Laver Sequences for Extendible and Super-Almost-Huge Cardinals

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Abstract. Versions of Laver sequences are known to exist for supercompact and strong cardinals. Assuming very strong axioms of infinity, Laver sequences can be constructed for virtually any globally defined large cardinal not weaker than a strong cardinal; indeed, under strong hypotheses, Laver sequences can be constructed for virtually any *regular class* of embeddings. We show here that if there is a regular class of embeddings with critical point κ, and there is an inaccessible above κ, then it is consistent for there to be a regular class that admits no Laver sequence. We also show that extendible cardinals are *Laver-generating,* i.e., that assuming only that κ is extendible, there is an extendible Laver sequence at κ. We use the method of proof to answer a question about *Laver-closure* of extendible cardinals at inaccessibles. Finally, we consider Laver sequences for super-almost-huge cardinals. Assuming slightly more than super-almost-hugeness, we show that there are super-almost-huge Laver sequences, improving the previously known upper bound for such Laver sequences. We also describe conditions under which the canonical construction of a Laver sequence fails for super-almost-huge cardinals.

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§1. Introduction.

Laver sequences were originally defined by Laver [La2] who showed that if κ is supercompact, there is a (supercompact) Laver sequence, and used this notion to prove that it is consistent, relative to supercompactness of κ, for κ to be indestructible. Since then, Laver sequences for supercompact cardinals have been used in a variety of other proofs, most notably the consistency of PFA relative to a supercompact (see [De]). In [GS], Gitik and Shelah observe that Laver sequences for strong cardinals can be constructed, assuming only a strong cardinal, and used these to obtain indestructibility results for strong cardinals. These results and others like them lead naturally to the question: Which large cardinals admit their own brand of Laver sequences, and which results (like consistency of indestructibility) that rely on such sequences go through for these other large cardinals?

In order to give precise meaning to "Laver sequence" for large cardinals in general, we developed in [C1] a generalized context for studying properties of globally defined large cardinal axioms—namely, classes of set embeddings of the form $j : V_\beta \to M$. Roughly, a globally defined large cardinal, like a supercompact, can be viewed as such a class of embeddings by considering (possibly a subcollection of) all restrictions of embeddings of the form $j : V \to M$ that define the large cardinal (note, on the other hand, that extendible cardinals are already defined in terms of a class of embeddings). We then provided a uniform definition of Laver sequences for large cardinals by defining this notion in the context of classes of embeddings. Since any reasonable definition of Laver sequence would require that for every x there is a j in the class whose codomain includes x, we restricted our consideration of classes to those that satisfy this requirement; we called such classes *regular* (a more precise definition is given in Section 2).

The following results from [C1] are relevant here:

1. *Assuming the Wholeness Axiom, if $\mathcal{E}$ is a regular class of embeddings "compatible" with j, then $\mathcal{E}$ admits an $\mathcal{E}$-Laver sequence.*
2. *If κ is superhuge, then κ admits an extendible Laver sequence and a super-almost-huge Laver sequence.*

In Section 2, we define and explain the terminology of (1). Roughly, the Wholeness Axiom (*WA*) is an axiom schema in the language $\{\in, j\}$ that asserts the existence of a nontrivial elementary embedding $j : V \to V$. The theory *ZFC+ WA* is not known to be inconsistent; Kunen's well-known inconsistency result does not apply since the embedding j is not required to be weakly definable (see Section 2). *WA* is known to have consistency strength strictly between "super-n-huge for every

n" and I_3. The hypothesis in (2) was improved in [C2] to "hyperextendible" in the extendible case. [C1] raised the following questions:

A. In light of (1), do there exist regular classes of embeddings that do *not* admit Laver sequences?

B. Can the hypothesis in (2) be improved? In particular, are extendible and super-almost-huge cardinals *Laver-generating*? That is, is the hypothesis "κ is extendible" ("κ is super-almost-huge") sufficient to prove the existence of a(n) extendible (super-almost-huge) Laver sequence?

In this paper, we show that the answer to (A) is consistently "yes" (relative to a mild extra large cardinal assumption). For (B), we prove that extendible cardinals are Laver-generating, and improve upon known results for the super-almost-huge case. The method of proof in the extendible case is also used to answer a question (also raised in [C1]) concerning whether extendible cardinals are *simply Laver-closed at inaccessibles* (defined in Section 2).

I would like to thank the organizing committee for the PhD Centennial Conference at UW Madison for inviting me to speak on the topics of this paper, and K. Kunen for interesting and helpful discussions on some of the questions discussed here. Thanks also go to the referee for simplifying one of the proofs in Section 5.

§2. Preliminaries.

In this section we review the basic definitions and theorems we'll need for the rest of the paper and state the questions we plan to address in later sections. Detailed information about large cardinals can be found in [Je],[Ka]. Our treatment of strong cardinals follows [MS]. Results on the Wholeness Axiom and regular classes of embeddings can be found in [C1].

Large cardinals.

In this paper, we will focus on three large cardinal notions: *strong, extendible,* and *super-almost-huge,* along with a weakening of I_3 which we call the *Wholeness Axiom.*

For $\lambda > \kappa$, κ is *λ-strong* if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $V_\lambda \subset M$; the map j is called a *λ-strong embedding.* κ is *strong* if κ is λ-strong for every $\lambda > \kappa$.

Strong cardinals can also be defined in terms of extenders; rather than give a full definition, we state the basic properties of extenders that we will need later. Suppose Y is a transitive set and κ is a cardinal. An extender with critical point κ and support Y is a sequence $E = \langle E_a : a \in {}^{<\omega}[Y]\rangle$ of κ-complete ultrafilters over ${}^aV_\kappa, a \in {}^{<\omega}[Y]$, satisfying several additional technical properties which we do not enumerate here. If $\lambda > \kappa$ and E is an extender with critical point κ and support V_λ one can define a well-founded ultrapower $Ult(V, E)$ (which we identify with its transitive collapse); in this case, there is a canonical elementary embedding $i_E : V \to Ult(V, E)$ having critical point κ and definable from $\{\kappa, E\}$ such that $i_E(\kappa) \geq \lambda$ and $V_\lambda \subset Ult(V, E)$. It is possible for $i_E(\kappa) = \lambda$, but, as is easily verified, we are assured that $i_E(\kappa) > \lambda$ whenever λ is a successor ordinal. (Although we will not need this here, it is possible to obtain, for each such extender E, and *every* λ, an elementary embedding $i^*_E : V \to M$ having the same properties as i_E and with $i^*_E(\kappa) > \lambda$; see [Ka, 26.7] or [C1, Section 2].) In [MS], the following propositions are proved:

2.1 Proposition. *Suppose $j : V \to N$ is an elementary embedding with N transitive and with critical point κ. Let Y be a transitive set with $\kappa \in Y \subseteq V_{j(\kappa)} \cap N$. Define $E = \langle E_a : a \in {}^{<\omega}[Y]\rangle$ as follows: For each a and all $X \subseteq {}^aV_\kappa$, let $E_a(X) = 1$ iff $j^{-1} \mid j(a) \in j(X)$. Then*

(a) E is an extender.

(b) Let $Ult(V, E)$ denote the ultrapower of V by E. Define $k : Ult(V, E) \to N$ by

$$k([F]) = \big(j(F)\big)\big(j^{-1} \mid j(a)\big)$$

whenever $F : {}^aV_\kappa \to V$. Then k is an elementary embedding, $k \mid Y = id_Y$, and $j = k \circ i_E$.∎

We call the extender E above the *extender derived from j with support Y*. In [MS] it is observed that every extender can be derived from some elementary embedding. In particular, if E is an extender with critical point κ and support Y, the extender derived from i_E with support Y is E itself. The next proposition is proved in [C1]; see also [Ka, 26.7b].

2.2 Proposition. *Suppose κ is an infinite cardinal and $\kappa < \lambda$. Then κ is λ-strong if and only if there is an extender E with critical point κ and support V_λ.* ■

For $\kappa \leq \lambda$, we let $Ext(\kappa, \lambda) = \{E : E \text{ is an extender with critical point } \kappa \text{ and support } V_\lambda\}$. If $\kappa \leq \lambda \leq \mu$ and $F \in Ext(\kappa, \mu)$, then we denote by $F|\lambda$ the extender in $Ext(\kappa, \lambda)$ derived from i_F.

We will need to consider ultrapowers of the form $Ult(V_\nu, E)$ for regular cardinals ν, where E is an extender in V_ν. The following special case of [MS,Lemma 1.7] will be useful in this regard:

2.3 Proposition [MS]**.** *Suppose ν is a regular cardinal, $\kappa \leq \gamma < \nu$, and*

$$V_\nu \models \text{“}E \text{ is an extender with critical point } \kappa \text{ and support } V_\gamma.\text{”}$$

Let $i_E^{V_\nu}$ denote the canonical embedding of V_ν into $Ult(V_\nu, E)$. Then

$$V_{i_E^{V_\nu}(\gamma)+1} \cap Ult(V_\nu, E) = V_{i_E(\gamma)+1} \cap Ult(V, E). \blacksquare$$

Next we review extendible, super-almost-huge and superhuge cardinals. For $\eta \geq 0$, κ is *η-extendible* if there are ζ and an elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ with critical point κ such that $\eta < j(\kappa) < \zeta$. κ is *extendible* if κ is η-extendible for every η. An important fact about extendible cardinals is the following:

2.4 Proposition. *If κ is extendible, there are arbitrarily large λ such that $V_\kappa \prec V_\lambda$.* ■

A cardinal κ is *almost huge* if there exists an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that for each $\alpha < j(\kappa)$, M is closed under α-sequences; $j(\kappa)$ is called the *a.h. target* of j and j is called an *a.h. embedding*. κ is *super-almost- huge* if, for each γ there are $\lambda > \gamma$ and $j_\lambda : V \to M_\lambda$ such that j_λ is an a.h. embedding with a.h. target λ.

