Consistency of $V = HOD$ With the Wholeness Axiom

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Abstract. The Wholeness Axiom (*WA*) is an axiom schema that can be added to the axioms of *ZFC* in an extended language $\{\in, j\}$, and that asserts the existence of a nontrivial elementary embedding $j : V \to V$. The well-known inconsistency proofs are avoided by omitting from the schema all instances of Replacement for j-formulas. We show that the theory $ZFC + V = HOD + WA$ is consistent relative to the existence of an I_1 embedding. This answers a question about the existence of Laver sequences for *regular* classes of set embeddings: Assuming there is an I_1-embedding, there is a transitive model of $ZFC + WA +$ "there is a regular class of embeddings that admits no Laver sequence."

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§1. Introduction

The Wholeness Axiom (*WA*) is an axiom schema in an extended language $\{\in, j\}$ that is intended to provide a near-minimal weakening of the assertion "there is a nontrivial elementary embedding $j : V \to V$" that is not obviously inconsistent with *ZFC*. In [C2], the details of this schema are developed; briefly, the axioms consist of all instances of Separation (but no instance of Replacement) for formulas having an occurrence of the symbol j, together with all axioms of the form $\phi(x_1, x_2, \ldots, x_n) \longrightarrow \phi(j(x_1), j(x_2), \ldots, j(x_n))$, and the axiom $\exists x\, j(x) \neq x$. Omitting from the schema all instances of Replacement for j-formulas provides a means to avoid a crucial step in Kunen's well-known inconsistency proof, because without Replacement for j-formulas, there is no guarantee that the sequence $\langle \kappa, j(\kappa), j^2(\kappa), \ldots \rangle$ has a supremum. Defining a *WA-embedding* $j : V \to V$ to be any witness to *WA*, we proved the following in [C2]:

Proposition (*Consistency Strength Lemma*).

(1) *Assume WA and let j denote the WA-embedding. If κ is the critical point of j, then κ is the κth cardinal that is super-n-huge for every n.*

(2) *If there is a nontrivial elementary embedding $i : V_\lambda \to V_\lambda$, for some limit λ, then $\langle V_\lambda, \in, i \rangle$ is a model of WA.*

Several years ago, the author was asked whether *WA* is consistent with $V = HOD$ (see [Ku1] or [Je] for an introduction to *HOD*); the question appears as Open Question 4.7 of [C2]. The question is quite natural: $V = HOD$ is known to be consistent with many globally defined large cardinal axioms, such as strong and supercompact cardinals, under mild hypotheses (see [Me]), but the proofs do not carry over to the *WA* case in any obvious way. The main result of this note is the proof that $V = HOD$ is indeed consistent with *WA*, modulo a strong large cardinal assumption. We will prove:

Theorem (*Main Theorem*). *Suppose there is an I_1-embedding. Then there is a model of $ZFC + WA + V = HOD$.*

Our Main Theorem answers another question raised in [C2] concerning Laver sequences. Laver sequences were originally defined by Laver in [La]: A function $f : \kappa \to V_\kappa$ is a Laver sequence if for each set x and each ordinal $\lambda \geq \max(\kappa, |TC(x)|)$, there is a normal ultrafilter U over $P_\kappa\lambda$ such that if i_U is the canonical embedding defined from U, then $i_U(f)(\kappa) = x$. In [C2], a uniform scheme for defining Laver sequences for virtually all globally defined large cardinal axioms was developed by considering classes of set embeddings of the form $j : V_\beta \to M$, all having the same critical point. Such a class can be considered a candidate for admitting its own brand of Laver sequence only if it has the property that each set x is in the codomain of at least one of the embeddings in the class. In [C2], such a class is called a *regular* class and is denoted $\mathcal{E}^\theta_\kappa$ (where θ is a 4-parameter formula that defines the class). A function $f : \kappa \to V_\kappa$ is then said to be a *$\mathcal{E}^\theta_\kappa$-Laver sequence at κ* if for each set x and each $\lambda > \max(\kappa, rank(x))$, there are $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ such that $i(\kappa) > \lambda$ and $i(f)(\kappa) = x$. In [C2] we proved the following:

Proposition (*Compatibility Lemma*)**.** *Assume WA. Let $j : V \to V$ be the WA-embedding. Then every regular class $\mathcal{E}_\kappa^\theta$ of embeddings that is compatible with j admits a $\mathcal{E}_\kappa^\theta$-Laver sequence at κ.*

For completeness, we give the definition of compatibility:

Definition (*Compatibility*). Suppose $j : V \to V$ is the *WA*-embedding with critical point κ and suppose $\mathcal{E}_\kappa^\theta$ is regular at κ. Then we say $\mathcal{E}_\kappa^\theta$ *is compatible with* j if, whenever $\kappa < \lambda < j(\kappa)$, there are $\beta > \lambda, i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, and $k : M \to V_{j(\beta)}$ such that

(1) $i(\kappa) > \lambda$;

(2) k is an elementary embedding with $k \circ i = j|V_\beta$;

(3) $k \mid V_\lambda \cap M = \mathrm{id}_{V_\lambda \cap M}$.

