\xrightarrow{s.i.}$ $SAH_4(\kappa)$.*

Proof. Let $j : V \to N$ be a huge embedding with critical point κ. It suffices to show that

$$N \models \mathrm{SAH}_4(\kappa).$$

The proof of Theorem (3) shows that $V_{j(\kappa)} \models \mathrm{SAH}_4(\kappa)$. It is easy to see that the statement $\mathrm{SAH}_4(\kappa)$ is globalized local and hence Π_3^{ZFC}. Thus, if $N \models \neg\mathrm{SAH}_4(\kappa)$, this fact would reflect down to $V_{j(\kappa)}$ (since, in N, $j(\kappa)$ is superhuge, whence $V_{j(\kappa)} \prec_3 V$), giving a contradiction. ∎



1.3 Theorem. *$SAH_4(\kappa)$ implies that for any well-ordering R of V_κ, the function f^R obtained in the construction $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ is $\mathcal{E}_\kappa^{sah}$-Laver.*

Proof. This follows from Theorem (3) and Theorem . ∎

Correction to Error #2.

The second error in [2] that we address is the proof of [2, Theorem 4.1], which asserts that, assuming only that κ is extendible, there is a $\mathcal{E}_\kappa^{ext}$-Laver sequence. The theorem is true; we provide a correct proof below. We begin with a definition:

1.4 Definition. A well-ordering R of V_δ is *rank-preserving* if, for all $x, y \in V_\delta$, if $rank(x) < rank(y)$ then $(x, y) \in R$.

1.5 Theorem. *If κ is extendible, there is an $\mathcal{E}_\kappa^{ext}$-Laver sequence at κ. In particular, assuming κ is extendible, for every rank-preserving well-ordering R and every $t : \kappa \to V_\kappa$ definable in $\langle V_\kappa, \in, R\rangle$, the function f^R constructed in $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{ext})$ is $\mathcal{E}_\kappa^{ext}$-Laver.*

Proof. Suppose x_0, λ_0 witness that f^R is not $\mathcal{E}_\kappa^{ext}$-Laver at κ. Let α be such that $V_\kappa \prec V_\alpha$ and $x, \lambda \in V_\alpha$. Let $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$. Let D denote the normal ultrafilter derived from j. As usual, the definition of f^R implies that exactly one of the following sets is in D:

$$S_1 = \{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \text{“}f^R \restriction \alpha \text{ is } \mathcal{E}_\alpha^{ext}\text{-Laver at } \alpha\text{”}\};$$
$$S_2 = \{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \exists\lambda\, \phi(f^R \restriction \alpha, f^R(\alpha), \lambda)\}.$$

If $S_1 \in D$, then, $\langle V_{j(\kappa)}, \in, j(R)\rangle \models$ “f^R is $\mathcal{E}_\kappa^{ext}$-Laver”. By adequate absoluteness of θ_{ext}, we can find $i : V_\xi \to V_\gamma \in \mathcal{E}_\kappa^{ext} \cap V_{j(\kappa)}$ for which $i(f^R)(\kappa) = x_0$ and $i(\kappa) > \lambda_0$, which contradicts our assumptions about x_0, λ_0.

Thus, $S_2 \in D$, whence

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \exists\lambda\, \phi(f^R, j(f^R)(\kappa), \lambda). \tag{1.1}$$

Let $x_1 = j(f^R)(\kappa)$ and $\lambda_1 < j(\kappa)$ be witnesses to this formula. By the formulation of the second case in the definition of f^R, x_1 must be the R-least set for which f^R fails to be $\mathcal{E}_\kappa^{ext}$-Laver. Since R is rank-preserving, this means that, since x_0 is another set witnessing Laver-failure of f^R,

$$rank(x_1) \le rank(x_0) < \alpha.$$

Since $V_\kappa \prec V_\alpha$, α must be a limit; thus, let β be such that $rank(x_1) < \beta < \alpha$. Then if $i = j \restriction V_\beta : V_\beta \to V_{j(\beta)}$, $(i, V_{j(\beta)}) \in \mathcal{E}_\kappa^{ext}$, $i(\kappa) > \lambda_1$ and $i(f^R)(\kappa) = x_1$. Since $\beta < \alpha$, it follows that $j(\beta) < \eta$ and $(i, V_{j(\beta)}) \in V_\eta$. Thus

$$V_\eta \models \exists e\, \exists\beta\, \exists\zeta\, \big[(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \text{cp}(e) = \kappa \wedge$$
$$\beta < \alpha \wedge e(\kappa) > \lambda_1 \wedge e(f^R)(\kappa) = x_1\big].$$