Almost-huge cardinals can be defined in terms of direct limits of ultrapowers as follows: if $\lambda > \kappa$, a sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ is *coherent* if for each η, U_η is a normal ultrafilter over $P_\kappa \eta$ and if $\kappa \leq \eta < \zeta < \lambda$, $U_\eta = U_\zeta | \eta$. (See [Ka].) For each such η, let M_η denote the transitive collapse of the ultrapower by U_η and let $j_\eta : V \to M_\eta$ denote the canonical embedding. Let $k_{\eta\zeta} : M_\eta \to M_\zeta$

denote the embeddings such that $j_\zeta = k_{\eta\zeta} \circ j_\eta$ and $k_{\eta\zeta}|\eta = \mathrm{id}_\eta$. We will call the following condition on a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ *Barbanel's Criterion* ($\mathcal{B}$):

> $\mathcal{B}(\kappa, \lambda)$**.** *$\lambda$ is inaccessible and for all η for which $\kappa \leq \eta < \lambda$ and all $h : P_\kappa \eta \to ON$, if $\{x \in P_\kappa \eta : h(x) < \kappa\} \in U_\eta$ then there is ζ such that $\eta \leq \zeta < \lambda$ and $\{x \in P_\kappa \zeta :$ ot $x = h(x \cap \eta)\} \in U_\zeta$.*

(What we have called Barbanel's Criterion is actually a slight modification of a criterion devised by Barbanel in [Ba]; see [C1, Section 2] for a discussion of several such equivalent criteria.)

2.5 Proposition ([Ba], [SRK])**.** *Suppose κ is an infinite cardinal and $\lambda > \kappa$. Then κ is almost huge with a.h. target λ if and only if there is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ satisfying $\mathcal{B}(\kappa, \lambda)$.*

Finally, suppose κ is almost huge and $\lambda < \mu$ are a.h. targets of κ. We call (λ, μ) an *a.h. coherent pair* if for each coherent sequence $\langle U_\eta : \kappa \leq \eta < \mu \rangle$ that satisfies $\mathcal{B}(\kappa, \mu)$, the coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ satisfies $\mathcal{B}(\kappa, \lambda)$.

Several of the results from [C1] that we will need concerning super-almost-huge cardinals require the notions of *huge cardinal* and *huge embedding*—notions with which we assume some familiarity (see [Ka], [C1] for more information). A cardinal κ is *superhuge* if for each α there is $\lambda > \alpha$ and a huge embedding $j : V \to M_\lambda$ with critical point κ and $j(\kappa) = \lambda$. When $j : V \to N$ is a huge embedding with critical point κ, the set $j(U)$, where U is any normal ultrafilter over κ, is itself a normal ultrafilter over $j(\kappa)$. In particular, if U is the normal ultrafilter derived from j, then for all $X \subseteq j(\kappa)$,

$$X \in j(U) \text{ iff } N_1 \models j(\kappa) \in j \cdot j(X),$$

where $N_1 = j \cdot j(N)$ and for all z, $j \cdot j(z) = j(j|V_{rank(z)+1})(z)$. (For more information on the embedding $j \cdot j$ see [C1] or [La1].) The following results are proven in [C1]:

2.6 Proposition [C1]**.** *Suppose $j : V \to N$ is a huge embedding with critical point κ and U is the normal ultrafilter over κ derived from j . Then*

1. *if κ is superhuge, $N \models V_{j(\kappa)} \prec_3 V$;*
2. *the following set is in $j(U)$: $\{\lambda < j(\kappa) : (\lambda, j(\kappa))$ is an a.h. coherent pair$\}$;*
3. *there is a set $T \in j(U)$ such that for all $\alpha < \beta$ both in T, (α, β) is an a.h. coherent pair.* ∎

The Wholeness Axiom

In [C1], we formulate an axiom of infinity which we call the *Wholeness Axiom*, or *WA*. The Wholeness Axiom (*WA*) is an axiom schema in an extended language $\{\in, \mathbf{j}\}$ that is intended to provide a near-minimal weakening of the assertion "there is a nontrivial elementary embedding $j : V \to V$" that is not obviously inconsistent with *ZFC*. In [C1], the details of this schema are developed; briefly, the axioms consist of all instances of Separation (but no instance of Replacement) for formulas having an occurrence of the symbol $\mathbf{j}$, together with all axioms of the form $\phi(x_1, x_2, \ldots, x_n) \longrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_n))$, and the axiom $\exists x\, \mathbf{j}(x) \neq x$. Omitting from the schema all instances of Replacement for $\mathbf{j}$-formulas provides a means to avoid a crucial step in Kunen's well-known inconsistency proof, because without Replacement for $\mathbf{j}$-formulas, there is no guarantee that the sequence $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots \rangle$ has a supremum. In the spirit of [E], if we call an $X \subset V$ *weakly definable in* V if $\langle V, \in, X \rangle$ satisfies all instances of Replacement in a language in which X is a predicate, then we can describe *WA* as the assertion that there is an elementary embedding $V \to V$ that is not weakly definable in V.

In a model of *WA*, the interpretation j of $\mathbf{j}$ is called the *WA-embedding* and its critical point is usually denoted κ. Henceforth, we will not attempt to continue observing the distinction between $\mathbf{j}$ and j; the fastidious reader may adopt the viewpoint that all arguments in which *WA* is applied take place within a model of *WA*. The following facts about *WA* were proved in [C1]:

1. If j is the *WA*-embedding with critical point κ, κ is the κth cardinal that is super-n-huge for every n; and there is an elementary chain

$$V_\kappa \prec V_{j(\kappa)} \prec \ldots \prec V_{j^n(\kappa)} \prec \ldots \prec V.$$

2. If $i : V_\lambda \to V_\lambda$ is an I_3 embedding, $\langle V_\lambda, \in, i \rangle$ is a model of *ZFC* + *WA*.

Regular Classes of Embeddings and $\mathcal{E}$-Laver Sequences

As described in the Introduction, regular classes of embeddings are to be thought of as classes of embeddings of the form $j : V_\beta \to M$ which satisfy the minimal requirements for defining a notion of Laver sequences—namely, that for each set x there should be a $j : V_\beta \to M$ in the class with $x \in M$. We give a quick outline here of the definitions needed to give a proper definition of "Laver-generating"; we mention some of the results we will need concerning these notions; and we will state and (briefly) motivate some of the open questions in this area that we will answer in this paper.

2.7 Definition (*Regular Classes*) Let $\theta(x,y,z,w)$ be a first-order formula. We will call θ a *suitable formula* if the following sentence is provable in *ZFC*:

$$\forall x,y,z,w\,[\theta(x,y,z,w) \Longrightarrow (w \text{ is a transitive set } \wedge z \in \text{ ON } \wedge$$
$$\wedge\ x : V_z \to w \text{ is an elementary embedding with critical point } y)].$$

For each cardinal κ and each suitable $\theta(x,y,z,w)$, let

$$\mathcal{E}^\theta_\kappa = \{(i,M) : \exists\beta\,\theta(i,\kappa,\beta,M)\}.$$

In order to ensure the proper degree of absoluteness for simple θ, we have had to pair elementary embeddings with their codomains in the definition of $\mathcal{E}^\theta_\kappa$; see [C1, Section 4] for a discussion. For convenience, when there is no chance of ambiguity, we will refer to the elements of $\mathcal{E}^\theta_\kappa$ as elementary embeddings (even though they are really ordered pairs) and we will understand an expression such as $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ to mean $(i,M) \in \mathcal{E}^\theta_\kappa$ and $i \in \mathcal{E}^\theta_\kappa$ to mean $(i,M) \in \mathcal{E}^\theta_\kappa$ for some M. Continuing with the definition:

$$\text{Dom } \mathcal{E}^\theta_\kappa = \{\beta : \exists i \in \mathcal{E}^\theta_\kappa\ (\text{dom } i = V_\beta)\};$$
$$\mathcal{E}^\theta_\kappa(\lambda) = \{i \in \mathcal{E}^\theta_\kappa : i(\kappa) > \lambda\}.$$

We shall call $\mathcal{E}^\theta_\kappa$ a *regular class of embeddings at* κ if

$$\forall\gamma > \kappa\,\exists\beta \geq \gamma\,\exists i\,\exists M\,[\theta(i,\kappa,\beta,M)\ \wedge\ i(\kappa) > \gamma\ \wedge\ V_\gamma \subset M].$$

Note in particular that a regular class is defined without any extra parameters (other than κ).

The regular classes that will concern us here, and that correspond to familiar large cardinal notions, are defined from the following suitable formulas; they correspond to the notions of strong, extendible and super-almost-huge cardinals, respectively.

$\theta_{str}(i,\kappa,\beta,M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\, \beta = \lambda + \omega\ \wedge i : V_\beta \to M$ is an elementary embedding with critical point $\kappa\ \wedge\ i(\kappa) > \lambda\ \wedge\ V_\lambda \subseteq M]$.