The natural question, left open in [C2], is whether there are regular classes that do not admit Laver sequences. In [C1], we proved the following:

Theorem (*Regular Non-Laver Theorem*)**.** *Suppose M is a transitive model of ZFC + V = HOD + "there exists a strong cardinal". Then M contains a regular class $\mathcal{E}_\kappa^\theta$ with no $\mathcal{E}_\kappa^\theta$-Laver sequence.*

The model M of the theorem can be obtained by forcing, starting with a model of a strong cardinal with an inaccessible above; see [Me]. The theorem answers the question raised in [C2] but still leaves open the possibility that, under *WA*, every regular class could admit a Laver sequence. However, under the hypothesis of an I_1-embedding, the Main Theorem, together with the Consistency Strength Lemma, provides a model M of *WA* that satisfies the hypotheses of the Regular Non-Laver Theorem, thereby closing the remaining gap. We state this as a corollary:

Corollary to the Main Theorem. *If there is an I_1-embedding, then there is a model of ZFC + WA+ "there is a regular class $\mathcal{E}_\kappa^\theta$ that admits no $\mathcal{E}_\kappa^\theta$-Laver sequence."*

The next section is devoted to the proof of the Main Theorem.

§**2. Proof of the Main Theorem**

In this section, we combine two well known forcing iterations to obtain a model of $WA + V = HOD$. Starting with an I_1-embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ, we use the standard technique (see [Ha2] for an excellent exposition) for preserving the embedding via reverse Easton forcing. At successor stages of the iteration, we use coding tricks as in [Me] to force larger and larger V_α's to be ordinal definable. In the final model $V[G]$, because we will have $\hat{j} : V[G]_{\lambda+1} \to V[G]_{\lambda+1}$, then $(V[G]_\lambda, \in, \hat{j}|V[G]_\lambda)$ will be a model of *WA*, and, because of coding, we will have $V[G]_\lambda \models V = HOD$.

Our forcing notation follows [Ba]; in particular, an iterated forcing P_α is completely specified by (the P_β-names for) its coordinate orderings $\dot{Q}_\beta$ and by the type of limit taken at each limit ordinal $\leq \alpha$.

For the most part, we will not need to delve into the specifics of the construction of names for the forcing language; nonetheless, certain arguments will require these details. In those cases we will rely on the treatment of names given in [Ku1]. In particular, we will make use of a useful technical lemma that extends the work in [Ku1], formulated by J. Hamkins in [H]:

Name-rank Lemma [Ha2]. If τ is a P-name, $rank(P) \leq \gamma$,and $\Vdash_P rank(\tau) \leq \beta$, then there is a name σ with $rank(\sigma) \leq \gamma + 3 \cdot \beta$ and $\Vdash_P \sigma = \tau$. ∎

We will be performing a reverse Easton iteration of an Easton forcing. We begin with a definition of these terms. A forcing iteration P_λ is a *reverse Easton iteration* iff direct limits are taken at all inaccessible cardinal stages and inverse limits are taken at all other limit stages. For sets I, J and cardinal λ, we let (as in [Ku1]) $Fn(I, J, \lambda)$ denote the partially ordered set of partial functions $p : I \to J$ having cardinality $< \lambda$, ordered by inclusion.

Suppose S is a set of regular cardinals and $f : S \to ON$ is defined so that for each ν, $f(\nu)$ is a cardinal, *cf*$(f(\nu)) > \nu$, and for all $\mu, \nu \in S$, $\mu < \nu$ implies $f(\mu) \leq f(\nu)$. (Such an f will be called an *Easton function*.) For such f, we let $E(f)$ denote the partial order whose underlying set consists of functions p on S such that for all $\nu \in S$, $p(\nu) \in Fn(f(\nu), 2, \nu)$, and such that for each regular λ, $|\{\nu < \lambda : \nu \in S \text{ implies } p(\nu) \neq 0\}| < \lambda$; $E(f)$ is ordered coordinate-wise. Recall that, if GCH holds, then $E(f)$ forces that $2^\nu = f(\nu)$ for all $\nu \in S$.

