The Axiom of Infinity And Transformations $j : V \to V$

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Abstract. We suggest a new approach for addressing the problem of establishing an axiomatic foundation for large cardinals. An axiom asserting the existence of a large cardinal can naturally be viewed as a strong Axiom of Infinity. However, it has not been clear on the basis of our knowledge of ω itself, or of generally agreed upon intuitions about the true nature of the mathematical universe, what the right strengthening of the Axiom of Infinity is — which large cardinals ought to be derivable? It was shown in the 1960s by Lawvere that the existence of an infinite set is equivalent to the existence of a certain kind of structure-preserving transformation from V to itself, not isomorphic to the identity. We use Lawvere's transformation, rather than ω, as a starting point for a reasonably natural sequence of strengthenings and refinements, leading to a proposed strong Axiom of Infinity. A first refinement was discussed in later work by Trnková-Blass, showing that if the preservation properties of Lawvere's tranformation are strengthened to the point of requiring it to be an *exact functor*, such a transformation is provably equivalent to the existence of a measurable cardinal. We propose to push the preservation properties as far as possible, short of inconsistency. The resulting transformation $V \to V$ is strong enough to account for virtually all large cardinals, but is at the same time a natural generalization of an assertion about transformations $V \to V$ known to be equivalent to the Axiom of Infinity.

• 2000 *Mathematics Subject Classification:* Primary 03E55; Secondary 03E40.

• *Keywords and phrases.* Axiom of Infinity, WA, Wholeness Axiom, large cardinal, exact functor, critical point, Lawvere, universal element.

[1]The results of this paper were presented at the 2008 Annual Meeting of the Association for Symbolic Logic, March 27–30, Irvine, California.

§0. Introduction

What is the right notion of "infinite" in mathematics? Prior to the work of Cantor, the only acceptable notion was "potential infinity": arbitrarily large finite sizes were allowed, but all such finite sizes could not be collected together to form a single, well-defined set. Both because of philosophical issues and seeming mathematical paradoxes, actual infinities were not considered legitimate objects of mathematical investigation. Cantor convinced the mathematical world that mathematics without an actual infinite is too impoverished. (See [15], [27], [28] for a discussion of these points.)

Part of Cantor's legacy is the presence of the Axiom of Infinity in the standard set theory ZFC. The Axiom of Infinity asserts that an infinite set exists, and this axiom, in conjunction with the other ZFC axioms, suffices to build Cantor's entire theory of transfinite cardinals.

Ironically, even as ZFC itself was crystallizing into its final form as the *de facto* universal foundation for mathematics, the first large cardinals were being discovered: Hausdorff discovered weakly inaccessible cardinals [18] in 1908 and Mahlo discovered (weakly) Mahlo cardinals [29] in 1911. As Gödel would later show, large cardinals are one type of mathematical entity that is *underivable* from ZFC.

It was clear even in Hausdorff's time that large cardinals represented new, stronger notions of infinity, giving rise to natural models of set theory (see [20, Introduction]). Cantor gave us a "minimal" notion of actual infinite, but perhaps a stronger type of infinity can truly be asserted to exist in the universe. How strong should the Axiom of Infinity be?

Since the time of Gödel's Second Incompleteness Theorem, which demonstrated among other things that large cardinals are not derivable from ZFC, the attitude in the community of set theorists toward large cardinals has been mixed.[2] Some have been unconcerned with existence of large cardinals. Others have actively attempted to prove that some or all large cardinals are in fact inconsistent.[3] And a third group attempted to devise heuristics, often based on Cantor's vision of the structure of the universe, to legitimize the presence of large cardinals in the universe. We consider two examples of such heuristics from this relatively early period in large cardinal history; both of these heuristics were introduced by Gödel and developed further by others (see [21]):

One of Cantor's perspectives about the mathematical universe is that the transition from finite to transfinite is reasonably smooth and consequently, the universe is fairly homogenous

[2] See [12] for a survey of a variety of contemporary views on these issues.

[3] A striking example of such an effort that bore great fruits was Silver's early efforts to prove the inconsistency of measurable cardinals; his efforts resulted in many of the foundational results on $0^{\#}$.

([15, Section 1.3], [20, Introduction]). From this point of view, it would be too "accidental" for ω to be the only cardinal having the large cardinal properties of (for example) inaccessibility and measurability, and so we conclude that there are inaccessible and measurable cardinals in the universe. This is an example of the *generalization* heuristic.

A second insight of Cantor's is that the Absolute Infinite is beyond mathematical determination. Neither the universe itself nor the class ON of ordinals should be able to be captured by a single first-order property [15, Section 1.3], [20, Introduction]. Therefore, if it can be legitimately claimed that V or ON satisfies such a property, some actual set or ordinal must also satisfy the property. This is an intuitive *reflection principle*; in [31], Reinhardt attempts to formalize this type of reflection. It can be argued on intuitive grounds that ON is inaccessible, Mahlo, and perhaps much more, and so by the *reflection principle*, at least inaccessibles and Mahlos should exist in the universe. See the work by Paul Bernays [1] where arguments of this kind are motivated and developed much further.

For the most part, while these early heuristics may have been useful in justifying some of the smaller large cardinals, they fail to motivate the stronger and often more complicated large cardinals that are widely used today, such as Woodin, strong, superstrong and supercompact cardinals. (See, however, [27], [28] for an excellent survey of heuristics for justifying even the strongest axioms.) Yet, despite the fact that completely satisfactory simple heuristics have not been found for these stronger types of infinities, there is wider acceptance than ever of large cardinals in the universe. Though this development may be due partly to long-time familiarity with the concepts, it is most likely due to the deep connections that were discovered by Woodin, Steel, Martin, Kunen, and others, between some of the very strongest largest cardinals and sets of small cardinality, particularly sets of reals. One of the pioneering theorems in this domain, due to Martin [30], established determinacy of all Π^1_2 sets of reals (from which it follows, for example, that all such sets of reals are Lebesgue measurable and have the property of Baire), assuming the existence of an I_2 embedding $j : V_\lambda \to V_\lambda$. In unpublished work, Woodin extended the technique to show that, assuming an I_0 embedding $j : \mathbf{L}(V_{\lambda+1}) \to \mathbf{L}(V_{\lambda+1})$, *all* sets of reals belonging to $\mathbf{L}(\mathbf{R})$ are determined. In a different direction, Kunen [23] showed how to collapse a huge cardinal to obtain in a forcing extension an ω_2-saturated ideal on ω_1. All these results were shown later to require much weaker large cardinal hypotheses, but the rich structural connections between these strong axioms and sets belonging to the earlier stages of the universe strengthened confidence in even these strongest large cardinal notions. (See [20, Chapter 31] for a survey of these results.) Reflecting similar sentiments, in his 2008 Gödel Lecture,[4] H. Woodin expressed his belief that the "hierarchy of large cardinal axioms

[4]Presented at the 2008 Annual Meeting of the Associaltion for Symbolic Logic, in Irvine, California.

has emerged as an intrinsic part of set theory."

Despite growing acceptance of large cardinals, more clear than ever is the fact that there is no axiomatic foundation for their study. ZFC has proven itself to be a nearly universal foundation for the rest of mathematics; it would be only natural to expect that an extension of ZFC should emerge that would provide a foundation for *all* of mathematics, including large cardinals. Large cardinals have been studied long enough and with enough rigor to warrant a theory similar in spirit to perhaps the theory of groups or topological spaces. Although the study of large cardinals continues on without such a formal theory, there are questions one simply cannot ask without such a theory. To make this point clearer, we consider here an example of such a question and discuss later in the paper how a well-formed theory could attempt to answer it:

Test Question: Which large cardinals have their own brand of Laver sequence?

Recall that a Laver sequence (originally defined by Laver [24]) is a function $f : \kappa \to V_\kappa$ such that for any x and any $\lambda \geq \max(\kappa, |\mathrm{TC}(x)|)$, there is a supercompact ultrafilter U on $P_\kappa\lambda$ such that if j_U is the canonical embedding, then $j_U(f)(\kappa) = x$.

In order to address the Test Question, certain points need to be clarified. First, one needs to have a notion of Laver sequence that could apply to other types of large cardinals. Laver sequences for strong and extendible cardinals have been defined and proven to exist [3], [4], [14]. But the question cannot really be answered in general without knowing which large cardinals exist in the first place, and then, beyond mere existence, some form for candidate large cardinals that would permit a general definition of "Laver sequence."

In earlier work on the Test Question [4], we attempted to address these issues. We used as our foundational axiom the axiom (schema) discussed in Parts 1 and 2 of this paper: the Wholeness Axiom (WA). We also provided a general definition of a broad class of large cardinals that would be candidates for admitting some form of Laver sequence; the background Wholeness Axiom ensured the existence of the classes defined in this way. In Part 1 we will review some of the results and point out some of the added value that resulted from having a strong axiomatic framework in which to work.

In devising an appropriate extension of ZFC to account for large cardinals, it is of course necessary to decide which large cardinals should be declared to "exist." From the trends we see in set theory research and Woodin's recent remarks, it is not perhaps too bold to seek an axiomatic foundation for all the large cardinals — not known to be inconsistent — that have been studied over the years. Rather than viewing such an axiom system as a commitment to a belief system, it could be viewed as meeting the practical need of providing a useful robust framework for studying the things that are already being studied.[5] Even if we agree to admit all of the better known large cardinals, it is still necessary to decide which sorts of axiom candidates ought to be taken seriously;

[5]An interesting parallel to this point of view is the fact that some set theorists are not personally

one would expect that a good axiom would provide not only the intended consequences, but in some way be "natural" enough to belong among the foundational axioms for all of mathematics. In this paper, we will offer one such axiom and make a case for its naturalness.

Our approach to the task of building an extension of ZFC that accounts for large cardinals will be to return to simple heuristics, but we will do so by examining an equivalent and rather different form of the Axiom of Infinity. In our view, simpler heuristics are more convincing, if they truly can account for all large cardinals; so far, as we mentioned earlier, simpler heuristics have fallen short, by quite a lot, of this objective. We will, in this paper, examine how a *global* version of the Axiom of Infinity can be strengthened in natural ways to lead to a candidate strong Axiom of Infinity that can indeed account for (virtually) all large cardinals.

The paper is divided into two parts. Part 1 surveys an equivalent form of the Axiom of Infinity, which is used as a starting point, and then quickly outlines the ideas for generalizing to another level, and ultimately to our final version of the axiom. In this part we will state theorems with only a hint about their proofs, and for relevant definitions the reader will in many cases be redirected to Part 2. Part 2 begins with definitions and background theorems that may be unfamiliar to the reader, particularly in category theory; it will then provide what we feel are the most interesting details of the proofs of the theorems mentioned in Part 1. Most of these proofs can be found in the literature, and whenever certain details are omitted, the references provided can be used to fill in gaps. We feel that collecting together the essential arguments from this diverse literature in one place and in a uniform context adds sufficient value to justify the extra length of the paper.

convinced of the consistency even of ZFC itself; some for example have argued that Σ_n-Replacement for $n > 2$ is unconvincing. (Solovay discusses this point of view briefly in his FOM post, August 17, 2007.) Nevertheless, these set theorists freely use ZFC without meticulous concern over the degree of Replacement involved in their arguments. A foundation for all large cardinals could be viewed in a similar fashion.

§1. In Search of the Right Axiom Of Infinity

The Lawvere Functor

We begin with an alternative version of the Axiom of Infinity that was discovered by W. Lawvere in the 1960s. After stating the result, we define possibly unfamiliar terms and discuss the significance of the result for our program to generalize to stronger axioms of infinity.

1.1 Lawvere's Theorem [25]**.** *Suppose V is a model of* ZFC − Infinity*. Then the following are equivalent:*

(1) *V satisfies the Axiom of Infinity*

(2) *There is a functor $j : V \to V$ that factors as a composition $G \circ F$ of functors satisfying:*

(A) *$F \dashv G$ (F is left adjoint to G)*

(B) *$F : V \to V^{\circlearrowright}$*

(C) *$G : V^{\circlearrowright} \to V$ is the forgetful functor, defined by $G(A \to A) = A$.*

In particular, F preserves all colimits and G preserves all limits.

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "F"'] & & V \\ & V^{\circlearrowright} \arrow[ur, "G"'] & \end{tikzcd}$$

In its original form, Lawvere's Theorem talked about cartesian closed categories, rather than models of ZFC − Infinity. A cartesian closed category is one that has all finite products (for sets, these are just cartesian products) and, for any pair of objects A, B, has an *exponential object* A^B, much as we have in a model of ZFC or ZFC − Infinity. As we discuss further in Part 2, any model of ZFC − Infinity can be viewed as a cartesian closed category, so the version of Lawvere's theorem given here is an immediate consequence of his original.

Quantification over classes that appears to occur in the statement of the theorem can be eliminated in the usual ways. In the proof of (1) ⇒ (2), F and G are given explicitly as class maps. For (2) ⇒ (1), the result can be viewed as a schema of theorems, one for each F. In either direction, j is simply the composition $G \circ F$.

We observe that in the theorem, the *factors* of j are endowed with strong preservation properties. For instance F preserves all colimits; examples of colimits in a category of sets are coproducts (which are disjoint unions) and direct limits of directed systems. Dually, G preserves all limits; limits include products (in the usual set-theoretic sense) and inverse limits. These are very strong preservation properties in the context of category theory, and arise because of the adjoint relationship between F and G. The class $V^{\circlearrowright}$ is the category of *endos* (endomorphisms) $A \to A$, where

A is a set in V. The statement $F \dashv G$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, means that for every object a of $\mathcal{C}$ and every b in $\mathcal{D}$, the arrows from $F(a)$ to b are in 1-1 correspondence with those from a to $G(b)$, and the bijection is natural in a and b. A familiar example of such an adjunction is found when $\mathcal{C}$ is the category of sets and $\mathcal{D}$ is the category of vector spaces over a fixed field. Then, if $F(X)$ is the vector space freely generated by X, and $G(V)$ is the set V without vector space structure, it follows that $F \dashv G$ and G is the forgetful functor, a situation exactly parallel to that in Lawvere's Theorem. Detailed definitions are provided in Part 2. We will henceforth call the functor $j : V \to V$ defined in part (2) of the theorem the *Lawvere functor*.

The proof of existence of j from the Axiom of Infinity in Lawvere's Theorem requires one to define a left adjoint F to the already defined forgetful functor G. A definition of F on objects is given by

$$F(A) = A \times \omega \stackrel{1_A \times \mathrm{succ}}{\longrightarrow} A \times \omega,$$

where succ is the successor function on ω. Defining F on arrows is straightforward. As we indicate in more detail in Part 2, a category-theoretic version of definition by recursion (derivable from the Axiom of Infinity) can be used to establish that $F \dashv G$.

Conversely, to establish the Axiom of Infinity from the existence of j, one can use the familiar trick of examining the critical point of j. Arguing category-theoretically, one shows that $j(0) = 0$ and $j(1) \cong \omega$, so 1 is the critical point of the mapping, and its image explicitly gives us an infinite set.

Lawvere's result is striking because it implies that to understand the Axiom of Infinity, which asserts simply that a certain set exists (such as ω), one must come to terms with tranformational dynamics of the universe as a whole. In fact, it tells us something that we are not perhaps accustomed to believing: that the universe comes equipped with an auxiliary map $j : V \to V$ that has strong structure-preserving characteristics.[6]

Lawvere's Theorem opens the door to generalizing to much stronger notions of infinity, in a way that the usual Axiom of Infinity does not. Recall that we are searching for the "right" version of the Axiom of Infinity. When limited to the usual version, the possible generalizations do not go very far. When instead we ask, "What is the *right* version of *Lawvere's* Axiom of Infinity?" we see new possibilities. The task then becomes one of formulating the preservation properties of j as clearly as possible and then strengthening them as much as possible. In Lawvere's Theorem, the strong preservation properties belong not to j itself, but to its factors F and G. So our first

[6]This point will be developed further in the paper. What we have in mind here is that the adjoint *factors* of j have strong structure-preserving properties (as in Lawvere's Theorem), and that this situation "points to" the possibility of a stronger axiomatic formulation in which j itself has these or similar strong preservation properties.

attempt at getting the "right" axiom is to ask the following:

1.2 Generalization Step 1. What happens if we endow j itself, rather than its factors, with the properties of preserving limits and colimits?

Before attempting an answer, we should address the following philosophical issue: Why should *stronger* versions of Lawvere's Theorem be *true*? When we considered *generalization*, the motivation for concluding that certain large cardinals exist on the basis of generalizing properties of ω was Cantor's insight concerning the *uniformity* of the universe — no single cardinal should have many special properties not found elsewhere in the universe. Here, a principle that motivates us to conclude that "more preservation is better" is Cantor's *principle of maximum possibility*: *As much as possible exists.* (See [15, pp. 20–23].) As we will see, when Lawvere's functor is endowed with ever stronger preservation properties, the consequence is increasing *combinatorial richness* in the universe. This principle suggests to us that moving in the direction of strengthened preservation will lead to a more complete rendering of the universe.

