

# THE GAP BETWEEN $I_3$ AND THE WHOLENESS AXIOM

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**Abstract.**  $\exists\kappa I_3(\kappa)$  is the assertion that there is an elementary embedding  $i : V_\lambda \rightarrow V_\lambda$  with critical point below  $\lambda$ , and with  $\lambda$  a limit. The Wholeness Axiom, or WA, asserts that there is a nontrivial elementary embedding  $j : V \rightarrow V$ ; WA is formulated in the language  $\{\in, \mathbf{j}\}$  and has as axioms an Elementarity schema, which asserts that  $\mathbf{j}$  is elementary; a Critical Point axiom, which asserts that there is a least ordinal moved by  $\mathbf{j}$ ; and includes every instance of the Separation schema for  $\mathbf{j}$ -formulas. Because no instance of Replacement for  $\mathbf{j}$ -formulas is included in WA, Kunen's inconsistency argument is not applicable. It is known that an  $I_3$  embedding  $i : V_\lambda \rightarrow V_\lambda$  induces a transitive model  $\langle V_\lambda, \in, i \rangle$  of ZFC + WA. We study here the gap in consistency strength between  $I_3$  and WA. We formulate a sequence of axioms  $\langle I_4^n : n \in \omega \rangle$  each of which asserts the existence of a transitive model of ZFC + WA having strong closure properties. We show that  $I_3$  represents the "limit" of the axioms  $I_4^n$  in a sense that is made precise.

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## §1. Introduction

$I_3(\kappa)$  is the statement that there is an elementary embedding  $i : V_\lambda \rightarrow V_\lambda$  having critical point  $\kappa$ , where  $\lambda$  is a limit ordinal. An  $I_3$  embedding is an embedding  $i : V_\lambda \rightarrow V_\lambda$  that witnesses  $I_3(\kappa)$  for some  $\kappa$ . The axiom  $\exists \kappa I_3(\kappa)$  arose in the work of Kunen [Ku] who observed that his proof in KM-set theory that there is no elementary embedding  $V \rightarrow V$  also showed that there could be no embedding  $V_{\lambda+2} \rightarrow V_{\lambda+2}$ ; he noted that his proof did not rule out the possible existence of embeddings  $V_\lambda \rightarrow V_\lambda$  ( $\lambda$  a limit) or  $V_{\lambda+1} \rightarrow V_{\lambda+1}$ .

The Wholeness Axiom, or WA, is an axiom schema asserting the existence of a nontrivial elementary embedding  $j : V \rightarrow V$ , formulated in the extended language  $\{\in, \mathbf{j}\}$ . WA is comprised of an Elementarity schema, which asserts that  $\mathbf{j}$  is an elementary embedding; a Nontriviality axiom, which asserts that for some  $x$ ,  $\mathbf{j}(x) \neq x$ ; and all instances of the Separation schema for instances of Separation in which the symbol  $\mathbf{j}$  occurs — we call the latter schema  $\text{Separation}_{\mathbf{j}}$ . The Wholeness Axiom was developed in an effort to formulate a schema asserting the existence of a  $j : V \rightarrow V$  that is not provably inconsistent with ZFC via Kunen’s methods, but that retains “most” of the strength of Reinhardt’s original (inconsistent) version of the axiom. WA is successful in this regard because, on the one hand, it is known to imply the existence of most large cardinals; yet, on the other hand, it is not known to be inconsistent since no instance of Replacement for  $\mathbf{j}$  formulas is assumed. (Such an instance is needed in order to arrive at Kunen’s inconsistency, because Replacement is what allows one to prove the critical sequence  $\kappa, j(\kappa), j^2(\kappa), \dots$  forms a set, and therefore has a supremum. Here, as usual,  $j^n$  denotes the  $n$ th iterate of  $j$  under composition.) The Wholeness Axiom has been studied in [Co1], [Co2], [Co3], [Co4], and [Ha]. It is known to have consistency strength strictly between that of a cardinal that is super- $n$ -huge for every  $n$  and that of the axiom  $\exists \kappa I_3(\kappa)$ .

The purpose of this paper is to see “how big” the gap is between WA and  $I_3$ . All the well-known large cardinal notions that are weaker than  $I_3$  are also known to be weaker than WA, so we would expect the gap to be “small” in a certain sense. Our approach to the problem is based on the simple observation that, if  $i : V_\lambda \rightarrow V_\lambda$  is an  $I_3$  embedding, then  $\langle V_\lambda, \in, i \rangle$  is a transitive model of ZFC + WA ([Co3]). Our perspective here is that  $V_\lambda$  is a highly closed transitive domain for a model of ZFC + WA. This perspective suggests a way to measure the gap between  $I_3$  and WA: Formulate a sequence of axioms asserting the existence of increasingly closed transitive models of WA so that in the “limit”, we arrive at  $I_3$ .

In the third section of this paper, we carry out this plan. We define, for each  $m \in \omega$ , an axiom  $I_4^m$  asserting the existence of a certain type of transitive model of ZFC + WA, and then we proceed to describe the sense in which  $I_3$  represents the limit of these new axioms. Before proceeding with the main results, we establish some preliminaries in the next section.

In concluding this Introduction, we gratefully acknowledge the efforts of the referee, which led

to clarifications in a number of proofs.

## §2. Preliminaries

We begin with several definitions and propositions, which form a proper subset of the material developed in [Co1]. The second half of this section provides a quick review of extenders.

When working with the theory ZFC+WA, we will always be working in the language  $\mathcal{L} = \{\in, \mathbf{j}\}$ , where  $\mathbf{j}$  is a unary function symbol.  $\mathbf{j}$ -formulas are  $\mathcal{L}$ -formulas that have an occurrence of the symbol  $\mathbf{j}$ . Like ordinary  $\in$ -formulas,  $\mathcal{L}$ -formulas can be classified into complexity classes; the class that will concern us here is the  $\Sigma_0$   $\mathcal{L}$ -formulas which consist of the  $\mathcal{L}$ -formulas in which every quantifier is bound.

We will need to refer to certain subtheories of ZFC + WA. For this purpose, it is helpful to list a number of axioms that pertain to the language  $\{\in, \mathbf{j}\}$  (see [Co1] for a complete list):

- **Elementarity.** Each of the following  $\mathbf{j}$ -sentences is an axiom, where  $\phi(x_1, x_2, \dots, x_m)$  is an  $\in$ -formula,

$$\forall x_1, x_2, \dots, x_m (\phi(x_1, x_2, \dots, x_m) \iff \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \dots, \mathbf{j}(x_m))).$$

- **Critical Point:** There is a least ordinal moved by  $\mathbf{j}$ .
- **BTEE:** Elementarity + Critical Point.
- **Separation<sub>j</sub>:** All instances of Separation for  $\mathbf{j}$  formulas.
- **$\Sigma_0$ -Separation:** All  $\Sigma_0$  instances of Separation for  $\mathbf{j}$  formulas.
- **Cofinal Axiom:**  $\forall \alpha \exists n \in \omega (\kappa_n \text{ exists and } \alpha \leq \kappa_n)$ .
- **WA:** Elementarity + Critical Point + Separation<sub>j</sub>
- **WA<sub>0</sub>:** Elementarity + Critical Point +  $\Sigma_0$ -Separation<sub>j</sub>

Whenever Critical Point is included as an axiom in one of our subtheories of ZFC + WA, the critical point of  $\mathbf{j}$  is denoted  $\kappa$ ;  $\kappa$  is understood to be a constant added to the language by definitional extension. The reader will notice that we refer to ‘ $\kappa$ ’ in our formulation of the Cofinal Axiom; we adopt the convention that the Cofinal Axiom is not included in any of our subtheories unless Critical Point is also included. Moreover, in the Cofinal Axiom, we define  $\kappa_0 = \kappa$  and, for  $n \geq 1$ , ‘ $\kappa_n$ ’ stands for  $\mathbf{j}^n(\kappa)$ , the  $n$ th iterate of  $\mathbf{j}$  applied to  $\kappa$ . To formulate this notion precisely, one defines the relation  $y = \mathbf{j}^n(x)$  as a three-place predicate with an appropriate formula  $\Phi(n, x, y)$ ; then the statement “ $\kappa_n$  exists” (as in the statement of the Cofinal Axiom) is simply an abbreviation for “ $\exists y \Phi(n, \kappa, y)$ ”. The formula  $\Phi(n, x, y)$  is discussed in detail in [Co1].