$\theta_{ext}(i,\kappa,\beta,M)$: $\exists\delta > 0\,\exists\zeta\ [\beta = \kappa + \delta\ \wedge\ M = V_\zeta\ \wedge\ i : V_\beta \to M$ is elementary with critical point $\kappa\ \wedge\ \beta < i(\kappa) < \zeta]$.

$\theta_{sah}(i,\kappa,\beta,M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\,\lambda$ is inaccessible $\wedge\,\beta = \lambda + \omega\ \wedge\ i : V_\beta \to M$ is elementary with critical point $\kappa\ \wedge\ i(\kappa) = \lambda\ \wedge\ \forall\mu(\kappa \leq \mu < \lambda)\forall f : \mu \to M\,\forall x \in M\ (\text{range}(f) \subseteq x \Longrightarrow f \in M)]$.

We write $\mathcal{E}^{str}_\kappa, \mathcal{E}^{ext}_\kappa,$ and $\mathcal{E}^{sah}_\kappa$ in place of $\mathcal{E}^{\theta_{str}}_\kappa, \mathcal{E}^{\theta_{ext}}_\kappa,$ and $\mathcal{E}^{\theta_{sah}}_\kappa$, respectively.

2.8 Definition (*$\mathcal{E}$-Laver Sequences*) Suppose $\mathcal{E}^\theta_\kappa$ is a regular class of embeddings at κ. A function $f : \kappa \to V_\kappa$ is said to be *$\mathcal{E}^\theta_\kappa$-Laver at* κ if for each set x and each $\lambda > \max(\kappa, rank(x))$ there are $\beta > \lambda$, and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ such that $i(\kappa) > \lambda$ and $i(f)(\kappa) = x$.

The definition differs from Laver's original defintion [La2] for supercompact cardinals in two small ways. First, we use $rank(x)$ instead of the transitive closure of x; in the general case, we have found ranks somewhat easier to work with. Secondly, we do not require that for each x and *every* large enough λ, a suitable i "based at" λ can be chosen so that $i(f)(\kappa) = x$, but rather, only that *arbitrarily large* λ of this kind can be found. This weakening makes the definition flexible enough to be useful for other large cardinals (beside supercompact and strong). In any case, our definition here can easily be shown to be equivalent to Laver's in the supercompact case (see [C1]).

For a given regular class $\mathcal{E}_\kappa^\theta$ we define a formula $\phi(f, x, \lambda)$, which says that f is not $\mathcal{E}_\kappa^\theta$-Laver at κ with witnesses x, λ:

> "there exists a cardinal α with $f : \alpha \to V_\alpha$ and $\lambda > \max(\alpha, rank(x))$, and for all $\beta > \lambda$ and all $i : V_\beta \to M \in \mathcal{E}_\alpha^\theta$, if $i(\alpha) > \lambda$ then $i(f)(\alpha) \neq x$."

2.9 Definition. Suppose $A(x)$ is a large cardinal property and θ is a suitable formula. We shall call $A(x)$ a *normal property* of κ (and we will call κ a *normal* large cardinal) with suitable formula θ if *ZFC* proves the following:

1. $A(\kappa) \Longleftrightarrow \forall\gamma > \kappa\, \exists\beta \geq \gamma\, \exists i\, \exists M \theta(i, \kappa, \beta, M)$; and

2. $A(\kappa) \Longrightarrow \mathcal{E}_\kappa^\theta$ is a regular class.

When $A(\kappa)$ holds along with statements 2.9.1 and 2.9.2, we call $\mathcal{E}_\kappa^\theta$ a *normal class of embeddings.* For such a class, if

$$A(\kappa) \Longrightarrow \text{"there is an } \mathcal{E}_\kappa^\theta\text{-Laver sequence at } \kappa\text{"},$$

we will call $A(\kappa)$ (and $\mathcal{E}_\kappa^\theta$) *Laver-generating.*

We prove in [C1] that the large cardinal properties "strong", "extendible", and "super-almost-huge" are normal properties with suitable formulas $\theta_{str}, \theta_{ext},$ and θ_{sah}, respectively.

In [C1] we prove that under sufficiently strong hypotheses, virtually any regular class admits a Laver sequence:

2.10 Theorem [C1]**.** *Assume WA. Then if $j : V \to V$ is the WA-embedding with critical point κ and $\mathcal{E}_\kappa^\theta$ is a regular class of embeddings compatible with j, then $\mathcal{E}_\kappa^\theta$ admits an $\mathcal{E}_\kappa^\theta$-Laver sequence.*

Below, we give a general definition of *compatibility* of embeddings which specializes to the case mentioned in the theorem. First we note that, regarding Theorem 2.10, the following question was left open in [C1]:

Question #1. Does there exist a regular class $\mathcal{E}_\kappa^\theta$ which does not admit an $\mathcal{E}_\kappa^\theta$-Laver sequence?

By Theorem 2.10, such a class would have to be incompatible with the *WA*-embedding in case *WA* holds. Theorem 3.3 below shows that at least an affirmative answer to the question is consistent, modulo a mild large cardinal assumption.

Once it is known that under *WA* $\mathcal{E}$-Laver sequences nearly always exist for regular $\mathcal{E}$, it is natural to try to find optimal hypotheses under which specific large cardinals (or other regular classes) admit their own brand of Laver sequence. As we mentioned in the Introduction, a general open question is, which (normal) large cardinal properties are Laver-generating? It is known that strong and supercompact cardinals are Laver-generating. We focus on the extendible and super-almost-huge cases in this paper:

Question #2. Is either $\mathcal{E}_\kappa^{ext}$ or $\mathcal{E}_\kappa^{sah}$ Laver-generating?

For extendibles, the weakest known hypothesis under which an $\mathcal{E}_\kappa^{ext}$-Laver sequence has been constructed is the existence of a *hyperextendible* (see [C2]); in Section 4 we improve this to show that $\mathcal{E}_\kappa^{ext}$ is Laver-generating. For super-almost-huge, the best upper bound has been the existence of a superhuge. In Section 5, we examine two separate conditions on the targets of a super-almost-huge, each of which implies the existence of an $\mathcal{E}_\kappa^{sah}$-Laver sequence.

We turn to the promised definition of compatibility:

2.11 Definition. *(Compatibility)* Suppose $\kappa < \lambda_0 < \beta \leq \gamma$, and $i_\beta : V_\beta \to N_0$, $i_\gamma : V_\gamma \to M_\gamma$ are elementary embeddings with critical point κ. Then i_β is *compatible with* i_γ *up to* V_{λ_0} if there is $k_\beta : N_0 \to M_\beta = V_{i_\gamma(\beta)}^{M_\gamma}$ such that $k_\beta \circ i_\beta = i_\gamma | V_\beta$ and $k_\beta | V_{\lambda_0} \cap N_0 = \mathrm{id}_{V_{\lambda_0} \cap N_0}$. Also, if $j : V \to N$ is elementary with critical point κ, we will say that $i_\beta : V_\beta \to N_0$ is compatible with j up to V_{λ_0} if i_β is compatible with $j|V_\beta$. Moreover, given such a j, suppose $\mathcal{E}_\kappa^\theta$ is a regular class. We will say that $\mathcal{E}_\kappa^\theta$ is *compatible with* j if for each $\lambda < j(\kappa)$ there is a $\beta \in \mathrm{Dom}\ \mathcal{E}_\kappa^\theta$ with $\beta > \lambda$ and there is $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ that is compatible with $j|V_\beta$ up to V_λ.

A typical example of compatibility is the following: Suppose $\kappa \leq \lambda \leq \mu$, λ, μ successor ordinals, and $F \in Ext(\kappa, \mu)$. Then if $E = F|\lambda$, $i_E|V_{\lambda+\omega}$ is compatible with $i_F|V_{\mu+\omega}$ up to V_λ (and $i_E(\kappa) > \lambda$).

Concerning the latter part of 2.11, the case in which the embedding $j : V \to N$ is actually the *WA*-embedding $j : V \to V$, our definition here differs somewhat from the definition of compatibility given in [C1]; however, a fairly easy proof shows that the two definitions are equivalent in this case.

The following easily proven proposition appears in [C1] in a slightly different form:

2.12 Proposition [C1]**.** *Suppose κ is a cardinal, $\kappa < \lambda < \beta < \gamma$, $i : V_\beta \to M, j : V_\gamma \to N$ are elementary embeddings with critical point κ, and $rank(j(f)(\kappa)) < \lambda$. Then if i is compatible with j up to V_λ, $i(f)(\kappa) = j(f)(\kappa)$.*

The Standard Construction of an $\mathcal{E}$-Laver Sequence

In this paper, we will make use of the construction in [C1] of Laver sequences that seems to admit the broadest possible generalization. Below, we isolate certain key features of the usual arguments that are used in conjunction with the construction. The process of finding ever improved hypotheses under which this construction provably yields $\mathcal{E}^\theta_\kappa$-Laver sequences for various θ involves successfully handling certain "standard" obstacles that typically arise. We make some general comments here about those obstacles that will lighten our load somewhat in Sections 4 and 5.