Suppose the inaccessible cardinals are unbounded in λ. We define a function $in = in_\lambda : \lambda \to \lambda$ by $in(\alpha) =$ least inaccessible $> \alpha$.

2.1 Lemma. *Suppose the inaccessible cardinals are unbounded in λ. Let P_λ be a reverse Easton iteration satisfying, for each $\alpha < \lambda$:*

(1) $|P_\alpha| < in(\alpha)$;

(2) $\Vdash_{P_\alpha} Q_\alpha$ *is* $< \alpha^+$*-directed closed.*

Then P_λ preserves inaccessible cardinals $\leq \lambda$.

Proof. Let $\gamma \leq \lambda$ be inaccessible. Suppose G is P_λ-generic over V.

Assume $V[G] \models$ "γ is not regular." In $V[G]$, let $f : \alpha \to \gamma$ be a cofinal map where α is a cardinal $< \gamma$. Write $P_\lambda \cong P_\alpha * P_{\alpha,\lambda}$. Since, in $V[G_\alpha]$, $P_{\alpha,\lambda}$ is $< \alpha^+$-directed closed, $f \in V[G_\alpha]$. However, since

$$\text{sat}(P_\alpha) \leq |P_\alpha|^+ < in(\alpha) \leq \gamma,$$

it follows that $V[G_\alpha] \models sup(f''\alpha) < \gamma$, and we have a contradiction.

Assume $V[G] \models$ "γ is not a strong limit". Let α be such that $\alpha < \gamma$ and $2^\alpha \geq \gamma$. Again note that since in $V[G_\alpha]$, $P_{\alpha,\lambda}$ is $< \alpha^+$-directed closed,

$$\left(2^\alpha\right)^{V[G_\alpha]} = \left(2^\alpha\right)^{V[G]}.$$

Thus, in $V[G_\alpha]$, $2^\alpha \geq \gamma$. But, using the usual bounds computation for 2^α in forcing extensions,

$$\left(2^\alpha\right)^{V[G_\alpha]} \leq \left(|P_\alpha|^{<\text{sat}(P_\alpha)}\right)^\alpha < in(\alpha)^\alpha = in(\alpha) \leq \gamma,$$

and we have a contradiction. ■

An iteration P_λ satisfying the hypotheses of the previous lemma will be called an *adequate* reverse Easton iteration. We will work with an adequate reverse Easton iteration $P_\lambda \subset V_{\lambda+1}$ where λ is a strong limit and there is an I_1 embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ. For such P_λ and j, we will say (see [Ha2]) that P_λ is *j-coherent* iff for each $\gamma < \lambda$, $j(P_\gamma) = P_{j(\gamma)}$; and that P_λ *admits a master condition for* j iff there is a condition $q \in P_\lambda$ such that $q \Vdash p \in G \to j(p) \in G$, where G is the name of the generic. Let $\kappa_0 = \kappa$ and, for each $n \geq 1$, let $\kappa_n = j^n(\kappa)$. We will need the following lemma:

2.2 Lemma [Ha2]. *If $j : V_{\lambda+1} \to V_{\lambda+1}$ is an elementary embedding with critical point κ and P_λ is a j-coherent adequate reverse Easton iteration, then P_λ admits a master condition for j.*

Proof. Although our hypotheses are slightly different, the proof is nearly identical to that of [Ha2, Lemma 5.2]; we give an outline of the proof, highlighting the places where our different hypotheses are used.

The master condition q is defined in stages, over intervals (κ_n, κ_{n+1}) and at their endpoints. $q|j(\kappa) \in P_{j(\kappa)}$ is the trivial condition. Let $\dot{G}_n$ be a P_{κ_n}-name for the generic G up to κ_n and let $\dot{G}^{(n)}$ denote the set $\{p(\kappa_n) : p \in G\}$. We let $q(\kappa_n)$ be a P_{κ_n}-name for a condition r such that, in $V[G_n]$,$r \in Q_{\kappa_n}$ and $r \leq j(p)(\kappa_n)$ for all $p \in G_n$. Such a name exists since

$$\Vdash_{\kappa_n} |j''G^{(n-1)}| = |G^{(n-1)}| < in(\kappa_{n-1}) < \kappa_n,$$

and

$$\Vdash_{\kappa_n} \dot{Q}_{\kappa_n} \text{ is } < \kappa_n^+\text{-directed closed.}$$

In a similar way, obtain $q|(\kappa_n, \kappa_{n+1})$ as a P_{κ_n}-name for a sequence $g \in P_{\kappa_n+1,\kappa_{n+1}}$ such that, in $V[G_n]$, $g(\beta) \leq j(p)(\beta)$ for all $p \in G_n$ and all $\beta \in (\kappa_n, \kappa_{n+1})$. The proof that this q satisfies the requirements of a master condition is the same as in [Ha2]. ■

The next lemma tells us that any j-coherent adequate reverse Easton forcing preserves $I_1(\kappa)$, whenever j is an I_1 embedding; the proof uses the master condition obtained in Lemma 2.2 to lift j to the forcing extension in the usual way.

