The Axiom of Infinity And Transformations $V \to V$

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- Ironically, as ZFC (including Cantor's Axiom of Infinity) emerged, so did the first large cardinals, which would eventually be shown to be underivable from ZFC. These can be viewed as "strong axioms of infinity".
- A long-standing question: How strong should the Axiom of Infinity be?

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- (*Generalization*) Some large cardinal properties, like inaccessibility and measurability, are already properties of ω. From Cantor's view, an intuition about the universe is that it is reasonably uniform, so these properties should not be present in just one infinite cardinal. Therefore, inaccessibles and measurables can reasonably be supposed to exist.

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- (*Reflection*) Some large cardinal properties can be argued to hold for the class Ord of ordinals, such as inaccessibility and Mahloness. From Cantor's view, since the "Absolute Infinite" is beyond mathematical determination, Ord couldn't be uniquely defined by a first-order formula, so some actual sets must also have these properties. Therefore, inaccessibles and Mahlos can reasonably be supposed to exist.

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- These days, there is wider acceptance among set theorists of a much larger range of large cardinals, mostly because of the deep relationships that have been discovered between larger large cardinals and sets of reals.
- Whether researchers "believe" in large cardinals or not, large cardinals are used without restriction in research.
- Despite evolving heuristics, there is to date no generally agreed upon extension of ZFC from which the large cardinals that are used in practice can be derived.

Our Objective

Take another look at the Axiom of Infinity and return to simple heuristics to motivate a reasonably natural extension of ZFC from which virtually all large cardinals can be derived.

Lawvere's Theorem (1969) Suppose V is a model of ZFC$-$Infinity. Then the following are equivalent:

(A) V satisfies the Axiom of Infinity

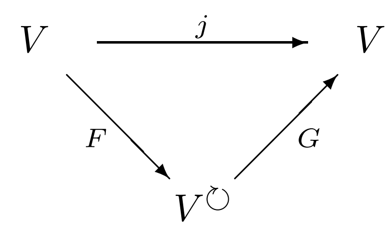
(B) There is a functor $j : V \to V$ that factors as a composition $G \circ F$ of functors satisfying:

   (1) $F \dashv G$ (F is left adjoint to G)

   (2) $F : V \to V^{\circlearrowright}$

   (3) $G : V^{\circlearrowright} \to V$ is the forgetful functor, defined by $G(A \to A) = A$.

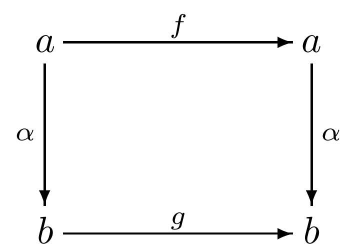
   In particular, F preserves all colimits and G preserves all limits.



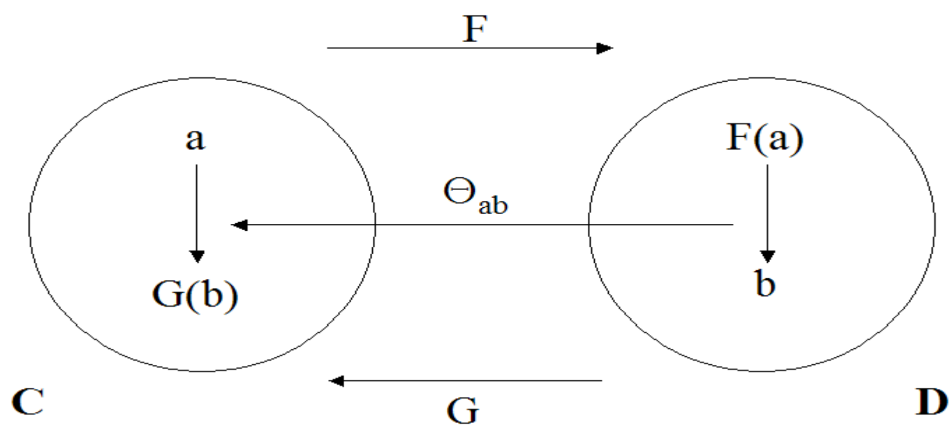
When it exists, we call j the *Lawvere functor*.

