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Indestructibility of Wholeness

by

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Abstract. The Wholeness Axiom (WA) is an axiom schema that asserts the existence of a nontrivial elementary embedding from V to itself. The schema is formulated in the language $\{\in, \mathbf{j}\}$, where $\mathbf{j}$ is a unary function symbol intended to stand for the embedding. WA consists of an Elementarity schema that asserts $\mathbf{j}$ is an elementary embedding, a Critical Point axiom that asserts existence of a least ordinal moved, and a schema Separation$_{\mathbf{j}}$ that asserts Separation holds for all instances of $\mathbf{j}$-formulas. The theory ZFC + WA has been proposed in the author's earlier papers as a natural axiomatic extension of ZFC to account for most of the known large cardinals. In this paper we offer evidence for the naturalness of this theory by showing that it is, like ZFC itself, indestructible by set forcing. We show first that if κ is the critical point of the embedding, then ZFC + WA is preserved by any notion of forcing that belongs to V_κ. This step is nontrivial because to prove Separation$_{\mathbf{j}}$ holds in the forcing extension after lifting the embedding, it is necessary to incorporate $\mathbf{j}$ into the definition of the forcing relation. Then for arbitrary notions of forcing, we introduce a different technique of lifting that lifts one of the original embedding's applicative iterates.

1. Introduction. The Wholeness Axiom (WA) is an axiom schema that asserts the existence of a nontrivial elementary embedding from V to itself. By formalizing the axiom as an axiom schema in the language $\{\in, \mathbf{j}\}$ that asserts $\mathbf{j}$ is an elementary embedding, and supplementing with axioms that assert existence of a least ordinal moved (Critical Point) and with another schema Separation$_{\mathbf{j}}$ that asserts Separation holds for $\mathbf{j}$-formulas, Kunen's famous inconsistency result [Ku71] is averted. The schema Elementarity for elementarity of $\mathbf{j}$ together with Critical Point is called BTEE (the *Basic Theory of Elementary Embeddings*). Then WA = BTEE + Separation$_{\mathbf{j}}$. We also

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define WA_0 to be the theory $\mathrm{BTEE} + \Sigma_0\text{-Separation}_{\mathbf{j}}$, where $\Sigma_0\text{-Separation}_{\mathbf{j}}$ is the schema of all Σ_0 instances of Separation for **j**-formulas.

The theory ZFC + WA is strong enough to show that the critical point κ of the WA-embedding is super-n-huge for every n, and that it is the κth such cardinal [C06]. Note that if $i : V_\lambda \to V_\lambda$ is an I_3 embedding, then $(V_\lambda, \in, i)$ is a model of ZFC + WA. The Wholeness Axiom, together with a number of related weaker theories, has been investigated in [C00a, C00b, C06, C07b, C10, C20] and in [H01].

We have argued elsewhere [C10, C20] that ZFC + WA provides a natural extension of ZFC for providing an axiomatic foundation for large cardinals. One test of "naturalness" of such a foundation is whether the theory is indestructible by set forcing; certainly ZFC itself has this characteristic. We show in this paper that this requirement is met:

Main Theorem 1.1. *Suppose* $(M, \in, j)$ *is a model of* ZFC+WA, $P \in M$, *and* $M \models$ *"P is a notion of forcing". Then for any generic extension N of M, there is a map $k : N \to N$ such that $(N, \in, k)$ is a model of* ZFC + WA. *In particular, the theory* ZFC + WA *is indestructible by set forcing.*

It follows that, for example, WA is consistent with CH, Martin's Axiom, PFA, and their negations.

In this Introduction, we outline the proof of the main result. A careful implementation of the logic given here involves more technical details than might be expected. As we explain in the Appendix (Section 4), when forcing from ZFC+WA, the usual justification for assuming that the ground model is countable and transitive fails. One alternative is to work entirely in Boolean-valued models; another is to begin with a possibly non-well-founded model (M, E, j) of ZFC + WA. Since the forcing machinery for the latter approach, developed along the lines of modern treatments of forcing using countable transitive models as in [J78] and [Ku80], has already been worked out [C07a], we have opted to follow this second route.

In Section 2 we fix notation and state the main results concerning the forcing machinery we will be using. We also review facts about the theories $\mathrm{ZFC} + \mathrm{WA}_0$ and ZFC + WA that we will need. In Section 3 we prove the Main Theorem. We include at the end, as Section 4, an Appendix in which we explain in some detail why countable transitive ground models may not be assumed in the present context.

We turn to an outline of the proof of the main result. Suppose $(M, \in, j)$ is a model of ZFC + BTEE, the critical point of j is κ, B is a complete Boolean algebra in M, G is B-generic over M, and, in M, $B \in V_\kappa$. Then, if we define a map $k : M[G] \to M[G]$ by $k(\sigma_G) = (j(\sigma))_G$, using familiar arguments one verifies that $(M[G], \in, k)$ is also a model of ZFC + BTEE. The map k is an example of a *lifting* of j in the sense that k is an elementary embedding and

$k \supseteq j$. This result can then be applied to show that "small forcing" (forcing with a complete Boolean algebra of size $< \kappa$) also preserves BTEE.

If $(M, \in, j)$ as in the previous paragraph is a model of ZFC + WA_0, we show that, for a particular $k : M[G] \to M[G]$, also obtainable as a lifting, $(M[G], \in, k)$ is also a model of ZFC + WA_0. To do this, we invoke [C06, Theorem 8.8], which asserts that, in M,

$$V_\kappa \prec V_{j(\kappa)} \prec \cdots \prec V, \tag{1.1}$$

to show that, for each n, $V^M_{\kappa_n}[G] = (V_{\kappa_n})^{M[G]}$ (where $\kappa_n = j^n(\kappa)$), and from this we show that $M[G] \models \exists z\, (z = j \restriction V_{\kappa_n})$. This is enough because of the fact [C06, Theorems 8.2, 8.6] that Σ_0-$\mathrm{Separation}_{\mathbf{j}}$ is equivalent to $\mathrm{Amenability}_{\mathbf{j}}$: for each set x, $j \restriction x$ is a set.

To establish the same result for ZFC + WA, we must show that $(M[G], \in, k)$ satisfies full $\mathrm{Separation}_{\mathbf{j}}$. The first step to obtain this result is to extend the usual Boolean evaluation map $[\![\,\cdot\,]\!]^{\mathcal{M}}_B$ to include terms defined by **j**-formulas, in the case in which $\mathrm{rank}(B) < \kappa$. This is accomplished by induction on formula complexity. The important case is atomic formulas. We show that there is a unique extension of $[\![\,\cdot\,]\!]^{\mathcal{M}}_B$, restricted to atomic formulas, that is definable in $(M, \in, j)$ and that respects elementarity, and then extend the domain of $[\![\,\cdot\,]\!]^{\mathcal{M}}_B$ so that names defined by any **j**-formula are included. Having taken this step, the property that $(M[G], \in, k)$ satisfies $\mathrm{Separation}_{\mathbf{j}}$ follows from the fact that it holds in $(M, \in, j)$.

To prove that ZFC + WA is indestructible by set forcing, let $(M, \in, j)$ be a model of ZFC + WA and let $B \in M$ be such that $M \models$ "B is a complete Boolean algebra". Using (1.1), we obtain $n \in \omega$ large enough so that $B \in V_{\kappa_n}$. We then replace j by its applicative iterate $j^{[n]}$ (recall from [C06] and [De00] that $j^{[1]} = j$ and $j^{[n+1]} = j \cdot j^{[n]}$ and the critical point of $j^{[n]}$ is κ_n). It is known (see Proposition 2.11) that $(V, \in, j^{[n]})$ is also a model of ZFC + WA, and now, by the choice of n, $B \in V_{\kappa_n}$. Our work above then gives us a lifting $k : M[G] \to M[G]$ of $j^{[n]}$ such that $(M[G], \in, k)$ satisfies ZFC + WA. This completes the outline of the proof that set forcing cannot destroy ZFC + WA.

2. Preliminaries

2.1. The theory $\mathrm{ZFC}_{\mathbf{j}}$ and some extensions. When working in the language $\mathcal{L} = \{\in, \mathbf{j}\}$, where **j** is a unary function symbol, our basic theory is $\mathrm{ZFC}_{\mathbf{j}}$, consisting of ZFC together with the first-order logic of $\mathcal{L}$. Formulas in which **j** does not occur will be called $\in$-*formulas*, whereas formulas having at least one occurrence of **j** will be called **j**-*formulas.*

Including the function symbol **j** means that we need to consider $\mathcal{L}$-terms (which we will call **j**-*terms* from now on). As usual, terms are defined by the clauses: (a) a variable is a term, and (b) if t is a term, so is $\mathbf{j}(t)$. The terms are of the form $\mathbf{j}^n(x)$ for variables x (assuming $\mathbf{j}^0(x)$ is taken to be x).

The $\mathcal{L}$-formulas can be classified by complexity in the usual way, though some of the usual theorems about the Lévy hierarchy of ZFC formulas do not hold in the present context ([1]). An *atomic* formula is any formula of the form $s = t$ or $s \in t$, where s and t are $\mathbf{j}$-terms. A *bounded* formula is one in which all quantifiers are bound. The collection of bounded formulas is denoted Σ_0 (or, equivalently, Π_0 or Δ_0). Continuing the inductive definition in the metatheory, Σ_{n+1} is the set of $\mathcal{L}$-formulas ϕ of the form $\exists x\ \psi$ where ψ is in Π_n, and similarly for Π_{n+1}. If T is an extension of $\mathrm{ZFC}_{\mathbf{j}}$ and ϕ is an $\mathcal{L}$-formula, we say that ϕ is Σ_n^T if, for some Σ_n $\mathcal{L}$-formula ψ, $T \vdash \phi \Leftrightarrow \psi$, and similarly for Π_n^T.