The three theories that will concern us in this paper are:

- (1) ZFC + BTEE

- (2) ZFC + WA<sub>0</sub>
- (3) ZFC + WA

To specify the strengths of these theories, and for later work, let us first recall the definition of  $n$ -huge cardinals and some variants (see [Ka]): For  $1 \leq n < \omega$ ,  $\kappa$  is  $n$ -huge if there exists an inner model  $M$  and an elementary embedding  $j : V \rightarrow M$  such that  $\text{crit}(j) = \kappa$  and  $M$  is closed under  $j^n(\kappa)$ -sequences;  $j(\kappa)$  is called the *target* of  $j$  and  $j$  is called an  $n$ -huge embedding.  $\kappa$  is super- $n$ -huge if, for every  $\lambda > \kappa$ , there is an  $n$ -huge embedding  $j$  such that  $\text{crit}(j) = \kappa$  and  $j(\kappa) > \lambda$ . An equivalent form of the definition of  $n$ -huge in terms of ultrafilters is well-known (see [Ka]). We state this equivalent form in a somewhat nonstandard way, in terms of  $n$ -huge indexes, for use in later sections.

**Definition 2.1.** Suppose  $1 \leq n < \omega$ ,  $\kappa$  is an uncountable cardinal, and  $\lambda > \kappa$ . Then  $\lambda$  is an  **$n$ -huge index for  $\kappa$**  if there is a  $\kappa$ -complete normal ultrafilter  $U$  over  $P(\lambda)$  and cardinals  $\kappa = \lambda_0 < \lambda_1 < \dots < \lambda_n = \lambda$  so that for each  $i < n$ ,  $\{x \in P(\lambda) : \text{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U$ .

Now, by the usual proof,  $\kappa$  is  $n$ -huge if and only if there is an  $n$ -huge index for  $\kappa$ . In that case, the witnessing ultrafilter is called an  $n$ -huge ultrafilter (over  $P(\lambda)$ ).

**Proposition 2.2.**

- (1) *The consistency strength of ZFC + BTEE lies between that of an ineffable cardinal and that of  $0^\#$ .*
- (2) *The consistency strength of ZFC + WA lies between that of a cardinal that is  $n$ -huge for every  $n$  and the existence of an  $I_3$  embedding.*

To give the consistency strength of ZFC + WA<sub>0</sub> we need to observe that it is possible in a model of ZFC + WA<sub>0</sub> that  $\kappa_n$  fails to exist for some nonstandard integer  $n$ . This peculiar problem (which does not arise in the theory ZFC + WA) is mollified considerably by part (2) of the following:

**Proposition 2.3.**

- (1) *The theory ZFC + WA proves*

$$V_{\kappa_0} \prec V_{\kappa_1} \prec \dots \prec V_{\kappa_n} \prec \dots \prec V.$$

- (2) *The theory ZFC + WA<sub>0</sub> proves that if  $\mathbf{A}$  is the class of  $n \in \omega$  for which  $\kappa_n$  exists, then  $\{\kappa_n : n \in \mathbf{A}\}$  is cofinal in ON, and the  $V_{\kappa_n}$  for which  $n \in \mathbf{A}$  forms an elementary chain whose union is  $V$ .*

As a corollary, we have:

**Proposition 2.4.** *Cofinal Axiom is derivable from each of the theories ZFC+WA and ZFC+WA<sub>0</sub>.*

Finally, we remark that different kinds of models of the language  $\{\in, \mathbf{j}\}$  — in particular, of the three main theories mentioned above — are possible, depending on one’s assumptions about the surrounding universe. In this paper (as in [Co1]), all models will live in a ZFC universe  $\langle V, \in \rangle$ , fixed once and for all, and in particular, if  $\langle M, E, i \rangle$  is a model of ZFC <sub>$\mathbf{j}$</sub> , we assume  $i$  is definable in  $V$ . Using the terminology of [Co1], we call such models **sharp-like**. An important consequence of this assumption is the following:

**Metatheorem 2.5.** *There is no proper class sharp-like transitive model of ZFC + WA<sub>0</sub>; that is, if  $\langle M, \in, i \rangle \models \text{ZFC} + \text{WA}_0$  and  $M$  is sharp-like and transitive, then  $M$  is a set.*

**Proof.** If  $\langle M, \in, i \rangle \models \text{ZFC} + \text{WA}_0$ , then in particular the Cofinal Axiom holds in  $M$ . If  $M$  is a sharp-like transitive model containing all the ordinals, (Cofinal Axiom) <sup>$M$</sup>  would imply that there is a cofinal  $\omega$ -sequence in the real class ON of ordinals, violating Replacement. Therefore, if  $M$  is transitive, it must only be a set. ■

Metatheorem 2.5 does not rule out the possibility that  $\langle V, \in, j \rangle$ , where  $j$  is an elementary embedding, is a model of ZFC + WA or ZFC + WA<sub>0</sub>; by the theorem, though, such a model would not be sharp-like.

We begin the second half of this section with a review of extenders. Our treatment follows [Ka, Section 26]. As in [Ka], we start with a discussion of extenders derived from an embedding, though here we restrict attention to embeddings  $i : M \rightarrow M$ , and to just the  $(\kappa, i(\kappa))$ -extenders derived from  $i$ . After fixing notation and stating some of the known results, we discuss the more general  $M$ - $(\kappa, \lambda)$  extenders, and conclude with some lemmas that will be needed in the next section.

For the next few paragraphs, we fix an elementary embedding  $i : M \rightarrow M$  with critical point  $\kappa$ , where  $M$  is a transitive set model<sup>1</sup> of ZFC. Let  $\lambda = i(\kappa)$ . For  $a \in [\lambda]^{<\omega}$ , define  $E_a$  from  $i$  by

$$E_a = \{X \subseteq P(\kappa^{|a|}) \cap M : a \in i(X)\}.$$

$\langle M, \in, E_a \rangle \models$  “ $E_a$  is a  $\kappa$ -complete ultrafilter over  $[\kappa]^{|a|}$ ”. For  $f, g \in M$  with  $f : [\kappa]^{|a|} \rightarrow M$  and  $g : [\kappa]^{|a|} \rightarrow M$ , we write  $f \sim_a g$  iff  $\{b \in [\kappa]^{|a|} : f(b) = g(b)\} \in E_a$ . We denote the equivalence class containing  $f$  by  $[f]_{E_a}$ . (Recall that Scott’s trick is used in this step.) Now the *ultrapower of  $M$  by  $E_a$*  is denoted  $\text{Ult}(M, E_a)$  and consists of all  $[f]_{E_a}$  for which  $f : [\kappa]^{|a|} \rightarrow M$  and  $f \in M$ . It has a membership relation  $\in_a$  defined by  $[f]_{E_a} \in_a [g]_{E_a}$  iff  $\{b \in [\kappa]^{|a|} : f(b) \in g(b)\} \in E_a$ . This relation is well-defined.  $(\text{Ult}(M, E_a), \in_a)$  is well-founded; we identify it with its transitive collapse  $(M_a, \in)$ . The canonical embedding  $j_a : M \rightarrow M_a$  is given by  $j_a(x) = [c_x^a]_{E_a}$ , where  $c_x^a : \kappa^{|a|} \rightarrow M$

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<sup>1</sup> In [Ka], extenders are developed relative to  $M$  that are *proper classes*, but it is pointed out in [MS] that the same treatment goes through for transitive set models of ZFC.

is the constant function with value  $x$ .  $k_a : M_a \rightarrow M$  is defined by  $k_a([f]_{E_a}) = i(f)(a)$ .  $k_a$  is a well-defined elementary embedding and  $i = k_a \circ j_a$ .