2.3 Lemma [Ha2]. *Suppose $j : V_{\lambda+1} \to V_{\lambda+1}$ is an elementary embedding with critical point κ and P_λ is a j-coherent adequate reverse Easton iteration. Let $q \in P_\lambda$ be a master condition for j. Then $q \Vdash_{P_\lambda} I_1(\kappa)$.*

Outline of Proof. We outline the proof of Lemma 5.3 of [Ha2] in the present context. First notice that each $x \in V[G]_\lambda$ has a name in V_λ: $x \in V[G]_{\kappa_n}$ for some n, and so by adequacy of P_λ, x has a name in $V_{\kappa_{n+1}}$. Likewise, each element of $V[G]_{\lambda+1}$ has a name in $V_{\lambda+1}$ (using the fact that the name of a union is essentially a union of names). Now suppose q is a master condition, $q \in G$, and G is P_λ-generic over V. As usual, we wish to define j on $V[G]_{\lambda+1}$ by $j(\tau_G) = [j(\tau)]_G$. By our observation, we may assume $\tau \in V_{\lambda+1}$, so the definition makes sense. To prove it's well-defined, we wish to show that

$p \Vdash \sigma = \tau$ implies $j(p) \Vdash j(\sigma) = j(\tau)$. We first observe that the statement makes sense: for each formula $\psi(x_1, x_2, \ldots, x_n)$, the relation $p \Vdash V[G]_{\lambda+1} \models \psi(\tau_1, \tau_2, \ldots, \tau_n)$, for all P_λ-names $\tau_1, \ldots, \tau_n$, is definable in $V_{\lambda+1}$ (by a straightforward but tedious induction that follows the proof of the Definability Theorem; see Kunen's treatment [Ku1, pp. 195-201]). The proof that j is well-defined now follows by a typical master condition argument, as does the proof that j is elementary. ■

In order to describe the Easton forcing that we will iterate, we need to fix some notation. Let $\pi : ON \times ON \to ON$ be the definable bijection given by Gödel's definable well-ordering of $ON \times ON$, having the property that for every cardinal ν, $\pi|\nu \times \nu$ is a bijection from $\nu \times \nu$ onto ν. In order to prove the consistency of $V = HOD$, we use the following coding scheme (exactly as in [Me]): For each beth fixed point ν and each set $A \subseteq \nu$, we write $A \sim V_\nu$ if there is a bijection $t : \nu \to V_\nu$ such that for all $(\beta, \alpha) \in \nu \times \nu$, $t(\beta) \in t(\alpha)$ iff $\pi(\beta, \alpha) \in A$. Now, as Menas observes in [Me], for any beth fixed point ν, if A is ordinal definable, then t, and hence every element of V_ν, is ordinal definable. (To find a set $A \sim V_\nu$, start with a bijection $f : \nu \to V_\nu$ and define $E \subseteq \nu \times \nu$ by putting $\beta \; E \; \alpha$ iff $f(\beta) \in f(\alpha)$, and then let $A = \pi'' E$.)

Given a beth fixed point ν and a set $A \subseteq \nu$ with $A \sim V_\nu$, we define a function $f = f_{\nu,A}$ on $S = S_{\nu,A} = \{\gamma : \gamma \text{ is a successor cardinal and } \nu = \omega_\nu < \gamma < \omega_{\nu+\nu}\}$ by setting, for each $\alpha < \nu$,

$$f(\omega_{\nu+\alpha+1}) = \begin{cases} \omega_{\nu+\alpha+3} & \text{if } \alpha \in A \\ \omega_{\nu+\alpha+2} & \text{if } \alpha \notin A \end{cases}$$

In this way, the pair (ν, A) determines a unique Easton function f; as in [Me], we write $E^*(\nu, A) = E(f)$.