The following proposition from [C06] says that Σ_0 formulas in the language $\mathcal{L}$ have the familiar absoluteness properties:

Proposition 2.1 (Absoluteness of Σ_0 formulas). *Suppose $\mathcal{M} = (M, E, j)$ is a model of $T = \mathrm{ZFC}_{\mathbf{j}}$. Suppose A is a transitive subset of M (that is, if $y \in A$, $x \in M$, and $\mathcal{M} \models x\,E\,y$, then $x \in A$) and $j{\restriction}A : A \to A$. Let $\mathcal{A} = (A, E, j{\restriction}A)$. Suppose $\phi(x_1, \ldots, x_n)$ is a Σ_0 $\mathcal{L}$-formula. Then for all $a_1, \ldots, a_n \in A$,*

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \iff \mathcal{A} \models \phi[a_1, \ldots, a_n].$$

Next, we mention some of the axioms studied in [C06] that will be relevant in this paper:

- **Elementarity**: Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, \ldots, x_m)$ is an $\in$-formula:

$$\forall x_1, \ldots, x_m\ \big(\phi(x_1, \ldots, x_m) \Leftrightarrow \phi(\mathbf{j}(x_1), \ldots, \mathbf{j}(x_m))\big).$$

- **Nontriviality**: $\exists x\ \mathbf{j}(x) \neq x$.
- **Critical Point**: There is a least ordinal moved by $\mathbf{j}$ (denoted κ).
- **BTEE**: Elementarity + Critical Point.
- **Amenability$_{\mathbf{j}}$**: For every set x, there is a set z such that $z = \mathbf{j}{\restriction}x$.
- **Separation$_{\mathbf{j}}$**: All instances of Separation for $\mathbf{j}$-formulas.
- **Σ_n-Separation$_{\mathbf{j}}$**: All Σ_n instances of Separation for $\mathbf{j}$-formulas.
- **Cofinal Axiom**: $\forall \alpha\ \exists n \in \omega\ (\mathbf{j}^n(\kappa)$ exists and $\alpha \leq \mathbf{j}^n(\kappa))$.
- **WA**: Elementarity + Critical Point + Separation$_{\mathbf{j}}$.
- **WA$_n$**: Elementarity + Critical Point + Σ_n-Separation$_{\mathbf{j}}$.
- **Σ_n-Replacement$_{\mathbf{j}}$**: Replacement for $\mathbf{j}$-formulas, restricted to Σ_n instances.

([1]) For instance, it is not generally the case that $\exists x \in y\ \phi$ is equivalent to a Π_n formula if ϕ is Π_n, nor that $\forall x \in y\ \psi$ is equivalent to a Σ_n formula if ψ is Σ_n since the usual proofs of these equivalences require applications of Replacement for $\mathbf{j}$-formulas, which is not available in the theories we discuss here; see [C06, p. 335].

As a notational convention, whenever we are dealing with an extension of the theory $\mathrm{ZFC}_{\mathbf{j}}+$Critical Point, we let κ (or κ_0) denote the critical point of $\mathbf{j}$, and we define $\kappa_n = \mathbf{j}^n(\kappa)$, $n \geq 1$, whenever it exists ([2]).

Although Critical Point clearly implies Nontriviality, the converse is false. In fact, we have the following:

Proposition 2.2 ([C06, pp. 336–338]).

(1) Con(ZFC) *implies that there is a model* (M, E, j) *of* ZFC+ Elementarity +Nontriviality.
(2) $\mathrm{ZFC} + \mathrm{BTEE} \vdash$ *"κ is inaccessible".*

The model (M, E, j) guaranteed to exist in Proposition 2.2(1) is necessarily ill-founded. Indeed:

Proposition 2.3 (Transitive models satisfy BTEE, [C06, Proposition 2.8]). *Any well-founded model of* ZFC + Elementarity + Nontriviality *also satisfies* Critical Point, *and hence* BTEE.

As usual, the critical point is always moved up by the embedding:

Proposition 2.4 ([C06, Proposition 2.6]).

$$\mathrm{ZFC} + \text{Elementarity} + \text{Critical Point} \vdash \mathbf{j}(\kappa) > \kappa.$$

The three theories that will concern us in this paper are

(1) ZFC + BTEE,
(2) $\mathrm{ZFC} + \mathrm{WA}_0$,
(3) ZFC + WA.

The strengths of these theories are given by the following proposition:

Proposition 2.5 (Strengths of common extensions of $\mathrm{ZFC}_{\mathbf{j}}$, [C06]).

(1) *The consistency strength of* ZFC+BTEE *lies between that of an ineffable cardinal and that of* $0^{\#}$.
(2) *The consistency strength of each of the theories* $\mathrm{ZFC}+\mathrm{WA}_0$ *and* ZFC+WA *lies between that of a cardinal that is super-n-huge for every n and the existence of an* I_3 *embedding* ([3]).
(3) *The theory* $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_1\text{-Replacement}_{\mathbf{j}}$ *is inconsistent.*

Part (2) leads to the question of whether ZFC + WA is consistencywise stronger than $\mathrm{ZFC} + \mathrm{WA}_0$. The answer is unknown at this time, though it is known that ZFC + WA is *stronger than* $\mathrm{ZFC} + \mathrm{WA}_0$ in the following

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([2]) It is possible that $\mathbf{j}^n(\kappa)$ fails to exist in certain models of $\mathrm{ZFC}_{\mathbf{j}}$. For instance, if there is a model of ZFC + BTEE at all, there is also a nonstandard model $\mathcal{M} = (M, E, j)$. Let $N = \{x \in M : \exists n \in \omega\ \mathcal{M} \models \mathrm{rank}(x) < \mathbf{j}^n(\kappa)\}$. Then N together with $k = j{\restriction}N$ is also a model of ZFC + BTEE, but if $\bar{n}$ is a nonstandard integer, then $k^{\bar{n}}(\kappa)$ does not exist. See [C06, p. 350 ff.] for more on this point.

([3]) Definitions may be found in [C06].

sense: There is a sentence σ that is provable from ZFC + WA but not from ZFC + WA_0 ([4]). Part (3) of this proposition is essentially Kunen's famous inconsistency result [Ku71].

The anomalous fact mentioned above that, in some models of ZFC + BTEE, $j^n(\kappa)$ may not exist for certain (nonstandard) n, is mollified considerably for models of ZFC + WA_0 by part (2) of the following:

Proposition 2.6.

(1) ([C00b, Propositions 3.11, 3.12]) *The theory* ZFC + WA *proves*

$$V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots \prec V_{\kappa_n} \prec \cdots \prec V.$$

(2) ([C06, Section 8]) *The theory* ZFC + WA_0 *proves that if* $\mathbf{A}$ *is the* $\mathbf{j}$*-class of* $n \in \omega$ *for which* κ_n *exists, then* $\{\kappa_n : n \in \mathbf{A}\}$ *is cofinal in* ON, *and the* V_{κ_n} *for which* $n \in \mathbf{A}$ *form an elementary chain whose union is* V.

As a corollary, we have:

Proposition 2.7 ([C06, Proposition 8.4]). Cofinal Axiom *is derivable from each of the theories* ZFC + WA *and* ZFC + WA_0.

For most purposes in this paper, the precise formalization of the statement "κ_n exists" will not concern us, but in Section 3 we will need to know that the complexity of the defining formula is quite low. First, we present the definition of the three-place predicate $\mathbf{j}^n(x) = y$, given by the formula $\Phi(n, x, y)$:

$$\Phi(n,x,y) \equiv n \in \omega \implies \exists f\ \Theta(f,n,x,y), \tag{2.1}$$

where

$$\begin{aligned}\Theta(f,n,x,y) \equiv{}& \text{“}f \text{ is a function”} \wedge \operatorname{dom} f = n+1 \wedge f(0) = x \\ &\wedge \forall i\ \big(0 < i \le n \Rightarrow f(i) = \mathbf{j}(f(i-1))\big) \wedge f(n) = y.\end{aligned} \tag{2.2}$$

Notice that the formula $\Theta(f,n,x,y)$ is Σ_0. Now, the statement "κ_n exists" is formalized as the Σ_1 statement

$$\exists f\ \exists y\ \Theta(f,n,\kappa,y), \tag{2.3}$$

and whenever this statement is provable, $y = \kappa_n$.

In Section 3, we also need the following lemma from [C06], which says that j^n always maps ordinals to ordinals:

Lemma 2.8. ZFC + BTEE $\vdash \forall f,n,x,y\ [\Theta(f,n,x,y) \wedge$ *"x is an ordinal"* $\Rightarrow$ *"y is an ordinal"*$]$.

([4]) One such σ, mentioned in [C06, p. 383], is the instance of the Induction schema for $\mathbf{j}$-formulas that establishes the existence of $\mathbf{j}^n(\kappa)$ for each $n \in \omega$. For another result of this kind, see [H01].

As we mentioned in the Introduction, the significance of Amenability$_{\mathbf{j}}$ for our work here is the following:

Proposition 2.9 (Amenability$_{\mathbf{j}}$ and Σ_0-Separation$_{\mathbf{j}}$, [C06, Section 8]). *The theory* ZFC + BTEE *proves the following:*

$$\text{Amenability}_{\mathbf{j}} \iff \Sigma_0\text{-Separation}_{\mathbf{j}}.$$

In the theory ZFC + WA, it is possible to define the *self-application* operation $\mathbf{j} \cdot \mathbf{j}$:

$$\mathbf{j} \cdot \mathbf{j} = \bigcup_{\alpha \in \mathrm{ON}} \mathbf{j}(\mathbf{j} \restriction V_\alpha).$$

This self-application operation can be iterated inside ZFC + WA. The nth iterate is denoted $\mathbf{j}^{[n]}$. In [C06, pp. 380–381] (see also [De00]) we prove:

Proposition 2.10. *The theory* ZFC + WA *proves the following:*

(1) $\forall n \in \omega \ \mathrm{crit}(\mathbf{j}^{[n]}) = \kappa_n$.
(2) $\forall n \in \omega \ \forall r \geq n \ \mathbf{j}^{[n]}(\kappa_r) = \kappa_{r+1}$.