The  $(\kappa, \lambda)$ -extender derived from  $i$  is now defined to be

$$E = \langle E_a : a \in [\lambda]^{<\omega} \rangle.$$

$E$  gives rise to a directed system of models as follows: Suppose  $a \subseteq b$  and that both are elements of  $[\lambda]^{<\omega}$  with  $|b| = n, |a| = m$ . Define  $\pi_{ba} : [\kappa]^n \rightarrow [\kappa]^m$  by

$$(2.1) \quad \pi_{ba}(\{\xi_1, \dots, \xi_n\}) = \{\xi_{i_1}, \dots, \xi_{i_m}\},$$

where we adopt the convention that elements  $\xi_1, \dots, \xi_n$  occur in increasing order and that if  $b = \{\alpha_1 < \dots < \alpha_n\}$ , then  $a = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ . Now for such  $a, b$ , define  $i_{ab} : M_a \rightarrow M_b$  by

$$(2.2) \quad i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}.$$

$i_{ab}$  is a well-defined elementary embedding, and we have that

- (a)  $i_{ab} \circ j_a = j_b$
- (b)  $k_b \circ i_{ab} = k_a$
- (c)  $\langle M_a; i_{ab} : a \subseteq b, a, b \in [\lambda]^{<\omega} \rangle$  is a directed system.

We let  $\langle M_E, \in_E \rangle$  denote the direct limit of this directed system. In order to fix notation, we review the elements of this construction. Let  $B = \{(a, [f]_{E_a}) : a \in [\lambda]^{<\omega} \text{ and } [f]_{E_a} \in M_a\}$  (the disjoint union of the classes  $\{a\} \times M_a$ ). Define  $\sim$  on  $B$  by

$$(a, [f]_{E_a}) \sim (b, [g]_{E_b}) \text{ iff } \exists c \supseteq a \cup b \text{ (} i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b}) \text{)}.$$

The equivalence class containing  $(a, [f]_{E_a})$  will be denoted by  $[a, [f]_{E_a}]$ . An easy computation shows that

$$(2.3) \quad (a, [f]_{E_a}) \sim (b, [g]_{E_b}) \text{ iff } \exists c \supseteq a \cup b \text{ (} \{d \in [\kappa]^{|c|} : f(\pi_{ca}(d)) = g(\pi_{cb}(d))\} \in E_c \text{)}.$$

The membership relation  $\in_E$  is defined similarly by

$$(a, [f]_{E_a}) \in_E (b, [g]_{E_b}) \text{ iff } \exists c \supseteq a \cup b \text{ (} i_{ac}([f]_{E_a}) \in_c i_{bc}([g]_{E_b}) \text{)}.$$

We have the following criterion for membership, analogous to (2.3):

$$(2.4) \quad (a, [f]_{E_a}) \in_E (b, [g]_{E_b}) \text{ iff } \exists c \supseteq a \cup b \text{ (} \{d \in [\kappa]^{|c|} : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c \text{)}.$$

The class  $M_E$  is defined by

$$M_E = \{[a, [f]_{E_a}] : a \in [\lambda]^{<\omega}, f : [\kappa]^{|a|} \rightarrow M, \text{ and } f \in M\}.$$

By an abuse of notation,  $M_E$  is also denoted  $\text{Ult}(M, E)$ , the *extender ultrapower of  $M$  by  $E$* .  $(M_E, \in_E)$  is well-founded and is therefore identified with its transitive collapse; both notations —  $M_E$  and  $\text{Ult}(M, E)$ —are used to denote the transitive collapse.

The embeddings  $k_{aE} : M_a \rightarrow M_E$ ,  $j_E : M \rightarrow M_E$ , and  $k_E : M_E \rightarrow M$  are defined by:

$$\begin{aligned} k_{aE}([f]_{E_a}) &= [a, [f]_{E_a}] \\ j_E(x) &= k_{aE}(j_a(x)) \text{ for some (any) } a \in [\lambda]^{<\omega} \\ k_E([a, [f]_{E_a}]) &= i(f)(a). \end{aligned}$$

These functions are well-defined elementary embeddings satisfying the following relations:

- (A)  $k_E \circ j_E = i$
- (B)  $k_{aE} \circ j_a = j_E$
- (C)  $k_E \circ k_{aE} = k_a$ .

An easy computation using (B) yields

$$(2.5) \quad j_E(x) = [a, [c_x^a]_{E_a}] \text{ for some (any) } a \in [\lambda]^{<\omega}.$$

We have the following standard lemma:

**Lemma 2.6** [Ka, Lemma 26.1(b),(c)]. *Suppose  $i : M \rightarrow M$  is an elementary embedding with critical point  $\kappa$ . Let  $\lambda = i(\kappa)$ . Let  $E$  be the  $(\kappa, \lambda)$ -extender derived from  $i$ . Let  $j_E : M \rightarrow M_E$  denote the canonical embedding. Then*

- (1)  $\lambda = j_E(\kappa)$ .
- (2) If  $(\lambda = |V_\lambda|)^M$ , then  $V_\lambda^M = V_\lambda^{M_E}$ .

We turn to a discussion of the abstract notion of an extender:

**Definition 2.7** *Extenders* [Ka, p. 354]. Suppose  $M$  is a transitive set model of ZFC,  $\kappa$  is a cardinal in  $M$ ,  $\lambda > \kappa$ , and  $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ . Then  $E$  is an  $M$ - $(\kappa, \lambda)$ -extender (or simply a  $(\kappa, \lambda)$ -extender whenever  $M = V$ ) if:

- (1) For each  $a \in [\lambda]^{<\omega}$ ,  $\langle M, \in, E_a \rangle \models$  “ $E_a$  is a  $\kappa$ -complete ultrafilter over  $[\kappa]^{|a|}$ ”, and:
  - (a) For at least one such  $a$ ,  $\langle M, \in, E_a \rangle \models$  “ $E_a$  is not  $\kappa^+$ -complete”.
  - (b) For each  $\xi \in \kappa$ , there is such an  $a$  satisfying  $\{s \in [\kappa]^{|a|} : \xi \in s\} \in E_a$ .
- (2) (Coherence) Suppose  $a \subseteq b$  are both in  $[\lambda]^{<\omega}$ . Then

$$X \in E_a \quad \text{iff} \quad \{s : \pi_{ba}(s) \in X\} \in E_b,$$

where  $\pi_{ba}$  is defined as in (2.1).

- (3) (Well-foundedness) Whenever  $a_m \in [\lambda]^{<\omega}$  and  $X_m \in E_{a_m}$  for  $m \in \omega$ , there is  $d : \bigcup_m a_m \rightarrow \kappa$  such that  $d''a_m \in X_m$  for every such  $m$ .

- (4) (Normality) Whenever  $a \in [\lambda]^{<\omega}$ ,  $f : [\kappa]^{|a|} \rightarrow M$  is an element of  $M$ , and  $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$ , there is a  $b \in [\lambda]^{<\omega}$  with  $a \subseteq b$  such that

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b,$$

where  $\pi_{ba}$  is defined as in (2.1).

With this definition, one builds a model  $M_E$  in much the same way as described earlier: For each  $a \in [\lambda]^{<\omega}$ , let  $\text{Ult}(M, E_a)$  be the ultrapower of  $M$  by  $E_a$ , again using only functions that lie in  $M$ . Since  $\text{Ult}(M, E_a)$  is well-founded, we again identify it with its transitive collapse  $M_a$ . We again have the canonical map  $j_a : M \rightarrow M_a$ , defined as before. We again obtain a directed system  $\langle M_a; i_{ab} : a \subseteq b \text{ both in } [\lambda]^{<\omega} \rangle$ , where the maps  $i_{ab}$  are defined as before as in (2.2). The direct limit  $M_E$  is obtained exactly as before, and its membership relation  $\in_E$  and the maps  $k_{aE}, j_E$  have the same definitions and properties as before (see (A) and (B) above). By an abuse of notation,  $M_E$  is sometimes referred to as  $\text{Ult}(M, E)$ , though  $M_E$  is not itself an ultrapower. As before (but for different reasons),  $(M_E, \in_E)$  is well-founded, and so we identify it with its transitive collapse. Again, both  $M_E$  and  $\text{Ult}(M, E)$  may be used to denote this transitive collapse.