A theme that will develop as we move from Lawvere's result to refinements is a strengthening of what we call here *critical point dynamics*. Historically, perhaps the first and most striking example of this notion of critical point dynamics occurs in the context of the canonical elementary embedding $j : V \to V^\kappa/U \cong M$, where M is the transitive collapse of the ultrapower V^κ/U and U is a nonprincipal κ-complete ultrafilter over an uncountable cardinal κ. The first point to observe is that the large cardinal strength of such an embedding is located precisely at the first ordinal moved by the embedding — namely, κ — which must be a measurable cardinal. The second point is that every set in M arises from dynamics that "live" in the vicinity of κ. In particular, for every $y \in M$, there is a function f with domain κ such that $y = j(f)(\kappa)$; here, the domain of f and the definition of j itself, being defined from an ultrafilter over κ, depend on sets in the vicinity of κ. In Hamkins' terminology [16], κ is a *seed* via j for U that generates M.

In studying critical point dynamics from both the set-theoretic and category-theoretic points of view, it will be natural to define a *critical point* of a mapping j to be the least cardinal γ for which $\gamma < |j(\gamma)|$. (The mappings $j : V \to V$ we consider here will never have the property that the least ordinal γ moved by j is either a non-cardinal or of cardinality $|j(\gamma)|$.)

An imprecise question that is nevertheless natural to ask as we consider critical point dynamics is, "*Why* does V come equipped with a $j : V \to V$? What purpose does it serve?" If we think of set theory with large cardinals as being *full* set theory, then one could consider this Lawvere functor, with which any ZFC universe is equipped, to be an intimation of a type of transformation that would have more fully developed properties in the presence of the spectrum of large cardinals; this fully developed version could be seen as the evolutionary pinnacle of j. We have already suggested that j embodies transformational dynamics of V, since this is what functions do. Since an obvious aspect of the transformational dynamics of V is the unfoldment of all sets, we hypothesize that part of what j is "designed to do" is to generate all sets, in the way (or by analogous means) that a seed of an elementary embedding $j : V \to M$ generates all sets in M. We formulate this hypothesis below and then, as a secondary theme as we seek strengthenings of Lawvere's Theorem, we track the extent to which this hypothesis is verified in stronger contexts.

1.3 Critical Point Dynamics Hypothesis. It should be possible to generate every set in V from the auxiliary functor j, its critical point κ, and sets in the vicinity of κ.

We will take a look at the critical point dynamics both for Lawvere's functor and for each of the strengthenings we will consider. In each case, we will see that the "type" of infinity that emerges from the embedding (whether it is a large cardinal or, in the case of the Lawvere functor, simply a countably infinite set) does so at its critical point, and that, on the basis of dynamics in the vicinity of this critical point, "all" sets can be generated. We will call this latter characteristic the *seed property* of the mapping.

Having at least roughly defined what we mean critical point dynamics, we can now examine these dynamics in the case of the Lawvere functor. As we observed earlier, in the presence of the Axiom of Infinity, the critical point of j is 1, and the image $j(1)$ tells us the "type" of infinity we are dealing with—in this case, simply ω itself. The seed property in this case arises from the fact that $0 \in j(1) = G(F(1))$ is a *universal element* for G (but not for j); this means that, for any endo $\alpha : B \to B$ and any $y \in G(\alpha)$, we can find a unique $f : F(1) \to \alpha$ so that $y = G(f)(0)$. Moreover, every set in V is expressible as $G(f)(0)$ for some $V^{\circlearrowright}$-arrow f. That is,

$$V = \{G(f)(0) \mid f \in \mathbf{Arr}(V^{\circlearrowright})\}.$$

We notice that the Critical Point Dynamics Hypothesis is only partially verified for the Lawvere functor since the seed property does not belong to j itself but rather to one of its factors. We also note that the "type" of infinity is given to us in this case as the *image* of the critical point rather than by the critical point itself.

The Trnková-Blass functor.

We now attempt an answer to Generalization Step 1. We note that, from a category-theoretic perspective, every set is a coproduct (i.e., disjoint union) of singleton sets. Therefore, if a functor preserves singletons (and a functor that preserves limits preserves singletons) and preserves all coproducts, it must preserve all sets; in other words, it must be naturally isomorphic to the identity functor. Therefore, we do not attempt to endow j with such strong preservation properties. Trnková, and independently Blass, discovered that requiring only that j preserve *finite* limits and colimits already has strong large cardinal consequences; functors that preserve all finite limits and colimits are called *exact*.

1.4 Trnková-Blass Theorem [2], [32]**.** *Suppose V is the universe of sets (a model of* ZFC *or even of* ZFC − Infinity*).*

(1) *Suppose there is a measurable cardinal in V and D is a nonprincipal κ-complete ultrafilter on κ. Then there is an exact functor $j : V \to V$, definable in V from D, that is not isomorphic to the identity.*

(2) *Suppose $j : V \to V$ is a definable (with parameters) exact functor not isomorphic to the identity functor. Then $V \models$ "there exists a measurable cardinal."*

As before, the universe V plays the role of the category of sets. Again, the functor j is a class in V, definable with parameters. Henceforth, we will call any $j : V \to V$ satisfying the properties given in (2) a *Trnková-Blass functor*.

The theorem given here is actually a slight weakening of the original theorems of Trnková and Blass, since we have replaced "natural isomorphism" in their results with "isomorphism" in ours. Functors F and G are considered in this context to be *isomorphic* if for every X, $F(X) \cong G(X)$, without the usual naturality requirement. We use this less natural version here because proofs and related results can be expressed more easily in the framework we are developing in this paper. In Part 2, we show that our version follows easily from theirs.

The idea behind the proof of (1) is that, given a measurable cardinal κ, if we let D be a nonprincipal κ-complete ultrafilter on κ, we can define a functor $j = j_D : V \to V$ as follows: For any X, Y and any $h : X \to Y$:

$$
\begin{aligned}
& j(X) = X^\kappa/D = \{[f] \mid f : B \to X \text{ and } B \in D\} \\
(*) \qquad & \qquad (f \sim g \text{ iff they agree on a set in } D) \\
& j(h)([g]) = [h \circ g]
\end{aligned}
$$

The proof of the Trnková-Blass Theorem shows that this j is exact, and also that $|j(\kappa)| > \kappa$, whence $j \ncong \text{id}$.

For the converse, given an exact functor $F : V \to V$, not isomorphic to the identity functor, one shows that the critical point κ of F must be at least as large as the least measurable cardinal (which must exist). Moreover, if κ happens to be the least possible critical point among all critical points of exact functors not isomorphic to the identity, then κ must in fact equal the least measurable cardinal. We prove this fact in Part 2.

For exact functors j defined as in $(*)$, it can be shown that the equivalence class $[\text{id}] \in j(\kappa)$ of the identity function $\text{id} : \kappa \to \kappa$, as a member of κ^κ/D, is a weakly universal element for j; that is, for any B and any $y \in j(B)$ there is $f : \kappa \to B$ such that $j(f)([\text{id}]) = y$. When D is a normal measure, as usual $[\text{id}]$ can be identified with κ. We will show that despite weak universality of κ, it is not the case here that every set in V is expressible as $j(f)(\kappa)$.

We pause to compare our formulation of a stronger Axiom of Infinity based on a Trnková-Blass functor

Trnková-Blass Axiom of Infinity: There is an exact functor $j : V \to V$ not isomorphic to the identity.

with the formulation due to Lawvere, provably equivalent to the (original) Axiom of Infinity:

Lawvere Axiom of Infinity: There is a functor $j : V \to V$ where $j = G \circ F$, $F \dashv G$, and G is the forgetful functor $V^{\circlearrowright} \to V$.

Certainly, the Trnková-Blass formulation is "cleaner" than Lawvere's since j itself (rather than its factors) is endowed with the key preservation properties. Moreover, the price we pay for this cleaner formulation is not great: we are simply requiring j to be exact, and exactness is a natural geometric property of functors.

Nevertheless, we are seeking to strengthen the Lawvere formulation to the fullest possible extent, and so we naturally seek further improvement. One aspect of the Trnková-Blass formulation that could be further refined, at least from the point of view of "critical point dynamics," is the fact that the critical point of an exact functor not isomorphic to the identity is not always guaranteed to be measurable (which is the large cardinal property associated with such functors), though the existence of the critical point κ does at least guarantee the existence of a measurable $\leq \kappa$.

Another point to notice is that the ZFC extension obtained by replacing the usual Axiom of Infinity with the Trnková-Blass formulation is equivalent to the theory ZFC + "there is a measurable cardinal". One problem with this theory as a foundation is its lack of robustness under set forcing: We might expect a truly foundational theory to be indestructible under set forcing (since ZFC itself has this property). But it is easy to destroy a measurable (by collapsing or via Prikry forcing, for example).

A natural question to ask as we seek to further refine our strong axioms of infinity, in the direction of a *maximal* strengthening of preservation properties of j, is, Can exactness be replaced by something stronger? This question leads to the next step of generalization:

1.5 Generalization Step 2. What happens if we replace "exact functor" with "elementary embedding" in the Trnková-Blass formulation?

The Wholeness Axiom.

As we address Generalization Step 2 and attempt to enhance the preservation properties of j much beyond exactness, we quickly encounter the limits imposed by Kunen's well-known theorem [22]. We run into trouble if we replace "exact functor" with "elementary embedding": there can be no elementary embedding $j : V \to V$ that is definable in V. However, if we weaken definability sufficiently, inconsistency can be avoided and elementarity of j does not cause a problem. This can be accomplished by working in the expanded language $\{\in, \mathbf{j}\}$ and formulating axioms asserting collectively that $\mathbf{j}$ is an elementary embedding $V \to V$ having a critical point; these axioms are known [7] as the *Basic Theory of Elementary Embeddings*, or BTEE. In working in the theory

ZFC + BTEE, no other axioms regarding $\mathbf{j}$ are assumed (in particular, no instances of Separation or Replacement for $\mathbf{j}$-formulas are assumed). This approach departs from the assumption that $\mathbf{j}$ is definable in V (or that $\mathbf{j}$ is a class in some kind of class theory) since, under such an assumption, all instances of Separation and Replacement for $\mathbf{j}$-formulas must automatically hold true.

1.6 Axioms of BTEE [7]

$(1)_\phi$ (*Elementarity Schema for $\in$-formulas*). Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula:

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right);$$

(2) (*Critical Point*). "There is a least ordinal moved by $\mathbf{j}$".

The theory ZFC + BTEE is not particularly strong: A transitive model can be found [7] assuming less than $0^\sharp$. This shows that the large cardinal strength of ZFC+ "there is an elementary embedding $j : V \to V$" is nowhere near inconsistency in the absence of instances of Separation and Replacement for $\mathbf{j}$-formulas.

We clarify a possibly confusing point here, having to do with the lack of definability of embeddings that we get from ZFC+BTEE. Suppose $j : V \to V$ is an elementary embedding with critical point κ given to us in a model of the theory ZFC+BTEE. In [7] it is shown that κ must be weakly compact, but not much more, because as we just observed, a transitive model of ZFC + BTEE can be obtained from (less than) $0^\#$. However, viewed as a functor, j is exact and not isomorphic to the identity. Therefore, one might be tempted to conclude that, by the Trnková-Blass Theorem, κ is measurable. This conclusion does not follow, though, because in this case j is *not definable* in the universe V as it must be for the Trnková-Blass Theorem to apply. In particular, the Trnková-Blass Theorem makes use of an instance of Separation for $\mathbf{j}$-formulas to define the ultrafilter D from j, and this instance is not derivable from ZFC + BTEE.

To provide j with additional strength as we search for maximal preservation properties of a $j : V \to V$, one further step we could take is to require $\mathbf{j}$ to satisfy all instances of $\text{Separation}_{\mathbf{j}}$. This additional requirement forces $\mathbf{j}$ to be very strong. Before investigating this "ultimate" theory, we first briefly survey the various strengths obtainable by adding certain natural instances of $\text{Separation}_{\mathbf{j}}$. We begin with several definitions; the reader is asked to consult [19] and [20] for unfamiliar terms. A cardinal κ is *superstrong* if there is an elementary embedding $j : V \to M$ such that κ is the critical point of j, M is an inner model, and $V_{j(\kappa)} \subseteq M$. A cardinal κ is *extendible* if, for every ordinal $\eta > \kappa$, there is an ordinal ζ and an elementary embedding $i : V_\eta \to V_\zeta$ such that $\mathrm{crit}(i) = \kappa$ and $\eta < i(\kappa) < \zeta$; the map i is called an *extendible embedding with critical point κ*. For each $n \in \omega$, κ is *n-huge* if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and M is closed under $j^n(\kappa)$-sequences; $j(\kappa)$ is called the *target* of j and j is called an *n-huge embedding*. κ is *super-n-huge* if, for every $\lambda > \kappa$, there is an n-huge embedding

j such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. The axiom $\mathrm{I}_3(\kappa)$ asserts there is an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point κ and $\lambda > \kappa$ a limit; in this case j is called an *I_3 embedding* and κ an *I_3 cardinal*. The axiom $\mathrm{I}_2(\kappa)$ asserts there is an elementary embedding $j : V \to M$ having critical point κ so that the inner model M includes as a subset V_λ where λ is the supremum of the critical sequence of j. The axiom $\mathrm{I}_1(\kappa)$ asserts there is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ where λ is a limit and $\kappa < \lambda$ is the critical point; in this case, j is an *I_1 embedding* and κ is an *I_1 cardinal*. The axiom $\mathrm{I}_0(\kappa)$ asserts there is an elementary embedding $\mathbf{L}(V_{\lambda+1}) \to \mathbf{L}(V_{\lambda+1})$ having critical point κ, where $\lambda > \kappa$ is a limit; in this case j is an *I_0 embedding* and κ is an *I_0 cardinal*.

> **Measurable Ultrafilter Axiom of Infinity**: The class $\{X \subseteq \kappa : \kappa \in \mathbf{j}(X)\}$ is a set.

Adding the Measurable Ultrafilter Axiom to ZFC + BTEE is the result of adding the single instance of $\text{Separation}_{\mathbf{j}}$ that asserts that the ultrafilter over κ defined by $\mathbf{j}$ is a *set*. Perhaps surprisingly, for a theory to assert in this way that the critical point κ of $\mathbf{j}$ must be measurable forces κ to be much stronger than measurable:

1.7 Measurable Ultrafilter Theorem [7]**.** *The following give lower and upper bounds on the theory* ZFC + BTEE + Measurable Ultrafilter Axiom*:*

(1) *The theory* ZFC + BTEE + Measurable Ultrafilter Axiom *proves that κ is measurable with Mitchell order $> \kappa$.*

(2) *If κ is 2^κ-supercompact, there is a transitive model of* ZFC + BTEE+ *Measurable Ultrafilter Axiom.*

The critical point κ of a $\mathbf{j} : V \to V$ will be forced to be a superstrong cardinal if we supplement ZFC + BTEE instead with the following axiom:

> **$P(\kappa)$-Amenability$_{\mathbf{j}}$ Axiom of Infinity**: The restriction $\mathbf{j} \restriction P(\kappa)$ is a set.

Given a model $j : V \to V$ of ZFC+BTEE+$P(\kappa)$-Amenability$_{\mathbf{j}}$ Axiom of Infinity, let $\gamma = j(\kappa)$. For $a \in [\gamma]^{<\omega}$, define E_a from j by

$$E_a = \{X \in P(\kappa^{|a|}) : a \in j(X)\}.$$

The important point is that for each a, $j \restriction P(\kappa^{|a|})$ is a set in V. Therefore, by the usual arguments, E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$—defined in V—and we can define in V the usual (κ, γ)-extender E derived from j by

$$E = \langle E_a : a \in [\gamma]^{<\omega} \rangle.$$

The usual arguments show that if $j_E : V \to M_E$ is the resulting canonical embedding, the critical point of j_E is κ, $j_E(\kappa) = \gamma = j(\kappa)$, and $V_\gamma \subseteq M_E$. (See [6] and [20] for details.) Once we know κ

is superstrong, it follows that $\{\alpha < \kappa : \alpha \text{ is superstrong}\}$ belongs to the normal measure derived from j.

1.8 Restriction To $P(\kappa)$ Theorem. *The theory* ZFC+BTEE+$P(\kappa)$*-Amenability*$_{\mathbf{j}}$ *Axiom proves that κ is the κth superstrong.*

The best known upper bound for this theory is a 2-huge cardinal [7]. We conclude this quick survey of intermediate axioms with the following:

> **Huge Amenability$_n$ Axiom of Infinity**: The restriction $\mathbf{j} \restriction P(P(\mathbf{j}^n(\kappa)))$ is a set.