Notes About Lawvere's Theorem

- Originally formulated for cartesian-closed categories (not for models of ZFC − Infinity) but our statement follows immediately from Lawvere's.
- The quantification of proper classes can be eliminated in the usual ways, without expanding to a class theory.
- The category $V^{\circlearrowright}$ has as objects all self-maps $f : a \to a$ from V; its arrows are of the form $u_\alpha : f \to g$, defined by commutative diagrams of the following form:



- Adjoints



Definition. (Adjoints) Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. Then F is *left adjoint to* G, and we write $F \dashv G$, if there is, for each $a \in \mathcal{C}, b \in \mathcal{D}$, a bijection $\theta_{ab} : \mathrm{Hom}_{\mathcal{D}}(F(a), b) \to \mathrm{Hom}_{\mathcal{C}}(a, G(b))$ that is natural in a and b. In this case $\langle F, G, \theta \rangle$ is said to be an *adjunction*.

Example. $F : Set \to Vect : X \mapsto U_X$, $G : Vect \to Set : U \to U$. $F \dashv G$ since every map $X \to U$ extends uniquely to $U_X \to U$.

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  - $j(0) = 0$ (F preserves colimits, hence 0; clearly $G(0 \to 0) = 0$)
  - $j(1)$ is infinite (properties of the adjunction allow us to conclude that $j(1)$ is a "natural numbers object", which must be infinite) and so the Axiom of Infinity follows.

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- Our Plan: Formulate stronger and equally natural versions of Lawvere's Theorem as candidates for Strong Axioms of Infinity.
- Why should stronger versions of Lawvere's Theorem be true? They will imply significantly enhanced combinatorial richness in the universe (because they will imply existence of stronger and stronger large cardinals). Cantor's Principle of Maximum Possibility: As much as possible exists.

What To Generalize?

- (*Preservation properties*) Strong preservation properties show up only in the *factors* G, F of j, but not j itself. What if we endow j with the preservation properties of G, F?

First Generalization: The Blass-Trnková Functor

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It is not hard to see that $j \cong \mathrm{id}$. Blass-Trnková showed that something interesting happens if j is required only to preserve *finite* limits and colimits; such a functor is said to be *exact*.

The Blass-Trnková Theorem (1976) Suppose V is the universe of sets (a model of ZFC or even of ZFC – Infinity).

(1) Suppose $j : V \to V$ is a definable (with parameters) exact functor not naturally isomorphic to the identity functor. Then

$$V \models \text{"there exists a measurable cardinal"}$$

(2) Suppose that in V, there is a measurable cardinal κ and D is a non-principal κ-complete ultrafilter on κ. Then there is an exact functor $j : V \to V$, definable in V from D, that is not naturally isomorphic to the identity.

Idea Behind The Proof of the Blass-Trnková Theorem

- Suppose κ is measurable and D is a normal measure on κ. Define a functor $j = j_D : V \to V$ as follows: For any X, Y and any $h : X \to Y$:

$$
\begin{aligned}
j(X) &= X^\kappa/D = \{[f] \mid f : B \to \kappa \text{ and } B \in D\} \\
&\qquad (f \sim g \text{ iff they agree on a set in } D) \\
j(h)([g]) &= [h \circ g]
\end{aligned}
$$

  Blass-Trnková shows that this j is exact, and also that $|j(\kappa)| > \kappa$, whence $j \ncong \mathrm{id}$.

- Assume $j : V \to V$ is an exact functor not naturally isomorphic to the identity and let κ be least such that $j(\kappa) \ncong \kappa$. One shows that κ is measurable in V.

Existence Of A Blass-Trnková Functor As A Strong Axiom of Infinity

Consider as our new (global) Axiom of Infinity the statement

> "There is an exact functor $j : V \to V$ not naturally isomorphic to the identity."

Is this an improvement over the version given to us by ZFC alone:

> "There is a functor $j : V \to V$ where $j = G \circ F$, $F \dashv G$ and G is the forgetful functor $V^{\circlearrowleft} \to V$"

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- *Possible improvements.*
  - ZFC + "there exists a measurable" is not robust under set forcing (e.g. Prikry forcing)
  - We are seeking a *maximal* type of preservation, so we can ask, Can "exactness" of the functor be replaced by something stronger?

Maximizing Preservation Properties in Blass-Trnková

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- (*BTEE*) Work in a language $\{\in, \mathbf{j}\}$ where $\mathbf{j}$ is a symbol for an elementary embedding. Use as axioms the usual ZFC axioms (for $\in$-formulas), an axiom schema Elementarity asserting that $\mathbf{j}$ is an elementary embedding, and an axiom Critical Point asserting that a least ordinal is moved by $\mathbf{j}$. In previous work, these axioms were given the name the *Basic Theory of Elementary Embeddings*, or BTEE.

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  - ZFC + BTEE = ZFC + Elementarity + Critical Point
  - Fact: ZFC + BTEE has consistency strength somewhat less than the existence of $0^{\#}$.