One difference in the constructions is that we no longer define the maps  $k_a, a \in [\lambda]^{<\omega}$  since in the present abstract context, we do not have an ambient embedding  $i : M \rightarrow M$ ; therefore, properties (c) and (C) above are no longer relevant here. Also, the analogue to Lemma 2.6 is somewhat weaker:

**Lemma 2.8** [Ka, Lemma 26.2(b)]. *Suppose  $M$  is a transitive set model of ZFC,  $\kappa$  is a cardinal in  $M$ ,  $\lambda > \kappa$ ,  $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$  is an  $M$ - $(\kappa, \lambda)$ -extender, and  $M_E$  is the direct limit of the corresponding ultrapowers. Then  $\text{crit}(j_E) = \kappa$  and  $j_E(\kappa) \geq \lambda$ .*

In Lemma 2.8, it is possible for  $j_E(\kappa) > \lambda$ . Recall that  $\kappa$  is *superstrong* if there is a nontrivial elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $V_{j(\kappa)} \subseteq M$ .

**Proposition 2.9** [Ka, Exercise 26.7(c)].  *$\kappa$  is superstrong iff there is a  $(\kappa, \lambda)$ -extender  $E$  for some  $\lambda > \kappa$  such that  $j_E(\kappa) = \lambda$  and  $V_{j_E(\kappa)} \subseteq M_E$ , where  $j_E$  is the canonical embedding.*

Actually, in [Ka], the condition “ $j_E(\kappa) = \lambda$ ” in Proposition 2.9 is dropped, since it is not strictly necessary; however, if  $E$  is derived from a superstrong embedding  $j : V \rightarrow M$  with critical point  $\kappa$ , it follows from [Ka, Lemma 26.1(c)] that  $j_E(\kappa) = j(\kappa)$ .

Because Lemma 2.8 is so weak, in order to verify that a given  $(\kappa, \lambda)$ -extender  $E$  satisfies  $j_E(\kappa) = \lambda$  and  $V_\lambda \subseteq M_E$  (whence  $\kappa$  is superstrong), the easiest approach (one that we will use) is to obtain  $E$  from another embedding, and then make use of Lemma 2.6, or some variation thereof.

The following proposition shows the connection between the two types of extenders we have discussed:



**Proposition 2.10** [Ka, Exercise 26.3(a)]. *Suppose  $i : M \rightarrow M$  is an elementary embedding with critical point  $\kappa$ , where  $M$  is a transitive set model of ZFC. Let  $E$  be the  $(\kappa, i(\kappa))$ -extender derived from  $i$ . Then  $E$  is an  $M$ - $(\kappa, i(\kappa))$ -extender.*

We introduce a definition pertaining to superstrong cardinals:

**Definition 2.11.** Suppose  $\kappa$  is superstrong and  $\lambda > \kappa$ . Then  $\lambda$  is a **target** for  $\kappa$  if there is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $V_{j(\kappa)} \subseteq M$  and  $\lambda = j(\kappa)$ . Equivalently,  $\lambda$  is a target for  $\kappa$  if there is a  $(\kappa, \lambda)$ -extender  $E$  such that if  $j_E : V \rightarrow M_E$  is the canonical embedding, then  $j_E(\kappa) = \lambda$  and  $V_\lambda \subseteq M_E$ .

Our last technical lemma of this section recalls the well-known fact that, in the definition of Normality (Definition 2.7(4)), one may restrict attention to functions  $f$  whose range lies in a sufficiently large rank. To state this in the form of a lemma, we introduce two variants of the Normality condition:  $\text{Normality}_V$  and  $\text{Normality}_{V_\gamma}$ , where  $\gamma \geq \kappa$ ;  $\text{Normality}_V$  is what one gets when  $M = V$  in the definition given above.

**Normality<sub>V</sub>:** Whenever  $a \in [\lambda]^{<\omega}$ ,  $f : [\kappa]^{|a|} \rightarrow V$ , and  $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$ , there is a  $b \in [\lambda]^{<\omega}$  with  $a \subseteq b$  such that, with  $\pi_{ba}$  as in (2.1),

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$

Suppose  $\gamma \geq \kappa$ .

**Normality<sub>V<sub>γ</sub></sub>:** Whenever  $a \in [\lambda]^{<\omega}$ ,  $f : [\kappa]^{|a|} \rightarrow V_\gamma$ , and  $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$ , there is a  $b \in [\lambda]^{<\omega}$  with  $a \subseteq b$  such that, with  $\pi_{ba}$  as in (2.1),

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$

**Lemma 2.12.** *Suppose  $\gamma \geq \kappa$ . Then the following are equivalent:*

- (1)  $\text{Normality}_V$
- (2)  $\text{Normality}_{V_\gamma}$

**Proof.** (1)  $\Rightarrow$  (2) is obvious. For (2)  $\Rightarrow$  (1), let  $a \in [\lambda]^{<\omega}$ ,  $f : [\kappa]^{|a|} \rightarrow V$ , and assume  $A \in E_a$  where

$$A = \{s \in [\kappa]^{|a|} : f(s) \in \max(s)\}.$$

We show that

$$(2.6) \quad \text{there is a } b \in [\lambda]^{<\omega} \text{ with } a \subseteq b \text{ such that } \{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$

Notice that for all  $s \in A$ ,  $\text{rank}(f(s)) < \gamma$ . Define  $g : [\kappa]^{|a|} \rightarrow V_\gamma$  by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ s & \text{if } s \notin A. \end{cases}$$

Clearly  $f \upharpoonright A = g \upharpoonright A$ . By Normality $_{V_\gamma}$ , there is a  $b \in [\lambda]^{<\omega}$  with  $a \subseteq b$  such that  $B \in E_b$ , where

$$B = \{s \in [\kappa]^{|b|} : g(\pi_{ba}(s)) \in s\}.$$

By Coherence, we also have that  $C \in E_b$ , where

$$C = \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in A\}.$$

Then, letting  $D = B \cap C$ , we have  $D \in E_b$  as well. We show that  $D \subseteq \{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\}$ , and this will establish (2.6) and complete the proof. Let  $s \in D$ . Then  $\pi_{ba}(s) \in A$  and  $g(\pi_{ba}(s)) \in s$ ; since  $f$  and  $g$  agree on  $A$ , it follows that  $f(\pi_{ba}(s)) \in s$ , as required. ■

### §3. Main Results

In this section, we formulate a sequence  $\langle I_4^n : n \in \omega \rangle$  of axioms, each asserting the existence of a transitive model of ZFC + WA having closure properties that increase in strength as  $n$  increases; we show that, in a sense to be made precise,  $I_3$  represents the limit of this sequence of axioms. To motivate the formulation of these axioms, we begin by considering several kinds of closure, typically imposed on transitive set models in studies of this kind. We start by showing that even countable closure of transitive models of WA is inconsistent, and then turn to three other familiar notions of closure, which we show are equivalent in this context. Motivated by this observation, we use these latter notions of closure to formulate our  $\omega$ -sequence of axioms  $I_4^n$ . We compare the strengths of these axioms with known large cardinals and finally demonstrate the sense in which the axioms  $I_4^n$  “converge” to  $I_3$ . We begin with an observation that was made in Section 2:

**Proposition 3.1.** *No transitive model  $\langle M, \in, i \rangle$  of ZFC + WA $_0$  satisfies  ${}^\omega M \subseteq M$ .*

**Proof.** Suppose  $\mathcal{M} = \langle M, \in, i \rangle \models \text{ZFC} + \text{WA}_0$  and  ${}^\omega M \subseteq M$ . By hypothesis,  $f : \omega \rightarrow M : n \mapsto i^n(\kappa)$  is a set in  $M$ . Therefore, the Cofinal Axiom fails. This contradicts Proposition 2.4. ■

**Theorem 3.2.** *Suppose  $\langle M, \in, i \rangle$  is a transitive model of ZFC + WA, and  $\text{crit}(i) = \kappa$ . Let  $U_M = \{X \in P(\kappa) \cap M : \kappa \in i(X)\}$ . Then the following are equivalent:*

- (1)  $P(\kappa) \subseteq M$ .
- (2)  $U_M$  is a  $\kappa$ -complete nonprincipal ultrafilter on  $\kappa$  (in  $V$ ).
- (3)  $V_{\kappa+1} \subseteq M$ .
- (4) Whenever  $f : \kappa \rightarrow X$  and  $X \subseteq V_\kappa^M$ , we have  $f \in M$ .