Note that we have one such axiom for each particular natural number n. Whenever ZFC + BTEE is supplemented with Huge Amenability$_n$, the critical point κ must be the κth n-huge, and more:

1.9 Huge Amenability Theorem [7]**.** *Bounds for* ZFC + BTEE + Huge Amenability$_n$ *are given by the following, for each particular n:*

(1) *The theory* ZFC+BTEE+Huge Amenability$_n$ *proves κ is n-huge and admits a normal measure that contains the set of n-huge cardinals below κ.*

(2) *Assuming an $n+2$-huge cardinal κ, there is a transitive M and an elementary embedding $j : M \to M$ such that $\langle M, \in, j\rangle \models$* ZFC + BTEE + Huge Amenability$_n$.

Each of the extensions of ZFC + BTEE just considered here asserts the existence of an elementary embedding $j : V \to V$ with certain additional properties. Any of these could be considered candidates for a foundational extension of ZFC. Still, our aim has been to strengthen the preservation properties of the Lawvere functor given at the beginning to the furthest possible extent. Our way of doing this is to introduce as additional axioms to ZFC + BTEE *all* instances of Separation for $\mathbf{j}$-formulas (denoted Separation$_{\mathbf{j}}$). The theory BTEE + Separation$_{\mathbf{j}}$ is known as the *Wholeness Axiom* or WA. A model of the theory ZFC+WA provides us with a very strong structure-preserving map of V to itself. We first establish that, relative to some of the very strongest large cardinals known, ZFC + WA is not inconsistent:

1.10 Consistency Theorem [4]**.** *If there is an I_3 embedding $i : V_\lambda \to V_\lambda$, then there is a transitive model of* ZFC + WA.

In this case, the model is $\langle V_\lambda, \in, i\rangle$. The following theorem summarizes some of the main results concerning the theory ZFC + WA; outlines of proofs will be given in Part 2. We begin with the following meta-definition (which, as we discuss below, may be expressed formally in the language of ZFC + WA): A cardinal κ is a WA-*cardinal* if there is a WA-embedding $\mathbf{j} : V \to V$ having critical point κ.

1.11 The Wholeness Axiom Theorem [4],[7]**.** *The following are characteristics of the theory* ZFC + WA*:*

(1) *Assume* ZFC + WA *and that* $\mathbf{j} : V \to V$ *is the* WA*-embedding, with critical point* κ*. Then* κ *is super-*n*-huge for every* n*; moreover, there is a proper class of cardinals that are super-*n*-huge for every* n*.*

(2) ZFC + WA *is indestructible under set forcing.*

(3) *The only "natural" inner model of* ZFC + WA*, if there is one at all, is* V *itself.*

(4) *The critical sequence* $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}(\mathbf{j}(\kappa)), \ldots \rangle$ *forms a* $\mathbf{j}$*-class of indiscernibles for* V*. That is, for any* $\in$*-formula* $\phi(x_1, \ldots, x_m)$ *and for any two finite subsequences* $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ *and* $\beta_1 < \beta_2 < \ldots < \beta_m$ *of the critical sequence of* $\mathbf{j}$*,*

$$V \models \phi[\alpha_1, \ldots, \alpha_m] \leftrightarrow \phi[\beta_1, \ldots, \beta_m].$$

(5) *(Self-Replication) If there is a* WA *cardinal* κ*, there are unboundedly many* WA *cardinals in the universe above* κ*.*

Part (1) shows that from WA, we obtain some of the very strongest large cardinals; note that among large cardinal notions that have been studied widely, a cardinal that is super-n-huge for every n is the strongest type known below an I_3 cardinal. Part (2) shows ZFC + WA has the desirable robustness under forcing. The concept of "natural" inner model, mentioned in part (3) of the theorem, arises because of the pathologies that are possible when considering models of the language $\{\in, \mathbf{j}\}$; the natural inner models are those that behave in the expected ways. In Part 2 of the paper, we give definitions and motivation for this concept. Part (3) of the Theorem of course depends on our definition of "natural model"; if the reader finds the intuitive motivation for this definition compelling, then part (3) of the theorem is quite striking: It suggests that a "good choice" for a model of ZFC + WA is V itself; that, in an unexpected way, V is even the *canonical* model, being the only reasonable inner model of the theory.

Part (4) of the theorem highlights the sense of "ultimacy" that accompanies a large cardinal axiom such as WA. It tells us for example (speaking somewhat loosely) that every large cardinal property that holds true of κ must also hold of every $\mathbf{j}^n(\kappa)$ in the critical sequence of $\mathbf{j}$; moreover, since the critical sequence is unbounded in ON (see Part 2 for more on this point), every large cardinal property true of κ holds true for unboundedly many cardinals in the universe.

A slightly stronger fact of this kind is mentioned in Part (5) of the theorem: The large cardinal property of κ that is derived from the WA embedding $\mathbf{j}$ also holds for each member of the critical sequence. We call this the *self-replicating* property of WA cardinals. This property can be formulated within the language $\{\in, \mathbf{j}\}$ as follows: Given that WA holds with embedding $\mathbf{j}$ and critical point κ, then *every* κ_n in the critical sequence of $\mathbf{j}$ is a WA-cardinal, where $\kappa_0 = \kappa$ and $\kappa_n = \mathbf{j}^n(\kappa)$.

The idea here is that the usual iterates by application[7] $\mathbf{j}$, $\mathbf{j}^{[2]} = \mathbf{j}\cdot\mathbf{j}$, $\mathbf{j}^{[7]} = \mathbf{j}\cdot(\mathbf{j}\cdot\mathbf{j}), \ldots$ are definable in ZFC + WA from $\mathbf{j}$ and, in general $(V, \in, \mathbf{j}^{[n+1]})$ is a model of ZFC + WA that witnesses the fact that κ_n is a WA cardinal. Thus the metatheorem can be stated formally as a ZFC + WA schema as follows: For each $\in$-formula $\phi(x_1, x_2, \ldots, x_m)$, ZFC + WA proves each of the following:

$$\forall n \in \omega\, \forall x_1, x_2, \ldots, x_m \left[\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}^{[n+1]}(x_1), \mathbf{j}^{[n+1]}(x_2), \ldots, \mathbf{j}^{[n+1]}(x_m))\right].$$

These consequences together with the fact that $\mathrm{ZFC} + \mathrm{WA} \vdash \forall n \in \omega(\mathrm{crit}(\mathbf{j}^{[n+1]}) = \kappa_n)$ state in a formal way that each term of the critical sequence for $\mathbf{j}$ is a WA-cardinal.

It could be argued that adding Separation$_{\mathbf{j}}$ to ZFC + BTEE is a bit artificial, even though the consequences are quite nice. An alternative to Separation$_{\mathbf{j}}$, which was in fact our first choice in our earliest attempts [8] at studying these axiom systems, is the following:

> **Amenabililty$_{\mathbf{j}}$ Axiom of Infinity**. For every set x, the restriction $\mathbf{j} \restriction x$ is also a set.

In the literature, the theory BTEE + Amenability$_{\mathbf{j}}$ is denoted WA_0; note that Amenability$_{\mathbf{j}}$ is provable from Separation$_{\mathbf{j}}$. It is shown in [7] that $\mathrm{ZFC} + \mathrm{WA}_0$ has the same strong large cardinal consequences as ZFC+WA; that is, part (1) of the Wholeness Axiom Theorem still holds. However, $\mathrm{ZFC} + \mathrm{WA}_0$ is not known to be indestructible under set forcing. This limitation leads us to believe that ZFC + WA is the better choice.

As for the critical point dynamics of an embedding j that comes from WA, part (1) shows that the critical point has nearly the strongest possible large cardinal properties. Such a $j : V \to V$ has a different sort of seed property. We will show in Part 2 of the paper that it *cannot* be the case that every set is expressible as $j(f)(\kappa)$ for $f \in V$. However, j has a much stronger property than is found in the other examples we have considered. We state the result here and outline the proof in Part 2:

1.12 WA **Critical Point Theorem** [4]**.** *Let* $\mathbf{j} : V \to V$ *be a* WA*-embedding with critical point* κ*. Then there is a function* $f : \kappa \to V_\kappa$ *such that for every set* x*, there is an elementary embedding* $i : V_\eta \to V_\zeta$ *satisfying*

[7] Application $\mathbf{j}\cdot\mathbf{j}$ is defined in the usual way: For any $\mathbf{j}$-class $\mathbf{C}$, define $\mathbf{j}\cdot\mathbf{C}$ by

$$\mathbf{j}\cdot\mathbf{C} = \bigcup_{\alpha\in\mathrm{ON}} \mathbf{j}(\mathbf{C}\cap V_\alpha).$$

In ZFC + WA, such definitions make sense since $\mathbf{C} \cap V_\alpha$ is a set. It is easy to see that

$$\mathbf{j}\cdot\mathbf{j} = \bigcup_{\alpha\in\mathrm{ON}} \mathbf{j}(\mathbf{j}\restriction V_\alpha).$$

(1) *κ is the critical point of i;*

(2) *$i(f)(\kappa) = x$.*

Indeed, we may write

$$V = \{i(f)(\kappa) \,|\, i \text{ is an extendible embedding with critical point } \kappa\}$$

The idea behind the WA Critical Point Theorem is that, in the spirit of Laver sequences for supercompact cardinals [24], one can show [3], [4] that if there is an extendible cardinal κ, there is an "extendible" Laver sequence $f : \kappa \to V_\kappa$ (definitions and details are given in Part 2). As the referee points out, the WA Critical Point Theorem, as given here, could be restated more simply by replacing η with $\kappa + 1$ and ζ with $i(\kappa) + 1$. We will give a slightly enhanced version of this theorem in Part 2 for which this simplification will not be possible.

Applying the Axiomatic Framework ZFC + WA **To The Test Question**. In this subsection we review some of the work done in [3], [4], [5], and [17] to address the Test Question ("Which large cardinals admit their own brand of Laver sequence?") in the context of the theory ZFC + WA. We offer this as an example to illustrate how a foundational axiom for large cardinals may contribute to the general program of large cardinal research. Here, we indicate how the use of WA as part of the axiomatic framework not only ensures the existence of the many different kinds of large cardinals that naturally arise, but also becomes an integral part of the solution and even suggests a direction for further refinements of the solution in which WA is replaced with weaker axioms.

We begin with a convenient uniform definition of a broad class of large cardinals that are candidates for admitting some type of Laver sequence. The intuition is that the large cardinals belonging to our class are those that arise as the critical point of some elementary embedding of the form $V_\beta \to M$ where M is a transitive set. Our background theory ZFC + WA ensures the existence of many such classes; as it happens, it also guides our research into the question. Here is a more precise definition of the classes of embeddings that concern us:

Let $\theta(x, y, z, w)$ be a first-order formula (in the language $\{\in\}$) with all free variables displayed. We call θ a *suitable formula* if the following sentence is provable in ZFC:

$$\begin{aligned} \forall x, y, z, w\, [\theta(x, y, z, w) \Longrightarrow \text{"}w \text{ is a transitive set"} \wedge \text{"}z \text{ is an ordinal"} \\ \wedge \text{"}x : V_z \to w \text{ is an elementary embedding with critical point } y\text{"}]. \end{aligned}$$

For each cardinal κ and each suitable $\theta(x, y, z, w)$, let

$$\mathcal{E}_\kappa^\theta = \{(i, M) : \exists \beta\, \theta(i, \kappa, \beta, M)\}.$$

Intuitively, $\mathcal{E}_\kappa^\theta$ consists of a class, defined by θ, of elementary embeddings $V_\beta \to M$; the need to pair up embeddings with their codomains in the definition of $\mathcal{E}_\kappa^\theta$ is a technicality that is explained in [4].

We show in [4] how many of the familiar globally defined large cardinal notions, such as strong, supercompact, extendible, superhuge and super-almost-huge cardinals, can be characterized as classes of embeddings of this kind. Of course, having ZFC + WA as the background theory ensures these and plenty of other such classes exist in the universe.

If such a class is to have any hope of admitting some kind of Laver sequence, it needs to have the additional property of *regularity*, which asserts that for any set x, there is some $i : V_\beta \to M$ in the class $\mathcal{E}$ for which $x \in M$. More precisely:

1.13 Definition. (*Regular Classes*) A class $\mathcal{E}^\theta_\kappa$ is *regular* if

$$\forall\gamma > \kappa\, \exists\beta \geq \gamma\, \exists i\, \exists M\, [\theta(i, \kappa, \beta, M) \wedge i(\kappa) > \gamma \wedge V_\gamma \subseteq M].$$

It is shown in [4] that the classes corresponding to each of the five global large cardinal notions mentioned above are in fact regular classes. We can now define a notion of "Laver sequence" for any class $\mathcal{E}^\theta_\kappa$; it can be shown that if such a class does admit such a Laver sequence, it must in fact be a regular class.

1.14 Definition. (*Laver sequences*) Suppose $\mathcal{E}^\theta_\kappa$ is a class of embeddings, where θ is a suitable formula. A function $g : \kappa \to V_\kappa$ is said to be *$\mathcal{E}^\theta_\kappa$-Laver at κ* if for each set x and each $\lambda > \kappa$ there are $\beta > \lambda$, and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ such that $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$.

We show in [4] that our definition coincides with the usual definitions of Laver sequences; for instance, if θ is a formula that defines a class $\mathcal{E}^\theta_\kappa$ equivalent to supercompactness of κ, then $f : \kappa \to V_\kappa$ is Laver in the usual sense if and only if it is $\mathcal{E}^\theta_\kappa$-Laver.

Now that we have a definition of Laver sequence, we can ask, in the spirit of the Test Question, "Which classes $\mathcal{E}^\theta_\kappa$ admit Laver sequences?"

In [4] we observe that an application of the WA-embedding $\mathbf{j}$ gives a hint about building Laver sequences generally: Let U be the normal ultrafilter over κ derived from $\mathbf{j}$: $U = \{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$. If $f : \kappa \to V_\kappa$ is a Laver sequence (in the usual sense), it follows that

$$\{\alpha < \kappa \mid f \restriction \alpha : \alpha \to V_\alpha \text{ is Laver at } \alpha\} \in U.$$

From this observation, one could attempt to build $f : \kappa \to V_\kappa$ generally by arranging it so $f(\alpha)$ is arbitrary if $f \restriction \alpha$ is Laver at α (and the intention is that this condition holds for "most" cardinals α), whereas $f(\alpha)$ is a witness to the failure of Laverness at α otherwise. This idea can be used to obtain the usual Laver sequences for supercompact and strong cardinals without the use of WA. However, in the presence of the WA-embedding, one shows the construction yields Laver sequences for virtually any globally defined large cardinal notion; for instance: superhuge, super-almost-huge, and extendible cardinals. In fact, we show [4] that whenever class $\mathcal{E}^\theta_\kappa$ is regular and is sufficiently "compatible" with the WA embedding, it must admit a Laver sequence. Here we observe the helpful role played by the embedding $\mathbf{j}$ that is now part of the background axiomatic framework. For completeness, we give a more precise definition of "compatibility."

1.15 Definition. (*Compatibility*) $\mathcal{E}^\theta_\kappa$ is said to be *compatible* with the WA-embedding $\mathbf{j}$ if for each λ, $\kappa < \lambda < \mathbf{j}(\kappa)$, there exist β, i such that $\lambda < \beta < \mathbf{j}(\kappa)$, $i : V_\beta \to M$, $(i, M) \in \mathcal{E}^\theta_\kappa$, and i is compatible with $\mathbf{j} \restriction V_\beta : V_\beta \to N = V_{\mathbf{j}(\beta)}$ in the sense that there is $k : M \to N$ such that $k \circ i = \mathbf{j} \restriction V_\beta$ and $k \restriction V_\lambda \cap M = \mathrm{id}_{V_\lambda \cap M}$.

$$\begin{tikzcd}
V \arrow[r, "\mathbf{j}"] & V \\
V_\beta \arrow[r, "\mathbf{j} \restriction V_\beta"] \arrow[d, "i"'] & V_{\mathbf{j}(\beta)} \\
M \arrow[ur, "k"'] &
\end{tikzcd}$$

To complete the picture, we show in [3] from ZFC + WA that not all regular classes necessarily admit a Laver sequence. Here is a summary of results:

1.16 Generalized Laver Theorem [3], [4]**.** *(*ZFC + WA*)* .

(1) *If $\mathcal{E}^\theta_\kappa$ is a regular class of emebeddings compatible with the* WA-*embedding, then there is an $\mathcal{E}^\theta_\kappa$-Laver sequence.*
(2) *There is a transitive model of* ZFC *in which there is a regular class $\mathcal{E}^\theta_\kappa$ of embeddings that does not admit an $\mathcal{E}^\theta_\kappa$-Laver sequence.*

Our purpose in reviewing these results is to illustrate the role of an axiomatic framework for large cardinals. In this example, the theory ZFC + WA guarantees the existence of the many classes of large cardinals that could admit Laver sequences. It then also plays a direct and natural role in the construction of a general notion of Laver sequence as well as providing a criterion to decide which classes $\mathcal{E}^\theta_\kappa$ admit Laver sequences. A third contribution of WA in this context is that it points the way for further refinements in which results of this kind are obtained under weaker hypotheses than WA. For instance, we obtained in [4] similar results by replacing $\mathbf{j}$ with a class of embeddings of the form $i : V \to N$, modulo several technical conditions. The point is that the axiomatic framework ZFC + WA not only provides in this case an "ontology" of large cardinals in the background but also plays a direct role in formulating a solution to the question and even serves to guide further research beyond the direct use of WA.[8]

[8]Even without a formal theory of large cardinals, the strongest large cardinal notions have already been used in a similar way. For instance, as we mentioned earlier, the first proof of determinacy of Π^1_2 sets was obtained by Martin assuming an I_2 embedding, and the first proof

The Naturalness of WA **As A Foundational Axiom.** Even if we think it is reasonable to admit into the formal universe all the known large cardinals that are used today in research, the question about how to state this acceptance axiomatically would remain. An axiom that simply asserts the existence of certain large cardinals lacks the cogency we might expect to find in a foundational axiom. In addition, as we have pointed out, an optimal extension of ZFC would have other desirable properties, such as indestructibility under set forcing. It would also be more in the spirit of foundations to introduce an axiom that is clearly motivated by our knowledge of the structure of the universe on the basis of ZFC. Certainly the early simple heuristics had this characteristic, using basic principles obtained from Cantor's early vision (and those of others) of the universe as guidelines for enhancing the richness of the universe via large cardinals.

