Fixed Points of Logs and a Characterization of e

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Abstract. We describe the set of all b for which the equation $x = \log_b x$ has a solution. A consequence of this computation is a new characterization of the transcendental number e.

Introduction

Is there some real b such that the graph of $y = \log_b x$ is tangent to $y = x$, as in the following diagram?

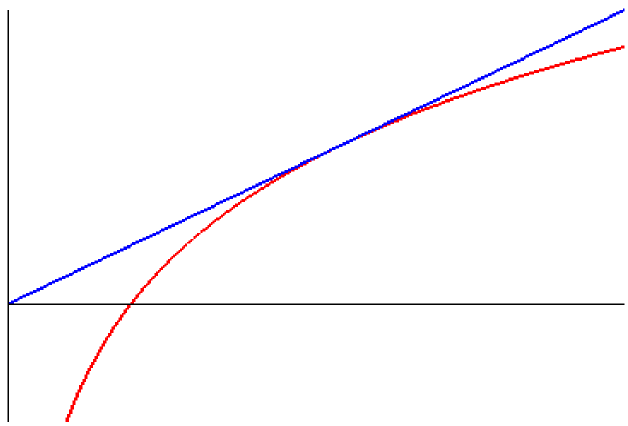

Figure 1. $y = x$ and $y = \log_b x$ for some value of b.

In working with logarithms, we are more accustomed to graphs like the following, where the log function lies strictly below $y = x$.

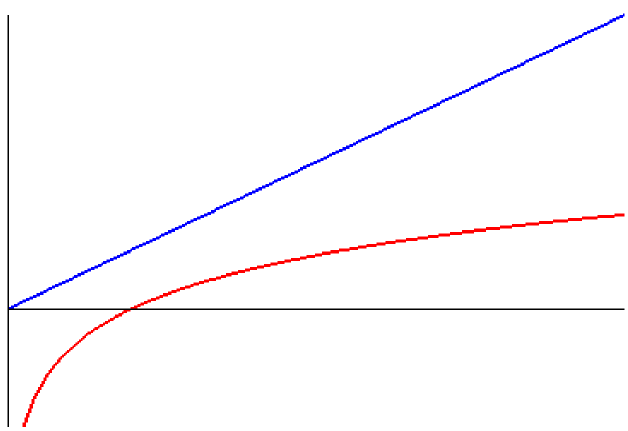

Figure 2. $y = x$ and $y = \log_e x$.

For a real-valued function f defined on the reals, we say x is a *fixed point* of f if $f(x) = x$. In this paper, we study the set of all b for which $\log_b$ has a fixed point; that is, we study the set B where

$$B = \{b > 0 : \log_b x = x \text{ for some } x\}.$$

Is B non-empty? Infinite? Do the elements of B admit a simple characterization? We will show the answer is "yes" in every case. One surprising consequence of our results is a new characterization of the number e, the base of natural logarithms.

I developed the material for this paper when, in preparing for a class, I found myself unable to prove that $x > \log_b x$ for every x and every $b > 1$. After working out the correct results, I discovered that they could be organized into an interesting student project (the math pre-requisite is just a firt course in calculus). To support this project format, I have included a number of exercises in the paper.

The Fixed Points of Logarithms

To study the set $B = \{b > 0 : \log_b \text{ has a fixed point}\}$, we begin by dispensing with some trivial cases. First, notice that $\log_1$ is not defined anywhere; even $\log_1 1$ is ambiguous, since $1^0 = 1 = 1^1$. We will therefore omit $b = 1$ from the range of possible elements of B. (However, as a curiosity, if we were to define $\log_1 1 = 1$, then $x = 1$ would be the only solution to $\log_x x = x$.)

Another easy case occurs when $0 < b < 1$.

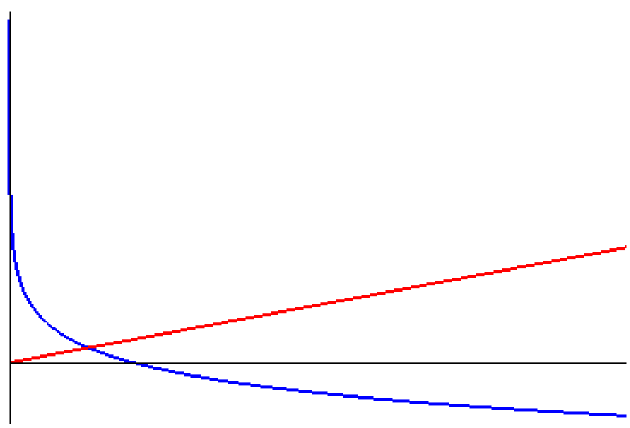

Figure 3. $y = x$ and $y = \log_{\frac{1}{2}} x$.