We also have the following:

Proposition 2.11. *Suppose* $\bar{\mathcal{M}} = (M, E, j) \models \mathrm{ZFC} + \mathrm{WA}$, *and suppose* $\bar{\mathcal{M}} \models n \in \omega$. *Let* $k : M \to M$ *be defined by*

$$k(x) = y \iff \bar{\mathcal{M}} \models j^{[n]}(x) = y.$$

Then $(M, E, k) \models \mathrm{ZFC} + \mathrm{WA}$.

Proposition 2.11 is a consequence of the following proposition, which is mentioned in [Ka94, p. 323] (see also [C15, Lemma 98]):

Proposition 2.12. *Suppose* $f : V \to V$ *is a function* (*not necessarily definable in* V) *for which* $f \restriction X$ *is a set for every set* X. *Suppose*

$$(V, \in, j) \models \mathrm{ZFC} + \mathrm{BTEE}.$$

Then for all formulas $\phi(x_1, \ldots, x_m)$ *in the language* $\{\in, \mathbf{f}\}$ (*where* $\mathbf{f}$ *is a unary function symbol*) *and all* $a_1, \ldots, a_m \in V$,

$$(V, \in, f) \models \phi[a_1, \ldots, a_m] \iff (V, \in, j \cdot f) \models \phi[j(a_1), \ldots, j(a_m)].$$

2.2. Forcing over possibly non-well-founded models of ZFC. Performing forcing arguments over a possibly non-well-founded model does not pose significant technical challenges, but sometimes care is needed in the formulation of results. We follow the framework developed for this purpose in [C07a]. That approach parallels closely the approach given in [J78] for well-founded models, in which forcing conditions are elements of a complete Boolean algebra; the partial order approach to forcing emerges as a by-product of this approach. The Boolean-valued model approach has the disadvantage of casting certain familiar results in terms of Boolean-valued expressions rather than by way of the forcing relation. However, it has the advantage of simplifying the verification that Separation$_{\mathbf{j}}$ holds when performing set forcing from a ground model of ZFC + WA; see the end of Section 3.

We review points from [C07a] and fix notation. We assume familiarity with the results of [C07a] but reference the results from that paper as necessary.

Let $\mathcal{M} = (M, E)$ be a (possibly non-well-founded) model of the language $\{\in\}$; in particular, we assume $\mathcal{M}$ is a model of ZFC. The symbol "$\in$" will be used both for the formal symbol of the language and for the "real" membership relation in the surrounding universe V ([5]).

We adopt the following notational conventions:

- For any $X \in M$, we let X_E denote the following extension of X: $X_E = \{x \in M : M \models x\, E\, X\}$.
- For any $t, X, Y \in M$ with $\mathcal{M} \models t : X \to Y$, we define $\mathrm{graph}(t) : X_E \to Y_E$ by $t(x) = y$ iff $\mathcal{M} \models t(x) = y$.
- By convention, $V_\omega \subseteq M$.
- We indicate that (M, E) satisfies an atomic formula $x \in y$ at (a, b) by writing
$$(M, E) \models a\, E\, b$$
rather than $(M, E) \models a \in b$.

Given a model $\mathcal{M} = (M, E)$ of ZFC and a $B \in M$ such that $\mathcal{M} \models$ "B is a complete Boolean algebra", we build the Boolean-valued model $\mathcal{M}^B$ in $\mathcal{M}$ in the usual way: $\mathcal{M}^B = \bigcup_{\alpha \in \mathrm{ON}} \mathcal{M}^B_\alpha$, where $\mathcal{M}^B_0 = \emptyset$; $\mathcal{M}^B_{\alpha+1}$ is the set of all functions $f \in M$ such that $\operatorname{dom} f \subseteq \mathcal{M}^B_\alpha$ and $\operatorname{ran} f \subseteq B$; and $\mathcal{M}^B_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}^B_\alpha$ when λ is a limit. For notational convenience, we will write M^B for both $\mathcal{M}^B$ and $(\mathcal{M}^B)_E$.

In $\mathcal{M}$, we also define sets $M_{B,\gamma}$ for γ an ordinal in $\mathcal{M}$, as follows:

$$\mathcal{M} \models M_{B,\gamma} = M^B \cap V_\gamma. \tag{2.4}$$

By [C07a, Theorem 6(6)], it follows easily (in $\mathcal{M}$) that, for any $\sigma \in M^B$, there are arbitrarily large $\alpha \in \mathrm{ON}$ for which

$$[\![\sigma \in V_\alpha]\!]_B = 1 \implies \exists \tau \in M_{B,\alpha}\ [\![\sigma = \tau]\!]_B = 1. \tag{2.5}$$

Moreover, one can define B-names $\dot{\mathbf{r}}_\alpha$ for ranks V_α so that, in $\mathcal{M}$, for any ordinal α and any strong limit λ greater than $\max\{\alpha, \mathrm{rank}(B)\}$,

$$\dot{\mathbf{r}}_\alpha \in V_\lambda. \tag{2.6}$$

([5]) Different kinds of models of ZFC$_{\mathbf{j}}$ are possible, depending on one's assumptions about the surrounding universe. In this paper (as in [C06]), all models will live in a ZFC universe $(V, \in)$, fixed once and for all, and in particular if (M, E, i) is a model of ZFC$_{\mathbf{j}}$, we assume i is definable in V. In [C06], such models are called *sharp-like*.

As usual, define a first-order language $\mathcal{L}^B = \mathcal{L}^{\mathcal{M},B}$ consisting of $\in$ together with a constant for each member of M^B. Formulas of $\mathcal{L}^B$ are coded in $\mathcal{M}$ so that the formulas form a definable class in $\mathcal{M}$. We refer to the formulas of $\mathcal{L}^B$ as *B-formulas.* As usual, there is a Boolean truth value map $[\![\cdot]\!] = [\![\cdot]\!]_B^{\mathcal{M}}$, depending on B and $\mathcal{M}$ and defined within $\mathcal{M}$ by recursion on a well-founded relation, which assigns a value in B to each B-formula. See [J78] for details.

Names of ground model elements x are denoted $\check{x}$. The canonical name for the generic ultrafilter is $\mathbf{u} = \{(\check{b}, b) : b \in B\}$. The usual forcing relation $\Vdash$ is defined in $\mathcal{M}$ by

$$b \Vdash \phi \quad \text{iff} \quad b \leq [\![\phi]\!]_B.$$

Suppose P is a partial order and $B = \mathrm{ro}(P)$. The canonical name for a generic filter in P is $\mathbf{g} = \{(\check{p}, p) : p \in P\}$. As noted in [C07a], in $\mathcal{M}$, for each $p \in P$, $[\![\check{p} \in \mathbf{g} \leftrightarrow \check{e}(\check{p}) \in \mathbf{u}]\!]_B = 1$ where $e : P \to B$ is a dense embedding.

To obtain two-valued models from $\mathcal{M}^B$ that extend the ground model $\mathcal{M}$, we collapse $\mathcal{M}^B$ by a generic ultrafilter (in the usual way). If B is a complete Boolean algebra in $\mathcal{M}$, an ultrafilter $U \subseteq B_E$ is *B-generic over* $\mathcal{M}$ if, whenever $\mathcal{M} \models X \subseteq B$ and $X_E \subseteq U$, we have $\bigwedge X_E \in U$. Equivalently, U is B-generic over $\mathcal{M}$ if and only if, for each $D \in M$ for which $\mathcal{M} \models$ "D is dense in $B - \{0\}$", $D_E \cap U \neq \emptyset$.

Given a B-generic ultrafilter U, define an equivalence relation $\sim_U$ on M^B by

$$\tau_1 \sim_U \tau_2 \quad \text{iff} \quad [\![\tau_1 = \tau_2]\!]_B^{\mathcal{M}} \in U.$$

We denote by $\tau_U = \tau_U^{\mathcal{M}}$ the $\sim_U$-equivalence class containing τ. We let

$$M_U = \{\tau_U : \tau \in M^B\}. \tag{2.7}$$

Define a membership relation E_U on M_U by

$$\sigma_U \, E_U \, \tau_U \quad \text{iff} \quad [\![\sigma \in \tau]\!]_B^{\mathcal{M}} \in U.$$

As usual, E_U respects equivalence classes. We have the following:

THEOREM 2.13. *Suppose $\phi(x_1, \ldots, x_n)$ is a formula and $\tau_1, \ldots, \tau_n \in M^B$. Then*

$$\mathcal{M}_U \models \phi\big((\tau_1)_U, \ldots, (\tau_n)_U\big) \quad \textit{iff} \quad [\![\phi(\tau_1, \ldots, \tau_n)]\!]_B^{\mathcal{M}} \in U.$$

In particular, $\mathcal{M}_U \models \mathrm{ZFC}$.

The analogues to the usual Forcing Theorems now follow as a corollary:

THEOREM 2.14 (Forcing Theorems). *Let ψ be a sentence of the B-language for $\mathcal{M}$.*

(1) *Suppose $b \in B_E$. Then $\mathcal{M} \models b \Vdash \psi$ if and only if, for every U that contains b and is B-generic over $\mathcal{M}$, we have $\mathcal{M}_U \models \psi$.*
(2) *$\mathcal{M}_U \models \psi$ if and only if there is $b \in U$ such that $\mathcal{M} \models b \Vdash \psi$.*

Next, we describe properties of the natural embedding of M into M_U. We revert for a moment to the more familiar context of transitive ground models to highlight the differences that need to be introduced in the present context; to avoid confusion, we denote the ground model N (rather than M). The usual mappings when forcing over a countable transitive ground model N with a generic ultrafilter U in B are given by

$$N \check{\to} N^B \xrightarrow{\eta_U} N^B/U \xrightarrow{m} N[U],$$

and $m \circ \eta_U$ is often denoted i_U. (This decomposition is given explicitly in [B85, Chapter 4]. The quotient N^B/U for transitive N is denoted M_U in the present paper, equation (2.7) above, where we are considering possibly nontransitive ground models M.)