We can now define the reverse Easton iteration P_λ that we will use. WLOG, we may assume GCH (see [Ha2, Corollary 5.4]). Fix an I_1 embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ. We define P_λ by first specifying a reverse Easton iteration $P_\kappa \subset V_\kappa$ and then letting P_λ be the (unique) reverse Easton iteration for which $P_{\kappa_{n+1}} = j(P_{\kappa_n})$ for all $n \geq 0$. Using this approach, it is straightforward to verify, by elementarity of j, that the resulting P_λ is j-coherent.

For the definition of P_κ, since we have specified the behavior of the iteration at limits, it suffices to define $\dot{Q}_\alpha$ for each $\alpha < \kappa$. Given $\alpha < \kappa$ and P_α, we let $\dot{e}$ be a P_α-name for the increasing enumeration of the beth fixed points $\leq \kappa$ in V^{P_α}. Let $\dot{A}$ be a P_α-name for a subset of $\dot{e}(\alpha)$ such that $\Vdash_{P_\alpha} \dot{A} \sim \dot{V}_{\dot{e}(\alpha)}$. Finally, let $\dot{Q}_\alpha$ be a P_α-name of least rank for $E^*(\dot{e}(\alpha), \dot{A})$. This completes the definition of P_κ and P_λ.

Note that, for each $\alpha < \lambda$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is $< \dot{e}(\alpha)^+$-directed closed". We prove that in fact, P_λ is an adequate reverse Easton iteration by showing that for each $\alpha \leq \lambda$, $|P_\alpha| < in(\alpha)$. We proceed by induction to show that for each α, $P_\alpha \in V_{in(\alpha)}$, and this will suffice. The base case and limit case are easy. We assume $P_\alpha \in V_{in(\alpha)}$ and show that $P_{\alpha+1} \in V_{in(\alpha)}$. Because $\dot{Q}_\alpha$ was chosen to be of least rank, it suffices to prove the existence of some P_α-name $\dot{Q} \in V_{in(\alpha)}$ which satisfies the definition of $\dot{Q}_\alpha$.

Let $\mu < in(\alpha)$ be such that P_α has the μ-cc. Because (as one shows by induction in V^{P_α}) $\Vdash_{P_\alpha} \dot{e}(\alpha) < in(\alpha)$, it follows from the μ-cc that there is $\nu < in(\alpha)$ such that

$$\Vdash_{P_\alpha} \dot{A} \subset \nu.$$

Using this bound, a straightforward computation in V^{P_α} yields:

$$\Vdash_{P_\alpha} |\dot{Q}_\alpha| \leq 2^{\omega_{\nu+\nu}} < in(\alpha),$$

and, moreover, there is $\gamma < in(\alpha)$ such that

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \subset V[G]_\gamma.$$

By Hamkins' name-rank lemma, there is a P_α-name $\dot{Q}$ of rank $< in(\alpha)$ such that $\Vdash_{P_\alpha} \dot{Q} = \dot{Q}_\alpha$, and we are done.

Our observations so far enable us to conclude that Lemma 2.3 holds for P_λ. We now show that our forcing extension gives us a model of $WA + V = HOD$, completing the proof of the Main Theorem.

2.4 Lemma. *If G is P_λ-generic and $q \in G$, where q is a master condition for j, then $V[G]_\lambda \models V = HOD$.*

Proof. As in [Me], note that for each $\alpha < \lambda$, we have, in $V[G_{\alpha+1}]$, that there is $A \subseteq e(\alpha)$ such that $A \sim V_{e(\alpha)}$, and such that for each $\gamma < e(\alpha)$,

$$(*) \qquad \gamma \in A \text{ iff } 2^{\omega_{e(\alpha)+\gamma+1}} = \omega_{e(\alpha)+\gamma+3}.$$

Thus, in $V[G_{\alpha+1}]$, every element of $V_{e(\alpha)}$ is hereditarily ordinal definable. Since in $V[G_{\alpha+1}]$, $P_{\alpha+1,\lambda}$ is $< e(\alpha+1)^+$-directed closed, later stages of the iteration do not add new subsets to $V_{e(\alpha+1)}$, so that the defining property of A given in $(*)$ continues to hold in $V[G]$. It follows that $V[G]_\lambda \models V = HOD$. ∎

A natural question left open by our work here is whether the hypothesis of the main theorem can be weakened from the existence of an I_1 embedding to *WA* only. I will state the problem as a conjecture because both M. Magidor (in 1996) and, independently, the referee of this paper have made this conjecture to the author:

2.5 Conjecture.* From a model of *ZFC*+ *WA*, a forcing extension can be found that satisfies *ZFC*+ *WA*+ $V = HOD$.

* While I was in the process of working out a proof for this conjecture, I received a communication from J. Hamkins informing me that he had already done so. His arguments will appear in [Ha1].

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