Using properties of logarithms, one verifies that for any $c > 1$,

$$\log_{\frac{1}{c}} x = -\log_c x.$$

Therefore, when $0 < b < 1$, $\log_b$ is the reflection in the x-axis of $\log_c$, where $c = 1/b$. Therefore, by a simple inspection of the graph (or by checking limits at 0 and ∞ and noticing that the derivative of $\log_b$ is always negative), one sees that $\log_b$ is strictly decreasing, extending from $+\infty$ as $x \to 0^+$ to $-\infty$ as $x \to +\infty$. One can then use the Intermediate Value Theorem (applied to the function $x - \log_b x$) to conclude that, for such b, $\log_b$ must always have a unique fixed point; moreover, the fixed point must lie in the interval $(0, 1)$.

Proposition 1. $(0,1) \subseteq B$. ∎

Therefore, the case of interest is $b > 1$. We let $B^* = \{b \in B : b > 1\}$. The interesting answers to the questions raised earlier pertain, as we shall see, to B^* rather than B.

We study the function $f(x) = f_b(x) = x - \log_b x$; a fixed point of $\log_b$ is then a zero of f. We begin with several straightforward computations:

$$\lim_{x \to 0^+} f(x) = +\infty$$
$$\lim_{x \to +\infty} f(x) = +\infty$$
$$f'(x) = 1 - (\log_b e)\frac{1}{x}$$
$$f''(x) = (\log_b e)\frac{1}{x^2}$$

Exercise. Prove the second of these limits. *Hint.* Use the identity

$$x - \log_b x = (\log_b x)(\frac{x}{\log_b x} - 1)$$

and apply L'Hôpital's Rule.

These observations about limits, together with the observation that $f''(x) > 0$ on $(0, +\infty)$, show that the graph of f may be one of three types, always having a minimum at $x = \log_b e$:

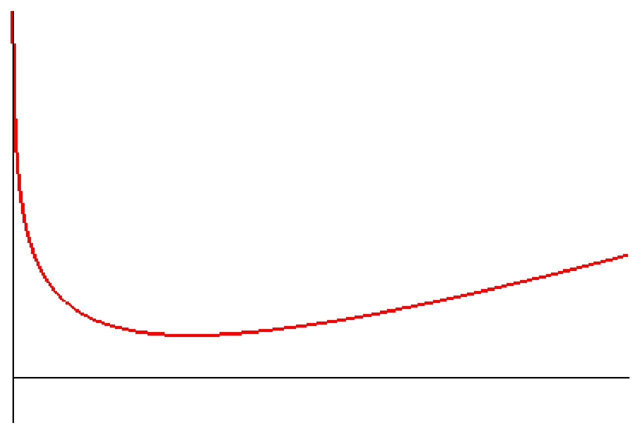

Figure 4. A Type 1 variant of $y = x - \log_b x$, $b > 1$.

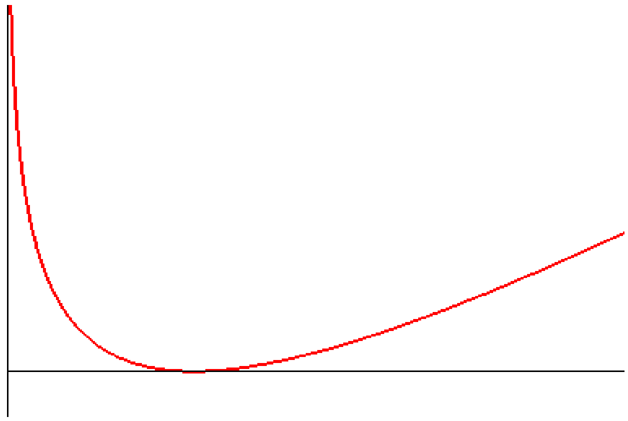

Figure 5. A Type 2 variant of $y = x - \log_b x$, $b > 1$.

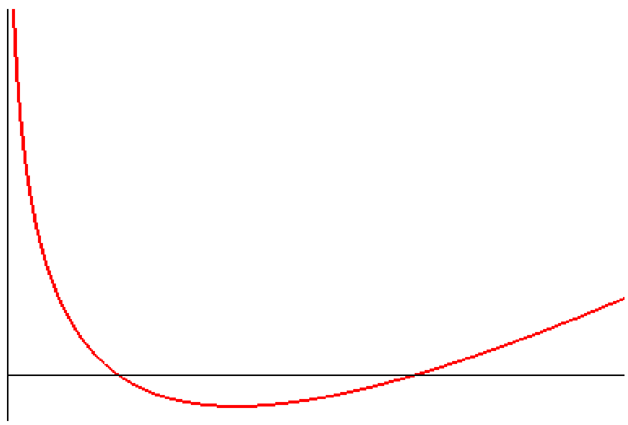

Figure 6. A Type 3 variant of $y = x - \log_b x$, $b > 1$.

Type I graphs arise when the usual bases for logarithms are used, such as 2, e, 10. For instance, when $b = e$, the minimum is achieved at

$$x = \log_b e = \log_e e = 1.$$

Since $f_e(1) = 1 > 0$, it follows that the natural log has no fixed points, as expected.

Exercise. Show that $y = x - \log_b x$ yields a Type I graph whenever $b = 2$ or 10.