**Remark.** We use the following elementary facts without special mention (see [Je, p. 21]):

- (a) There is a definable bijection  $\Gamma : ON \times ON \rightarrow ON$  having the property that for any ordinal  $\alpha$  for which

$$(3.1) \quad \alpha = \omega^\alpha \quad (\text{ordinal exponentiation}),$$

we have  $\Gamma''\alpha \times \alpha = \alpha$ .

(b) Every cardinal satisfies (3.1).

(c) The property (3.1) is absolute for transitive models of ZFC.

**Proof of (1) $\Rightarrow$ (2).** Assuming (1), we first show that  $\kappa$  is a regular uncountable cardinal. Suppose  $\alpha < \kappa$  and  $f : \alpha \rightarrow \kappa$  ( $f$  is a potential witness to “ $\kappa$  is not a cardinal” or “ $\kappa$  is singular”). We do not know yet that  $\kappa$  is a cardinal; however, as  $\kappa$  is a cardinal in  $M$ , we have that “ $\kappa = \omega^\kappa$ ” (under ordinal exponentiation) holds in  $M$  and therefore in  $V$ . Thus,  $f$  can be coded as a subset of  $\kappa$  using  $\Gamma$ , and hence must already lie in  $M$ . But in  $M$ ,  $\kappa$  is a regular cardinal, so  $f$  must be bounded in  $M$ , and hence in  $V$  by absoluteness. Thus,  $\kappa$  is a regular cardinal in  $V$ . Finally, since  $i(\omega) = \omega$ ,  $\kappa$  must be uncountable.

By (1),  $U_M = \{X \subseteq \kappa : \kappa \in i(X)\}$ ; also,  $U_M$  is a nonprincipal  $M$ - $\kappa$ -complete ultrafilter. To prove that it is  $\kappa$ -complete, suppose  $S = \{X_\alpha : \alpha < \gamma\} \subseteq U_M$  (where  $\gamma < \kappa$ ). We can code  $S$  as the disjoint union of its elements: Let  $\bar{S} = \{(\alpha, x) \in \kappa \times \kappa : \alpha < \gamma \text{ and } x \in X_\alpha\}$ .  $\bar{S}$  in turn can be coded as a subset of  $\kappa$  using  $\Gamma$ . Thus,  $S \in M$ , and by  $M$ - $\kappa$ -completeness,  $\bigcap S \in U_M$ , as required. ■

**Proof of (2) $\Rightarrow$ (3).** We begin by observing that  $P(\kappa) \subseteq M$ : If  $X \in P(\kappa) \setminus M$ , then certainly  $\kappa \setminus X \notin M$ , and  $U_M$  would fail to be an ultrafilter in  $V$ , violating (2). Next, we prove that (3) holds. We show by induction that for all  $\alpha \leq \kappa + 1$ ,  $V_\alpha \in M$ . Assuming  $V_\alpha \in M$ , the inaccessibility of  $\kappa$  implies that there is a  $\gamma \leq \kappa$  and a bijection  $f : \gamma \rightarrow V_\alpha$ , with  $f \in M$ . Let  $X \subseteq V_\alpha$ . There is, in  $V$ , a set  $A \subseteq \gamma$  such that  $A = f^{-1}(X)$ . Because  $P(\kappa) \subseteq M$ ,  $A \in M$ . Thus  $X = f''A \in M$ . This shows that  $V_{\alpha+1} \subseteq M$ , whence  $V_{\alpha+1} \in M$ . The limit case is easy, since unions are absolute. This completes the induction, and we have  $V_{\kappa+1}^M = V_{\kappa+1}$ , as required. ■

**Proof of (3) $\Rightarrow$ (4).** Given  $f : \kappa \rightarrow X \subseteq V_\kappa^M$ , notice  $f \in V_{\kappa+1}$ . The result follows by (3). ■

**Proof of (4) $\Rightarrow$ (1).** Let  $X \subseteq \kappa$  and let  $f : \kappa \rightarrow X$  be a surjection. Since  $X \subseteq V_\kappa^M$ , by (4),  $f \in M$ . But then  $X = f''\kappa \in M$ . It follows that  $P(\kappa) \subseteq M$ . ■

The theorem can be proven for transitive models of ZFC+BTEE — the full strength of WA (or even  $WA_0$ ) is unnecessary. (Note that in the proof of (1)  $\Rightarrow$  (2), we do not need to know whether or not  $U_M$  is a set in  $M$ ; to establish that it is (apparently) requires an application of Separation<sub>j</sub>, but set-hood of  $U_M$  is not needed for the proof.)

The theorem leads us to the definition of the following sequence  $\{I_4^n : n \in \omega\}$  of axioms:

$I_4^n(\kappa)$ :      there exist sets  $M, i$  such that  $M$  is transitive,  $\langle M, \in, i \rangle \models \text{ZFC} + \text{WA}$ ,  $\text{crit}(i) = \kappa$ ,  
                          and  $V_{i^n(\kappa)+1} \subseteq M$ .

In this context, we interpret  $i^0(\kappa)$  as  $\kappa$ .

Clearly,  $I_3(\kappa)$  implies  $\forall n \in \omega I_4^n(\kappa)$ . Also notice that  $I_4^0(\kappa)$  is already slightly stronger than WA since it asserts the existence of a transitive model of WA.

Before proving the large cardinal consequences of each of these axioms, we first verify that for each  $n$ , the axiom  $I_4^n(\kappa)$  is strong enough to ensure that  $i^n(\kappa)$  is a measurable cardinal:

**Proposition 3.3.** *Assume  $I_4^n(\kappa)$  is true, with witness  $\langle M, \in, i \rangle$ . Then  $i^n(\kappa)$  is a measurable cardinal.*

**Proof.** The case  $n = 0$  follows from the fact that (3)  $\Rightarrow$  (2) in Theorem 3.2. For  $n > 0$ , let  $\langle M, \in, i \rangle$  satisfy the properties of  $I_4^n(\kappa)$ . As in the proof of (1)  $\Rightarrow$  (2) in Theorem 3.2, note that any function  $f : \alpha \rightarrow i^n(\kappa)$  that is a potential witness to  $i^n(\kappa)$  being either a non-cardinal or a singular cardinal must already lie in  $M$  by coding. Thus  $i^n(\kappa)$  is, in  $V$ , a regular cardinal. Let  $U = \{X \subseteq \kappa : \kappa \in i(X)\}$ , defined in  $M$ . In  $M$ ,  $U$  is a  $\kappa$ -complete nonprincipal ultrafilter on  $\kappa$ . By elementarity, in  $M$ ,  $i^n(U)$  is an  $i^n(\kappa)$ -complete nonprincipal ultrafilter on  $i^n(\kappa)$ . Since  $i^n(U) \subseteq P(i^n(\kappa)) \subseteq M$ , one can use a coding argument as in the proof of (1)  $\Rightarrow$  (2) of Theorem 3.2 to conclude that  $i^n(U)$  is an  $i^n(\kappa)$ -complete nonprincipal ultrafilter on  $i^n(\kappa)$  in  $V$ . The result follows. ■

**Proposition 3.4.**  $I_4^0(\kappa)$  implies that  $\kappa$  is measurable, and that the set  $\{\alpha < \kappa : \alpha \text{ is measurable}\}$  has normal measure 1.

**Proof.** Proposition 3.3 shows  $\kappa$  is measurable. Let  $\langle M, \in, i \rangle$  be a witness to  $I_4^0(\kappa)$ . Since  $U = \{X \subseteq \kappa : \kappa \in i(X)\} \in M$ , and since, for all  $\alpha < \kappa$ , “ $\alpha$  is measurable” is absolute for  $M$ , the second clause of the proposition follows, where  $U$  is the required normal measure. ■

**Theorem 3.5.**  $I_4^1(\kappa)$  implies that  $\kappa$  is superstrong, and that the set  $\{\alpha < \kappa : \alpha \text{ is superstrong}\}$  has normal measure 1.