In the present context, the map m, which is the Mostowski collapsing function, is not generally an isomorphism since E_U is typically non-well-founded, but all the other maps are defined and used in the usual way. Consequently, it is not generally true that M is a subset of the forcing extension M_U. Therefore, in our present context, we define the insertion map that gives the canonical isomorphism $s_U : \mathcal{M} \to \mathcal{M}_U$ by $s_U = \eta_U \circ \check{}$; in other words, for all $x \in M$,

$$s_U(x) = \check{x}_U.$$

For our purposes, it is safe to assume that s_U is the identity ([6]). Using this identification, for each ordinal γ, we recursively define $i_{\gamma,\mathbf{u}_U} = i_\gamma$ on $s_U(M_{B,\gamma})$ by

$$i_\gamma(\tau) = \{i_\gamma(\sigma) : \sigma\, E_U\, \mathrm{dom}\, \tau \wedge \tau(\sigma) \in \mathbf{u}_U\}. \tag{2.8}$$

The maps i_γ provide approximations to the collapsing map $i : N^B \to N[G]$ that is typically used to obtain the generic extension $N[G]$ when the ground model N is transitive. We show in [C07a] that these maps i_γ form a "coherent" collection in the following sense:

$$\mathcal{M}_U \models \forall\alpha, \beta\ (\alpha < \beta \Rightarrow i_\alpha = i_\beta {\restriction} M_{B,\alpha}). \tag{2.9}$$

See discussion following [C07a, Theorem 15] for more about the role of the maps i_γ.

3. Proof of indestructibility of WA**.** In this section, we prove the Main Theorem, that set forcing preserves ZFC + WA. We follow the outline given in the Introduction. We first show that if $\mathcal{M} = (M, E, j)$ is a model of ZFC + BTEE and $\mathrm{crit}(j) = \kappa$, and if in M, B is a complete Boolean algebra that belongs to V_κ, and if U is a B-generic ultrafilter over $\mathcal{M}$, then defining $k : M_U \to M_U$ by $k(\sigma_U) = (j(\sigma))_U$ yields an elementary embedding, and if $\mathcal{M}$

([6]) This point is elaborated in [C07a].

satisfies $\mathrm{ZFC}+\mathrm{WA}_0$ ($\mathrm{ZFC}+\mathrm{WA}$), then so does (M_U, E_U, k). As we attempt to show that the same kind of lifting preserves $\mathrm{ZFC}+\mathrm{WA}$, we observe that the proof that $\mathrm{Separation}_{\mathbf{j}}$ holds in the extension requires some care since the meaning of $[\![\phi]\!]$ for arbitrary $\mathbf{j}$-formulas ϕ is not immediately apparent. To handle this issue, we will develop an extended forcing methodology in which the symbol $\mathbf{j}$ is part of the forcing language, at least in the case of notions of forcing having rank $< \kappa$.

3.1. Small forcing preserves BTEE. We begin with a few preliminaries. In Section 2, we mentioned that when forcing over (M, E) to obtain an extension (M_U, E_U), we identify M with its image $s_U''M$ in M_U (see [C07a]). In the present context, where we will be dealing with models of the language $\{\in, \mathbf{j}\}$, this identification is also justified when s_U turns out to be an isomorphism of $\{\in, \mathbf{j}\}$-structures (and not just of $\{\in\}$-structures). The obvious criterion for this to happen—that is, for $s_U : (M, E, j) \to (s_U''M, E_U, k\restriction s_U''M)$ to be an isomorphism—is the following:

$$k \circ s_U = s_U \circ j. \tag{3.1}$$

For the examples that will concern us, this criterion will always be satisfied.

We will call a function $k : M_U \to M_U$ satisfying (3.1) a *weak lifting of* j, and, since in this case s_U is an isomorphism of $\{\in, \mathbf{j}\}$-structures, we will identify M with its image under s_U and also write $k \supseteq j$. A weak lifting k of j that is also an elementary embedding $M_U \to M_U$ will be called a *lifting of* j.

A notational convention that we adopt is the following: We will indicate models of the language $\{\in, \mathbf{j}\}$ with a bar, and models of $\{\in\}$ without a bar. Thus, a typical model of $\mathrm{ZFC}_{\mathbf{j}}$ will be denoted $\bar{\mathcal{M}} = (M, E, j)$, whereas the corresponding $\in$-model will be denoted $\mathcal{M} = (M, E)$. We will switch between these models without special mention.

We record some observations about (weak) liftings.

Proposition 3.1. *Let* $\bar{\mathcal{M}} = (M, E, j)$ *be a model of* $\mathrm{ZFC} + \mathrm{BTEE}$. *Suppose that* $\mathcal{M} \models$ *"B is a complete Boolean algebra" and U is B-generic over* $\mathcal{M}$. *Suppose also that* $k : M_U \to M_U$ *is a weak lifting of* j *and* $\bar{\mathcal{M}}_U = (M_U, E_U, k)$ *is a model of* $\mathrm{ZFC}_{\mathbf{j}}$. *Then we have the following:*

(1) $(M_U, E_U, k) \models$ Critical Point.

(2) *Suppose* $\phi(x_1, \dots, x_n)$ *is a* Σ_0 $\mathcal{L}$*-formula. Then for all* $a_1, \dots, a_n \in M$,

$$\bar{\mathcal{M}} \models \phi[a_1, \dots, a_n] \iff \bar{\mathcal{M}}_U \models \phi[s_U(a_1), \dots, s_U(a_n)].$$

(3) *The following equivalence holds:*

$$\bar{\mathcal{M}} \models \text{“}j^n(\kappa) \text{ exists”} \iff \bar{\mathcal{M}}_U \models \text{“}k^n(\kappa) \text{ exists”}.$$

Moreover, if $\bar{\mathcal{M}} \models j^n(\kappa) = \beta$, *then* $\bar{\mathcal{M}}_U \models k^n(\kappa) = \beta$.

(4) *The theory* $\mathrm{ZFC}_{\mathbf{j}} + \text{Cofinal Axiom}$ *holds in* $\bar{\mathcal{M}}_U = (M_U, E_U, k)$.

(5) *Suppose that* $k : \mathcal{M}_U \to \mathcal{M}_U$ *is in fact a lifting. Then*

$$\bar{\mathcal{M}} \models \mathrm{ZFC} + \mathrm{BTEE} \implies \bar{\mathcal{M}}_U \models \mathrm{ZFC} + \mathrm{BTEE}.$$

Proof of (1). This follows from the fact that s_U is the identity on ordinals and $\bar{\mathcal{M}} \models$ Critical Point. ■

Proof of (2). This follows from Proposition 2.1 because $s''_U M$ is a transitive subset of M_U and ϕ is Σ_0. ■

Proof of (3). This follows from Lemma 2.8 and from (2), using the Σ_0 formula $\Theta(f, n, \kappa, \beta)$ that asserts $\mathbf{j}^n(\kappa) = \beta$ (see (2.2)). ■

Proof of (4). Since $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_U$ have the same ordinals, this follows immediately from the Cofinal Axiom and part (3). ■

Proof of (5). Since k is a weak lifting, Critical Point holds. Since k is elementary, Elementarity holds. ■

The most familiar way of obtaining liftings when preserving embeddings of the form $V \to M$ or $M \to N$ also works here to produce liftings, as long as the rank of the notion of forcing is small and the ground model satisfies at least ZFC + BTEE. As usual, suppose we are given $\mathcal{M} = (M, E, j), B, U$. We define k on $\mathcal{M}_U$ by

$$k(\sigma_U) = (j(\sigma))_U \tag{3.2}$$

for all $\sigma \in M^B$. The map k is called the *standard extension of* j *by* U. Certainly, if k is well-defined, then k must at least be a weak lifting since for all $x \in M$,

$$k(s_U(x)) = k(\check{x}_U) = (j(\check{x}))_U = [j(x)^\vee]_U = s_U(j(x)).$$

Proposition 3.2 (Small rank proposition). *Suppose* $\bar{\mathcal{M}} = (M, E, j)$ *is a model of* ZFC + BTEE *and* B *is a complete Boolean algebra in* M. *If* $\mathcal{M} \models B\, E\, V_\kappa$, *then for any* U *that is* B*-generic over* $\mathcal{M}$, *the standard extension* k *of* j *by* U *is a well-defined lifting. Therefore, the theory* ZFC + BTEE *is preserved by forcings having small rank.*

Proof. Because $j(B) = B$ and $j \restriction B = \mathrm{id}_B$, we have $j'' M^B \subseteq M^B$. Therefore, elementarity of j guarantees that in $\mathcal{M}$,

$$[\![\sigma = \tau]\!]_B = [\![j(\sigma) = j(\tau)]\!]_B \quad \text{and} \quad [\![\sigma \in \tau]\!]_B = [\![j(\sigma) \in j(\tau)]\!]_B. \tag{3.3}$$

It follows that k is well-defined and isomorphic to its range. Also, since $\mathcal{M} \models j(\sigma)\, E\, M^B$, the range of k is included in M_U. Now, using (3.3), the usual argument by induction on complexity of formulas shows that $k : \mathcal{M}_U \to \mathcal{M}_U$ is an elementary embedding. The final clause now follows by Proposition 3.1(5). ■

It is now easy to show that ZFC + BTEE is preserved by *small* forcings as well, since any such forcing B is isomorphic to a forcing C of low rank. The elementary embedding on the forcing extension by B is obtained as a conjugate of the elementary embedding that works for the extension by C.