Suppose $\mathcal{E}^\theta_\kappa$ is a regular class and R is a well-ordering of V_κ. The "canonical construction" of an $\mathcal{E}^\theta_\kappa$-Laver sequence (depending on R) is as follows: In $\langle V_\kappa, \in, R\rangle$, define $f : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} 0, & \text{if } f|\alpha \text{ is a } \mathcal{E}^\theta_\alpha\text{-Laver sequence at } \alpha; \\ x \in V_\kappa, & \text{if } \alpha \text{ is a cardinal and } f|\alpha \text{ is not } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha, \\ & \text{where } x \text{ is } R\text{-least such that } \exists\lambda\, \phi(f|\alpha, x, \lambda); \\ 0, & \text{otherwise.} \end{cases}$$

In [C1] we show the following:

2.13 Proposition. *Suppose κ is a strong cardinal. Then the function f obtained in the canonical construction is a $\mathcal{E}^{str}_\kappa$-Laver sequence at κ.*

We take a moment here to consider the general structure of a proof that the canonical construction yields a $\mathcal{E}^\theta_\kappa$-Laver sequence, when $\mathcal{E}^\theta_\kappa$ is regular. Suppose f is not $\mathcal{E}^\theta_\kappa$-Laver at κ. Let x, λ witness this failure; i.e., $\phi(f, x, \lambda)$ holds. The difficult part is to obtain an appropriate elementary embedding either of the form $j : V \to N$ or $j : V_\alpha \to M$ with which $\mathcal{E}^\theta_\kappa$ is sufficiently compatible and such that N or M contains enough information about f, x, and λ. At the very least, we choose j so that $x, \lambda \in V_{j(\kappa)} \cap \text{codomain } j$; usually, more conditions are required. We'll return to these in a moment. Proceeding, let D be the normal ultrafilter over κ derived from j. We have two cases, based on the definition of f:

Case I: $\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \text{"}f|\alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\text{"}\} \in D$;

Case II: $\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \exists\lambda\, \phi(f|\alpha, f(\alpha), \lambda)\} \in D$.

Notice that if Case I holds, then

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \text{"}f \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa\text{."}$$

Thus, we can find $i \in \left(\mathcal{E}_\kappa^\theta\right)^{V_{j(\kappa)}}$ such that $i(\kappa) > \lambda$ and $i(f)(\kappa) = x$. As long as $\left(\mathcal{E}_\kappa^\theta\right)^{V_{j(\kappa)}} = \mathcal{E}_\kappa^\theta \cap V_{j(\kappa)}$, we have a contradiction. In order to distinguish regular classes that have this property, we will say that a suitable formula θ is *adequately absolute* if for all ρ for which $|V_\rho| = \rho$ and all $i, \kappa, \beta, M \in V_\rho$, we have $\theta^{V_\rho}(i, \kappa, \beta, M) \iff \theta(i, \kappa, \beta, M)$. It is easy to see that $\theta_{str}, \theta_{ext}$, and θ_{sah} are adequately absolute. (Note that adequate absoluteness is enough to complete the proof of Case I precisely because of the somewhat awkward definition we are using for the classes $\mathcal{E}_\kappa^\theta$.)

Assuming Case II holds and that θ is adequately absolute, it follows that there is $\lambda < j(\kappa)$ such that

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \phi(f, j(f)(\kappa), \lambda).$$

Note that if there happen to be β, M such that $i = j|V_\beta : V_\beta \to M$ and $(i, M) \in \mathcal{E}_\kappa^\theta \cap V_{j(\kappa)}$, we would have a contradiction since then $\langle V_{j(\kappa)}, \in, j(R)\rangle \models i(\kappa) > \lambda$ and $i(f)(\kappa) = j(f)(\kappa)$.

In practice, for each $\beta < j(\kappa)$, the restriction $j|V_\beta$ (together with its codomain) may be a member of neither $V_{j(\kappa)}$ nor $\mathcal{E}_\kappa^\theta$. However, if there is a β such that $j|V_\beta$ is a member of $\mathcal{E}_\kappa^\theta \setminus V_{j(\kappa)}$, and $V_{j(\kappa)}$ has sufficient closure, we can often find another embedding $i' : V_\beta \to M' \in \mathcal{E}_\kappa^\theta$ with $(i', M') \in V_{j(\kappa)}$ that approximates $j|V_\beta$ in the sense that $i'(\kappa) > \lambda$ and $i'(f)(\kappa) = j(f)(\kappa)$. The following definition from [C1] is useful in this regard:

2.14 Definition. We call a regular class $\mathcal{E}_\kappa^\theta$ *simply Laver-closed at inaccessibles* if for each $f : \kappa \to V_\kappa$ and all x, λ, ρ for which $\kappa < \lambda < \rho$, ρ is inaccessible, and $x \in V_\lambda$, if there is $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $\lambda < \beta < \rho$, $i(\kappa) > \lambda$, and $i(f)(\kappa) = x$, then there is $i' : V_\beta \to N \in \mathcal{E}_\kappa^\theta \cap V_\rho$ such that $i'(\kappa) > \lambda$ and $i'(f)(\kappa) = x$.

Thus, continuing with the argument, if $j|V_\beta \in \mathcal{E}_\kappa^\theta$ and $\mathcal{E}_\kappa^\theta$ is simply Laver-closed at inaccessibles, we are able to obtain a contradiction in this case.

As we shall see in Section 4, the situation just described—whereby the given j has restrictions to at least one $\beta < j(\kappa)$ in $\mathcal{E}_\kappa^\theta$ that is not necessarily a member of $V_{j(\kappa)}$—fairly well describes the case in which $\theta = \theta_{ext}$. Unlike virtually all other globally defined large cardinals, however, $\mathcal{E}_\kappa^{ext}$ is not simply Laver-closed at inaccessibles (as we show in Example 4.2); but it does satisfy a somewhat weaker property that is good enough to obtain the contradiction (Proposition 4.3).

Another strategy is required when no suitable restriction of the given j lies in $\mathcal{E}_\kappa^\theta$. In that case, it is natural to try to find an $i' \in \mathcal{E}_\kappa^\theta$ and a $\beta < j(\kappa)$ so that i' is compatible with $j|V_\beta$ up to $V_{\lambda+1}$. If we are successful and $\mathcal{E}_\kappa^\theta$ is also simply Laver-closed at inaccessibles and $j(\kappa)$ is inaccessible, we again obtain the desired contradiction. The scenario just described is, roughly, the approach that was taken for the class $\mathcal{E}_\kappa^{sah}$ in [C1,Section 6]; we outline this argument in Theorem 5.3. The effort in this case was only partially successful since we needed to assume extra hypotheses in order to guarantee the existence of i' (and in order to obtain an appropriate j as a starting point).

Returning to the general setting, in the case in which there is no restriction $j|V_\beta, \beta < j(\kappa)$, that lies in $\mathcal{E}_\kappa^\theta$, it is quite possible that the search for a compatible embedding that does lie in $\mathcal{E}_\kappa^\theta$ fails. In that case, the following proposition is sometimes useful; it describes the minimal conditions under which the canonical construction can succeed. We use the proposition in obtaining Laver sequences for $\mathcal{E}_\kappa^{sah}$ under new hypotheses, in Theorem 5.4. The proposition also provides a technique for obtaining counterexamples, as we show in Theorem 5.5.

2.15 Proposition. *Suppose $\mathcal{E}_\kappa^\theta$ is regular, θ is adequately absolute, $\mathcal{E}_\kappa^\theta$ is simply Laver-closed at inaccessibles, and for each $i \in \mathcal{E}_\kappa^\theta, i(\kappa)$ is inaccessible. Suppose $f : \kappa \to V_\kappa$ is obtained by the canonical construction for Laver sequences relative to a well-ordering R of V_κ. Then the following are equivalent:*

1. *f is $\mathcal{E}_\kappa^\theta$-Laver at κ.*
2. *For all $\lambda > \kappa$ there are $i, j \in \mathcal{E}_\kappa^\theta$ such that $\lambda < i(\kappa) < j(\kappa)$, dom $i \in V_{j(\kappa)}$, and $i(f)(\kappa) = j(f)(\kappa)$.*

Proof of (1) $\Rightarrow$ (2)**.** Assume f is $\mathcal{E}_\kappa^\theta$-Laver. Given $\lambda > \kappa$, we can find $i, j \in \mathcal{E}_\kappa^\theta$ such that $i(f)(\kappa) = 0 = j(f)(\kappa)$ and $i(\kappa) > \lambda$ and $j(\kappa) > \lambda$. Without loss of generality, we can pick j so that $\max(i(\kappa), rank(\text{dom } i)) < j(\kappa)$, and we are done. ∎

Proof of (2) $\Rightarrow$ (1). Assume f is not $\mathcal{E}_\kappa^\theta$-Laver and let x, λ witness. Let $j \in \mathcal{E}_\kappa^\theta$ be such that $x, \lambda \in V_{j(\kappa)}$, and let D denote the normal ultrafilter over κ derived from j. As in the discussion above, there are two cases, and because θ is adequately absolute, Case I is impossible.