**Proof.** Let  $\langle M, \in, i \rangle$  be a model given by  $I_4^1(\kappa)$ . Let  $\lambda = i(\kappa)$ . By Proposition 2.10,  $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$  is an  $M$ - $(\kappa, \lambda)$ -extender, where  $E_a = \{X \in P([\kappa]^{|a|}) \cap M : a \in i(X)\}$ . We wish to prove:

- (A)  $E$  is a  $(\kappa, \lambda)$ -extender (in  $V$ )
- (B) the canonical embedding  $j_E : V \rightarrow \text{Ult}(V, E)$  satisfies  $j_E(\kappa) = \lambda$ , and
- (C)  $V_\lambda \subseteq \text{Ult}(V, E)$ .

The first part of the theorem will then follow from (A)–(C), by Proposition 2.9.

To prove (A), we verify that conditions (1)–(4) of Definition 2.7 hold true, with  $M$  in each case being replaced by  $V$ . Several of the conditions follow immediately from absoluteness between  $M$  and  $V$ ; in particular, conditions (2) and (3) can be verified in this way. Also, for each  $E_a$  that is a  $\kappa$ -complete ultrafilter in  $V$ , conditions (1a) and (1b) also hold by absoluteness. What remains is to verify the following:

- (i) Each  $E_a$ ,  $a \in [i(\kappa)]^{<\omega}$ , is a  $\kappa$ -complete ultrafilter (in  $V$ )
- (ii) Normality $_V$

Part (i) follows immediately from our assumption that  $V_{i(\kappa)+1} \subseteq M$ . For (ii), Lemma 2.12 tells us that it suffices to demonstrate Normality $_{V_\lambda}$ . But again, Normality $_{V_\lambda}$  follows immediately from the fact that  $V_{i(\kappa)+1} \subseteq M$ .

To prove (B) and (C), we begin with a number of easy observations. For the next several paragraphs, until indicated otherwise, we let  $\text{Ult}(V, E)$ ,  $\text{Ult}(M, E)$ , and  $\text{Ult}^M(V, E)$  denote the “raw” direct limits, prior to applying the Mostowski collapsing function. (To eliminate possible confusion here, the notation  $\text{Ult}^M(V, E)$  signifies the extender ultrapower construction carried out entirely in  $M$ ; in this context, we understand  $V$  to be the “extension” of the predicate  $x = x$ , and so  $V^M = M$ .) We explicitly refer to the relevant collapsing functions as follows:  $p_E : \text{Ult}(V, E) \rightarrow V_E$ ,  $p_{M,E} : \text{Ult}(M, E) \rightarrow M_E$ , and  $p_E^M : \text{Ult}^M(V, E) \rightarrow V_E^M$ , respectively. We denote the corresponding ultrapower membership relations by  $\in_E, \in_{M,E}$  and  $\in_E^M$ , respectively. Our strategy is to show that initial segments of all three of these extender ultrapowers are isomorphic and that, in fact, their transitive collapse is in every case equal to  $V_\lambda$ . As we will show, this will establish (B) and (C).

We isolate the relevant “initial segments” as follows. Let

$$\begin{aligned} A &= \{[a, [f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \rightarrow V_\kappa\} \subseteq \text{Ult}(V, E) \\ A_M &= \{[a, [f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \rightarrow V_\kappa, f \in M\} \subseteq \text{Ult}(M, E) \\ A^M &= \left( \{[a, [f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \rightarrow V_\kappa\} \right)^M \subseteq \text{Ult}^M(V, E). \end{aligned}$$

Before proving that  $A, A_M$  and  $A^M$  are equal, we first mention that they are not equal merely by definition. For instance, to verify that  $A_M = A^M$ , we wish to verify the fact that for all  $a \in [\lambda]^{<\omega}$  and  $f : [\kappa]^{|a|} \rightarrow V_\kappa$  in  $M$ ,  $[a, [f]_{E_a}] = [a, [f]_{E_a}]^M$ .

Because Scott’s trick is used in the formation of equivalence classes of the form  $[f]_{E_a}$  in forming the ultrapowers by  $E_a$ , all functions  $g : [\kappa]^{|a|} \rightarrow N$  (where  $N = V$  or  $N = M$ , depending on which model we are building) that belong in the class  $[f]_{E_a}$ , including  $f : [\kappa]^{|a|} \rightarrow V_\kappa$  itself, must have the least possible rank, namely  $\kappa$ . Thus, as a set,  $[f]_{E_a} \subseteq V_{\kappa+1}$ . Since the  $a$ ’s lie in  $[\lambda]^{<\omega}$ , each pair  $(a, [f]_{E_a})$  has rank below  $\lambda$ , and the class  $[a, [f]_{E_a}]$  belongs to  $V_{\lambda+1}$ . Since  $V_{\lambda+1} \subseteq M$ , it follows that, for each  $a \in [\lambda]^{<\omega}$  and each  $f : [\kappa]^{|a|} \rightarrow V_\kappa$ ,  $[a, [f]_{E_a}]$  is the same set as  $([a, [f]_{E_a}])^M$ . Moreover, it is clear that the sets  $A, A_M$  and  $A^M$  are equal. It also follows that each of the structures  $\mathcal{A} = \langle A, \in_E \rangle$ ,  $\mathcal{A}_M = \langle A_M, \in_{M,E} \rangle$ , and  $\mathcal{A}^M = \langle A^M, \in_E^M \rangle$  is well-founded in  $V$ . This observation is obvious for  $\mathcal{A}$  and  $\mathcal{A}_M$  since  $\text{Ult}(V, E)$  and  $\text{Ult}(M, E)$  are themselves well-founded (see the discussion in Section 2 or in [Ka, Chapter 26]). It is perhaps less obvious in the case of  $\mathcal{A}^M$  because  $M$  is not even countably closed, as we showed in Lemma 3.1. However, if  $F : \omega \rightarrow A^M$  witnesses ill-foundedness, with  $F(n) = [a_n, [f_n]_{E_{a_n}}]$ , then

$$(3.2) \quad \exists \{c_n : n \geq 1\} \forall n \{d : f_n(\pi_{c_n a_n}(d)) \in f_{n-1}(\pi_{c_{n-1} a_{n-1}}(d))\} \in E_{c_n}.$$

In fact, (3.2) is equivalent to ill-foundedness not only of  $\mathcal{A}^M$ , but of  $\mathcal{A}_M$  as well. This contradicts the fact that  $\mathcal{A}_M$  is well-founded.

Next we observe that, as  $\in$ -structures,  $\mathcal{A}$ ,  $\mathcal{A}_M$ , and  $\mathcal{A}^M$  are isomorphic. The proof in each case is easy. We show how to verify  $\mathcal{A}_M \cong \mathcal{A}^M$  and leave the other cases to the reader. We must show that for all  $a, b \in [\lambda]^{<\omega}$  and all  $f : [\kappa]^{|a|} \rightarrow V_\kappa$  and  $g : [\kappa]^{|b|} \rightarrow V_\kappa$  in  $M$

$$(3.3) \quad [a, [f]_{E_a}] \in_{M,E} [b, [g]_{E_b}] \text{ iff } \left( [a, [f]_{E_a}] \in_E^M [b, [g]_{E_b}] \right)^M.$$

Using the membership criterion (2.4), the display (3.3) is equivalent to

$$\begin{aligned} & \text{for some } c \supseteq a \cup b, \{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c \\ & \text{iff } \left( \text{for some } c \supseteq a \cup b, \{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c \right)^M. \end{aligned}$$

Since the statement “for some  $c \supseteq a \cup b, \{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c$ ” is absolute for  $M$  (since  $V_{\lambda+1} \subseteq M$ ), the displayed equivalence is true.

The fact that  $\mathcal{A}$ ,  $\mathcal{A}_M$ , and  $\mathcal{A}^M$  are isomorphic leads to the conclusion that their transitive collapses are all equal. We show that these in turn are equal to  $V_\lambda$ . We do this by showing that the transitive collapse of  $\mathcal{A}_M$  is  $V_\lambda^{\text{Ult}(M,E)}$ ; this will suffice because, by Lemma 2.6(2),  $V_\lambda^{\text{Ult}(M,E)} = V_\lambda$  (note that  $\lambda = |V_\lambda|$  because  $\lambda$  is inaccessible).