The Wholeness Axiom (WA)

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$(1)_\phi$ (*Elementarity Schema for $\in$-formulas*). Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula,

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right);$$

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$$\text{WA} = \text{BTEE} + \text{Separation}_{\mathbf{j}}$$

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(C) The only "natural" inner model of ZFC + WA, if there is one at all, is V itself.

Note: If in the definition of WA, we replace Separation$_{\mathbf{j}}$ with the following simpler axiom:

$$\text{for every set } x, \mathbf{j} \restriction x \text{ is also a set}$$

the resulting theory is called WA_0. ZFC + WA_0 has the same large cardinal consequences as ZFC + WA, but it is not known to be indestructible by set forcing.

Summary of Results:

Consider as our final (global) Axiom of Infinity the statement

> "There is a nontrivial elementary embedding $j : V \to V$ such that Separation holds relative to all j-formulas"

or possibly,

> "There is a nontrivial elementary embedding $j : V \to V$ such that for all x, $j \restriction x$ is a set."

Compare this with

> "There is an exact functor $j : V \to V$ not naturally isomorphic to the identity"

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- The new version maximizes preservation properties
- The new version proves the existence of virtually all large cardinals
- The resulting theory ZFC + WA is indestructible by set forcing
- ZFC + WA has the same "desirable" consequences that have already been observed in the context of other large cardinals — e.g. Projective Determinacy

Final Thoughts

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We have arrived at WA by

- starting from a version of the Axiom of Infinity, something known to be true and
- strengthening the axiom in a "natural" way, in the direction of maximizing combinatorial richness of the universe, short of inconsistency

Appendix: Critical Point Dynamics

ω arises as the image of the critical point of j in one direction of the proof of Lawvere's Theorem. Based on acquaintance with critical point dynamics at a measurable cardinal, we may track

- The type of infinity (large cardinal) that characterizes the functor j by looking at its critical point (here, the critical point of j is the least ordinal α such that $|j(\alpha)| > \alpha$, relativized to V if necessary)
- "Seed behavior"– to what extent do parameters in the vicinity of the critical point give rise to all sets?

Canonical Example of Critical Point Dynamics

- Canonical Example: $j : V \to V^\kappa/U \cong M$, where M is the transitive collapse of the ultrapower V^κ/U and U is a nonprincipal κ-complete ultrafilter over an infinite cardinal κ.
- The large cardinal strength is determined by the critical point κ of the embedding (namely, the strength of the embedding j is that of a measurable cardinal)
- Every set in M arises from dynamics in the vicinity of κ. In particular, for every $y \in M$, there is a function f with domain κ such that $y = j(f)(\kappa)$; here, the domain of f and the definition of j itself, being defined from an ultrafilter over κ, depend on sets in the vicinity of (and definable from) κ. In Hamkins' terminology, κ is a *seed* via j that generates M. We can write:

$$M = \{j(f)(\kappa) \mid \mathrm{dom}\ f = \kappa\}.$$

Critical Point Dynamics of the Lawvere Functor

- We have seen that the "large cardinal strength" given by the critical point of j is simply ω
- Significant "seed behavior" is found in G but not in j itself. It can be shown that $0 \in G(F(1))$ is a *universal element* for G. In particular

$$V = \{G(f)(0) \mid f \text{ is a } V^{\circlearrowright}\text{-arrow}\}$$

Critical Point Dynamics of the Blass-Trnková Functor

Let $j : V \to V$ be a Blass-Trnkova functor.

- We've seen that the critical point of j is a measurable cardinal.
- Suppose now j is one of the "canonical" functors, defined as before from a measurable κ and a normal measure D on κ. A "seed" for j is given by the element $[\mathrm{id}] \in j(\kappa)$, where $\mathrm{id} : \kappa \to \kappa$ is the identity. (It can be shown that $[\mathrm{id}] \in j(\kappa)$ is a weakly universal element for j.) If

  $$M = \{j(f)([\mathrm{id}]) \mid f : \kappa \to V\},$$

  can show $M \neq V$.

Critical Point Dynamics For A WA **Embedding** $j : V \to V$

It can be shown that $V \neq \{\mathbf{j}(f)(\kappa) \,|\, f \in V\}$. However, we have something even better:

Theorem Let $\mathbf{j} : V \to V$ be a WA-embedding with critical point κ. Then there is a function $f : \kappa \to V_\kappa$ such that for every set x, there is an elementary embedding $i : V_\alpha \to V_\eta$ satisfying
(a) κ is the critical point of i
(b) $i(f)(\kappa) = x$.

In other words, we may write

$$V = \{i(f)(\kappa) \,|\, i \text{ is an extendible embedding}\}$$