Assume Case II holds; then, as we observed earlier, there is $\lambda < j(\kappa)$ such that

$$\langle V_{j(\kappa)}, \in, j(R) \rangle \models \phi(f, j(f)(\kappa), \lambda).$$

Pick $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $i(\kappa) > \lambda$, $i(f)(\kappa) = j(f)(\kappa)$, and $\beta < j(\kappa)$. We are not quite done, however, since (i, M) may not be in $V_{j(\kappa)}$. But since dom $i \in V_{j(\kappa)}, j(\kappa)$ is inaccessible, and $\mathcal{E}_\kappa^\theta$ is simply Laver-closed at inaccessibles, we can find an $(i', M') \in V_{j(\kappa)} \cap \mathcal{E}_\kappa^\theta$ such that i' approximates i (in the sense described earlier), yielding the desired contradiction. ∎

To conclude this section, we consider an alternative construction for $\mathcal{E}^\theta_\kappa$-Laver sequences, built as in the canonical construction, but without requiring the construction to be defined within a structure of the type $\langle V_\kappa, \in, R\rangle$. More specifically, assume $\mathcal{E}^\theta_\kappa$ is regular and define $f : \kappa \to V_\kappa$ by:

$$f(\alpha) = \begin{cases} 0, & \text{if } f|\alpha \text{ is a } \mathcal{E}^\theta_\alpha\text{-Laver sequence at } \alpha; \\ x \in V_\kappa, & \text{if } \alpha \text{ is a cardinal and } f|\alpha \text{ is not } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha, \\ & \text{where } \exists \lambda < \kappa\ \phi(f|\alpha, x, \lambda); \\ 0, & \text{otherwise.} \end{cases}$$

We used this construction to establish 2.10; however, efforts in [C1] to prove that the construction yields $\mathcal{E}^\theta_\kappa$-Laver sequences under more reasonable hypotheses than *WA* were largely unsuccessful. In particular, it was left open in that paper whether the f just described can be proven to be $\mathcal{E}^\theta_\kappa$-Laver, where θ corresponds to extendible, super-almost-huge, or superhuge, using hypotheses weaker than *WA* (see [C1, Question 5.43.2]). We give a positive answer in the extendible case in Section 4, making use of the techniques developed for proving that the canonical construction works. For future reference, we record two definitions and a theorem from [C1] concerning this construction that we will make use of in our discussion in Section 4.

2.16 Definition. *(Reflecting Laver Sequences)* Suppose $\mathcal{E}^\theta_\kappa$ is a regular class and $\rho > \kappa$. Then $\mathcal{E}^\theta_\kappa$ is *Laver reflecting in* V_ρ if, whenever $g : \kappa \to V_\kappa$ is $\mathcal{E}^\theta_\kappa$-Laver at κ, we have

$$V_\rho \models \text{“}g \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa.\text{”}$$

2.17 Definition. *(Localized Laver Failures)* Suppose $\mathcal{E}^\theta_\kappa$ is a regular class and $\rho > \kappa$. Then *$\mathcal{E}^\theta_\kappa$-Laver failures are localized below* ρ if for each $g : \kappa \to V_\kappa$,

$$g \text{ is } \textit{not } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa \iff \exists x \in V_\rho\, \exists \lambda < \rho\, \phi(g, x, \lambda).$$

2.18 Theorem [C1]**.** *Suppose $\mathcal{E}^\theta_\kappa$ is a regular class, θ is adequately absolute, and κ is globally superstrong. Let $f : \kappa \to V_\kappa$ be defined as in the alternative construction. Assume that, for each γ, there is a superstrong embedding $j : V \to N$ with critical point κ such that $j(\kappa) > \gamma$ and the following statements hold in N:*

1. *$\forall \lambda < j(\kappa)\, \neg\phi(f, j(f)(\kappa), \lambda)$;*
2. *$\mathcal{E}^\theta_\kappa$-Laver failures are localized below $j(\kappa)$;*
3. *$\mathcal{E}^\theta_\kappa$ is Laver-reflecting in $V_{j(\kappa)}$.*

Then f is $\mathcal{E}^\theta_\kappa$-Laver at κ.

§**3.** ***WA*** **and Regular Classes.**

In this section, we show that it is consistent, relative to mild hypotheses, for there to be regular classes that do not admit Laver sequences, partially answering Question #1. The main tool is the construction of a *locally coherent* class of extenders.

3.1 Definition Suppose κ is a strong cardinal, $\lambda > \kappa$, and $\mathbf{C} = \langle E^\eta : \eta \geq \lambda \rangle$ is a class of extenders. Then $\mathbf{C}$ is *locally coherent at* λ if for every $\eta \geq \lambda$, $E^\eta \in Ext(\kappa, \eta)$ and $E^\eta | \lambda = E^\lambda$.

3.2 Lemma. *Suppose κ is a strong cardinal and $V = HOD$. Then for each $\lambda \geq \kappa$, there is a class of extenders which is locally coherent at λ and definable from $\{\kappa, \lambda\}$.*

Remark. As we will observe in the proof of Theorem 3.3, the hypotheses of the lemma are consistent with the existence of a strong cardinal with an inaccessible above.

Proof. Let $\mathbf{R}$ be a definable global well-ordering of the universe. To obtain E^λ, first observe that there must be some $E \in Ext(\kappa, \lambda)$ with the property that

$$(*) \qquad \forall \beta > \lambda \, \exists F \in Ext(\kappa, \beta) \big(F|\beta = E \big).$$

If not, for each $E \in Ext(\kappa, \lambda)$, let $\beta_E > \lambda$ be such that for all $F \in Ext(\kappa, \beta_E)$, $F|\lambda \neq E$. Let $\gamma > \sup \{\beta_E : E \in Ext(\kappa, \lambda)\}$, let $F \in Ext(\kappa, \gamma)$, and let $E = F|\lambda$. But now, letting $F_E = F|\beta_E$, we have $F_E \in Ext(\kappa, \beta_E)$ and $F_E|\lambda = E$, which is impossible.

Thus, let E^λ be the $\mathbf{R}$-least member of $Ext(\kappa, \lambda)$ that satisfies $(*)$. Then for each $\eta > \lambda$, let E^η be $\mathbf{R}$-least in $Ext(\kappa, \eta)$ such that $E^\eta|\lambda = E^\lambda$. Clearly the resulting class sequence of extenders is locally coherent and definable from $\{\kappa, \lambda\}$. ∎

We can now use a locally coherent class of extenders to consistently build a regular class of embeddings having no correponding Laver sequence:

3.3 Theorem. *Con(ZFC+ there is a regular class of embeddings with critical point κ and an inaccessible $\lambda > \kappa$) implies Con(ZFC+ there is a regular class $\mathcal{E}_\kappa^\theta$ that admits no $\mathcal{E}_\kappa^\theta$-Laver sequence at κ).*

Proof. Start with a model V in which there is a regular class of embeddings with critical point κ. In [C1, Section 4], it is shown that κ must be a strong cardinal. In [Me], Menas shows that if κ is *supercompact* and there is an inaccessible λ above, then there is a forcing extension of the model V_λ in which κ is still supercompact and $V = HOD$ holds; the same proof can be used to show in the present context that there is a forcing extension $V_\lambda[G]$ in which κ is strong and $V = HOD$.

Working in $V_\lambda[G]$, let $\mathbf{R}$ denote a definable well-ordering of the universe. As in the previous lemma, let $\mathbf{C} = \langle E^\eta : \eta \geq \kappa+2 \rangle$ be a class of extenders that is locally coherent at $\kappa+2$ and definable from $\{\kappa\}$; let $\psi(x, y, \kappa)$ define $\mathbf{C}$. Let

$$\mathcal{E} = \{ \big(i_{E^\eta} | V_{\eta+\omega}, M_\eta \big) : \eta \geq \kappa+2 \text{ and } \eta \text{ is a successor ordinal} \},$$

where $M_\eta = i_{E^\eta}(V_{\eta+\omega})$. We allow η to vary only over successor ordinals to ensure that $i_{E^\eta}(\kappa) > \eta$; see the discussion preceding Proposition 2.1. We show that $\mathcal{E}$ is a regular class with no corresponding Laver sequence.

First, we obtain a suitable formula θ for which $\mathcal{E} = \mathcal{E}^\theta_\kappa$. We would like $\theta(i, \kappa, \beta, M)$ to be a formula that says

$$\theta_{str}(i, \kappa, \beta, M) \wedge \text{"}i \text{ is the restriction of } i_{E^\eta} \text{ to } V_{\eta+\omega}, \quad \text{where } \beta = \eta+\omega \text{ and } \eta \text{ is a successor ordinal"},$$

but then we would have to refer to the proper class i_{E^η} as if it were a set. We can work around this inconvenience by replacing restrictions of class embeddings i_{E^η} with restrictions of set embeddings $i^{\mathcal{F}(\eta)}_{E^\eta}$, where the latter denotes the canonical embedding defined on the rank $V_{\mathcal{F}(\eta)}$, and $\mathcal{F}$ is some appropriate class function, defined without parameters. Here are the details: Define a function $\mathcal{F}$ on the ordinals by $\mathcal{F}(\eta) = |V_\eta|^+$. Let $j = i_{E^\eta}$ and let $j^\eta = i^{V_{\mathcal{F}(\eta)}}_{E^\eta}$. By Lemma 2.3 (where $\nu = \mathcal{F}(\eta)$ and $\gamma = \eta+\omega$), we have, for each $\eta \geq \kappa+2$,

$$Ult(V_{\mathcal{F}(\eta)}, E^\eta) \cap V_{j^\eta(\eta+\omega)} = Ult(V, E^\eta) \cap V_{j(\eta+\omega)},$$

whence,

$$j|V_{\eta+\omega} = j^\eta|V_{\eta+\omega}.$$

Thus, we let $\theta(i, \kappa, \beta, M)$ be the formula

$$\theta_{str}(i, \kappa, \beta, M) \ \wedge \ \exists \eta \left[\beta = \eta+\omega \ \wedge \ succ(\eta) \ \wedge \ \psi(\eta, E^\eta, \kappa) \ \wedge \ i = i^{V_{\mathcal{F}(\eta)}}_{E^\eta} | V_\beta \right],$$

where $succ(\eta)$ says that η is a successor ordinal. Now, because $\mathcal{E}^\theta_\kappa$ contains embeddings in $\mathcal{E}^{str}_\kappa$ whose codomains contain arbitrarily large ranks, it is easy to verify that $\mathcal{E}^\theta_\kappa$ is a regular class. We show that this class has no Laver sequence by showing that

$$\{ i_{E^\eta}(f)(\kappa) : \eta \geq \kappa+2 \text{ and } f \in {}^\kappa V_\kappa \} \not\supseteq V_{\kappa+2}.$$