Recalling that  $p_{M,E} : \text{Ult}(M,E) \rightarrow M_E$  denotes the collapsing function, we show now that  $p''_{M,E} \mathcal{A}_M = V_\lambda^{\text{Ult}(M,E)}$ . Recall  $\lambda = i(\kappa) = (p_{M,E} \circ i_E)(\kappa)$ , where  $i_E : M \rightarrow \text{Ult}(M,E)$  is the canonical embedding and  $i_E(\kappa) = [a, [c_\kappa^a]]$  for some (any)  $a \in [\lambda]^{<\omega}$ . Thus,  $V_\lambda^{\text{Ult}(M,E)} = p_{M,E}([a, [c_{V_\kappa}^a]])$  for some (any)  $a \in [\lambda]^{<\omega}$ .

Suppose  $x \in V_\lambda^{\text{Ult}(M,E)}$ . Let  $b, g$  be such that  $x = p_{M,E}([b, [g]_{E_b}])$ . Since  $x \in V_\lambda^{\text{Ult}(M,E)}$ , it follows that  $[b, [g]_{E_b}] \in_{M,E} [a, [c_{V_\kappa}^a]_{E_a}]$ . Thus, for some  $c \supseteq a \cup b$ ,

$$(3.4) \quad D = \{d \in [\kappa]^{|c|} : g(\pi_{cb}(d)) \in V_\kappa\} \in E_c.$$

Let  $B = \{e \in [\kappa]^{|b|} : g(e) \in V_\kappa\}$ . By Coherence,  $B \in E_b$  iff  $D' \in E_c$  where  $D' = \{d \in [\kappa]^{|c|} : \pi_{cb}(d) \in B\}$ . But this follows easily since  $D \subseteq D'$ . We have shown that  $g$  has values in  $V_\kappa$  on the set  $B \in E_b$ . Define  $g'$  by  $g' \upharpoonright B = g \upharpoonright B$  and  $g'(e) = 0$  if  $e \notin B$ . Clearly,  $(b, [g']_{E_b}) \sim (b, [g]_{E_b})$ , and so we can write  $x = p_{M,E}([b, [g']_{E_b}])$  where now the representative function  $g'$  has range in  $V_\kappa$ . We have shown that  $V_\lambda^{\text{Ult}(M,E)} \subseteq p''_{M,E} \mathcal{A}_M$ . For the converse, suppose  $[b, [g]_{E_b}] \in \mathcal{A}_M$ , with  $g : [\kappa]^{|b|} \rightarrow V_\kappa$ . From our work above, it suffices to prove that for some  $a \in [\lambda]^{<\omega}$  and  $c \supseteq a \cup b$ , equation (3.4) holds. Let  $a \in [\lambda]^{<\omega}$  and  $c \supseteq a \cup b$  be chosen arbitrarily. Defining  $B$  as above by  $B = \{e \in [\kappa]^{|b|} : g(e) \in V_\kappa\}$ , it is clear that  $B \in E_b$ . By Coherence, we must have  $D' \in E_c$ , where  $D' = \{d \in [\kappa]^{|c|} : \pi_{cb}(d) \in B\}$ . As is easily checked,  $D' \subseteq D$ , whence  $D \in E_c$ , as required.

We have therefore shown that  $V_\lambda$  is a subset of each of the (transitive collapses of the) extender ultrapower models, namely  $V_E$ ,  $M_E$ , and  $V_E^M$ . Next we wish to show that the canonical embeddings all take on the same value at  $\kappa$ . For the rest of the proof of the theorem, we return to the point of

view that each of  $\text{Ult}(V, E)$ ,  $\text{Ult}(M, E)$  and  $\text{Ult}^M(V, E)$  is identified with its transitive collapse. As a matter of notation, we denote the canonical embeddings as follows:

$$\begin{aligned} j_E &: V \rightarrow \text{Ult}(V, E) \\ i_E &: M \rightarrow \text{Ult}(M, E) \\ j_E^M &: M \rightarrow \text{Ult}^M(V, E). \end{aligned}$$

By Lemma 2.6,  $i_E(\kappa) = \lambda$ . We compute  $j_E(\kappa)$ : Suppose  $[b, [g]_{E_b}] \in_E [a, [c_\kappa^a]]$ . For some  $c \supseteq a \cup b$ ,  $\{d : g(\pi_{cb}(d)) \in \kappa\} \in E_c$ . Reasoning as above, it follows that, on a set in  $E_b$ ,  $g$  takes values in  $V_\kappa$ , and therefore the (transitive collapse of)  $[b, [g]_{E_b}]$  lies in  $V_\lambda$ . Therefore  $j_E(\kappa) \leq \lambda$ . But by Lemma 2.8,  $j_E(\kappa) \geq \lambda$ , and the result follows. Exactly the same reasoning shows that  $j_E^M(\kappa) = \lambda$ .

We have proven (B) and (C). To prove that the set  $\{\alpha < \kappa : \alpha \text{ is superstrong}\}$  has normal measure 1, we first observe that  $M \models \text{“}\kappa \text{ is superstrong”}$ : Because  $V_{\lambda+1} \subseteq M$ ,  $M \models \text{“}E \text{ is a } (\kappa, \lambda)\text{-extender”}$ . As we observed above,  $j_E^M(\kappa) = \lambda$  and  $V_\lambda \subseteq \text{Ult}^M(V, E)$ . It follows from Proposition 2.9 that  $\kappa$  is superstrong in  $M$ .

The fact that  $\kappa$  is superstrong in  $M$  is enough to ensure that those  $\alpha$  below  $\kappa$  that are superstrong in  $M$  form a normal measure 1 set; however, to ensure that such  $\alpha$  are also superstrong in  $V$ , we need to make fuller use of elementarity. Recalling Definition 2.11, we observe that

$$M \models \text{“}\kappa \text{ is superstrong and has an inaccessible target } < i(\lambda)\text{”}.$$

(In fact,  $\lambda$  is such a target in  $M$ .)

It follows therefore that if  $U = \{X \subseteq \kappa : \kappa \in i(X)\}$  and  $X = \{\alpha < \kappa : (\alpha \text{ is superstrong and has an inaccessible target } < \lambda)^M\}$ , then  $X \in U$ . The following Claim will complete the proof of the theorem; note that the required normal measure (mentioned in the statement of the theorem) is  $U$ .

**Claim.** For each  $\alpha$  in  $X$ ,  $\alpha$  is superstrong.

**Proof of Claim.** Let  $\alpha \in X$  and let  $\gamma < \lambda$  be, in  $M$ , an inaccessible target for  $\alpha$ . Moreover, let  $F$  be, in  $M$ , an  $(\alpha, \gamma)$ -extender for which the canonical embedding  $j_F^M$  (defined in  $M$ ) maps  $\alpha$  to  $\gamma$  and for which  $V_\gamma^M = V_\gamma$  is a subset of the resulting extender ultrapower  $\text{Ult}^M(V, F)$ . By considering sets analogous to  $A$ ,  $A_M$ , and  $A^M$  as we did earlier, one can perform a similar analysis of the models  $\text{Ult}(V, F)$ ,  $\text{Ult}(M, F)$  and  $\text{Ult}^M(V, F)$  to conclude that  $V_\gamma$  is a subset of all of them and that the canonical embedding in each case maps  $\alpha$  to  $\gamma$ . It follows that  $\alpha$  is superstrong, and the Claim is proved. ■

**Theorem 3.6.** For each  $n \geq 1$ ,  $I_4^{n+1}(\kappa)$  implies that  $\kappa$  is  $n$ -huge and that the set  $\{\alpha < \kappa : \alpha \text{ is } n\text{-huge}\}$  has normal measure 1.

**Proof.** Let  $\langle M, \in, i \rangle$  witness  $I_4^{n+1}(\kappa)$ . By Proposition 3.3,  $i^{n+1}(\kappa)$  is inaccessible. As usual, define an  $n$ -huge ultrafilter  $U \subseteq P(P(i^n(\kappa)))$  by putting  $X \in U$  if and only if  $X \subseteq P(i^n(\kappa))$  and

$i''(i^n(\kappa)) \in i(X)$ . The definition makes sense since  $i''(i^n(\kappa)) \in M$ . (Note that  $i''(i^n(\kappa))$  lies in a proper initial segment of  $i^{n+1}(\kappa)$ , and so must lie in  $V_{i^{n+1}(\kappa)}$ .) Verification that  $U$  is an  $n$ -huge normal measure involves verifying that  $U$  is closed under diagonal intersections, is fine, and contains all collections of the form  $\{x \in P(i^n(\kappa)) : \text{ot}(x \cap i^{m+1}(\kappa)) = i^m(\kappa)\}$ , for  $m < n$ ; all these arguments go through because the relevant sets lie in  $V_{i^{n+1}(\kappa)+1}$ .