Taking into account issues of this kind, the solution provided by $\mathrm{ZFC} + \mathrm{WA}$ has many advantages. As we have seen, it is strong enough to derive virtually all the large cardinals that arise in research and it is indestructible by set forcing. It also bears the character of the "ultimate" large cardinal axiom because of the self-replicating feature that if there is a WA cardinal κ, there must be unboundedly many such cardinals above κ. Moreover, it has long been recognized [22], [28] that an axiom of the form "there is a nontrivial elementary embedding $j : V \to V$" represents a kind of ultimate large cardinal axiom in another sense: The strongest large cardinals are defined using elementary embeddings of the form $j : V \to M$; and one of the characteristics of the stronger large cardinals is that the image model M of the corresponding embedding(s) tends to include more of V. For instance, a superstrong embedding requires $V_{j(\kappa)} \subseteq M$ and a huge embedding requires ${}^{j(\kappa)}M \subseteq M$. Therefore, in a sense already familiar to large cardinal researchers, some strong but not inconsistent version of an elementary embedding $j : V \to V$ would be the natural culmination of all large cardinal axioms.

We consider "naturalness" of WA from another point of view in this paper. Recall that WA is the last of a series of refinements of a certain form of the Axiom of Infinity — an axiom that we already take to be incontrovertible. That first form of the axiom, obtained from Lawvere's Theorem, already tells us that once we accept the actual infinite into our universe, the universe becomes equipped with a certain map $j : V \to V$ that exhibits strong preservation properties. Our analysis began with this natural and incontrovertible starting point and proceeded by asking, "How can this axiom be optimized?" Moreover, refinements of the axiom always proceeded in the direction

of projective determinacy was obtained by Woodin using an I_0 embedding. Many of the insights in these proofs were refined to obtain the optimal large cardinal hypotheses for these results that are known today. The difference is that the strong large cardinal axioms used to obtain the early results had to be invoked as *ad hoc* and somewhat amazing hypotheses rather than as part of a formal axiomatic system.

of increasing combinatorial richness of the universe, adhering fairly closely to Cantor's *principle of maximum possibility*. Therefore, since the form of the axiom WA, asserting the existence of a certain $j : V \to V$, originates with an axiom of ZFC, and since the theme of refinement is a natural one, we suggest that WA meets the requirement of "naturalness" of a new foundational axiom.

Despite our belief that $\mathrm{ZFC} + \mathrm{WA}$ is a strong candidate for an extension of ZFC that accounts for large cardinals, one can reasonably argue that there is still room for improvement, and also that there are other acceptable alternatives; we consider these points in the next section.

Limitations Of The Theory $\mathrm{ZFC}+\mathrm{WA}$ **And An Alternative Axiom Schema.** The following is a list of legitimate concerns about the theory $\mathrm{ZFC} + \mathrm{WA}$. We make some remarks about each point, and then discuss an alternative to WA, introduced by Woodin, which offers improvements in some of these areas.

1.17 Limitations of $\mathrm{ZFC} + \mathrm{WA}$.

(1) From $\mathrm{ZFC} + \mathrm{WA}$, cardinals that are super-n-huge for every n are derivable, but none of the cardinals given by the axioms I_3-I_0 are derivable, though, as was discussed earlier, some of these have played a significant role in the early proofs of core theorems concerning properties of sets of reals.

(2) $\mathrm{ZFC}+\mathrm{WA}$ is formulated in the expanded language $\{\in, \mathbf{j}\}$. This diminishes its intuitive appeal.

(3) In a $\mathrm{ZFC} + \mathrm{WA}$ universe $(V, \in, \mathbf{j})$, the critical sequence of $\mathbf{j}$ is cofinal in the ordinals. This again detracts from its intuitive appeal since it violates our intuition that the universe is "inaccessible" — an intuition that is formalized by the Replacement Axiom.

Points (1)–(3) seem to the author to be legitimate concerns; we make a first attempt to address them here, though we do not claim to have entirely resolved these issues yet.

For (1), one could invoke Cantor's *reflection principle* to motivate the existence of an I_3 cardinal in the universe, assuming $\mathrm{ZFC}+\mathrm{WA}$ holds true; even this approach, however, does not account for the axioms I_2–I_0.

For (2), a form of this problem has been present as long as the question "Is there a nontrivial elementary embedding $j : V \to V$?" has been around. Since it is not possible to formalize the question in the language of ZFC, even with the use of proper classes (without quantification), it is inevitable that some extended framework must be used to study this question. In Kunen's theorem [22], in which he proved that no such embedding could exist, he worked in a class theory. Doing so forces us to regard j as a class, which then must obey the instances of Separation and Replacement; this additional axiomatic baggage plays a crucial role in the proof of inconsistency. To examine the actual strength of elementarity therefore requires a different context for studying the question. The only way we know to do this is to expand the language in the way that we have. Therefore, though the language $\{\in, \mathbf{j}\}$ may not be appealing, it may be unavoidable if we wish to study elementary embeddings $V \to V$ without the impact of class theory.

To address point (3), we suggest an alternative perspective, arguing that the spirit behind Replacement is not truly being violated in ZFC + WA after all. The original intuition about Replacement was that, because the universe is vast and, in Cantor's words (cf. [15, p. 44]), "beyond mathematical determination," in forming a countable (or longer) sequence of sets, the resulting sequence should not be cofinal in the universe. This intuition continues to be realized in a ZFC + WA universe relative to $\in$-formulas, since Replacement for such formulas continues to hold true in ZFC + WA. Indeed, the only sort of violation of this intuition that could occur is via $\mathbf{j}$-classes. And, of course, the sets that arise in "ordinary" mathematical practice would never require the use of $\mathbf{j}$. Because $\mathbf{j}$ is a very different type of entity, which encodes (via an extendible Laver sequence) all sets in the universe, it may not be too heretical to consider that V could appear different from $\mathbf{j}$'s point of view, and that this point of view is not accessible to inhabitants of V. Analogous situations in set theory are commonly encountered. For instance, consider $\mathbf{L}$'s view of the V-cardinal ω_ω, assuming $0^\#$ exists. Let $\delta = (\omega_\omega)^V$. Since $0^\#$ exists, δ is a Silver indiscernible, hence inaccessible in $\mathbf{L}$. This means that from "inside" the world of $\mathbf{L}$, V_δ appears to be a universe for mathematics, a transitive model of ZFC. The knowledge that there is a cofinal ω-sequence $f : \omega \to V_\delta^{\mathbf{L}}$ is not available to $\mathbf{L}$ inhabitants. This ω-sequence is an altogether different type of entity relative to $V_\delta^{\mathbf{L}}$: though it is a subcollection of $V_\delta^{\mathbf{L}}$, it is neither a set nor a (definable) class relative to this universe. What grants f its special status is the existence of $0^\#$—something that is beyond the comprehension of $\mathbf{L}$. In a similar way, in a ZFC + WA universe, what grants the ω sequence $\kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots$ its special status in V (the status of being a cofinal ω sequence that is not $\in$-definable in V) is the presence of $\mathbf{j}$—transformational dynamics that are outside the comprehension of V. Therefore, any non-Replacement-like phenomena we might find in a ZFC+WA universe remain forever unknown to the inhabitants of V. For the business of the mathematics of sets and $\in$-classes, no violation of Replacement can ever occur in a ZFC + WA universe.

To conclude this section, we give a brief discussion of a very strong axiom schema due to Woodin which could be used in place of WA.[9]

> **Weak Reinhardt Axiom (WRA)**. There is a $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ such that $V_\kappa \prec V$.

Woodin calls the critical point κ in this axiom a *weak Reinhardt cardinal*. The notation

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[9] We hasten to point out that Woodin has not (as far as we know) proposed his axiom as the "right" or "best" strong large cardinal axiom or as a means of filling the need for an optimal extension of ZFC to account for large cardinals. His axiom was simply presented as an example of one of the strongest possible axioms from which large cardinals can be derived.

WR(κ) means that κ is a weak Reinhardt cardinal. The axiom WRA is technically motivated in the following ways: Kunen's inconsistency proof shows that (in ZFC) there is no nontrivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. Kunen observed that no inconsistency appears to be derivable from a nontrivial $j : V_{\lambda+1} \to V_{\lambda+1}$, i.e., an I_1 embedding. As a foundational axiom, however, an I_1 embedding is unsatisfactory since the strongest large cardinals cannot be proven from this axiom to exist outright in V, but only in the model $V_{\lambda+1}$ (or in V_λ, which is a model of ZFC). Intuitively, the difficulty is that the model V_λ does not sufficiently reflect the truth of V. This shortcoming is remedied by requiring $V_\kappa \prec V$. This extra requirement greatly enhances the strength of the axiom; for completeness, we include a proof of the following proposition:

1.18 Proposition. *From the theory* ZFC + WRA*, one can prove that there are arbitrarily large* I_1 *cardinals in the universe, and also arbitrarily large cardinals that are super-n-huge for every n.*

Proof. Let κ be a weak Reinhardt cardinal. For the first part, let $\phi(x)$ be the following formula:

$$\phi(x) \equiv \text{“}x \text{ is an ordinal”} \wedge \exists\gamma > x\, \exists\delta > \gamma\, \exists e : V_{\delta+1} \to V_{\delta+1}\, (\mathrm{crit}(e) = \gamma).$$

First we claim that $V_\kappa \models \forall\alpha\, \phi(\alpha)$: Fix $\alpha < \kappa$. Since κ is an I_1 cardinal, $V \models \phi(\alpha)$. Since $V_\kappa \prec V$, $V_\kappa \models \phi(\alpha)$. Since $\alpha < \kappa$ was arbitrary, the claim follows. But now, from the claim and the fact that $V_\kappa \prec V$ again, it follows that $V \models \forall\alpha\, \phi(\alpha)$, and we are done.

For the second part, let $i : V_\lambda \to V_\lambda$ be the restriction of an I_1 embedding to V_λ, where λ is a limit and $\mathrm{crit}(i) = \kappa$. One can argue as in part (1) of the Wholeness Axiom Theorem (given in part 2 of this paper), inside V_λ, to show that the statement “there are arbitrarily large cardinals that are super-n-huge for every n” holds in V_λ. Since $V_\kappa \prec V_\lambda$ (see [4, Proposition 3.12] or [20, Exercise 24.5]), the statement holds in V_κ, and because $V_\kappa \prec V$, the statement holds in V. ∎

Therefore, from ZFC+ WRA, the very strongest largest cardinals, even beyond those derivable from WA, can be accounted for. And it accomplishes this without any kind of violation of Replacement. Also, it has similar critical point dynamics to those of WA, though κ itself does not play the role of “seed” in this case (see below). Nevertheless, since there are extendible cardinals throughout the universe (by Proposition 1.18), one can prove that there are α and $f : \alpha \to V_\alpha$ such that

$$V = \{i(f)(\alpha) \mid i \text{ is an extendible embedding with critical point } \alpha\}.$$

One disadvantage of the strong large cardinal consequences of WRA is that it is difficult to assess whether WRA is consistent; unlike WA, it is not known to be consistent relative to any of the usual very strong large cardinal axioms.

The only well-known large cardinal axiom that is not known to be derivable from WRA is I_0. Thus, even this seemingly all-encompassing axiom is not quite able to account for *all* the known large cardinals.

One could argue that the language in which WRA is formulated is more natural than $\{\in, \mathbf{j}\}$, but note that WRA is not formalizable in ZFC: If it were, let κ be the least weakly Reinhardt cardinal; then V_κ — and therefore V itself — satisfies the sentence "there is no weakly Reinhardt cardinal", which is impossible. WRA can be formulated in the language $\{\in, \mathbf{c}\}$ where $\mathbf{c}$ is a new constant symbol intended to denote the critical point κ of an embedding $j : V_{\lambda+1} \to V_{\lambda+1}$. An infinite collection of axioms could be used to assert formally that $V_\kappa \prec V$.

Weak Reinhardt cardinals do not have the self-replicating property that we found for WA cardinals. Indeed, the existence of two such cardinals is strictly stronger than the existence of just one, as one can see from classical reasoning: If $\kappa < \delta$ are weak Reinhardt cardinals, notice that since $V_\delta \prec V$, we must have $V_\kappa \prec V_\delta$ and also that, in V_δ, there is an I_1 embedding having critical point κ. Therefore, V_δ is a transitive model of $\mathrm{ZFC} + \mathrm{WR}(\kappa)$. (This argument can be formalized using two constant symbols with appropriate axioms.) Therefore, the existence of two weak Reinhardt cardinals implies the existence of a transitive model of one weak Reinhardt cardinal. In a similar vein, from a model of ZFC together with arbitrarily large weak Reinhardt cardinals, one can obtain a model of ZFC in which the weak Reinhardt cardinals are bounded in the universe: Starting with a model (M, E) of arbitrarily large weak Reinhardt cardinals — formalized in some expansion of the language of set theory — one can obtain a model (N, E, κ, δ) (in the language $\{\in, \mathbf{c}_1, \mathbf{c}_2\}$) in which κ and δ are both weak Reinhardt cardinals, $\kappa < \delta$, and for all γ with $\kappa < \gamma < \delta$, γ is not weak Reinhardt. Then, in N, V_δ satisfies $\mathrm{ZFC} + \mathrm{WRA}$ and also "the weak Reinhardt cardinals are bounded in the universe."

The conclusion that we draw from these bits of reasoning is that the existence of a weak Reinhardt cardinal does not imply the existence of others above it. This fact leads us to conjecture that $\mathrm{ZFC} + \mathrm{WRA}$ can be destroyed by set forcing — moreover we conjecture that collapsing κ to a countable ordinal would be enough to destroy WRA. Of course, as Proposition 1.18 shows, no such forcing can alter the fact that I_1 cardinals and cardinals that are super-n-huge for every n must pervade the universe.

The reasoning just given also shows that a weak Reinhardt cardinal typically fails to have most of the strong, globally defined large cardinal properties, like extendibility and superhugeness. Suppose for example that κ is both extendible and weak Reinhardt with I_1 embedding $j : V_{\lambda+1} \to V_{\lambda+1}$. Let δ be an inaccessible above λ with $V_\kappa \prec V_\delta$ (this follows from extendibility). Now V_δ is a transitive model of WRA. Thus, $\mathrm{WR}(\kappa)$ + "κ is extendible" is strictly stronger than $\mathrm{WR}(\kappa)$. (We do not know if the same can be said for supercompactness. It does follow from $\mathrm{WR}(\kappa)$ that κ is *strongly* compact since it must be a measurable limit of supercompacts.) Therefore, as mentioned above, it does not follow from $\mathrm{WR}(\kappa)$ that there is an $f : \kappa \to V_\kappa$ such that $V = \{i(f)(\kappa) \mid i \text{ is an extendible embedding with critical point } \kappa\}$ since the existence of such an f implies κ is extendible.

Perhaps the biggest drawback to WRA in our view is that its motivation (as far as we know) is purely technical: It arises from taking the strongest form of an elementary embedding from a rank to a rank not known to be inconsistent, and ensuring a global effect by requiring $V_\kappa \prec V$. It is not clear from "first principles" why an axiom such as WRA should hold true in the universe. And since WRA is not known to be consistent relative to other very strong (and well-known) large cardinal axioms, it is difficult to be entirely confident about it as a candidate for a new foundational axiom.