Corollary 3.3 (Small forcing proposition). *Suppose* $\bar{\mathcal{M}} = (M, E, j)$ *is a model of* ZFC + BTEE *and* B *is a complete Boolean algebra in* M *such that* $\mathcal{M} \models |B| < \kappa$. *Then for any* B*-generic ultrafilter* U *over* $\mathcal{M}$, *there is a lifting* $k : M_U \to M_U$ *such that* $(M_U, E_U, k) \models$ ZFC + BTEE. *In particular, small forcing preserves* BTEE.

Proof. Given $\mathcal{M}$ and B as in the hypothesis, let C and i be in M such that, in $\mathcal{M}$, $C\,E\,V_\kappa$ is a complete Boolean algebra and $i : B \to C$ is an isomorphism. Let U be B-generic over $\mathcal{M}$ and let $W = \text{graph}(i)''U$ be the C-generic ultrafilter induced by i [C07a, Proposition 18(1)]. Let $k_W : M_W \to M_W$ be the standard extension of j, as in Proposition 3.2. Let $\hat{\imath} : \mathcal{M}_U \to \mathcal{M}_W$ be the isomorphism induced by i [C07a, Proposition 18(2)]; since we are assuming $M \subseteq M_U$ and $M \subseteq M_W$, it follows that $\hat{\imath}$ fixes M (see [C07a, proof of Proposition 18(1)]).

$$\begin{array}{ccc} \mathcal{M}_U & \xrightarrow{k_U} & \mathcal{M}_U \\ {\scriptstyle \hat{\imath}}\downarrow & & \downarrow{\scriptstyle \hat{\imath}} \\ \mathcal{M}_W & \xrightarrow{k_W} & \mathcal{M}_W \end{array} \tag{3.4}$$

Letting $k_U = \hat{\imath}^{-1} \circ k_W \circ \hat{\imath}$ yields an elementary embedding $\mathcal{M}_U \to \mathcal{M}_U$. As $\hat{\imath}$ fixes M, for each $x \in M$ we have $k_U(x) = k_W(x) = j(x)$; that is, $k \supseteq j$.

Since k is a lifting, it follows from Proposition 3.1(1) that $(M_U, E_U, k_U) \models$ Critical Point. ∎

3.2. Small forcing preserves WA_0**.** We show next that if small forcing is performed over a model (M, E, j) of ZFC + WA_0, then j can be lifted to a WA_0-embedding defined on the extension.

Theorem 3.4 (Small forcing preserves WA_0). *Suppose* $\bar{\mathcal{M}} = (M, E, j) \models$ ZFC + WA_0. *Suppose that in* $\mathcal{M}$, B *is a complete Boolean algebra and* $|B| < \kappa$. *Then for any ultrafilter* U *that is* B*-generic over* $\mathcal{M}$, *there is a lifting* $k : \mathcal{M}_U \to \mathcal{M}_U$ *such that* $(M_U, E_U, k) \models$ ZFC + WA_0.

Proof. Given $\bar{\mathcal{M}} = (M, E, j)$, B, and U as in the hypothesis, we let $\bar{\mathcal{M}}_U = (M_U, E_U, k_U)$ be the model obtained in Corollary 3.3, where $k_W : M_W \to M_W$ is the standard extension of j and $k_U = \hat{\imath}^{-1} \circ k_W \circ \hat{\imath}$ (see diagram (3.4)).

We observe first that Cofinal Axiom holds in $\bar{\mathcal{M}}_U = (M_U, E_U, k_U)$; this follows immediately from the fact that k_U is a lifting (Proposition 3.1(4)).

We show next that $\bar{\mathcal{M}}_U \models \text{Amenability}_{\mathbf{j}}$. We first prove the following claim:

Claim 3.5. *Define* $\phi : (M_W, E_W, k_W) \to (M_U, E_U, k_U)$ *by* $\phi(x) = \hat{\imath}^{-1}(x) \in M_U$. *Then* ϕ *is an isomorphism of* $\{\in, \mathbf{j}\}$*-structures. In particular,* (M_W, E_W, k_W) *and* (M_U, E_U, k_U) *are elementarily equivalent.*

Proof of Claim. We observe first that ϕ has an inverse $\psi : M_U \to M_W$ defined by $\psi(y) = \hat{\imath}(y)$. To see that ϕ is an isomorphism, it suffices to prove that ϕ preserves the unary operation. We have, for each $x \in M_W$,

$$\phi(k_W(x)) = \hat{\imath}^{-1}(k_W(x)) = \hat{\imath}^{-1} k_W \hat{\imath}\hat{\imath}^{-1} x = k_U \hat{\imath}^{-1} x = k_U(\phi(x)),$$

as required. ∎

To prove $\bar{\mathcal{M}}_U \models \text{Amenability}_{\mathbf{j}}$, it suffices by Claim 3.5 to prove $\bar{\mathcal{M}}_W \models \text{Amenability}_{\mathbf{j}}$. Recall that by Proposition 3.1(3), $\bar{\mathcal{M}}_W \models$ "$k_W^n(\kappa)$ exists" if and only if $\bar{\mathcal{M}} \models$ "$j^n(\kappa)$ exists", and that, in $\bar{\mathcal{M}}$, $k_W^n(\kappa) = j^n(\kappa)$ whenever they both exist. We write $\kappa_n = (j^n(\kappa))^{\mathcal{M}}$.

Since the κ_n are cofinal (by Proposition 2.6(2)), to prove $\text{Amenability}_{\mathbf{j}}$, it suffices to show that

$$\bar{\mathcal{M}}_W \models \forall n\ (k_W^n(\kappa) \text{ exists } \Rightarrow \exists e\ e = k_W \restriction V_{k_W^n(\kappa)}). \tag{3.5}$$

The idea for the proof is to define e by $e(\sigma_W) = (j(\sigma))_W$ whenever σ_W belongs to $(V_{\kappa_n})^{\mathcal{M}_W}$, as in the definition of the standard extension. The point that needs to be verified is that the range of such a map e is never unbounded. This, however, is guaranteed by the fact that each element of $(V_{\kappa_n})^{\mathcal{M}_W}$ has a name in $(V_{\kappa_n})^{\mathcal{M}}$, ensuring that images by e are uniformly bounded. Here are the details:

Since $\bar{\mathcal{M}} \models \text{WA}_0$, we may make the following definition in $\bar{\mathcal{M}}$: We define i_n for each $n \geq 1$ for which κ_n exists by $i_n = j \restriction V_{\kappa_n}$. We define $R_{\kappa_n} \in \mathcal{M}_W$ by (see (2.6))

$$\kappa_n \text{ exists} \implies R_{\kappa_n} = (\check{V}_{\kappa_n})_W. \tag{3.6}$$

That is, in $\mathcal{M}_W$, R_{κ_n} denotes $(V_{\kappa_n})^{\mathcal{M}}$.

In $\bar{\mathcal{M}}_W$, fix an n for which κ_n exists. Using the set of names $M_{B,\kappa_n} = M^B \cap V_{\kappa_n}$ defined in $\mathcal{M}$, we can obtain, inside $\mathcal{M}_W$, the usual transitive forcing extension $R_{\kappa_n}[\mathbf{u}_W]$ of R_{κ_n}. Since, by (2.6), every set in $(V_{\kappa_n})^{\mathcal{M}_W}$ has a name in M_{B,κ_n}, it follows that $\mathcal{M}_W \models V_{\kappa_n} = R_{\kappa_n}[\mathbf{u}_W]$.

We now define, in $\bar{\mathcal{M}}_W$, the restriction $k \restriction V_{\kappa_n}$ as a class in $R_{\kappa_{n+1}}[\mathbf{u}_W]$ by

$$e = \{(\sigma_W, y)\, E_W\, V_{\kappa_{n+1}} : y = (i_n(\sigma))_{\mathbf{u}_W} \text{ and } \sigma\, E_W\, M_{B,\kappa_n}\}.$$

Certainly $\mathcal{M}_W \models e\, E_W\, V_{(\kappa_{n+1})+1}$. But now $\bar{\mathcal{M}}_W \models e = k \restriction V_{\kappa_n}$. Thus we have established (3.5), and this completes the proof of Theorem 3.4. ∎

3.3. Set forcing preserves WA. We conclude this section with a proof of the main result, that set forcing cannot destroy WA. As a first step, we show that forcing by a notion of forcing of rank below the critical point of the embedding cannot destroy WA. Though the latter is not a surprising result, its proof is necessarily quite different from the proof of the corresponding result for WA_0. In the latter case we were able to rely on the equivalence of $\mathrm{Amenability}_{\mathbf{j}}$ with Σ_0-$\mathrm{Separation}_{\mathbf{j}}$; this reliance allowed us to avoid dealing with the problem of showing directly that Σ_0 instances of $\mathrm{Separation}_{\mathbf{j}}$ hold in the extension. What makes the latter approach seem difficult at first is that it is not immediately clear how to evaluate $[\![\phi]\!]_B$ for arbitrary $\mathbf{j}$-formulas ϕ. Now to prove that WA holds in a forcing extension, we must address this issue. We do this as a preliminary to the proof of the main result. For certain simple $\mathbf{j}$-formulas ϕ, the meaning of $[\![\phi]\!]_B$ is clear, as long as B lies in $(V_\kappa)^{\mathcal{M}}$. For instance, $[\![j(\sigma) = j(\tau)]\!]_B$ must have value $[\![\sigma = \tau]\!]_B$ by elementarity of j. Using a similar insight, one can give meaning to $[\![\phi]\!]_B$ for each atomic $\mathbf{j}$-formula ϕ, and then proceed by induction on the complexity of ϕ to define $[\![\phi]\!]_B$ in all cases.

As a starting point, we assume $\bar{\mathcal{M}} = (M, E, j) \models \mathrm{ZFC} + \mathrm{WA}$ and $\mathcal{M} \models$ "B is a complete Boolean algebra in V_κ". Our inductive procedure for formally extending $[\![\cdot]\!]$ to all $\mathbf{j}$-formulas is carried out in V (standard $\mathbf{j}$-formulas can be viewed as having been coded as sets in V).