Let $S_{\kappa+2} = \{ i_{E^{\kappa+2}}(f)(\kappa) : f \in {}^\kappa V_\kappa \}$. Note that $|S_{\kappa+2}| \leq 2^\kappa < |V_{\kappa+2}|$. Suppose there are $\eta > \kappa+2$, $f \in {}^\kappa V_\kappa$ and $X \in V_{\kappa+2} \setminus S_{\kappa+2}$ such that

$$i_{E^\eta}(f)(\kappa) = X.$$

But now $rank(X) < \kappa + 2$ and $i_{E^{\kappa+2}}$ is compatible with i_{E^η} up to $V_{\kappa+2}$; so, by 2.12,

$$\begin{aligned} X &= i_{E^\eta}(f)(\kappa) \\ &= i_{E^{\kappa+2}}(f)(\kappa) \in S_{\kappa+2}, \end{aligned}$$

which is impossible. This completes the proof that $\mathcal{E}_\kappa^\theta$ has no $\mathcal{E}_\kappa^\theta$-Laver sequence. ■

This theorem, being only a consistency result, leaves open the following question:

3.4 Open Question. Is it consistent with a strong cardinal that every regular class of embeddings admits a Laver sequence?

§4. Extendible Laver Sequences.

The main result of this section is the proof that extendibility is Laver-generating, answering Question #2 for extendibles. As promised in Section 2, we also show that $\mathcal{E}_\kappa^\theta$ is not simply Laver-closed at inaccessibles (answering Question 5.5 of [C1]), but describe a property similar to simple Laver-closure that does hold for extendibles (and could have been used in our proof of Theorem 4.1). We also prove that the alternative construction of a Laver sequence (given at the end of Section 2) is also $\mathcal{E}_\kappa^{ext}$-Laver assuming only that κ is extendible.

4.1 Theorem. *$\mathcal{E}_\kappa^{ext}$ is Laver-generating.*

Proof. Let $f : \kappa \to V_\kappa$ be obtained via the canonical construction (Section 2). Following the reasoning template of Section 2, assume x, λ witness that f is not $\mathcal{E}_\kappa^{ext}$-Laver; in other words, assume $\phi(f, x, \lambda)$. Using 2.4, let $\alpha > \kappa$ be large enough so that $x, \lambda \in V_\alpha$ and $V_\kappa \prec V_\alpha$, and let $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$.

Proceeding to the two cases mentioned in Section 2, since θ_{ext} is adequately absolute, we can assume Case I fails and assume Case II holds. Let β be such that $\lambda < \beta < \alpha$ and let $i = j|V_\beta : V_\beta \to V_\zeta$. Now $(i, V_\zeta) \in \mathcal{E}_\kappa^{ext} \setminus V_{j(\kappa)}$. Note, however, that $(i, V_\zeta) \in V_\eta$. Letting $x = j(f)(\kappa)$, we have:

$$V_\eta \models \exists e\, \exists \zeta\, [(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \beta > \lambda \wedge e(f)(\kappa) = x].$$

Note that since $V_\kappa \prec V_\alpha$, then $V_{j(\kappa)} \prec V_\eta$; since $\lambda, \beta \in V_{j(\kappa)}$,

$$V_{j(\kappa)} \models \exists e\, \exists \zeta\, [(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \beta > \lambda \wedge e(f)(\kappa) = x].$$

Thus (as in our template in Section 2), we have obtained $(i', V_\zeta) \in \mathcal{E}_\kappa^{ext} \cap V_{j(\kappa)}$ for some ζ, contradicting $\phi(f, j(f)(\kappa), \lambda)$, and we are done. ∎

In [C1], we ask whether it is consistent for $\mathcal{E}_\kappa^{ext}$ to be simply Laver-closed at inaccessibles, assuming κ is extendible. An easy counterexample can be constructed as follows:

4.2 Example. *$\mathcal{E}_\kappa^{ext}$ is not simply Laver-closed.* Assume κ is extendible. We show that for every $\beta > \kappa$ there is an inaccessible cardinal $\mu_\beta > \beta$ for which

1. there are μ_β inaccessibles in the interval (β, μ_β);
2. $\mathcal{E}_\kappa^{ext}$ is not simply Laver-closed at any inaccessible in (β, μ_β).

Given $\beta > \kappa$ and given $f : \kappa \to V_\kappa, x, \lambda$ as in the definition for simply Laver-closed, let $\mu_\beta = \min\{j(\kappa) : j \in \mathcal{E}_\kappa^{ext}$ and $\text{dom } j = V_\beta\}$. Since μ_β is measurable, (1) holds. And (2) follows from the fact that μ_β is the least target for extendible embeddings having domain V_β. ∎

We can extract from the proof of Theorem 4.1 the following modification of simple Laver-closure that holds for extendibles:

4.3 Proposition. *Assume κ is extendible. There are arbitrarily large α such that for each $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$, there is an inaccessible $\rho < \eta$ such that for every $i : V_\beta \to V_\nu, x, \lambda,$ and $f : \kappa \to V_\kappa$ for which $i \in \mathcal{E}_\kappa^{ext}$, $\kappa < \beta \leq \alpha$, $\nu < \eta$, $i(\kappa) > \lambda$ and $i(f)(\kappa) = x$, there exists an $i' : V_\beta \to V_{\nu'} \in \mathcal{E}_\kappa^{ext} \cap V_\rho$ such that $i'(\kappa) > \lambda$ and $i'(f)(\kappa) = x$.*

Proof. Using the proof of Theorem 4.1, the result follows if we set $\rho = j(\kappa)$. ∎

We conclude this section by showing—assuming only that κ is extendible—that the alternative construction of a Laver sequence, given at the end of Section 2, is $\mathcal{E}_\kappa^{ext}$-Laver; this answers Questions 5.17 and 5.22 in [C1] for the extendible case. We begin with a modification of 2.18 which can be proved exactly as in [C1, Theorem 5.13]; for the rest of this section, we let f denote the function obtained from the alternative construction of Laver sequences.

4.4 Theorem. *Suppose κ is extendible. Assume that, for each $\gamma > \kappa$, there are α, j such that $\alpha > \gamma$, α is inaccessible, $V_\kappa \prec V_\alpha$, $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$, and the following statements hold in V_η:*

1. *$\forall \lambda < j(\kappa) \neg\phi(f, j(f)(\kappa), \lambda)$;*
2. *$\mathcal{E}_\kappa^{ext}$-Laver failures are localized below $j(\kappa)$;*
3. *$\mathcal{E}_\kappa^{ext}$ is Laver-reflecting in $V_{j(\kappa)}$.*

Then f is $\mathcal{E}_\kappa^{ext}$-Laver at κ.∎

4.5 Proposition. *Suppose κ is extendible and $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$ where α is inaccessible and $V_\kappa \prec V_\alpha$. Then V_η satisfies properties (1) - (3) of Theorem 4.4. Hence, f is $\mathcal{E}_\kappa^{ext}$-Laver at κ.*

Proof. For (1), given any $g : \kappa \to V_\kappa$, notice as in other proofs in this section, that for any $\beta < \alpha$, $j|V_\beta : V_\beta \to V_\zeta \in \mathcal{E}_\kappa^{ext} \cap V_\eta$. But clearly a restriction of this kind witnesses the failure of $\exists\lambda\, \phi(g, j(g)(\kappa), \lambda)$. For (2), certainly if x, λ witness in V_η that a function $g : \kappa \to V_\kappa$ is not $\mathcal{E}_\kappa^{ext}$-Laver at κ, then since $V_{j(\kappa)} \prec V_\eta$, such witnesses can be found in $V_{j(\kappa)}$, as required. Finally, (3) follows immediately from the fact that $V_{j(\kappa)} \prec V_\eta$. ∎

§5. Super-almost-huge Laver Sequences.

In this section, we show that the standard construction for Laver sequences yields $\mathcal{E}_\kappa^{sah}$-Laver sequences when we make certain assumptions about the a.h. targets of κ (roughly, that they contain enough of their limit points or that they exhibit a natural kind of coherence). The existence of a super-almost-huge cardinal having any one of these properties will turn out to be strictly stronger than the existence of a super-almost-huge without extras conditions. We shall also describe conditions under which the standard construction does not yield a $\mathcal{E}_\kappa^{sah}$-Laver sequence; although we do not have a proof of the consistency of these conditions with a super-almost-huge, they suggest a direction of research in the case in which the class of a.h. targets contains none of its limit points.

Our results suggest that it is natural to single out certain conditions on the class of a.h. targets and consider them as large cardinal properties themselves, lying between super-almost-huge and superhuge in consistency strength; we begin with a description of these and a discussion of their relative strengths. For the rest of this section, if κ is super-almost-huge, let $\Lambda = \{\lambda : \lambda \text{ is an a.h. target for } \kappa\}$, and, for any class $\mathbf{C}$, $\mathbf{C}' = \{\nu : \nu \text{ is a limit point of } \mathbf{C}\}$.

$\mathrm{SAH}_0(\kappa)$: κ is super-almost-huge.

$\mathrm{SAH}_1(\kappa)$: κ is super-almost-huge and the class $\Lambda \cap \Lambda'$ is bounded.