We conclude this section with a table comparing relative strengths and weaknesses of WA and WRA.

Property of Axiom Schema	**WA**	**WRA**
Consistency	Consistent relative to an I_3 embedding.	Not known to be consistent relative to other strong, well-known axioms.
Naturalness	Motivated by strengthening preservation properties of a map $j : V \to V$ that is provably equivalent to the Axiom of Infinity. Also, a WA-embedding $j : V \to V$ is a natural culmination of embeddings $j : V \to M$ that define most strong large cardinal notions.	Primarily technically motivated: Weaken inconsistent $j : V_{\lambda+2} \to V_{\lambda+2}$ by lowering $\lambda + 2$ to $\lambda + 1$ and then globalize by requiring $V_\kappa \prec V$.
Large cardinal consequences	Critical point κ is super-n-huge for every n, and there is a proper class of such cardinals.	There is a proper class of I_1 cardinals and cardinals that are super-n-huge for every n. However, though critical point κ is strongly compact, it lacks many other global large cardinal properties, such as extendibility and superhugeness.
Robustness under set forcing	WA is indestructible under set forcing.	WRA is not known to be indestructible under set forcing.
"Inaccessibility" of the universe	In ZFC + WA, the length-ω critical sequence of $\mathbf{j}$ is cofinal in ON. Replacement continues to hold for $\in$-formulas, so the universe for "ordinary mathematics" continues to be inaccessible.	A ZFC + WRA universe is inaccessible, without qualification.
Language in which theory is formalized	$\{\in, \mathbf{j}\}$	$\{\in, \mathbf{c}\}$
"Ultimacy" of the axiom schema	WA cardinals have the self-replicating property: If κ is a WA cardinal, unboundedly many cardinals above κ are also WA cardinals.	WRA implies there is a proper class of I_1 cardinals and of cardinals super-n-huge for every n. But the existence of one WR cardinal does not imply existence of others above it.
Critical point dynamics	Critical point κ generates all sets.	There is α (different from critical point) that generates all sets.

§2. Selected Proofs Of The Main Results

Preliminaries.

In this section we give a concise review of category-theoretic concepts that are used in the paper. We assume the reader knows the definition of a category and the assortment of standard constructions that are done in categories: (finite) products, (finite) coproducts, equalizers, co-equalizers, terminal objects, initial objects, and exponentials. We also assume the reader knows the definitions of *monic*, *epic* and *iso* arrows, and of *functor* and *natural transformation*. See [26] as necessary.

For any category $\mathcal{C}$ and objects a, b in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(a, b)$ is the set of all arrows $a \to b$ in $\mathcal{C}$.

A *diagram* D in a category $\mathcal{C}$ is a collection of $\mathcal{C}$ objects $\{d_i \,|\, i \in I\}$ together with some $\mathcal{C}$ arrows $d_i \to d_j$ between some (or all) pairs of the objects in D. A *cone* for a diagram D consists of a $\mathcal{C}$-object c together with a $\mathcal{C}$-arrow $f_i : c \to d_i$ for each object d_i in D such that the diagram

$$\begin{tikzcd}
d_i \arrow[rr, "k"] & & d_j \\
& c \arrow[ul, "f_i"] \arrow[ur, "f_j"'] &
\end{tikzcd}$$

commutes, whenever k is an arrow in the diagram D. The notation $\{f_i : c \to d_i\}$ signifies a cone for D. The dual concept is a *co-cone* $\{g_i : d_i \to c\}$, consisting of an object c and arrows $g_i : d_i \to c$ for each d_i in D.

A *limit* for a diagram D is a D-cone $\{f_i : c \to d_i\}$ with the property that for any other D-cone $\{f_i' : c' \to d_i\}$, there is exactly one arrow $f : c' \to c$ such that the diagram

$$\begin{tikzcd}
& d_i & \\
c' \arrow[ur, "f_i'"] \arrow[rr, "f"] & & c \arrow[ul, "f_i"']
\end{tikzcd}$$

commutes for every d_i in D.

Dually, a *colimit* for a diagram D is a co-cone $\{g_i : d_i \to c\}$ with the property that for any other co-cone $\{g_i' : d_i \to c'\}$, there is exactly one arrow $g : c \to c'$ such that the diagram

$$\begin{tikzcd}
& d_i \arrow[dl, "g_i"'] \arrow[dr, "g_i'"] & \\
c \arrow[rr, "g"] & & c'
\end{tikzcd}$$

commutes for every d_i in D.

A diagram is *finite* if it has only finitely many objects and finitely many arrows. A limit (or colimit) is *finite* if its diagram is finite.

The constructions of products of finitely many objects, equalizers, and terminal objects are examples of finite limits. The constructions of coproducts of finitely many objects, coequalizers, and initial objects are examples of finite colimits. A category is said to be *cartesian closed* if it has all finite products and has exponentiation.

If D is a diagram in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, $F(D)$ is the $\mathcal{D}$-diagram whose objects are $F(d)$, $d \in D$, and whose arrows are $F(f_i)$ whenever f_i is an arrow in D. F is said to *preserve (finite) limits* if for every (finite) diagram D and limit cone $\{f_i : c \to d_i\}$ for D in $\mathcal{C}$, $\{F(f_i) : F(c) \to F(d_i)\}$ is a limit cone for $F(D)$ in $\mathcal{D}$. The dual notion of *preserving (finite) colimits* is defined similarly.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *left (right) exact* if F preserves all finite limits (colimits). F is *exact* if it is both left exact and right exact.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor and $c \in \mathcal{C}, d \in \mathcal{D}$. A $\mathcal{D}$-arrow $u : d \to F(c)$ is a *universal arrow* if for any $x \in \mathcal{C}$ and any $g : d \to F(x)$ in $\mathcal{D}$, there is a unique $f : c \to x$ in $\mathcal{C}$ such that $g = F(f) \circ u$.

$$\begin{tikzcd} c \arrow[d, "f"] \\ x \end{tikzcd} \qquad\qquad \begin{tikzcd} d \arrow[r, "u"] \arrow[dr, "g"'] & F(c) \arrow[d, "F(f)"] \\ & F(x) \end{tikzcd}$$

Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. Then F is *left adjoint to* G, and we write $F \dashv G$, if there is, for each $c \in \mathcal{C}, d \in \mathcal{D}$, a bijection $\theta_{cd} : \mathrm{Hom}_{\mathcal{D}}(F(c), d) \to \mathrm{Hom}_{\mathcal{C}}(c, G(d))$ that is natural in c and d. In this case $\langle F, G, \theta \rangle$ is said to be an *adjunction*. For each $c \in \mathcal{C}$, let η_c denote $\theta_{c,F(c)}(1_{F(c)})$; that is, η_c is the image under θ of the identity arrow living in $\mathrm{Hom}_{\mathcal{D}}(F(c), F(c))$. One shows that $\eta_c : c \to G(F(c))$ is a universal arrow for each c and that the collection $\{\eta_c \,|\, c \in \mathcal{C}\}$ are the components of a natural tranformation $\eta : 1_{\mathcal{C}} \to G \circ F$. η is called the *unit of the adjunction*.

An adjunction is completely determined by its unit. That is, given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\eta : 1_{\mathcal{C}} \to G \circ F$ such that, for each $c \in \mathcal{C}$, $\eta_c : c \to G(F(c))$ is a universal arrow, then $F \dashv G$.

Whenever $F \dashv G$, F preserves all colimits of $\mathcal{C}$ and G preserves all limits of $\mathcal{D}$.

The *category of sets*, denoted **Set**, has as objects all sets and as arrows all functions between sets. For any category $\mathcal{C}$, the *category of endos from* $\mathcal{C}$, denoted $\mathcal{C}^{\circlearrowleft}$, has as objects all $\mathcal{C}$-arrows $c \to c$. Given $f : c \to c, g : d \to d \in \mathcal{C}^{\circlearrowleft}$, an arrow $\alpha : f \to g$ is a $\mathcal{C}$-arrow $e_\alpha : c \to d$ that makes the following diagram commute:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "e_\alpha"'] & c \arrow[d, "e_\alpha"] \\ d \arrow[r, "g"] & d \end{tikzcd}$$

Suppose $F : \mathcal{C} \to \mathbf{Set}$ is a functor and $c \in \mathcal{C}$. An object $u \in F(c)$ is a *weakly universal element for F* if for each $d \in \mathcal{C}$ and each $y \in F(d)$ there is an $f_d : c \to d$ in $\mathcal{C}$ so that $F(f_d)(u) = y$; more verbosely, F is said to be *weakly represented by c with weakly universal element u*. Moreover, if f_d is unique for each choice of d, then u is a *universal element* for F; again, one also says in this case that F is *represented by c with universal element u*.

From a foundational point of view, one siginficant feature of a (weakly) universal element u for a functor F is that it provides a way of reaching a vast expanse of sets from a single "seed" u. For our purposes, it will be useful to know whether *every* set in **Set** can be reached in this way. We will declare that a functor $F : \mathcal{C} \to \mathbf{Set}$ is *cofinal* if for every $x \in \mathbf{Set}$ there is $c \in \mathcal{C}$ such that $x \in F(c)$. One easily verifies that if u is a weakly universal element for a functor $F : \mathcal{C} \to \mathbf{Set}$, then every set is expressible as $F(f)(u)$ for some arrow f in $\mathcal{C}$ if and only if F is cofinal. (From the category-theoretic point of view, this definition of "cofinal" is rather unnatural (as the referee has pointed out) because it is not preserved by natural transformations. This notion and its (set-theoretic) connection to a universal element has turned out to be conceptually helpful, so we have used it advisedly; see Theorem 2.17.)

Any model (M, E) of ZFC $-$ Infinity can be turned into a cartesian closed category $\bar{M}$ as follows: The objects are the elements of M. Given $a, b \in M$, $a \xrightarrow{f} b$ is an arrow in the category if and only if $M \models$ "$f : a \to b$ is a function". Since, internal to M, the usual set-theoretic product $a \times b$ and exponentiation a^b operations can be carried out, $\bar{M}$ is cartesian closed. For convenience, we will denote this category M instead of $\bar{M}$. We will refer to a model of this kind as *a category of sets*.

Lawvere's Theorem

As we mentioned in Part 1, Lawvere's Theorem is in fact a theorem about cartesian closed categories. For this paper, though, we restrict our attention to categories obtained from models of ZFC $-$ Infinity, as described in the section on Preliminaries. We fix a model V of ZFC $-$ Infinity, which we treat as a category of sets. Recall that the theory ZFC $-$ Infinity $+ \neg$Infinity is *equivalent* to first-order Peano Arithmetic (PA) in the sense that a model of one can be interpreted in a model of the other (see [10] for a discussion); as in PA, we may invoke the principle of induction and also define classes via recursion.

As a preliminary to the proof of Lawvere's Theorem, we need to introduce a category-theoretic construction for the set of natural numbers. Recall that if a category admits a terminal object, then all terminal objects in the category are isomorphic and that, in a category of sets, any singleton is a terminal object. Terminal objects are typically denoted with the numeral 1. A *natural numbers object (NNO)* in a category $\mathcal{C}$ is a triple (X, z, s) where z and s are arrows forming a diagram

$$1 \xrightarrow{z} X \xrightarrow{s} X,$$

with the following universal property: For any diagram $1 \xrightarrow{y} U \xrightarrow{t} U$ in $\mathcal{C}$, there is a unique arrow $X \xrightarrow{h} U$ such that the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[r, "s"] \arrow[d, "h"] & X \arrow[d, "h"] \\
1 \arrow[ur, "z"] \arrow[dr, "y"'] & & \\
 & U \arrow[r, "t"] & U
\end{tikzcd}$$

Clearly, the definition of an NNO at least makes sense in any category that has a terminal object, though not every such category actually has such an object. If each of (U, u, s) and (W, w, t) is an NNO, there is an isomorphism between them; indeed, the unique $h : U \to W$ guaranteed by the universality of NNOs is an iso in the ambient category.

We outline briefly why the assumption that the Axiom of Infinity holds in V implies there is an NNO in V. The Axiom of Infinity implies ω exists, and we can define the usual successor function $\mathrm{succ} : \omega \to \omega : n \mapsto n \cup \{n\}$, as well as the map $z : 1 \to \omega : 0 \mapsto 0$. Now, the assertion that $(\omega, z, \mathrm{succ})$ is an NNO is equivalent to definition by primitive recursion:

$$\begin{tikzcd}
 & \omega \arrow[r, "\mathrm{succ}"] \arrow[d, "h"] & \omega \arrow[d, "h"] \\
1 \arrow[ur, "z"] \arrow[dr, "y"'] & & \\
 & U \arrow[r, "t"] & U
\end{tikzcd} \tag{1}$$

The diagram can be seen as defining a (unique) sequence $h : \omega \to U$ from $t : U \to U$ by primitive recursion:

$$h(0) = y(0)$$
$$h(n+1) = t(h(n)).$$

The commutativity of the triangle in the diagram ($h \circ z = y$) is a statement of the base case of the recursion; the commutativity of the square in the diagram ($h \circ \mathrm{succ} = t \circ h$) is a statement of the induction step of the recursion. Since definition by recursion is provable from the existence of ω, it follows that one can obtain an NNO (namely, $(\omega, z, \mathrm{succ})$) from the Axiom of Infinity, working in ZFC $-$ Infinity.

To prove the converse in ZFC $-$ Infinity — that the existence of an NNO implies the Axiom of Infinity — start with an NNO (X, z, s). We show that X must be infinite. Define by recursion the class sequence $\mathbf{F} = \langle x_0, x_1, x_2, \ldots, x_n, \ldots\rangle$ by

$$x_0 = z(0) = s^0(x_0)$$
$$x_{n+1} = s(x_n) = s^{n+1}(x_0).$$

By Separation, the range of $\mathbf{F}$ is a *set* (being a subclass of the set X). One now uses induction and the universal property of NNOs to show that, for each n, $x_0, x_1, x_2, \ldots, x_n$ are distinct. (For the induction, formally, one uses the formula that defines $\mathbf{F}$ to define the induction $P(x)$: $P(n)$ asserts that the range of $\mathbf{F} \restriction n+1$ has $n+1$ distinct elements.) Assume the result for n, and hence that $x_0, x_1, x_2, \ldots, x_n$ are distinct. Define a set $U = X \cup \{u\}$ where $u \notin X$. Define $y : 1 \to U$ to be z. Define $t : U \to U$ by

$$t(x) = \begin{cases} x_{i+1} & \text{if } x = x_i \text{ and } 0 \le i < n \\ u & \text{if } x = x_n \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Now, letting $t^i(x)$ denote the ith iterate of t at x, and letting $t^0(x) = x$, it follows that for $0 \le i \le n$, $t^i(x_0) = x_i = s^i(x_0)$, but $s^{n+1}(x_0) \ne u = t^{n+1}(x_0)$.

By the universal property of NNOs, there is a unique $h : X \to U$ making (1) commute. To show that $x_0, x_1, \ldots, x_n, x_{n+1}$ are distinct, assume instead that $x_{n+1} = x_i$ for some $i, 0 \le i \le n$; in other words, $s^{n+1}(x_0) = s^i(x_0)$ for some i, $0 \le i \le n$. Tracing through the diagram, we have

$$x_i = t^i(x_0) = t^i(h(x_0)) = h(s^i(x_0)) = h(s^{n+1}(x_0)) = t^{n+1}(h(x_0)) = t^{n+1}(x_0) = u,$$

which is impossible. Therefore, $x_0, x_1, \ldots, x_n, x_{n+1}$ are indeed distinct. This completes the induction and establishes that X must be infinite. We have shown in ZFC $-$ Infinity that the existence of an NNO implies the Axiom of Infinity.

These observations are summarized in the following lemma:

2.1 NNO Lemma. *The theory* ZFC $-$ Infinity *proves that the following statements are equivalent:*

(1) *The Axiom of Infinity.*

(2) *There exists a natural numbers object.*

P. Freyd noticed how to enhance the diagram for an NNO to capture definition by recursion with a parameter. His result holds in any cartesian closed category having an NNO. We will need this result in our outline of the proof of Lawvere's theorem. We state the result just for our restricted context of models of set theory.