To handle atomic ϕ (the most important case), we first recall that $[\![\cdot]\!]_B^{\mathcal{M}}$, restricted to atomic $\in$-formulas, is definable in $\mathcal{M}$ by an $\in$-formula, which is obtained by induction on a suitable well-founded relation in $\mathcal{M}$ (see [B85, p. 14]). We claim that there is a unique extension of this restriction of $[\![\cdot]\!]_B^{\mathcal{M}}$ that is definable in $\bar{\mathcal{M}}$ and that respects elementarity of j. Moreover, as we will show, it suffices to show that this extension of $[\![\cdot]\!]$ computes the values of atomic $\mathbf{j}$-formulas in important special cases according to the following scheme:

$$[\![\sigma \in j^n(\tau)]\!]_B^{\mathcal{M}} = \bigvee_{t\, E\, \mathrm{dom}(j^n(\tau))} (j^n(\tau)(t) \wedge [\![\sigma = t]\!]_B),$$

$$[\![j^n(\sigma) \in \tau]\!]_B^{\mathcal{M}} = \bigvee_{t\, E\, \mathrm{dom}(\tau)} (\tau(t) \wedge [\![j^n(\sigma) = t]\!]_B),$$

$$[\![\sigma = j^n(\tau)]\!]_B^{\mathcal{M}} = \bigwedge_{s\, E\, \mathrm{dom}(\sigma)} (\sigma(s) \to [\![s \in j^n(\tau)]\!]_B) \wedge \bigwedge_{t\, E\, \mathrm{dom}(j^n(\tau))} (j^n(\tau)(t) \to [\![t \in \sigma]\!]_B),$$

for all $n \in \omega^V$.

The scheme given above appears to be the usual recursive definition of $[\![\cdot]\!]_B$, but it is important to recognize that such a definition is not generally permissible in $\mathrm{ZFC} + \mathrm{WA}$ if it involves $\mathbf{j}$-formulas. Instead, as we mentioned above, these formulas are obtained by using elementarity of j; in particular, we have applied j^n to the following formulas, each of which holds in $\mathcal{M}$:

$$\forall x\, E\, M^B \Big[[\![x \in \tau]\!]_B = \bigvee_{t\, E\, \mathrm{dom}(\tau)} (\tau(t) \wedge [\![x = t]\!]_B) \Big],$$

$$\forall x\, E\, M^B \Big[[\![\sigma \in x]\!]_B = \bigvee_{t\, E\, \mathrm{dom}(x)} (x(t) \wedge [\![\sigma = t]\!]_B) \Big],$$

$$\forall x\, E\, M^B \Big[[\![x = \tau]\!]_B = \bigwedge_{s\, E\, \mathrm{dom}(x)} (x(s) \to [\![s \in \tau]\!]_B) \wedge \bigwedge_{t\, E\, \mathrm{dom}(\tau)} (\tau(t) \to [\![t \in x]\!]_B) \Big].$$

To see that our scheme of specialized atomic formulas suffices for the computation of all atomic **j**-formulas, which are of the form

$$j^m(\sigma) = j^n(\tau) \quad \text{and} \quad j^m(\sigma) \in j^n(\tau),$$

we show how to obtain the computation of these more general formulas in a typical case. Assume $m < n$; we show how $[\![j^m(\sigma) \in j^n(\tau)]\!]_B$ is computed in $\mathcal{M}$: Let $\mu = j^{n-m}(\tau)\, E\, M^B$. The usual computation of $[\![\sigma \in \mu]\!]_B$ in $\mathcal{M}$ is given by

$$[\![\sigma \in \mu]\!]_B = \bigvee_{t\, E\, \mathrm{dom}(\mu)} (\mu(t) \wedge [\![\sigma = t]\!]_B).$$

Applying j^m to this formula (and replacing μ by $j^{n-m}(\tau)$) gives, in $\mathcal{M}$,

$$[\![j^m(\sigma) \in j^n(\tau)]\!]_B = \bigvee_{t\, E\, \mathrm{dom}(j^n(\tau))} (j^n(\tau)(t) \wedge [\![j^m(\sigma) = t]\!]_B).$$

Having extended the definition of $[\![\,\cdot\,]\!]_B$ to include all atomic **j**-formulas, we can now complete the induction on the complexity of the **j**-formula ϕ. But the $\neg$, $\wedge$, and $\exists$ cases are all identical to the corresponding clauses in the usual definition of $[\![\phi]\!]_B$ when ϕ is an $\in$-formula ([7]). This, therefore, completes the definition of $[\![\phi]\!]_B^{\mathcal{M}}$ for all **j**-formulas ϕ.

Using elementarity of j, one shows that all axioms of first-order logic for the language $\mathcal{L} = \{\in, \mathbf{j}\}$ have Boolean value 1, and that $[\![\,\cdot\,]\!]$ continues to preserve modus ponens (all this follows from the simple observation that, for all σ, τ, $[\![\sigma = \tau]\!]_B^{\bar{\mathcal{M}}} = [\![j(\sigma) = j(\tau)]\!]_B^{\bar{\mathcal{M}}}$; see [J89]). Of course, all axioms of ZFC—restricted to $\in$-formulas—continue to have Boolean value 1. We next show that each instance of Separation$_{\mathbf{j}}$ must also have Boolean value 1:

([7]) For easy reference, we reproduce this definition from [C07a]:

$$[\![\sigma \in \tau]\!]_B = \textstyle\bigvee_{t\, E\, \mathrm{dom}(\tau)} (\tau(t) \wedge [\![\sigma = t]\!]_B),$$

$$[\![\sigma = \tau]\!]_B = \textstyle\bigwedge_{s\, E\, \mathrm{dom}(\sigma)} (\sigma(s) \to [\![s \in \tau]\!]_B) \wedge \bigwedge_{t\, E\, \mathrm{dom}(\tau)} (\tau(t) \to [\![t \in \sigma]\!]_B),$$

$$[\![\psi \wedge \phi]\!]_B = [\![\psi]\!]_B \wedge [\![\phi]\!]_B,$$

$$[\![\neg\psi]\!]_B = ([\![\psi]\!]_B)^*,$$

$$[\![\exists x\ \psi(x)]\!]_B = \textstyle\bigvee_{t\, E\, M^B} [\![\psi(t)]\!]_B.$$

Lemma 3.6 (Preservation of $\text{Separation}_{\mathbf{j}}$). *Suppose* $\bar{\mathcal{M}} = (M, E, j) \models$ ZFC + WA. *Suppose that*

$$\mathcal{M} \models \text{“}B \text{ is a complete Boolean algebra and } B\,E\,V_\kappa\text{”.}$$

Suppose $\phi(x, \vec{y})$ *is a* $\mathbf{j}$*-formula. Then*

$$\llbracket \forall \vec{a}\ \forall A\ \exists X\ \forall z\ [z \in X \Leftrightarrow z \in A \wedge \phi(z, \vec{a})] \rrbracket_B^{\bar{\mathcal{M}}} = 1. \tag{3.7}$$

Remark. The formula (3.7) is the instance of $\text{Separation}_{\mathbf{j}}$ obtained from ϕ.

Proof of Lemma 3.6. Now that we are able to compute $\llbracket \psi \rrbracket_B^{\bar{\mathcal{M}}}$ for $\mathbf{j}$-formulas ψ, the usual proof of Separation in a Boolean-valued model of set theory can be used without modification. In particular, arguing in $\bar{\mathcal{M}}$, given $\tau, \vec{a}\,E\,M^B$, we define σ by

$$\sigma = \{(t, b) E \operatorname{dom} \tau \times B : b = \tau(t) \wedge \llbracket \phi(t, \vec{a}) \rrbracket\}.$$

By $\text{Separation}_{\mathbf{j}}$, σ is a set. Certainly $\sigma\,E\,M^B$, and by the usual argument [J78], one shows that for all $t\,E\,M^B$,

$$\llbracket t \in \sigma \rrbracket_B = \llbracket t \in \tau \rrbracket_B \wedge \llbracket \phi(t, \vec{a}) \rrbracket_B. \ \blacksquare$$

Now suppose U is B-generic over $\mathcal{M}$, $k : \mathcal{M}_U \to \mathcal{M}_U$ is the standard extension of j, and $\bar{\mathcal{M}}_U = (M_U, E_U, k)$. We know from Theorem 3.4 that k is in fact a WA_0-embedding. To prove $\bar{\mathcal{M}}_U \models \text{ZFC} + \text{WA}$, we would like to use Lemma 3.6; however, for an argument of this kind to be valid, we need to know that usual the Łoś-like theorem for collapsing Boolean-valued models (Theorem 2.13) extends to $\mathbf{j}$-formulas; this result in turn depends on *fullness* of M^B with respect to $\mathbf{j}$-formulas. These verifications are straightforward, but we give the highlights of the proofs for the sake of completeness; proofs are modeled after the proofs of similar results that appear in [J78]:

Lemma 3.7. *Assume* $\bar{\mathcal{M}}$ *and* $\bar{\mathcal{M}}_U$ *are as above.*

(1) *In* $\bar{\mathcal{M}}$, M^B *is full with respect to* $\mathcal{L}$*-formulas. That is, for any* $\mathcal{L}$*-formula* $\phi(x, x_1, \ldots, x_n)$, *and all* $\tau_1, \ldots, \tau_n \in M^B$, *there is* $\tau \in M^B$ *such that*

$$\bar{\mathcal{M}} \models \llbracket \phi(\tau, \tau_1, \ldots, \tau_n) \rrbracket_B = \llbracket \exists x\ \phi(x, \tau_1, \ldots, \tau_n) \rrbracket_B.$$

(2) *For any* $\mathcal{L}$*-formula* $\phi(x_1, \ldots, x_n)$, *and all* $\tau_1, \ldots, \tau_n \in M^B$,

$$\bar{\mathcal{M}}_U \models \phi((\tau_1)_U, \ldots, (\tau_n)_U) \quad \textit{iff} \quad \llbracket \phi(\tau_1, \ldots, \tau_n) \rrbracket_B^{\bar{\mathcal{M}}} \in U.$$

Proof of (1). We argue in $\bar{\mathcal{M}}$. We will obtain τ such that

$$\llbracket \exists x\ \phi(x, \tau_1, \ldots, \tau_n) \rrbracket_B \leq \llbracket \phi(\tau, \tau_1, \ldots, \tau_n) \rrbracket_B. \tag{3.8}$$

This will suffice since verification of $\geq$ is immediate.