$\mathrm{SAH}_2(\kappa)$: κ is super-almost-huge, and the class $\Lambda \cap \Lambda'$ is unbounded, and there is μ such that for all regular $\rho > \mu$ the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is nonstationary in ρ.

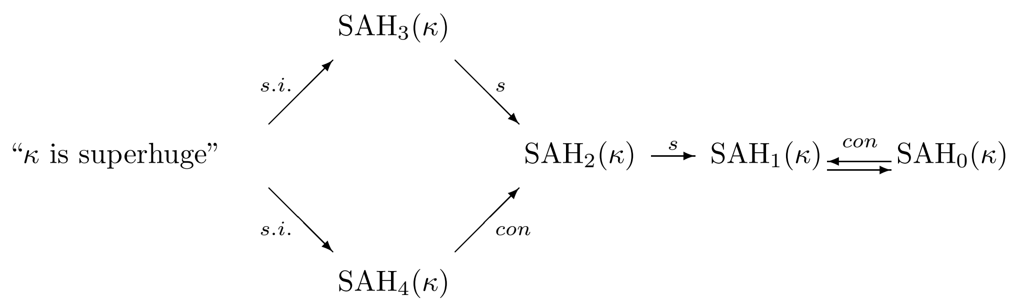
$\mathrm{SAH}_3(\kappa)$: κ is super-almost-huge and for arbitrarily large regular ρ, the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is stationary in ρ.

$\mathrm{SAH}_4(\kappa)$: κ is super-almost-huge and for arbitrarily large a.h. targets μ, the set $\{\lambda < \mu : \lambda \text{ is an a.h. target of } \kappa \text{ and } (\lambda, \mu) \text{ is an a.h. coherent pair}\}$ is unbounded in μ.

To compare relative strengths of these hypotheses, we introduce the following notation: Given properties $A(\kappa), B(\kappa)$ that depend on an infinite cardinal κ, we write:

$$
\begin{array}{rcl}
A(\kappa) \xrightarrow{ZFC} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ implies } B(\kappa)\text{”} \\
 & \text{iff} & ZFC \vdash A \longrightarrow B; \\
A(\kappa) \xrightarrow{con} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ is consistency-wise at least as strong as } B(\kappa)\text{”} \\
 & \text{iff} & Con(ZFC + A(\kappa)) \xrightarrow{ZFC} Con(ZFC + B(\kappa)); \\
A(\kappa) \xrightarrow{s} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ is strictly consistency-wise stronger than } B(\kappa)\text{”} \\
 & \text{iff} & A(\kappa) \xrightarrow{con} B(\kappa) \text{ and } A(\kappa) \xrightarrow{ZFC} Con(ZFC + B(\kappa)); \\
A(\kappa) \xrightarrow{si} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ strongly implies } B(\kappa)\text{”} \\
 & \text{iff} & A(\kappa) \xrightarrow{ZFC} B(\kappa) \text{ and } \{\alpha < \kappa : B(\alpha)\} \text{ has normal measure 1.}
\end{array}
$$

The terminology *strongly implies* was introduced in [SRK].



5.1 Theorem. *Suppose κ is an infinite cardinal.*

1. $SAH_0(\kappa) \overset{con}{\longleftrightarrow} SAH_1(\kappa)$.
2. $SAH_2(\kappa) \xrightarrow{s} SAH_1(\kappa)$.
3. $SAH_3(\kappa) \xrightarrow{s} SAH_2(\kappa)$.
4. $SAH_4(\kappa) \xrightarrow{con} SAH_2(\kappa)$.
5. *“κ is superhuge”* $\xrightarrow{si} SAH_3(\kappa) \wedge SAH_4(\kappa)$.

Proof. The following proves both (1) and (2): Given a model V of $\mathrm{SAH}_0(\kappa)$, if $\Lambda \cap \Lambda'$ is unbounded, let λ denote its least element and note that $V_\lambda \models \mathrm{SAH}_1(\kappa)$.

To prove (3), assume $\mathrm{SAH}_3(\kappa)$ and note that if λ is the least regular cardinal for which $\Lambda \cap \rho$ is stationary in ρ, then $V_\lambda \models \mathrm{SAH}_2(\kappa)$. A similar observation takes care of (4).

For (5), we begin by noting that by 2.6.2, if κ is superhuge, then both $\mathrm{SAH}_3(\kappa)$ and $\mathrm{SAH}_4(\kappa)$ hold. Let $j : V \to N$ be a huge embedding with critical point κ and let U be the normal ultrafilter over κ derived from j. For the $\mathrm{SAH}_3(\kappa)$ case, let

$$S = \{\rho < j(\kappa) : \rho \text{ is an a.h. target for } \kappa \text{ and}$$
$$\{\alpha < \rho : \alpha \text{ is an a.h. target for } \kappa\} \text{ is stationary in } \rho\}.$$

We show $S \in j(U)$. As is easily verified (see [C1] for details), $\Lambda \in j(U)$. For each cardinal $\nu > \kappa$, let R_ν denote the set of regular cardinals below ν. Define $W : R_{j(\kappa)} \to P(j(\kappa))$ by $W(\rho) = \{\alpha < \rho : \alpha \text{ is an a.h. target for } \kappa\}$. Let $\hat{\Lambda} = \{\rho < j(\kappa) : W(\rho) \text{ is stationary in } \rho\}$. Now, as $S = \Lambda \cap \hat{\Lambda}$, it suffices to show that $\hat{\Lambda} \in j(U)$. First observe that for all $\rho \in R_{j^2\kappa}$,

$$(j \cdot j(W))(\rho) = \{\alpha < \rho : \alpha \text{ is an a.h. target for } \kappa\}^{N_1}$$
$$= \{\alpha < \rho : \alpha \text{ is an a.h. target for } \kappa\},$$

since $V_{j(\kappa)+1} = V^N_{j(\kappa)+1} = V^{N_1}_{j(\kappa)+1}$. Now,

$$\hat{\Lambda} \in j(U) \iff j(\kappa) \in j \cdot j(\hat{\Lambda})$$
$$\iff N_1 \models \text{“}(j \cdot j(W))(j(\kappa)) \text{ is stationary in } j(\kappa)\text{”}$$
$$\iff \{\alpha < j(\kappa) : \alpha \text{ is an a.h. target for } \kappa\} \text{ is stationary in } j(\kappa)$$
$$\iff \Lambda \text{ is stationary in } j(\kappa),$$

and the last of these statements is true, and we are done.

To prove that $\{\alpha < \kappa : \mathrm{SAH}_4(\alpha)\}$ has normal measure 1, we show that for any huge embedding $j : V \to N$ with critical point κ, $N \models \mathrm{SAH}_4(\kappa)$. We begin by showing that for such j, $V_{j(\kappa)} \models \mathrm{SAH}_4(\kappa)$. This is enough, for if $N \models \neg\mathrm{SAH}_4(\kappa)$, by 2.6.1 and the fact that $\neg\mathrm{SAH}_4(\kappa)$ is Σ_3, we would have a contradiction. (To see that $\neg\mathrm{SAH}_4(\kappa)$ is Σ_3, note that $\mathrm{SAH}_4(\kappa)$ can be expressed in the form $\forall y\, \exists \beta \geq rank(y)\, (V_\beta \models \psi(x))$, which is a kind of normal form for Π_3 formulas—see comments after Theorem 2.17 in [C1].) Fix a huge embedding j with critical point κ and let U be the normal ultrafilter over κ derived from j. It suffices to show that $S \in j(U)$ where

$$S = \{\rho < j(\kappa) : V_{j(\kappa)} \models \text{“}\rho \text{ is an a.h. target for } \kappa \text{ and}$$
$$\{\lambda < \rho : (\lambda, \rho) \text{ is a.h. coherent}\} \text{ is unbounded in } \rho\text{”}\}.$$

Let $T \in j(U)$ be such that for all $\alpha < \beta$ both in T, (α, β) is an a.h. coherent pair (as in 2.6.3). Let

$$S_0 = \{\rho < j(\kappa) : V_{j(\kappa)} \models \text{“}\rho \text{ is an a.h. target for } \kappa\text{”}\}.$$

Then $S = \{\rho \in S_0 : V_{j(\kappa)} \models \text{“}\{\lambda < \rho : (\lambda, \rho) \text{ is an a.h. coherent pair}\} \text{ is unbounded in } \rho\text{”}\}$. We will be done if we show that $T \cap T' \subseteq S$. Let $\rho \in T \cap T'$; $\rho \in S_0$ since $\rho \in T$ and $\{\lambda \in T : \lambda < \rho\}$ is unbounded in ρ since ρ is a limit point of T, and we are done. ■

One question that we leave open here is the relationship between $\mathrm{SAH}_3(\kappa)$ and $\mathrm{SAH}_4(\kappa)$:

5.2 Open Question. Is either of the following true?

a. $\mathrm{SAH}_3(\kappa) \xrightarrow{ZFC} \mathrm{SAH}_4(\kappa)$;

b. $\mathrm{SAH}_4(\kappa) \xrightarrow{ZFC} \mathrm{SAH}_3(\kappa)$.