2.2 Freyd's Theorem [13]**.** *Suppose A is a set, $x \in A$, $x_0 : A \to A \times \omega$ is a function defined by $x_0(a) = (a, 0)$ and $1_A \times \mathrm{succ} : A \times \omega \to A \times \omega$ is defined by $(1_A \times \mathrm{succ})(a, n) = (a, \mathrm{succ}(n)) = (a, n+1)$. Then, given any functions $f : A \to B$, $g : B \to B$, there is a unique $h : A \times \omega \to B$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & A \times \omega \arrow[r, "1_A \times \mathrm{succ}"] \arrow[dd, "h"] & A \times \omega \arrow[dd, "h"] \\
A \arrow[ur, "x_0"] \arrow[dr, "f"'] & & \\
 & B \arrow[r, "g"] & B
\end{tikzcd}$$

We do not give the proof here. The main idea rests on the connection to the concept of definition by recursion. In the diagram, if we let $B = A$, define $x_0 : A \to A \times \omega$ by $x_0(a) = (a, 0)$, and define $f : A \to A = 1_A$, then the unique $h : A \times \omega \to A$ that is given by the theorem is the unique h given by the Definition By Recursion Theorem for the data

$$h(x, 0) = x$$

$$h(x, n+1) = g(h(x, n)),$$

where $x \in A$ and $g : A \to A$ are given.

We proceed now to the proof of Lawvere's Theorem. For the reader's convenience, we restate the theorem:

Lawvere's Theorem. *Suppose V is a model of* $\mathrm{ZFC} - \mathrm{Infinity}$*. Then the following are equivalent:*

(1) *V satisfies the Axiom of Infinity*

(2) *There is a functor $j : V \to V$ that factors as a composition $G \circ F$ of functors satisfying:*

   (A) *$F \dashv G$ (F is left adjoint to G)*

   (B) *$F : V \to V^{\circlearrowleft}$*

   (C) *$G : V^{\circlearrowleft} \to V$ is the forgetful functor, defined by $G(A \to A) = A$.*

   In particular, F preserves all colimits and G preserves all limits.

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[dr, "F"'] & & V \\
 & V^{\circlearrowleft} \arrow[ur, "G"'] &
\end{tikzcd}$$

Proof of Lawvere's Theorem (Outline). As before, we fix a model V of ZFC $-$ Infinity. Let $G : V^{\circlearrowright} \to V$ denote the *forgetful functor*. If $f : a \to a$, $g : b \to b$ are objects in $V^{\circlearrowright}$, and $\alpha : f \to g$ is an arrow in $V^{\circlearrowright}$, note that $G(f) = a, G(g) = b$, and $G(\alpha) = e_\alpha : a \to b$. Suppose F is a left adjoint to G, defined in V. Let $j = G \circ F$ be the Lawvere functor. We exhibit arrows z, s such that $\langle j(1), z, s\rangle$ is an NNO, from which it follows that the Axiom of Infinity holds (by the NNO Lemma). Moreover, we show that the critical point for j is 1.

First we observe that $j(0) = 0$: Being a left adjoint, F preserves all colimits, and so, in particular, preserves the initial object $\emptyset$ of V. It is easy to verify that the initial object of $V^{\circlearrowright}$ is the empty map $\emptyset \to \emptyset$. It follows, therefore, that $j(0) = G(F(\emptyset)) = G(\emptyset \to \emptyset) = 0$.

Next, we apply j to 1 and show that $j(1)$ is the underlying set of an NNO and, in particular, that $|j(1)| > 1$. $F(1)$ is an object $s : X \to X$ in $V^{\circlearrowright}$ and $G(F(1)) = X$. Let $\eta : 1_V \to G \circ F$ be the unit of the adjunction of F and G, and let $z = \eta_1 : 1 \to G(F(1))$ be its component at the object 1. We use the fact that z is a universal arrow to show that

$$1 \xrightarrow{z} X \xrightarrow{s} X$$

is an NNO: Suppose we are given V-maps $1 \xrightarrow{y} U \xrightarrow{g} U$. Note that s and g are objects in $V^{\circlearrowright}$. By universality of z, there is a unique $\alpha : s \to g$ such that $y = G(\alpha) \circ z = e_\alpha \circ z$.

$$\begin{tikzcd} s \arrow[d, "\alpha"] \\ g \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "y"'] & GF(1) \arrow[d, "G(\alpha)"] \\ & G(g) \end{tikzcd}$$

In other words, e_α is unique such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[r, "s"] \arrow[d, "e_\alpha"] & X \arrow[d, "e_\alpha"] \\ 1 \arrow[ur, "z"] \arrow[dr, "y"'] & & \\ & U \arrow[r, "g"] & U \end{tikzcd}$$

In particular, (X, z, s) is an NNO, as required. As $j(0) = 0$ and $j(1) = X$, which is infinite, we have also established that 1 is the critical point for j.

We now prove the other direction of the theorem. Assuming the truth of the Axiom of Infinity, our proof of the NNO Lemma gives us that $(\omega, x, \mathrm{succ})$ is an NNO, where $x : 1 \to \omega : 0 \mapsto 0$ and succ is the usual successor function. We define a left adjoint $F : V \to V^{\circlearrowleft}$ for the forgetful functor $G : V^{\circlearrowleft} \to V$ as follows: For any set A, $F(A)$ is defined to be the endo $A \times \omega \overset{1_A \times \mathrm{succ}}{\longrightarrow} A \times \omega$. Given a V-arrow $f : A \to B$, let $F(f)$ be the $V^{\circlearrowleft}$-arrow $f \times 1_\omega : 1_A \times \mathrm{succ} \to 1_B \times \mathrm{succ}$ making the following diagram commute:

$$\begin{tikzcd}
A \times \omega \arrow[r, "1_A \times \mathrm{succ}"] \arrow[d, "f \times 1_\omega"] & A \times \omega \arrow[d, "f \times 1_\omega"] \\
B \times \omega \arrow[r, "1_B \times \mathrm{succ}"] & B \times \omega
\end{tikzcd}$$

For each $A \in V$ and each $u_B : B \to B \in V^{\circlearrowleft}$, we obtain a bijection

$$\Theta_{A,u_B} : \mathrm{Hom}_{V^{\circlearrowleft}}(F(A), u_B) \cong \mathrm{Hom}_V(A, G(u_B))$$

natural in A and u_B. It is convenient to define the inverse map Θ^{-1}_{A,u_B}. Given $f : A \to G(u_B) = B$, we seek a unique $\phi : 1_A \times \mathrm{succ} \to u_B$, as in the following:

$$\begin{tikzcd}
A \times \omega \arrow[r, "1_A \times \mathrm{succ}"] \arrow[d, "e_\phi"] & A \times \omega \arrow[d, "e_\phi"] \\
B \arrow[r, "u_B"] & B
\end{tikzcd}$$

Defining $x_0 : A \to A \times \omega$ by $a \mapsto (a, 0)$, we can let e_ϕ be the unique map h guaranteed by Freyd's Theorem. It is routine to verify that this correspondence is a bijection that is natural in A and u_B. ■

As we observed in the proof, the critical point of the Lawvere functor $j : V \to V$ is 1, and $j(1) \cong \omega$. The seed behavior of j is given by the following:

2.3 Lawvere Seed Proposition. *Working in* ZFC, *let $j = G \circ F$ be the Lawvere functor. Then $0 \in G(\omega \overset{\mathrm{succ}}{\longrightarrow} \omega)$ is a universal element for G. In particular, for every endo $B \overset{u_B}{\to} B$ and every $y \in G(u_B) = B$ there is a unique $\alpha : \mathrm{succ} \to u_B$ so that $G(\alpha)(0) = y$. Moreover,*

$$V = \{G(\alpha)(0) \mid \alpha \text{ is an arrow in } V^{\circlearrowleft}\}.$$

Proof. Let $x : 1 \to \omega : 0 \mapsto 0$. Identify the element $y \in G(u_B)$ with the map $1 \to B : 0 \mapsto y$, and identify $F(1)$ with succ via the canonical isomorphism. Then the universality of the component $x = \eta_1 : 1 \to G(F(1))$ of the unit η of the adjunction $F \dashv G$ gives us as before a unique $\alpha : \mathrm{succ} \to u_B$ so that the following triangle commutes:

$$\begin{tikzcd} \mathrm{succ} \arrow[d, "\alpha"] \\ u_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "x"] \arrow[dr, "y"'] & GF(1) \arrow[d, "G(\alpha)"] \\ & G(u_B) \end{tikzcd}$$

The equation $G(\alpha) \circ x = y$ immediately gives the desired conclusion.

We give (in this unpublished version of the paper) more of the details for this argument here. First, we verify that if η is the unit of the adjunction $F \dashv G$, then $\eta_1 : 1 \to G(F(1)) = 1 \times \omega$ is the function z defined by $z(0) = (0, 0)$. Computing $\Theta^{-1}_{1,F(1)}(z)$ as in the proof of Lawvere's Theorem yields a $V^{\circlearrowright}$-map ϕ where e_ϕ is the unique function making Freyd's diagram commute, where (in that diagram) $A = 1, f = z, B = A \times \omega$, and $g = 1_1 \times \mathrm{succ}$. Since setting $h = 1_{1\times\omega}$ does make the diagram commute, by uniqueness h must equal $1_{1\times\omega}$. It follows that $\phi = 1_{F(1)}$ and

$$\eta_1 = \Theta_{1,F(1)}(1_{F(1)}) = z.$$

Next, given $u_B : B \to B$ and $y : 1 \to G(u_B)$, note that since z is a universal arrow, there is a unique $\beta : 1_1 \times \mathrm{succ} \to u_B$ in $V^{\circlearrowright}$ such that $G(\beta) \circ z = y$.

$$\begin{tikzcd} 1_1 \times \mathrm{succ} \arrow[d, "\beta"] \\ u_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "z"] \arrow[dr, "y"'] & GF(1) \arrow[d, "G(\beta)"] \\ & G(u_B) \end{tikzcd}$$

We wish to use the canonical isomorphism between succ and $F(1) = 1_1 \times \mathrm{succ}$ in $V^{\circlearrowright}$ to establish the result.

Define $\psi : \mathrm{succ} \to 1_1 \times \mathrm{succ}$ by letting $e_\psi(n) = (0, n)$; clearly ψ is a $V^{\circlearrowright}$-iso.

$$\begin{tikzcd} \omega \arrow[r, "\mathrm{succ}"] \arrow[d, "e_\psi"] & \omega \arrow[d, "e_\psi"] \\ 1 \times \omega \arrow[r, "1_1 \times \mathrm{succ}"] & 1 \times \omega \end{tikzcd}$$

Notice that $G(\psi) : \omega \to 1 \times \omega$ is the iso e_ψ defined above. Let $i = e_\psi^{-1}$. i is an iso $G(F(1)) \to G(\text{succ})$.

Let $\alpha = \beta \circ \psi$. We wish to show that if $x : 1 \to \omega$ is defined by $x(0) = 0$, then α is unique such that $G(\alpha) \circ x = y$.

$$\begin{tikzcd}
\text{succ} \arrow[rr, "\psi"] \arrow[dr, "\alpha"'] & & 1_1 \times \text{succ} \arrow[dl, "\beta"] \\
& u_B &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "y"'] & GF(1) \arrow[r, "i"] \arrow[d, "G(\beta)"] & G(\text{succ}) = \omega \arrow[dl, "G(\alpha)"] \\
& G(u_B) &
\end{tikzcd}$$

$$\begin{aligned}
G(\alpha) \circ x &= G(\alpha) \circ i \circ z \\
&= G(\beta) \circ G(\psi) \circ i \circ z \\
&= G(\beta) \circ e_\psi \circ i \circ z \\
&= G(\beta) \circ i^{-1} \circ i \circ z \\
&= G(\beta) \circ z \\
&= y.
\end{aligned}$$

It remains to show that α is unique. Suppose $\alpha' : \text{succ} \to u_B$ also has the property that $G(\alpha') \circ x = y$. Let $\beta' = \alpha' \circ \psi^{-1}$. We show that β' satisfies $G(\beta') \circ z = y$, violating uniqueness of β.

$$\begin{aligned}
G(\beta') \circ z &= G(\alpha') \circ G(\psi^{-1}) \circ i^{-1} \circ x \\
&= G(\alpha') \circ e_\psi^{-1} \circ i^{-1} \circ x \\
&= G(\alpha') \circ i \circ i^{-1} \circ x \\
&= G(\alpha') \circ x \\
&= y.
\end{aligned}$$

Therefore, we have shown that, if we define $x : 1 \to \omega$ by $x(0) = 0$, then, for any $u_B : B \to B$ in $V^{\circlearrowleft}$ and any $y : 1 \to G(u_B) = B$, there is a unique $\alpha : \text{succ} \to u_B$ for which $G(\alpha) \circ x = y$.■

The Trnková-Blass Theorem

In this section we review some of the work of Blass [2], showing that there is a measurable cardinal if and only if there is an exact functor from V to V not (naturally) isomorphic to the identity (the reader is also referred to Trnková's work [32] for a somewhat different argument). In some cases we have re-formulated his results to emphasize the framework of generalization that we pursue in this paper; we include additional discussion or proofs for these points of departure when necessary. An example of such a re-formulation is our omission of the naturality condition in the original statement of the Trnková-Blass Theorem. This omission was discussed in Part 1, and the proof of its harmlessness is given in the comments following Theorem 2.12.

We begin with several definitions and results that we state without proof (proofs can be found in [2]). As was done for Lawvere's Theorem, we treat V (a model of ZFC or even ZFC $-$ Infinity) as the category of sets. Recall that a functor is left exact if it preserves all finite limits; equivalently, if it preserves all finite products and all equalizers. Dually, a functor is right exact if it preserves all finite co-limits; equivalently, if it preserves all finite co-products and all co-equalizers.

A helpful observation (which was also made in Part 1) is that, since every set is (isomorphic to) a coproduct of 1's, if a functor preserves 1 (as it must if it preserves all finite limits) and all set-indexed coproducts, it is naturally isomorphic to the identity functor. As is the case with elementary embeddings $V \to M$, a measurable cardinal "tends to" arise as the first break in this type of symmetry: if κ is the least cardinal for which $F(\kappa) \ncong \kappa$ (where F is exact), κ will turn out in many cases to be measurable, and will always be at least as big as the least measurable (which must exist). Conversely, given a measurable cardinal κ, one can define an exact functor F such that κ is the critical point of F.

Notice that, if $F : V \to V$ is an exact functor that is not isomorphic to the identity, then if X is any set for which $F(X) \ncong X$, it follows that X is infinite (since F preserves all finite coproducts of 1). This observation can be established in the theory ZFC$-$Infinity, and allows us to conclude that the Axiom of Infinity is provable from

$$\text{ZFC}-\text{Infinity} + \text{“there is an exact functor from } V \text{ to } V$$
$$\text{not isomorphic to the identity.”}$$

We will often state in hypotheses of theorems that "F is an exact functor"; to be precise, we think of F as a class defined possibly with parameters. A theorem that assumes the existence of such a functor should be viewed as a schema of theorems, one for each class-defining formula F.

Suppose $F : V \to V$ is left exact and $F(\emptyset) = \emptyset$. For any set A and any $a \in F(A)$, define

$$F_{A,a}(X) = \{F(f)(a) \mid f : A \to X\}.$$

Formally, from a formula that defines F, we have a finitistic procedure for obtaining a formula defining $F_{A,a}$, with extra parameters A, a.

Suppose D is a filter on a nonempty set A. If f and g are partial functions on A, we write $f \sim g$ if and only if the set of α for which f and g are both defined and equal at α belongs to D. Define a functor D-Prod by

$$D\text{-Prod}(X) = X^A/D = \{[f] \mid f \text{ is a partial function } A \to X \text{ with domain in } D\}.$$

For any $h : X \to Y$,

$$D\text{-Prod}(h) : D\text{-Prod}(X) \to D\text{-Prod}(Y) : [g] \mapsto [h \circ g].$$

D-Prod is called a *reduced power mod D*. It is straightforward to show that D-Prod is a left exact functor. We state the next several lemmas without proof:

2.4 Subfunctor Lemma. *Suppose $F : V \to V$ is left exact and $F(\emptyset) = \emptyset$. Suppose X is a set.*

(1) *For every A, a, $F_{A,a}(X) \subseteq F(X)$*

(2) *For every $x \in F(X)$, $x \in F_{X,x}(X)$.*

(3) *For every A, a, $F_{A,a} : V \to V$ is left exact.*

(4) *Suppose I is a set. Then F preserves all I-indexed coproducts if and only if $F_{A,a}$ preserves all I-indexed coproducts, for all A, a. In fact, for any collection $\{B_i \mid i \in I\}$,*

$$F(\coprod_i B_i) \cong \coprod_i F(B_i) \quad \textit{if and only if, for each } A, a \quad F_{A,a}(\coprod_i B_i) \cong \coprod_i F_{A,a}(B_i).$$

Parts (1) and (2) of the Lemma imply that, for each X, $F(X)$ is the union over all A, a of $F_{A,a}(X)$. Given a left exact $F : V \to V$, suppose A is a set and $a \in F(A)$. Define a filter D over A by

$$D = \{X \subseteq A \mid a \in \operatorname{ran} F(i_X)\},$$

where $i_X : X \hookrightarrow A$ is the inclusion map. D is the filter *derived from F, A, a*. Notice that this is essentially the familiar definition of a measurable ultrafilter from an elementary embedding: if F happens to preserve the subset relation (more precisely, all inclusion maps), the expression "$a \in \operatorname{ran} F(i_X)$" becomes the more familiar "$a \in F(X)$".