Because $V_\kappa \prec V_\alpha$, it follows that $V_{j(\kappa)} \prec V_\eta$. Since $\kappa, f^R, \alpha, \lambda_1, x_1 \in V_{j(\kappa)}$, we have:

$$V_{j(\kappa)} \models \exists e\, \exists\beta\, \exists\zeta\, \big[(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \text{cp}(e) = \kappa \wedge$$
$$\beta < \alpha \wedge e(\kappa) > \lambda_1 \wedge e(f^R)(\kappa) = x_1\big].$$

But now any witness $(e, V_\zeta) \in \left(\mathcal{E}_\kappa^{ext}\right)^{V_{j(\kappa)}} = \mathcal{E}_\kappa^{ext} \cap V_{j(\kappa)}$ contradicts (1.1), and we have a contradiction. ∎

Correction to Error #3. The goal of [9, Theorems 4.4, 4.5] was to show that the function defined in $\mathbf{CC}(t, \mathcal{E}_\kappa^{ext})$ is $\mathcal{E}_\kappa^{ext}$-Laver, assuming only that κ is extendible. This result is basically true, but the two theorems cited are incorrect.

Theorem 4.4 in [9] claimed that, if in Theorem 5.13 of the present paper ([1]), we replace “superstrong embeddings having arbitrarily large targets” with embeddings $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$ for which α is an arbitrarily large inaccessible, then the conclusion of Theorem will hold for $\mathcal{E}_\kappa^{ext}$.

However, this replacement is not sufficient for the proof to go through. Indeed, conditions (1) and (2) would guarantee that for each such $j : V_\alpha \to V_\eta$,

$$\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}^{ext}_\alpha\text{-Laver}\} \in D, \tag{1.2}$$

where D is the normal ultrafilter over κ derived from j; clearly this would imply that κ is the κth extendible cardinal, a conclusion that is too strong to be obtained from the hypothesis.

Theorem 4.5 in [9] asserted that properties (1) - (3) of Theorem must hold in V_η whenever $j : V_\alpha \to V_\eta \in \mathcal{E}^{ext}_\kappa$ is such that $V_\kappa \prec V_\alpha$ and α is inaccessible. Again this is impossible because properties (1) and (2) would again imply (1.2) where D is the normal ultrafilter derived from j, and the consequences are too strong for the hypothesis.

Nonetheless, the f obtained from $\mathbf{CC}(t, \mathcal{E}^{ext}_\kappa)$ can be shown to be $\mathcal{E}^{ext}_\kappa$-Laver, assuming only that κ is extendible, if we make the following slight modification in the definition of f: In the second clause, we now require that if α is a cardinal but $f \restriction \alpha$ is not $\mathcal{E}^{ext}_\alpha$-Laver, then $f(\alpha)$ is defined to be a set $x \in V_\kappa$ *of least possible rank* that witnesses this failure. Let us call the function defined in this way $f' = f'_t$. Then the following is true:

1.6 Theorem. *Suppose κ is extendible. Then for any choice of the parameter t, the function f' is $\mathcal{E}^{ext}_\kappa$-Laver at κ.*

Proof. The idea of the proof is basically the same as the proof of Theorem 1.5; we give an outline. Assume x_0, λ_0 witness that f' is not $\mathcal{E}^{ext}_\kappa$-Laver. Pick an inaccessible α so that $x_0, \lambda_0 \in V_\alpha$ and $V_\kappa \prec V_\alpha$, and pick any $j : V_\alpha \to V_\eta \in \mathcal{E}^{ext}_\kappa$. If D is the normal ultrafilter over κ derived from j, either $S_1 \in D$ or $S_2 \in D$ where

$$S_1 = \{\alpha < \kappa : \text{“}f' \restriction \alpha \text{ is } \mathcal{E}^{ext}_\alpha\text{-Laver at } \alpha\text{”}\};$$
$$S_2 = \{\alpha < \kappa : \exists\lambda\, \phi(f' \restriction \alpha, f'(\alpha), \lambda)\}.$$

Reasoning as in Theorem 1.5, one shows $S_1 \notin D$. Assuming $S_2 \in D$, we have that for some $\lambda_1 < j(\kappa)$,

$$V_\eta \models \phi(f', j(f')(\kappa), \lambda_1). \tag{1.3}$$

Let $x_1 = j(f')(\kappa)$. As in Theorem 1.5, $rank(x_1) < \alpha$, so, as in that proof, we can use $j \restriction V_\beta$ as a counterexample to (1.3), giving the desired contradiction. ∎

References

[1] Corazza, P., *The wholeness axiom and Laver sequences,* accepted for publication in **Annals of Pure and Applied Logic**, 108 pages.

[2] ———, *Laver sequences for extendible and super-almost-huge cardinals*, **Journal of Symbolic Logic**, Vol 64, No 3, Sep 1999.

[3] Solovay, R., Reinhardt,W., Kanamori, A., *Strong axioms of infinity and elementary embeddings*, **Annals of Mathematical Logic,** 13, 1968, pp. 63-116.