We turn now to methods for constructing an $\mathcal{E}_\kappa^{sah}$-Laver sequence; in particular, we will show that the standard construction (defined in Section 2) yields an $\mathcal{E}_\kappa^{sah}$-Laver sequence whenever $\mathrm{SAH}_3(\kappa)$ holds (and we recall that this is also the case when $\mathrm{SAH}_4(\kappa)$ holds). On the other hand, assuming $\mathrm{SAH}_1(\kappa)$ is true, we demonstrate that it is at least plausible that the standard construction fails to be $\mathcal{E}_\kappa^{sah}$-Laver by describing a condition that implies this failure and that is not obviously inconsistent with $\mathrm{SAH}_1(\kappa)$. For the rest of this section, we let f denote the function obtained in the canonical construction, as in Section 2.

We begin with a result from [C1]:

5.3 Theorem [C1, 7.33]. *$SAH_4(\kappa)$ implies that f is $\mathcal{E}_\kappa^{sah}$-Laver at κ.*

Outline of Proof. Reasoning as in the template of Section 2, assume x, λ witness that f is not $\mathcal{E}_\kappa^{sah}$-Laver, and hence that $\phi(f, x, \lambda)$ holds. By $\mathrm{SAH}_4(\kappa)$, pick a huge embedding $j : V \to N$ with critical point κ so that $A = \{\alpha < j(\kappa) : (\alpha, j(\kappa)) \text{ is an a.h. coherent pair}\}$ is unbounded in $j(\kappa)$. As in our reasoning template, by adequate absoluteness we may assume Case II holds, whence $\langle V_{j(\kappa)}, \in, j(R)\rangle \models \phi(f, j(f)(\kappa), \lambda)$. Pick $\alpha \in A$ such that $(x, \lambda) \in V_\alpha$. Let $\beta = \alpha + \omega$ and $\gamma = j(\kappa) + \omega$. Now $\{\beta, \gamma\} \subset$ Dom $\mathcal{E}_\kappa^{sah}$, and clearly $j|V_\gamma \in \mathcal{E}_\kappa^{sah}$. As in [C1, Theorem 6.4], since $(\alpha, j(\kappa))$ is an a.h. coherent pair, there is an $i : V_\beta \to M$, with $(i, M) \in \mathcal{E}_\kappa^{sah}$, which is compatible with $j|V_\gamma$ up to $V_{\lambda+1}$. It follows that $i(\kappa) > \lambda$, and by Proposition 2.12, $i(f)(\kappa) = j(f)(\kappa)$. Because $\mathcal{E}_\kappa^{sah}$ is simply Laver-closed at inaccessibles, we may assume that such a pair (i, M) can be found in $V_{j(\kappa)}$, contradicting $\phi(f, x, \lambda)$. ∎

We now obtain the same result using $\mathrm{SAH}_3(\kappa)$:

5.4 Theorem. *$SAH_3(\kappa)$ implies that f is $\mathcal{E}_\kappa^{sah}$-Laver at κ.*

Proof. We use the criterion given in Proposition 2.15. Given $\lambda > \kappa$, we find $i, j \in \mathcal{E}_\kappa^{sah}$ such that $\lambda < i(\kappa) < j(\kappa)$ and $i(f)(\kappa) = j(f)(\kappa)$. As a notational convention, for each $j \in \mathcal{E}_\kappa^{sah}$, if $\lambda = j(\kappa)$ and for each η, $\kappa \leq \eta < \lambda$, U_η is the normal ultrafilter over $P_\kappa\eta$ derived from j, then we will let $j_\eta : V \to M_\eta$ denote the canonical embedding into the ultrapower of V by U_η. It is easy to verify that for each such j, there is a least ordinal η_j such that for all η, $\eta_j \leq \eta < j(\kappa)$, $j(f)(\kappa) = j_\eta(f)(\kappa)$. Let $\mathbf{A}_\lambda = \{\nu > \lambda : \nu \text{ is an a.h. target of } \kappa\}$. Define $\mathbf{F} : \mathbf{A}_\lambda \to ON$ by

$$\mathbf{F}(\nu) = \text{least } \eta \text{ such that } \exists j \in \mathcal{E}_\kappa^{sah}\,(j(\kappa) = \nu \,\wedge\, \eta = \eta_j).$$

By the definition of $\mathrm{SAH}_3(\kappa)$, there is a regular cardinal ρ such that $\mathbf{A}_\lambda \cap \rho$ is stationary in ρ. As $\mathbf{F}|\rho$ is regressive on this stationary set, there is a stationary $A_0 \subseteq \mathbf{A}_\lambda \cap \rho$ and an ordinal $\eta_0 > \lambda$ such that $\mathbf{F}'' A_0 = \{\eta_0\}$. For each $\nu \in A_0$, let $j^{(\nu)} \in \mathcal{E}_\kappa^{sah}$ be such that

$$j^{(\nu)}(\kappa) = \nu \text{ and } j^{(\nu)}(f)(\kappa) = j_{\eta_0}^{(\nu)}(f)(\kappa).$$

Let $U_{\eta_0}^{(\nu)}$ denote the normal ultrafilter over $P_\kappa \eta_0$ derived from $j^{(\nu)}$. Since there are fewer than ρ normal ultrafilters over $P_\kappa \eta_0$, there are U, ν_0, ν_1 such that U is a normal ultrafilter over $P_\kappa \eta_0$ and $\nu_0 < \nu_1$, both in A_0, are such that both $j_{\eta_0}^{(\nu_0)}$ and $j_{\eta_0}^{(\nu_1)}$ are determined by U; in other words, $j_{\eta_0}^{(\nu_0)} = j_{\eta_0}^{(\nu_1)}$. Thus, on the one hand,

$$\lambda < j^{(\nu_0)}(\kappa) < j^{(\nu_1)}(\kappa),$$

and on the other hand,

$$j^{(\nu_0)}(f)(\kappa) = j_{\eta_0}^{(\nu_0)}(f)(\kappa) = j_{\eta_0}^{(\nu_1)}(f)(\kappa) = j^{(\nu_1)}(f)(\kappa),$$

and we are done. ∎

We have been unable to obtain this result under weaker hypotheses. However, an approach that uses a similar technique, and appears to require less overhead, is the following: In the above proof, assume we are working in a model V_γ in which κ is super-almost-huge, and define, for each $j \in \mathcal{E}_\kappa^{sah}$ having target $\nu > \lambda$, a set $E(\eta_j, \nu)$ consisting of all U_η for which $\eta_j \leq \eta < \nu$ and j_η is the canonical embedding defined from U_η. If we could prove that the collection $\mathcal{C} = \{E(\eta_j, j(\kappa)) : j(\kappa) > \lambda \wedge j \in \mathcal{E}_\kappa^{sah}\}$ admits a subcollection $\mathcal{D}$ of the same cardinality such that $\mathcal{D}$ is a Δ-system with nonempty root, we would be able to proceed as above to show that f is $\mathcal{E}_\kappa^{sah}$-Laver. (However, since $|\mathcal{C}| \leq \gamma$, we can't use a Δ-system lemma, so some other approach to the proof would be required.)

We turn now to a condition under which f fails to be $\mathcal{E}_\kappa^{sah}$-Laver. As in the proof of 5.4, we let, for each $j \in \mathcal{E}_\kappa^{sah}$, η_j be least such that for all η, $\eta_j \leq \eta < j(\kappa)$, $j(f)(\kappa) = j_\eta(f)(\kappa)$.

5.5 Proposition. *Assume $SAH_1(\kappa)$. Let $\{\lambda_\xi : \xi \in ON\}$ be the increasing enumeration of a.h. targets. Let*

$$\eta_\xi = \min\{\eta_j : j \in \mathcal{E}_\kappa^{sah} \,\wedge\, j(\kappa) = \lambda_\xi\}.$$

Assume that for all $\xi < \zeta$,

$$\eta_\xi < \lambda_\xi < \eta_\zeta.$$

Then f is not $\mathcal{E}_\kappa^{sah}$-Laver at κ.

Proof. Again we use the criterion given in Proposition 2.15. Given $\lambda > \kappa$, we show that for any $i \in \mathcal{E}_\kappa^{sah}$ for which $i(\kappa) > \lambda$, there is no $j \in \mathcal{E}_\kappa^{sah}$ for which $i(\kappa) < j(\kappa)$ and $i(f)(\kappa) = j(f)(\kappa)$. Given $i \in \mathcal{E}_\kappa^{sah}$ with target $> \lambda$, let ξ be such that $i(\kappa) = \lambda_\xi$. Clearly, $rank(i(f)(\kappa)) < \lambda_\xi$. Let $j \in \mathcal{E}_\kappa^{sah}$ be such that $j(\kappa) = \lambda_\zeta > \lambda_\xi$. To complete the proof, it suffices to show that $rank(j(f)(\kappa)) \geq \lambda_\xi$.

Let $x = j(f)(\kappa)$. It is easy to verify that for all $\alpha < \eta_\zeta$, $j_\alpha(f)(\kappa) \neq j(f)(\kappa)$. In particular,

$$(*) \qquad j_{\lambda_\xi}(f)(\kappa) \neq x.$$

By way of contradiction, assume $rank(x) < \lambda_\xi$. Abusing notation somewhat, let k_ξ be the usual embedding such that

$$j = k_\xi \circ j_{\lambda_\xi} \text{ and } k_\xi | \lambda_\xi = \mathrm{id}_{\lambda_\xi}.$$

Then, as $k_\xi(x) = x$, we have

$$\begin{aligned} k_\xi(x) &= x \\ &= j(f)(\kappa) \\ &= \big(k_\xi \circ j_{\lambda_\xi}\big)(f)\big(k_\xi(\kappa)\big), \end{aligned}$$

from which it follows that

$$x = j_{\lambda_\xi}(f)(\kappa),$$

contradicting $(*)$. This completes the proof. ∎

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