2.5 Equivalence Lemma. *Suppose $F : V \to V$ is left exact with $F(\emptyset) = \emptyset$. Let A be a nonempty set and $a \in F(A)$. Let D be the filter derived from F, A, a. Then*

$$F_{A,a} \textit{ is naturally isomorphic to } D\text{-Prod}.$$

2.6 Reduced Power Filter Lemma. *Suppose D is a filter.*

(1) D-Prod *preserves finite coproducts if and only if D is an ultrafilter.*

(2) *Suppose λ is an infinite cardinal.* D-Prod *preserves λ-indexed coproducts if and only if every intersection of λ elements of D is also in D.*

2.7 Coproduct Coequalizer Lemma. *Suppose $F : V \to V$ is left exact with $F(\emptyset) = \emptyset$. F preserves coequalizers if and only if F preserves ω-indexed coproducts.*

2.8 Nonprincipal Filter Lemma. *Suppose κ, λ are infinite cardinals and κ is uncountable. Suppose D is an ultrafilter over λ.*

(1) *If D-*Prod*$(\kappa) \ncong \kappa$, then D is non-principal.*

(2) *If $\lambda = \kappa$ and D is nonprincipal and κ-complete, then D-*Prod*$(\kappa) \ncong \kappa$.*

Proof. For (1), assume D is principal with generator $\gamma < \lambda$. For each $\alpha < \kappa$, let $c_\alpha : \lambda \to \kappa$ be the constant function with value α. Now observe that for any total $f : \lambda \to \kappa$, if $\alpha = f(\gamma)$, then

$[f] = [c_\alpha]$. Also notice that for every partially defined $g : \lambda \to \kappa$, there is a total $f : \lambda \to \kappa$ such that $[g] = [f]$. Therefore $|D\text{-Prod}(\kappa)| = \kappa$, as required.

For (2), assume D is nonprincipal and suppose $\langle f_\alpha : \alpha < \kappa \rangle$ are such that for each α, $[f_\alpha]$ is an element of D-Prod(κ) and if $\alpha \neq \beta$, $[f_\alpha] \neq [f_\beta]$. Moreover, WLOG, assume each f_α is total. Define $g : \kappa \to \kappa$ by letting $g(\beta)$ be the least element not belonging to $\{f_\alpha(\beta) \,|\, \alpha < \beta\}$. Then for every α and every $\beta > \alpha$, $g(\beta) \neq f_\alpha(\beta)$. Since D is a κ-complete nonprincipal ultrafilter, it follows that $[g]$ is different from each element of $\langle [f_\alpha] : \alpha < \kappa \rangle$. Therefore, $|D\text{-Prod}(\kappa)| > \kappa$; in particular, $D\text{-Prod}(\kappa) \not\cong \kappa$. ∎

2.9 Trnková-Blass Theorem (First Half). *Suppose there is a measurable cardinal κ in V and D is a nonprincipal κ-complete ultrafilter on κ. Then D-*Prod *is an exact functor $V \to V$ that is not isomorphic to the identity. In particular, κ is the critical point of D-*Prod.

Outline of Proof. As we remarked earlier, D-Prod is left exact. Since D is an ultrafilter, D-Prod preserves finite coproducts. Since D is κ-complete and κ is uncountable, D-Prod preserves countably indexed coproducts; hence, by the Coproduct Coequalizer Lemma, D-Prod preserves coequalizers. Therefore, D-Prod is right exact. Morever, by the Reduced Power Filter Lemma and κ-completeness of D, D-Prod preserves all λ-indexed coproducts, for $\lambda < \kappa$, and so $D\text{-Prod}(\lambda) \cong \lambda$ for all such λ. By the Nonprincipal Filter Lemma, $D\text{-Prod}(\kappa) \not\cong \kappa$, and so D-Prod is not isomorphic to the identity; indeed, κ is the critical point of D-Prod. ∎

2.10 Trnková-Blass Theorem (Second Half). *Suppose $F : V \to V$ is an exact class functor not isomorphic to the identity functor. Then $V \models$ "there exists a measurable cardinal". Indeed, if κ is the least cardinal for which $F(\kappa) \not\cong \kappa$, then, in V, κ is at least as big as the least measurable (which must exist). Moreover, if κ is the least element of $\{\alpha : \alpha$ is the critical point of an exact functor $V \to V$ that is not isomorphic to the identity$\}$, then κ is the least measurable cardinal in V.*

Outline of Proof. Let κ be the smallest cardinal for which $F(\kappa) \not\cong \kappa$. By our earlier remarks, κ must be infinite. By the Coproduct Coequalizer Lemma, since F preserves coequalizers, κ must be uncountable. By part (4) of the Subfunctor Lemma, since F does not preserve the coproduct $\kappa = \coprod_\kappa 1$, for some A, a, $F_{A,a}$ also fails to preserve $\coprod_\kappa 1$. Therefore, if D is the filter derived from F, A, a,

$$D\text{-Prod}(\kappa) \cong F_{A,a}(\kappa) \not\cong \kappa. \tag{2.1}$$

Since F preserves coequalizers, so must D-Prod, and by the Reduced Power Filter Lemma, D is ω_1-complete. By the Nonprincipal Filter Lemma, D must also be nonprincipal. It follows that there must exist a measurable cardinal in V (see for example [11, Corollary 2.12]). (Note that the entire argument takes place inside V since both F and $F_{A,a}$ are definable over V.)

Let γ denote the least measurable cardinal in V. It follows that D is γ-complete (see for example [11, Theorem 2.11]). By the Reduced Power Filter Lemma again, it follows that D-Prod preserves ρ-indexed coproducts for all $\rho < \gamma$. Since D-Prod does not preserve $\coprod_\kappa 1$, it follows that $\gamma \leq \kappa$.

Finally, let us suppose that κ happens to be the least element of $\{\alpha : \alpha$ is the critical point of an exact functor $V \to V$ that is not isomorphic to the identity$\}$. We show that $\kappa = \gamma$. It suffices to show that $\kappa \leq \gamma$. Let U be a nonprincipal γ-complete ultrafilter on γ. By the proof of the First Half of the Trnková-Blass Theorem, U-Prod is an exact functor $V \to V$ with $U\text{-Prod}(\gamma) \ncong \gamma$. By leastness of κ, $\kappa \leq \gamma$, as required. ∎

We remark here that the definition of κ as the least element of the set $S = \{\alpha : \alpha$ is the critical point of an exact functor $V \to V$ that is not isomorphic to the identity$\}$ is formally expressible in ZFC. Consider the following set, defined in ZFC:

$$\begin{aligned} T = \{\alpha : \exists A, D\, (&D \text{ is an } \omega_1\text{-complete nonprincipal ultrafilter} \\ &\text{on } A \text{ and } \alpha \text{ is least such that } D\text{-Prod}(\alpha) \ncong \alpha)\}. \end{aligned}$$

We show κ is the least element of T. To begin, we let κ_0 denote the least element of T with witnesses A, D, so that $D\text{-Prod}(\kappa_0) \ncong \kappa_0$. Since $D\text{-Prod} : V \to V$ is exact, $\kappa \leq \kappa_0$. Conversely, let $F : V \to V$ be an exact functor witnessing membership of κ in S. Since $F(\kappa) \ncong \kappa$, then, as we argued before, there must be A, a such that $F_{A,a}(\kappa) \ncong \kappa$ and a corresponding D-Prod such that D is ω_1-complete and $D\text{-Prod}(\kappa) \ncong \kappa$. Therefore, $\kappa_0 \leq \kappa$. We have shown κ is the least element of T.

We also mention here that the proof of Trnková-Blass Theorem (Second Half) makes essential use of the fact that F is a class over V. In order for the derived filter D to be a *set* in V, F must be given by a formula. Later, we will discuss other functors that are not definable over V and it will be clear that the analogue to D will not provide us with a measurable cardinal in V. (A rough example along these lines that can be given at this point is an elementary embedding $j : \mathbf{L} \to \mathbf{L}$ with critical point κ, where $\mathbf{L}$ is the constructible universe and j is given by the existence of $0^\sharp$. Though one can define (in V) the subcollection $D = \{X \in P^{\mathbf{L}}(\kappa) \mid \kappa \in j(X)\}$ of $\mathbf{L}$, D is not a *set* in $\mathbf{L}$ precisely because it is defined in terms of the external embedding j.)

In the proof of Trnková-Blass Theorem (Second Half), one might hope to prove that the critical point of *any* exact functor $V \to V$ is measurable. The obvious strategy to accomplish this is to observe, as in the proof, that, given an exact $F : V \to V$, not isomorphic to the identity, there is, over some A, an ω_1-complete nonprincipal ultrafilter D for which $D\text{-Prod}(\kappa) \ncong \kappa$ (where κ is the critical point of F). By leastness of κ, we have that F preserves λ-indexed coproducts of 1 for all uncountable $\lambda < \kappa$. The natural hope is that F preserves *all* λ-indexed coproducts (in which case, D would be κ-complete), but this is not generally the case, as A. Blass pointed out to the author. We formulate the issue as an open question:

2.11 Open Question. Suppose $F : V \to V$ is exact and not isomorphic to the identity. Without assuming other conditions on F, is it necessarily true that the critical point of F is a large cardinal?

As we mentioned in Part 1, our version of the Trnková-Blass Theorem follows easily from the original theorems which referred to "natural isomorphism" instead of merely "isomorphism" in the theorem statement. We prove this now.

2.12 Original Trnková-Blass Theorem [2], [32]. There is an exact functor $V \to V$ not naturally isomorphic to the identity if and only if there is a measurable cardinal.

Assume the truth of Original Trnková-Blass Theorem; we show how the version used in this paper follows directly from the original formulation. Suppose there is an exact functor $F : V \to V$ not isomorphic to the identity. Then F is not *naturally* isomorphic to the identity, and so by the "Original" version, there is a measurable cardinal. Conversely, if there is a measurable cardinal, then the proof given above of the "First Half" of the Trnková-Blass Theorem is identical to the original proof, which establishes the existence of an exact functor not *naturally isomorphic* to the identity. The proof in fact shows that if D is a κ-complete nonprincipal ultrafilter over a measurable cardinal κ, D-Prod $: V \to V$ is an exact functor with $D\text{-Prod}(\kappa) \ncong \kappa$, which demonstrates the existence of an exact functor not *isomorphic* to the identity, as required.

As our proof of the Trnková-Blass Theorem indicates, it is often the case that the critical point of an exact functor $j : V \to V$ that is not isomorphic to the identity is a measurable cardinal κ. We show, however, that such a functor is unable to generate all sets in V with seed κ; in other words, $V \neq \{j(f)(\kappa) \,|\, \mathrm{dom}\ f = \kappa\}$. The reason is summarized in the following Proposition: First, if such a j does have a weakly universal element at all, j must be naturally isomorphic to one of the reduced product functors D-Prod. Also, whenever D is ω_1 complete, the class $\bigcup_X D\text{-Prod}(X)$ is the usual well-founded ultrapower V^κ/D whose transitive collapse M does not contain D. In particular, when D is normal over κ, if we identify the ultrapower with its transitive collapse in the usual way, κ itself is the seed, and we have $D \neq j(f)(\kappa)$ for any f for which $\mathrm{dom}\ f = \kappa$.

2.13 Restricted Seed Proposition. *Suppose $j : V \to V$ is an exact functor not isomorphic to the identity.*

(1) If j is naturally isomorphic to a reduced power D-Prod, then j admits a weakly universal element.

(2) If j admits a weakly universal element, then j is naturally isomorphic to a reduced power.

(3) If D is a normal measure on κ, $\kappa \in D\text{-Prod}(\kappa)$ is weakly universal for D-Prod (via the usual identification of $D\text{-Prod}(\kappa)$ with its transitive collapse), but for some set Y, $Y \neq D\text{-Prod}(f)(\kappa)$ for any f.

Proof of (1) We show that for any reduced power D-Prod, the element $a = [\mathrm{id}] \in D\text{-Prod}(A)$ (where $A = \bigcup D$) is weakly universal for D-Prod: Suppose $[f] \in D\text{-Prod}(X) = X^A/D$. Without loss of generality, assume $f : A \to X$ is total; but now f is a candidate to witness weak universality: We must verify that D-Prod$(f)([\mathrm{id}]) = [f]$; but this follows immediately from the definition of D-Prod on functions.

Proof of (2) Suppose $F = j : V \to V$ is exact, not isomorphic to the identity, and has a weakly universal element $a \in F(A)$. We show that $F = F_{A,a}$. By the Equivalence Lemma, this suffices to prove (2). Suppose $x \in F(X)$. By weak universality, there is $f : A \to X$ such that $F(f)(a) = x$. By definition, then, $x \in F_{A,a}(X)$. Conversely, suppose $F(f)(a) \in F_{A,a}(X)$. Clearly, since $F(f); F(A) \to F(X)$, $F(f)(a) \in F(X)$.

Proof of (3) We have seen that [id] is weakly universal for D-Prod in general. If D is a normal measure, we have that each of the ultrapowers D-Prod(X) is well-founded and that κ is represented by id in D-Prod(κ); so, identifying ultrapowers with their transitive collapse in the usual way, it follows that $\kappa \in D$-Prod(κ). Therefore $\kappa \in D$-Prod(κ) is weakly universal for D-Prod. As usual (see [20, Proposition 5.7] for example), if M denotes the transitive collapse of $V^\kappa/D = \bigcup_X D$-Prod(X), then $D \notin M$. So, via our identification, D belongs to no D-Prod(X), whence $D \neq D$-Prod$(f)(\kappa)$ for any f.∎

The Wholeness Axiom

As we discussed in Part 1, as we attempt to strengthen the Trnková-Blass functor j to an elementary embedding, it is helpful to work in the context of the language $\{\in, \mathbf{j}\}$ and the theory $\mathrm{ZFC} + \mathrm{BTEE}$. We recall the axioms of BTEE:

$(1)_\phi$ (*Elementarity Schema for* $\in$*-formulas*). Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula,

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right);$$

(2) (*Critical Point*). "There is a least ordinal moved by $\mathbf{j}$".

In [7], we showed that a transitive model for $\mathrm{ZFC} + \mathrm{BTEE}$ can be obtained from an ω-Erdös cardinal (which is weaker than $0^\#$). By adding to BTEE the axiom schema $\mathrm{Separation}_{\mathbf{j}}$ we obtain the Wholeness Axiom or WA.

$(3)_\phi$ (*Separation Schema for* $\mathbf{j}$*-formulas*). Each instance of the usual Separation schema involving ϕ is an axiom (where ϕ is a $\mathbf{j}$-formula).

Different kinds of models of $\mathrm{ZFC}+\mathrm{BTEE}$ are possible, depending on one's assumptions about the surrounding universe. To avoid pathologies, it is helpful to identify a class of *natural models*. A canonical example of a natural model arises from Kunen's classical result that the existence of $0^\#$ is equivalent to the existence of a nontrivial elementary embedding $j : \mathbf{L} \to \mathbf{L}$. The important point for us is that the j in Kunen's result must be definable in V. In particular, if we are given a $j : \mathbf{L} \to \mathbf{L}$, in order to carry out a proof (see [19]) that $0^\#$ exists, j must be sufficiently definable in V. Indeed, in [7] we give an example of a model M for which there is an elementary embedding $i : \mathbf{L}^M \to \mathbf{L}^M$ and $0^\#$ does not exist. The point of the example is that i does not satisfy the instances of Separation and Replacement necessary for the standard argument to go through.

The Kunen embedding can be represented in our present context (though it cannot be formalized in ZFC) as a model $\langle \mathbf{L}, \in, j\rangle$ of $\mathrm{ZFC} + \mathrm{BTEE}$ where j is definable in V. In [7] we call such a model *sharp-like*; our intention here is that the sharp-like models (models (M, E, i) of $\mathrm{ZFC}+\mathrm{BTEE}$ for which i is definable in V) should be included among the *natural models* for $\mathrm{ZFC} + \mathrm{BTEE}$.