To prove the second part of the theorem, first note that, as  $U \in V_{i^n(\kappa)} \subseteq M$  (and since  $i^n(\kappa)$  is inaccessible), we have that  $M \models$  “ $\kappa$  is  $n$ -huge”. Let  $\lambda = i^n(\kappa)$ . Recalling Definition 2.1, we also have that

$$M \models \text{“}\kappa \text{ is } n\text{-huge and has an } n\text{-huge index below } i(\lambda)\text{.”}$$

(In particular,  $\lambda$  itself is an  $n$ -huge index for  $\kappa$  in  $M$ .) It follows therefore that if  $D = \{X \subseteq \kappa : \kappa \in i(X)\}$ , then  $X \in D$ , where

$$X = \{\alpha < \kappa : M \models \text{“}\alpha \text{ is } n\text{-huge and has an } n\text{-huge index below } \lambda\text{”}\}.$$

Since  $D$  is a normal measure, the proof of the second part of the theorem will be complete after we have demonstrated the following claim:

**Claim.** For all  $\alpha \in X$ ,  $\alpha$  is  $n$ -huge (in  $V$ ).

**Proof of Claim.** Given  $\alpha \in X$ , let  $\gamma_\alpha$  be, in  $M$ , an  $n$ -huge index for  $\alpha$  such that  $\gamma_\alpha < \lambda$ . It follows that there is a  $U_\alpha \in M$  such that

$$M \models \text{“}U_\alpha \text{ is an } n\text{-huge ultrafilter on } P(\gamma_\alpha)\text{.”}$$

Since  $U_\alpha \in V_{i^n(\kappa)} \subseteq M$  and  $i^n(\kappa)$  is inaccessible,  $U_\alpha$  is an  $n$ -huge ultrafilter on  $P(\gamma_\alpha)$  in  $V$ . It follows that  $\alpha$  is  $n$ -huge in  $V$ . ■

We do not believe that  $\forall n I_4^n(\kappa)$  implies  $I_3(\kappa)$ , but we do not have a proof. We can show, however, that if we have embeddings  $i_n : M_n \rightarrow M_n$ ,  $n \in \omega$ , each witnessing  $I_4^n(\kappa)$  and collectively exhibiting enough “coherence”, then we can obtain  $I_3(\kappa)$ . We will need the following definition:

**3.7 Definition** (*Coherence of  $I_4$  embeddings*). Suppose  $I = \{i_n : M_n \rightarrow M_n \mid n \in \omega\}$  is an enumeration of elementary embeddings, all having critical point  $\kappa$ . Then  $I$  is called  $I_4(\kappa)$ -coherent if for each  $n$ ,  $\langle M_n, \in, i_n \rangle$  witnesses  $I_4^n(\kappa)$ , and for each  $m < n$ ,

$$(3.5) \quad i_m \upharpoonright V_{\kappa(m)+1} = i_n \upharpoonright V_{\kappa(m)+1},$$

where  $\kappa(m) = i_m^m(\kappa)$  for each  $m \geq 1$  and  $\kappa(0) = \kappa$ .

In the theorem below, we show that an  $I_3$  embedding can be obtained from an  $I_4(\kappa)$ -coherent set of embeddings by taking an appropriate “limit.”



**Theorem 3.8.** *The following are equivalent:*

- (1)  $I_3(\kappa)$ .
- (2) *There is an  $I_4(\kappa)$ -coherent set of embeddings.*

**Proof.** For (1)  $\Rightarrow$  (2), let  $M_n = V_\lambda$  and  $i_n = i$  for each  $n$ , where  $i : V_\lambda \rightarrow V_\lambda$  is an  $I_3$  embedding. For (2)  $\Rightarrow$  (1), let  $\{i_n : M_n \rightarrow M_n \mid n \in \omega\}$  be an  $I_4(\kappa)$ -coherent set of embeddings, and, as in the definition of coherence, let  $\kappa_{(m)} = i_m^m(\kappa)$  for each  $m \geq 1$ , and  $\kappa_{(0)} = \kappa$ . Let  $\lambda = \sup\{\kappa_{(m)} : m \in \omega\}$ . Then  $V_\lambda = \bigcup_{m \in \omega} V_{\kappa_{(m)}}$ .

**Claim.**  $V_{\kappa_{(0)}} \prec V_{\kappa_{(1)}} \prec \dots \prec V_{\kappa_{(n)}} \prec \dots \prec V_\lambda$  forms an elementary chain.

**Proof of Claim.** It suffices to show that  $V_{\kappa_{(n)}} \prec V_{\kappa_{(n+1)}}$  for all  $n$ . Clearly,

$$(3.6) \quad V_\kappa \prec V_{\kappa_{(1)}}.$$

We wish to apply the embedding  $i_n^n$  to (3.6). First, we observe that by (3.5),

- (a) for all  $n \geq 1$ ,  $i_n(i_1(\kappa)) = i_{n+1}(i_1(\kappa))$ ; so by induction on  $n$ ,  $i_n^n(i_1(\kappa)) = i_{n+1}^n(i_1(\kappa))$ .
- (b) for all  $n \in \omega$ ,  $i_1(\kappa) = i_{n+1}(\kappa)$ .

Applying (a) and (b), we obtain

$$i_n^n(\kappa_{(1)}) = i_n^n(i_1(\kappa)) = i_{n+1}^n(i_1(\kappa)) = i_{n+1}^n(i_{n+1}(\kappa)) = \kappa_{(n+1)}.$$

Now, applying  $i_n^n$  to (3.6) yields

$$V_{\kappa_{(n)}} \prec V_{\kappa_{(n+1)}},$$

as required. ■

**Continuation of Proof of Theorem.** We define  $i : V_\lambda \rightarrow V_\lambda$  by  $i(x) = i_m(x)$  for some (any)  $m$  for which  $x \in V_{\kappa_{(m)}}$ . That  $i$  is well-defined follows from (3.5). To prove that  $i$  is elementary, we use the Tarski-Vaught criterion: For a given  $\phi(x, y)$  and set  $c \in V_\lambda$ , we show that if, for some  $b \in V_\lambda$ ,  $V_\lambda \models \phi[b, i(c)]$ , then there is  $a \in V_\lambda$  such that  $V_\lambda \models \phi[i(a), i(c)]$ . Let  $n$  be such that  $b, c \in V_{\kappa_{(n)}}$ . By the definition of  $i$ ,  $i(c) = i_{n+1}(c)$ . By the claim,  $V_{\kappa_{(n+1)}} \prec V_\lambda$ . Hence  $V_{\kappa_{(n+1)}} \models \phi[b, i_{n+1}(c)]$ . Using (3.5), it is easy to verify that  $i_{n+1} \upharpoonright V_{\kappa_{(n)}} : V_{\kappa_{(n)}} \rightarrow V_{\kappa_{(n+1)}}$  is elementary. Thus, there is  $a \in V_{\kappa_{(n)}}$  such that  $V_{\kappa_{(n+1)}} \models \phi[i_{n+1}(a), i_{n+1}(c)]$ . It follows that  $V_\lambda \models \phi[i_{n+1}(a), i_{n+1}(c)]$  and  $V_\lambda \models \phi[i(a), i(c)]$ . Thus  $i : V_\lambda \rightarrow V_\lambda$  is elementary and the result follows. ■

#### §4. Open Questions.

The axioms  $I_4^n$  clearly appear to grow in strength as  $n$  increases, but we are unable to prove this:

**Question 1.** Show that if  $m < n$ , then  $\text{ZFC} + \text{I}_4^n(\kappa) \vdash \text{Con}(\text{ZFC} + \text{I}_4^m(\kappa))$ .

In a similar vein:

**Question 2.** Is there a model of  $\text{ZFC} + (\forall n \in \omega \text{I}_4^n(\kappa)) + \neg \text{I}_3(\kappa)$ ?

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