Let $b = \llbracket \exists x\ \phi(x, \tau_1, \ldots, \tau_n) \rrbracket_B$. Let

$$D_b = \{a \leq b : \text{there is } \sigma_a \text{ such that } a \leq \llbracket \phi(\sigma_a, \tau_1, \ldots, \tau_n) \rrbracket_B\}.$$

Clearly, D_b is open and dense below b. Let A be a maximal antichain below b included in D_b. Obviously, $\bigvee A \geq b$. Use the Mixing Lemma to obtain σ such that for each $a\,E\,A$, $a \leq [\![\sigma = \sigma_a]\!]_B$. It follows that for each $a \in A$, $a \leq [\![\phi(\sigma_a, \tau_1, \ldots, \tau_n)]\!]_B$, whence $b \leq [\![\phi(\sigma, \tau_1, \tau_1, \ldots, \tau_n)]\!]_B$. This establishes (3.8). ∎

Proof of (2). The proof is by induction on the complexity of the $\mathcal{L}$-formula ϕ. We argue in $\bar{\mathcal{M}}$ (recall that we are assuming that $(V_\omega)^V \subseteq M$). As in the usual proof, fullness of M^B with respect to **j**-formulas is used to handle the existential quantifier case, and the cases $\neg, \wedge$ are straightforward. We prove the result for atomic $\mathcal{L}$-formulas.

As a preliminary, we prove that for every $n\,E\,\omega^V$,

$$k^n(\sigma_U) = (j^n(\sigma))_U. \tag{3.9}$$

The base case $n = 1$ holds since k is the standard extension of j. But now

$$\begin{aligned} k^{n+1}(\sigma_U) &= k(k^n(\sigma_U)) = k((j^n(\sigma)_U) \\ &= k(\mu_U) \quad (\text{where } \mu = j^n(\sigma)) \\ &= (j(\mu))_U = (j^{n+1}(\sigma))_U. \end{aligned}$$

We prove the following three equivalences for each $n\,E\,\omega^V$:

$$\begin{aligned} \sigma_U\,E_U\,k^n(\tau_U) &\iff [\![\sigma \in j^n(\tau)]\!]_B \in U, \\ k^n(\sigma_U)\,E_U\,\tau_U &\iff [\![j^n(\sigma) \in \tau]\!]_B \in U, \\ \sigma_U = k^n(\tau_U) &\iff [\![\sigma = j^n(\tau)]\!]_B \in U. \end{aligned}$$

The proofs in each case are similar; we prove the first. Let $\mu = j^n(\tau)$. By the standard theorem,

$$\sigma_U\,E_U\,\mu_U \iff [\![\sigma \in \mu]\!]_B \in U.$$

Therefore, it suffices to show $\mu_U = (j^n(\tau))_U$; but this follows from (3.9).

Again, we have not directly dealt with the more general atomic formulas

$$\mathbf{j}^m(\sigma) \in \mathbf{j}^n(\tau) \quad \text{and} \quad \mathbf{j}^m(\sigma) = \mathbf{j}^n(\tau).$$

However, the result for these cases follows from the result for the three simpler atomic formulas above. We show how this works for the formula $\mathbf{j}^m(\sigma) \in \mathbf{j}^n(\tau)$ when $m < n$. Let $\mu = k^m(\sigma)$. By the result just proven,

$$\mu_U\,E_U\,k^{n-m}(\tau_U) \iff [\![\mu \in j^{n-m}(\tau)]\!]_B \in U.$$

But since $k^m(\sigma) = (j^m(\sigma))_U$ (by (3.9)), the result follows. ∎

Our work in the last few paragraphs amounts to an abbreviated development of Boolean-valued models of the theory ZFC + WA in the language $\{\in, \mathbf{j}\}$ for the special case in which the Boolean algebra B has rank below κ; in other words, for this special case, we have shown how to incorporate **j** into the forcing language. A complete treatment would require a definition of Boolean-valued models for the language $\{\in, \mathbf{j}\}$ (which is given in [J89]); M^B would be defined in the usual way; the operator $[\![\,\cdot\,]\!]$ would be defined recursively as usual for $\in$-formulas and then extended in the way described above for all $\mathbf{j}$-formulas; one would then verify that M^B satisfies all axioms of first-order logic for the language $\mathcal{L}$, and that all axioms of ZFC, restricted to $\in$-formulas, hold in M^B. The rest of the development would then proceed as above.

We can now show that forcings of low rank do not destroy WA.

Theorem 3.8 (Low rank forcing preserves WA). *Forcing by a complete Boolean algebra of low rank does not destroy* WA. *In particular, given a model* $\bar{\mathcal{M}} = (M, E, j)$ *of* $\mathrm{ZFC} + \mathrm{WA}$, *with* $\mathcal{M} \models$ *"B is a complete Boolean algebra and* $\mathrm{rank}(B) < \kappa$*", and given any U that is B-generic over $\mathcal{M}$, if $k : M_U \to M_U$ is the standard lifting of j, then* $(M_U, E_U, k) \models \mathrm{ZFC} + \mathrm{WA}$.

Proof. By Theorem 3.4, $\bar{M}_U = (M_U, E_U, k) \models \mathrm{ZFC} + \mathrm{WA}_0$. By Lemmas 3.6 and 3.7(2), $\bar{\mathcal{M}}_W \models \mathrm{Separation}_{\mathbf{j}}$. Thus, $\bar{\mathcal{M}}_U \models \mathrm{ZFC} + \mathrm{WA}$. ■

We turn to a proof of the Main Theorem.

Theorem 3.9 (Set forcing preserves WA). *Suppose* $\bar{\mathcal{M}} = (M, E, j) \models \mathrm{ZFC} + \mathrm{WA}$. *Suppose $B \in M$ is such that $\mathcal{M} \models$ "B is a complete Boolean algebra". Suppose U is B-generic over $\mathcal{M}$. Then there is a $k : M_U \to M_U$ such that* $(M_U, E_U, k) \models \mathrm{ZFC} + \mathrm{WA}$.

Proof. Let B and U be as in the hypothesis. Let n be such that $\mathcal{M} \models B\,E\,V_{\kappa_n}$. Recall from Proposition 2.10 that the nth self-applicative iterate $j^{[n]}$ is also an elementary embedding $\mathcal{M} \to \mathcal{M}$ and has critical point κ_n. Moreover, it follows from Proposition 2.11 that, if we define $\ell : M \to M$ by

$$\ell(x) = y \iff \bar{\mathcal{M}} \models j^{[n]}(x) = y,$$

then $(M, E, \ell) \models \mathrm{ZFC} + \mathrm{WA}$. Since $\mathcal{M} \models B\,E\,V_{\kappa_n}$, it follows from Theorem 3.8 that $(M_U, E_U, k) \models \mathrm{ZFC}{+}\mathrm{WA}$, where k is the standard lifting of ℓ. ■

4. Appendix: Unsuitability of countable transitive ground models of extensions of $\mathbf{ZFC_j}$. In this Appendix we address the question whether it is really necessary to use the at times cumbersome machinery for forcing over *nonstandard* models. Why cannot we carry out forcing arguments in the usual way? Start with a countable transitive ground model and build generic extensions? The seemingly obvious answer is that the theory ZFC + Elementarity + Nontriviality has the same consistency strength as ZFC, while existence of a transitive model of ZFC + Elementarity + Nontriviality has consistency strength at least as that of an inaccessible cardinal (Propositions 2.2 and 2.3). One could counter (8) that this disparity is no different from the disparity we find when forcing over ZFC: Consistency of ZFC does not imply existence of a transitive model of ZFC; and there are tricks that allow us to circumvent this disparity. The purpose of this Appendix is to demonstrate that such tricks do not solve the problem in the present context; that, in the context of extensions of the theory ZFC + Elementarity + Nontriviality, assuming we can carry out a forcing argument starting from a transitive ground model is never justified.

Ordinarily, in carrying out an argument with forcing, when one starts with a countable transitive model of ZFC, one implicitly uses the fact, derivable from Reflection, that one may prove from ZFC there is a transitive model of any finite list of ZFC axioms (cf. [Ku80, VII.1]). A folklore result (9) is that from ZFC − Replacement one can prove the equivalence of Reflection with Replacement + Infinity. The fact that Replacement (even Σ_1-Replacement) for **j**-formulas is inconsistent with ZFC + BTEE (recall Proposition 2.5) already adequately demonstrates that the usual logic for assuming transitive ground models is not applicable in forcing from ZFC + Elementarity + Nontriviality or any of its extensions. This limitation becomes strikingly apparent when one wishes to force from the theory ZFC + WA_0 since this theory is known [H01] to be finitely axiomatizable.

As we show next, one may not apply the usual logic to justify starting with a countable transitive model even for the weaker theory ZFC + Elementarity + Nontriviality.