We would also like to declare that any model of $\mathrm{ZFC} + \mathrm{BTEE}$ of the form $\langle V, \in, j\rangle$ is natural. Certainly such a model is not sharp-like because the embedding $j : V \to V$ would be required to be definable in V and hence inconsistent. Nevertheless, our motivation for viewing this type of model as natural also comes from our canonical example $j : \mathbf{L} \to \mathbf{L}$. To see why, suppose we attempt to mimic a standard proof that such an embedding implies the existence of $0^\#$; that is, suppose we attempt to prove in a standard way that

if $\langle \mathbf{L}, \in, j\rangle$ *is a natural model of* $\mathrm{ZFC} + \mathrm{BTEE}$*, then* $0^\#$ *exists.*

If we restrict ourselves to just *sharp-like* models (so that the only natural models are the sharp-like models), then, at the outset, we must assume that $V \neq \mathbf{L}$ (since $\langle \mathbf{L}, \in, j\rangle$ is supposed to be sharp-like). Although "$V \neq \mathbf{L}$" does indeed follow from Kunen's result, it is unnecessarily restrictive to *assume* this as part of the hypothesis. A more faithful representation of Kunen's result would therefore leave open the possibility that $V = \mathbf{L}$ and then let the proof itself demonstrate that this is not possible. (The fact that $V \neq \mathbf{L}$ falls out of the proof is easy to see: Given $j : \mathbf{L} \to \mathbf{L}$ with critical point κ, form in V the $\mathbf{L}$-ultrafilter $D = \{X \in P^{\mathbf{L}}(\kappa) \mid \kappa \in j(X)\}$. Then $\mathbf{L}$ is equal to the transitive collapse — formed in V — of the ultrapower $\mathbf{L}^{\kappa}/D$, which, for the usual reason, cannot contain D as an element (the ultrapower embedding $j_D : \mathbf{L} \to \mathbf{L}$ plays a central role in the rest of the proof). Therefore $V \neq \mathbf{L}$.) These considerations lead to the following definition of *natural model* and (we believe) a faithful representation of Kunen's result in our present context:

2.14 Definition (*Natural models*). A model $\langle M, E, j\rangle$ of ZFC + BTEE is a *natural model* of ZFC + BTEE if either $j : M \to M$ is definable in V or $M = V$ and E is the standard membership relation $\in$.

2.15 Kunen's $0^{\#}$ Theorem. *If there is a natural model of* ZFC + BTEE *of the form* $\langle \mathbf{L}, \in, j\rangle$, *then* $0^{\#}$ *exists.*

Although the theory ZFC + WA seems dangerously close to being an inconsistent theory via Kunen's inconsistency result, as we remarked in Part 1, this inconsistency argument does not apply to WA because WA does not include any instances of Replacement for $\mathbf{j}$-formulas. Confidence in the likelihood of consistency of ZFC + WA is further increased by the fact that, as we mentioned before, any I_3 embedding $i : V_\lambda \to V_\lambda$ gives rise to a model $\langle V, \in, i\rangle$ of ZFC + WA.

As a preliminary to the main theorem about WA, we list some known consequences of the theory ZFC + WA.

2.16 Wholeness Lemma [4]**.** *Assume* ZFC + WA*, where* $\mathbf{j} : V \to V$ *is the* WA*-embedding, with critical point* κ.

(1) *For every ordinal* α*, there exists* $n \in \omega$ *such that* $\mathbf{j}^n(\kappa) \geq \alpha$.

(2) *For every* $n \in \omega$,

$$V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec \ldots \prec V_{\mathbf{j}^n(\kappa)}$$

(3) *(Metatheorem)*

$$V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec V_{\mathbf{j}^2(\kappa)} \prec \ldots \prec V$$

We now restate and outline a proof of the Wholeness Axiom Theorem.

Wholeness Axiom Theorem [4], [7], [9]**.** *The following are characteristics of the theory* ZFC + WA*:*

(1) *Assume* ZFC + WA *and that* $\mathbf{j} : V \to V$ *is the* WA*-embedding, with critical point* κ*. Then* κ *is super-*n*-huge for every* n*; moreover, there is a proper class of cardinals that are super-*n*-huge for every* n*.*

(2) ZFC + WA *is indestructible under set forcing.*

(3) *The only "natural" inner model of* ZFC + WA*, if there is one at all, is* V *itself.*

(4) *The critical sequence* $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}(\mathbf{j}(\kappa)), \ldots \rangle$ *forms a* $\mathbf{j}$*-class of indiscernibles for* V*, relative to* $\in$*-formulas. That is, for any* $\in$*-formula* $\phi(x_1, \ldots, x_m)$ *and for any two finite subsequences* $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ *and* $\beta_1 < \beta_2 < \ldots < \beta_m$ *of the critical sequence of* $\mathbf{j}$*,*

$$V \models \phi[\alpha_1, \ldots, \alpha_m] \leftrightarrow \phi[\beta_1, \ldots, \beta_m].$$

(5) *(Self-Replication) If there is a* WA *cardinal* κ*, there are unboundedly many* WA *cardinals in the universe above* κ*.*

Outline of Proof of (1) Let $\kappa_0 = \kappa$ and for each $n \geq 1$, let $\kappa_n = \mathbf{j}^n(\kappa)$. We first verify that κ is n-huge for every n. But this is easy since, for each n, the normal ultrafilter over $P(\mathbf{j}^n(\kappa))$ derived from $\mathbf{j}$ witnesses that κ is n-huge.

Now, to prove super-n-hugeness for every n, it suffices to show that for all $m, n \in \omega$, κ is n-huge with κ_m targets. We will first show that there is a stationary subset S_1 of $\mathbf{j}(\kappa)$ each of whose elements is the target of an n-huge embedding with critical point κ; then we apply a suitable elementary embedding repeatedly to S_1 to show that similar stationary sets exist below each κ_m.

Let $D = \{X \subseteq \mathbf{j}(\kappa) : \mathbf{j}(\kappa) \in \mathbf{j} \cdot \mathbf{j}(X)\}$. D is a normal ultrafilter over $\mathbf{j}(\kappa)$. Let $S_1 = \big\{ \alpha < \mathbf{j}(\kappa) :$ α is a target of some n-huge embedding having critical point $\kappa \big\}$. Then $S_1 \in D$ since $\mathbf{j}(\kappa)$ is a target of an n-huge embedding having critical point κ, as we just showed. Hence, S_1 is stationary.

Now for each $m > 0$, inductively define

$$S_{m+1} = \mathbf{j} \cdot \mathbf{j}(S_m).$$

By elementarity, S_m is a stationary subset of κ_m each of whose elements is a target of an n-huge embedding with critical point κ.

Finally, to see that there is a proper class of super-n-huge cardinals, apply the Wholeness Lemma. ∎

Outline of Proof of (2) The proof of (2) is like the more easily proven fact that, whenever $i : V_\lambda \to V_\lambda$ is an I_3 embedding with critical point κ, the statement $\exists \kappa \, \mathrm{I}_3(\kappa)$ is preserved by any notion of forcing that belongs to V_λ. To see this, first note that if the notion of forcing P is an element of V_κ (in the ground model V), j can be lifted to $\hat{j} : V[G]_\lambda \to V[G]_\lambda$ in the standard way, by defining $\hat{j}(\dot{x}_G) = j(\dot{x})_G$. Next, if the notion of forcing P is an element of $V_\lambda \setminus V_\kappa$, then for some n, $P \in V_{\kappa_n}$, where $\kappa_n = j^n(\kappa)$, and one can obtain another elementary embedding $k : V_\lambda \to V_\lambda$

definable from j and having critical point κ_n (namely, let $k = j^{[n+1]}$, as in the discussion in Part 1 of the Wholeness Axiom Theorem, part (5)). But now, as before, k can be lifted in the standard way since $P \in V_{\kappa_n}$, and so in the extension, $\exists\kappa\, \mathrm{I}_3(\kappa)$ continues to hold.

To prove the parallel result for ZFC + WA, the argument just given has to be "internalized." The hard part is, in proving indestructibility of ZFC+WA by notions of forcing belonging to V_κ, to show that Separation$_{\mathbf{j}}$ still holds in the extension. To do this, boolean values of $\mathbf{j}$-formula need to be defined; the effort to do this amounts to extending the usual forcing methodology to the context of the expanded language $\{\in, \mathbf{j}\}$. Details are worked out in [9]. ■

Proof of (3) Suppose $\langle M, E, j\rangle$ is a model of ZFC + WA and M is a natural inner model; in particular, E is the standard membership relation $\in$. By part (1) of the Wholeness Lemma, the critical sequence $f = \langle \kappa, j(\kappa), j^2(\kappa), \ldots\rangle$ is cofinal in the ordinals of M, and hence in the ordinals of V as well (recall that inner models always include all the ordinals). If $V \neq M$, then by naturalness of the model M, j is definable in V. But now the formula defining j can be used in an instance of ordinary Replacement (for V) to show that the range of f must be a *set* in V, and this contradicts the fact that f is cofinal in the ordinals. It follows therefore that $M = V$. ■

Proof of (4). The key technical lemma for the proof is the following, which is proven in [4]:

> **Lemma.** Suppose $n_1 < n_2 < \ldots < n_s$ and $r > \max(\{n_{m+1} - n_m : 1 \leq m < s\})$. A $\mathbf{j}$-class function $\mathbf{i}$, defined from $\mathbf{j} \cdot \mathbf{j}$, can be specified having the following properties:
> (1) $\mathbf{i} : V \to V$ is an elementary embedding;
> (2) $\mathrm{crit}(\mathbf{i}) > \kappa_{n_1}$;
> (3) for $1 < m \leq s$, $\mathbf{i}(\kappa_{n_m}) = \kappa_{n_1+(m-1)r}$.

The Lemma says that, given sequences $\kappa_{m_1} < \kappa_{m_2} < \ldots < \kappa_{m_s}$ and $\kappa_{n_1} < \kappa_{n_2} < \ldots < \kappa_{n_s}$, one can push these cardinals up high enough with the appropriate choice of $\mathbf{i}$ so that their transformed values agree; indiscernibility follows easily from this observation.■

Proof of (5). The proof was given in the discussion of this Theorem given in Part 1 of the paper. ■

We turn to the critical point dynamics given by a WA-embedding. Part (1) of the Wholeness Axiom Theorem indicates the strength of the critical point of the WA-embedding. We also observe here a significant strengthening in the seed behavior of this type of mapping. We begin by showing that it is *not* the case that all sets in V are expressible as $\mathbf{j}(f)(\kappa)$ for f a function in V, but then demonstrate that $\mathbf{j}$ exhibits a different but much stronger type of seed behavior.

2.17 Restricted Seed Theorem For WA. *Suppose* $\mathbf{j} : V \to V$ *is the* WA*-embedding having critical point* κ. *Then there is a set* x *such that, for any function* f, $\mathbf{j}(f)(\kappa) \neq x$. *Moreover,* κ *is not a weakly universal element for* $\mathbf{j}$.

Proof. The "moreover" clause follows from the rest of the proposition because, as a functor, $\mathbf{j}$ is cofinal (see the Preliminaries section). Let $D = \{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$. We first observe that for every $f : A \to V$ for which $\kappa \in \text{dom } \mathbf{j}(f)$, there is a $g : \kappa \to V$ such that $\mathbf{j}(f)(\kappa) = \mathbf{j}(g)(\kappa)$: Let $S = \{\alpha < \kappa \mid \alpha \in \text{dom } f\}$; note that $S \in D$. Define $g : \kappa \to V$ by

$$g(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in S \\ \emptyset & \text{otherwise} \end{cases}$$

Clearly, $\{\alpha \mid f(\alpha) = g(\alpha)\}$ includes S and therefore belongs to D. Therefore, $\mathbf{j}(f)(\kappa) = \mathbf{j}(g)(\kappa)$.

To complete the proof, it suffices to show that there is $z \in V_{\mathbf{j}(\kappa)}$ such that $\mathbf{j}(g)(\kappa) \neq z$ for any $g : \kappa \to V$. Let $R = \{\mathbf{j}(g)(\kappa) \mid g : \kappa \to V\}$ and $T = \{\mathbf{j}(h)(\kappa) \mid h : \kappa \to V_\kappa\}$. Notice that whenever $g : \kappa \to V$ and $\mathbf{j}(g)(\kappa) \in V_{\mathbf{j}(\kappa)}$, then $\{\alpha \mid \text{rank}(f(\alpha)) < \kappa\} \in D$. It therefore follows that

$$R \cap V_{\mathbf{j}(\kappa)} \subseteq T.$$

By inaccessibility of $\mathbf{j}(\kappa)$, we have

$$|T| \leq |V_\kappa^\kappa| < |V_{\mathbf{j}(\kappa)}|.$$

It follows that some $z \in V_{\mathbf{j}(\kappa)}$ does not belong to R, as required. ∎

Although κ is not a weakly universal element for $\mathbf{j}$, $\mathbf{j}$ exhibits a stronger sort of seed behavior at κ, which arises because of the presence of Laver sequences (of various kinds) defined at κ. We consider perhaps the most natural type of Laver sequence for the present context: an *extendible Laver sequence*, first defined in [4].

2.18 Definition (*Extendible Laver Sequences*). A function $f : \kappa \to V_\kappa$ is a *weakly extendible Laver sequence at* κ if for any set x, there are ordinals η, ζ and an elementary embedding $i : V_\eta \to V_\zeta$ such that

(a) κ is the critical point of i;

(b) $\text{rank}(x) < \eta < i(\kappa) < \zeta$;

(c) $i(f)(\kappa) = x$.

f is an *extendible Laver sequence at* κ if, for any set x and any $\lambda > \kappa$, there are η, ζ and $i : V_\eta \to V_\zeta$ such that (a) and (c) hold as well as

(b′) $\text{rank}(x) < \lambda < \eta < i(\kappa) < \zeta$.

We will consider only *weakly* extendible Laver sequences here; extendible Laver sequences, as defined above, were studied in [4] in a more general context and more directly generalize Laver's original construction. The extra parameter λ forces there to be witnessing extendible embeddings the targets of whose critical point are cofinal in the ordinals. See Definition 1.14.

In our context here, where we consider critical point dynamics for a WA embedding, Laver sequences for *extendible cardinals*—rather than for supercompact or strong cardinals, or other types—are natural to consider because the notion of extendibility follows immediately from WA: one obtains a sufficiently large collection of extendible embeddings (with critical point κ) to meet the requirements in the definition of "extendible cardinal" by considering all restrictions of $\mathbf{j}$ and its iterates to ranks above V_κ.

In [3] (and [7, Appendix]), we show that if κ is extendible, κ admits an extendible Laver sequence. In the presence of a WA-embedding the proof can be simplified considerably; we give this simpler version here to establish existence of a weakly extendible Laver sequence. Essentially the same proof shows that, under WA, κ admits an extendible Laver sequence.

2.19 WA **Seed Theorem** [7]. *Let $\mathbf{j} : V \to V$ be a* WA-*embedding with critical point κ. Then there is a weakly extendible Laver sequence at κ. In particular, $V = \{i(f)\kappa : i$ is an extendible embedding with critical point $\kappa\}$.*

Proof. Let $\phi(g, x)$ be the formula

$$\begin{aligned} &\exists\alpha\,\big[\alpha \text{ is a cardinal} \wedge g : \alpha \to V_\alpha \wedge \\ &\quad \forall\eta\forall\zeta\,\forall i : V_\eta \to V_\zeta\big([\text{“}i \text{ elementary”} \wedge \mathrm{crit}(i) = \alpha \wedge \mathrm{rank}(x) < \eta < i(\alpha) < \zeta] \\ &\qquad\qquad \to i(g)(\alpha) \neq x\big)\big]. \end{aligned}$$

The formula $\phi(g, x)$ says that g is *not* weakly extendible Laver, with witness x. Define $f : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} \emptyset & \text{if } f \restriction \alpha \text{ is weakly extendible Laver at } \alpha \text{ or } \alpha \text{ is not a cardinal} \\ x & \text{otherwise, where } x \text{ satisfies } \phi(f \restriction \alpha, x). \end{cases}$$

Let D be the normal ultrafilter over κ that is derived from $\mathbf{j}$; that is:

$$D = \{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}.$$

D is a set by Separation for $\mathbf{j}$-formulas. Define sets S_1 and S_2 by

$$\begin{aligned} S_1 &= \{\alpha < \kappa \mid f \restriction \alpha \text{ is weakly extendible Laver at } \alpha\} \\ S_2 &= \{\alpha < \kappa \mid \phi(f \restriction \alpha, f(\alpha)\}. \end{aligned}$$

Clearly, $S_1 \cup S_2 \in D$. To complete the proof, it suffices to prove that $S_1 \in D$, and for this, it suffices to show $S_2 \notin D$.

Toward a contradiction, suppose $S_2 \in D$. Then, $\phi(f, \mathbf{j}(f)(\kappa))$ holds in V. Let $x = \mathbf{j}(f)(\kappa)$. Pick η so that $\mathrm{rank}(x) < \eta < \mathbf{j}(\kappa)$. Let $i = \mathbf{j} \restriction V_\eta : V_\eta \to V_\zeta$, where $\zeta = \mathbf{j}(\eta)$. Because of Separation$_{\mathbf{j}}$, i is a set, and is an elementary embedding with critical point κ. Clearly, in V, $\mathrm{rank}(x) < \eta < i(\kappa) < \zeta$ and $i(f)(\kappa) = x$, contradicting the fact that $\phi(f, \mathbf{j}(f)(\kappa))$ holds in V. Therefore $S_2 \notin D$, as required. ∎

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