We review in more detail the usual justification for assuming existence of a countable transitive model of ZFC, and show that application of parallel logic for the theory ZFC + Elementarity + Nontriviality leads to a proof that ZFC is inconsistent. We follow the treatment of this topic given in [Ku80, VII.9(1b)]. By Reflection, given a model M of ZFC, one may argue that the theory

$$T = \mathrm{ZFC} + \{\sigma^{\mathbf{c}} : \sigma \text{ is a ZFC axiom}\} + \text{“}\mathbf{c} \text{ is countable and transitive”}$$

in the language $\{\in, \mathbf{c}\}$ is finitely satisfiable (10) and hence consistent. Thus a forcing proof of the relative consistency of a sentence θ with ZFC can be understood to have the form

(8) Indeed, the reason for this Appendix is to address such concerns, presented to the author by more than one well-established set-theorist.

(9) See for example the Stanford Encyclopedia of Philosophy article on set theory, available online at https://plato.stanford.edu/entries/set-theory/.

(10) Let $F = \{\sigma_{m(1)}, \dots, \sigma_{m(r)}, \sigma^c_{n(1)}, \dots, \sigma^c_{n(s)}\}$ denote a finite set of axioms of the new theory. Since $M \models$ Reflection, there is $A \in M$ such that $M \models$ “A is countable and transitive” and $M \models \sigma^A_{n(1)} \wedge \dots \wedge \sigma^A_{n(s)}$. Now $(M, A) \models F$.

$$\text{Con(ZFC)} \Rightarrow \text{Con}(T) \Rightarrow \text{Con}(\text{ZFC} + \theta), \tag{4.1}$$

where the second implication is demonstrated by the forcing argument.

To see where this logic goes wrong for the theory ZFC + Elementarity + Nontriviality, we begin by obtaining an analogue to the theory T described above, relative to extensions of $\text{ZFC}_{\mathbf{j}}$. In the construction of T above, there were three sorts of axioms:

(1) Axioms that describe the outer universe (namely, the axioms of ZFC).
(2) Axioms that describe the inner universe (namely, the axioms $\sigma^{\mathbf{c}}$ for $\sigma \in \text{ZFC}$).
(3) Axioms that describe the sort of object $\mathbf{c}$ stands for (namely, that $\mathbf{c}$ is countable and transitive).

In the setting of models of extensions of $\text{ZFC}_{\mathbf{j}}$, we formulate appropriate analogues $(1')$–$(3')$ for (1)–(3). In order to observe our convention of considering only *sharp-like* models of extensions of $\text{ZFC}_{\mathbf{j}}$ (see footnote (5), or [C06]), the axioms for $(1')$ must be ZFC only. In order to formulate $(2')$, we need to have an appropriate structure in which to interpret relativized sentences. For T, relativizing ZFC axioms to the constant $\mathbf{c}$ was adequate, but it would not be adequate in this new setting since we must still interpret the symbol $\mathbf{j}$. Thus, we introduce a second constant symbol $\mathbf{i}$. For any $\mathcal{L}$-formula ϕ, the formula $\phi^{\mathbf{c},\mathbf{i}}$ is obtained by restricting quantifiers to $\mathbf{c}$ and replacing each occurrence of $\mathbf{j}$ with $\mathbf{i}$. Since we want the "inner universe" to be a model of the extended theory S, the axioms for $(2')$ must be $\{\sigma^{\mathbf{c},\mathbf{i}} : \sigma \in S\}$. Finally, for $(3')$, we assert that $\mathbf{c}$ is countable and transitive and that $\mathbf{i}$ is a function $\mathbf{c} \to \mathbf{c}$.

Summarizing, given a theory $S \supseteq \text{ZFC}_{\mathbf{j}}$, the theory $T(S)$ in the language $\{\in, \mathbf{j}, \mathbf{c}, \mathbf{i}\}$ consists of the following:

$$\text{ZFC} + \{\sigma^{\mathbf{c},\mathbf{i}} : \sigma \in S\} + \text{“}\mathbf{c} \text{ is countable and transitive”} + \text{“}\mathbf{i} \text{ is a function } \mathbf{c} \to \mathbf{c}\text{”}.$$

Now we turn to the theory $S = \text{ZFC} + \text{Elementarity} + \text{Nontriviality}$. We show that it is impossible to prove

$$\text{Con}(S) \Rightarrow \text{Con}(T(S)) \tag{4.2}$$

using methods formalizable in ZFC, unless ZFC itself is inconsistent; in other words, the first step of logic in (4.1) to justify the use of a transitive ground model is not valid for such theories S.

Our argument is modeled after the proof [Ku80, IV.10] that the consistency of an inaccessible is not provable from ZFC. We begin with an outline of the structure of the proof: Suppose (4.2) can be proven with methods formalizable in ZFC, so

$$\text{ZFC} \vdash \ulcorner \text{Con}(S) \Rightarrow \text{Con}(T(S)) \urcorner. \tag{4.3}$$

We will show that

$$T(S) \vdash \ulcorner\text{Con(ZFC)}\urcorner. \tag{4.4}$$

Recall from Proposition 2.2(1) that

$$\text{ZFC} \vdash \ulcorner\text{Con(ZFC)} \Rightarrow \text{Con}(S)\urcorner. \tag{4.5}$$

Combining (4.5) and (4.3) with (4.4), we see that

$$T(S) \vdash \text{Con}(T(S)),$$

whence $T(S)$ is inconsistent. But by (4.2), S must also be inconsistent. Finally, by Proposition 2.2(1), it follows that ZFC itself must be inconsistent. Thus, to obtain the result, it suffices to prove (4.4).

To prove (4.4), we must prove *more* than the fact that, for all finite subsets $F \subseteq \text{ZFC}$, $F = \{\sigma_1, \ldots, \sigma_k\}$, we have $T(S) \vdash \exists M\ (\sigma_1^M, \wedge \cdots \wedge \sigma_k^M)$. We must rather prove that $T(S) \vdash \exists M\ \forall p \in \ulcorner\text{ZFC}\urcorner\ \text{Sat}(p, M, \emptyset)$ ([11]). Recall also that ZFC $\vdash$ "λ is inaccessible" $\Rightarrow \forall p \in \ulcorner\text{ZFC}\urcorner\ \text{Sat}(p, V_\lambda, \emptyset)$.

Working in the extended language $\{\in, \mathbf{j}, \mathbf{c}, \mathbf{i}\}$, we can argue as in Proposition 2.3 to show

$$T(S) \vdash (\text{Critical Point})^{\mathbf{c},\mathbf{i}}.$$

Here is an outline of the proof: Suppose $\mathcal{M} = (M, E, A, i) \models T(S)$. In $\mathcal{M}$, define x, κ such that $x \in A$, $i(x) \neq x$, and κ is least for which $x\,E\,V_{\kappa+1}$. (This step requires further explanation. Since we are not assuming that (A, i) satisfies Separation$_{\mathbf{j}}$, there is no reason to suppose that, inside A, we can find such a "least κ". However, we can find such a κ in $\mathcal{M}$ since only ordinary Separation for $\in$-formulas is required to establish this.) Now, in $\mathcal{M}$, x, κ satisfy the same definition relativized to (A, E, i) since, in $\mathcal{M}$, $V_{\kappa+1}^{(A,i)} = V_{\kappa+1} \cap A$, by transitivity of A in M. Therefore,

$$\mathcal{M} \models \text{“}(A, E, i) \models \kappa \text{ is the critical point of } i\text{”}.$$

Next, we can argue as in Proposition 2.2(2) ([12]) to show

$$T(S) \vdash (\text{“crit}(\mathbf{j}) \text{ is inaccessible”})^{\mathbf{c},\mathbf{i}}.$$

Here is an outline of the proof: Suppose $\mathcal{M} = (M, E, A, i) \models T(S)$. Without loss of generality, assume $V_\omega^V \subseteq M$. Suppose $\mathcal{M} \models (\kappa = \text{crit}(i))^{(A,i)}$. The map i fixes each standard integer and also fixes ω. Arguing in $\mathcal{M}$, by leastness of κ and elementarity of i (with respect to standard-length $\in$-formulas), $i(f) = f$ whenever $f : \alpha \to \kappa$ and $\alpha < \kappa$. This establishes that κ is a regular

([11]) Here, Sat is the formalized satisfaction relation: In general, $\text{Sat}(u, M, b)$ asserts that u encodes the $\in$-formula $\phi(x_1, \ldots, x_m)$ and $\langle M, E(M)\rangle \models \phi(b(1), \ldots, b(m))$, where b is a function defined on ω that specifies set parameters. See [C06] and [Dr74].

([12]) See [C06, p. 338] for details.

cardinal in (A, i). A similar argument shows that for all $\alpha < \kappa$ in (A, i),

$$(\text{there is no surjection } P(\alpha) \to \kappa)^{(A,i)},$$

which can be used to show that κ is a strong limit in (A, i). Therefore, in $\mathcal{M}$, $(\kappa \text{ is inaccessible})^{(A,i)}$.

Next, we derive, in the usual way ([13]),

$$T(S) \vdash \left[\exists\lambda \left(\lambda = \mathrm{crit}(\mathbf{j}) \wedge \forall p \in \ulcorner\mathrm{ZFC}\urcorner \ \mathrm{Sat}(p, V_\lambda, \emptyset)\right)\right]^{\mathbf{c},\mathrm{i}}.$$

Let $\mathcal{M} = (M, E, A, i) \models T(S)$ and let $(\lambda = \mathrm{crit}(i))^{(A,i)}$. Then

$$\mathcal{M} \models \forall p \in \ulcorner\mathrm{ZFC}\urcorner \ (\mathrm{Sat}(p, V_\lambda, \emptyset))^A,$$

whence

$$\mathcal{M} \models \forall p \in \ulcorner\mathrm{ZFC}\urcorner \ \mathrm{Sat}^A(p, V_\lambda^A, \emptyset).$$

Since Sat is a standard-length formula and A is, in $\mathcal{M}$, a transitive model of standard-length ZFC axioms, Sat is absolute for $A, \mathcal{M}$. Thus

$$\mathcal{M} \models \mathrm{Con}(\ulcorner\mathrm{ZFC}\urcorner),$$

as required. ∎

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([13]) See for example [Ku80, Theorem 6.6].

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