We now show that a Type II graph is indeed possible. A Type II graph must contain the point $(\log_b e, 0)$. But this happens if and only if each of the following is true:

$$\begin{aligned} f(\log_b e) &= 0 \\ \log_b e &= -\log_b(\log_b e) \\ e &= \log_b e \\ b^e &= e \end{aligned}$$

But now, the equation $b^e = e$ has a solution

$$b = e^{1/e}.$$

Therefore, when $b = e^{1/e}$ and $x = \log_b e$, then $\log_b x = x$. It follows that e itself is also a fixed point of $\log_b$.

In this case, the fixed point represents a point of tangency of $y = x$ to $\log_b x$ — we shall therefore call such a fixed point a *tangent fixed point*. The fixed points we discovered for $\log_b$ when $0 < b < 1$ are not tangent fixed points.

Exercise. Verify that $x = e$ is a tangent fixed point for $y = x^2 - (2e - 1)x + e^2$.

We summarize our findings so far with the following:

Proposition 2. *When $b = e^{1/e}$, $\log_b$ has a fixed point. Indeed, e is a tangent fixed point of $\log_b$.*■

We have established that B^* is nonempty. This result leads to the next question:

Question. Is B^* infinite?

It is reasonable to hope that more candidates for elements of B^* can be found using powers of e in place of e. If we try

$$b = (e^2)^{\frac{1}{e^2}}, x = e^2$$

we again find that $\log_b x = x$. This observation generalizes to:

Proposition 3. *For each natural number $n \geq 1$, if*

$$b = (e^n)^{\frac{1}{e^n}},$$

e^n is a fixed point of $\log_b$.■

We wish to conclude from the Proposition that B^* is infinite; however, one has to check whether any of the numbers $(e^n)^{\frac{1}{e^n}}$ are duplicates. This concern is addressed in the following exercise:

Exercise. Prove the following: The sequence

$$e^{1/e}, (e^2)^{(1/e^2)}, (e^3)^{(1/e^3)}, \ldots$$

is strictly decreasing and converges to 1.

The Exercise gives us the next Proposition:

Proposition 4. *There are infinitely many $b > 1$ for which $\log_b$ has a fixed point. That is, B^* is infinite.*■

So far, our only examples of $b \in B^*$ lie in the half-open interval $(1, e^{1/e}]$ (note that $e^{1/e} \approx 1.4$).

Question. Must every $b \in B^*$ lie in $(1, e^{1/e}]$?

Reasoning as we did to obtain a Type II graph, it is straightforward to show that, whenever $b^e > e$, $x - \log_b x$ yields a Type I graph; therefore the answer to the question is "yes". However, using a different approach, we can arrive at this answer in a way that reveals more of the structure of B^*. We begin by asking another question: Suppose $b, c \in B$, $b \neq c$, and $\log_b x = x$. Is it possible that $\log_c x = x$? If this were possible, we would have

$$x = \log_c x = \frac{\log_b x}{\log_b c} = \frac{x}{\log_b c}.$$

But this would imply $\log_b c = 1$, which contradicts the fact that $b \neq c$. Therefore, $\log_c x \neq x$.

We call this principle the *Changing Base Property* of logarithmic fixed points: Different fixed points correspond to different bases.

Now we more closely examine the elements of B^* that we have found so far. These all have the form $b = q^{1/q}$ for some q. A reasonable conjecture is that *all* elements of B^* are of this form. To test the conjecture, we let $b \in B^*$; there must be an x such that

$$\log_b x = x.$$

Letting $q = b^x$ gives us immediately that $\log_b q = x$, and since $\log_b$ is one-one, we conclude that $q = x$. With this choice of q, we have therefore

$$(*) \qquad b^q = q.$$

We wish to conclude from $(*)$ that $b = q^{1/q}$, but it might be possible that some b different from $q^{1/q}$ also satisfies equation $(*)$. However, if such a b exists, we would have

$$\log_b q = q,$$
$$\log_{q^{1/q}} q = q, \text{ and}$$
$$b \neq q^{1/q}$$

in violation of the Changing Base Property. Therefore, we must have $b = q^{1/q}$, and we have shown that every $b \in B^*$ has the desired form. A quick review of the proof shows that, for $b \in B^*$, the q that works is > 1; also, the same proof goes through for *any* $b \in B$, but we may require only that $q > 0$. Summarizing,

Proposition 5. *For every $b \in B$, there is a $q > 0$ such that $b = q^{1/q}$. Moreover, for every $b \in B^*$, there is a $q > 1$ such that $b = q^{1/q}$.*■

So how does Proposition 5 help us prove that every $b \in B$ must be $\leq e^{1/e}$? Since every $b \in B$ has the form $x^{1/x}$ it is helpful to consider the graph of $y = x^{1/x}$. Using calculus, we can identify the graph's main features:

Exercise. Verify the following properties of the function $g(x) = x^{1/x}$:

(A) $\lim_{x \to 0^+} x^{1/x} = 0$

(B) $\lim_{x\to+\infty} x^{1/x} = 1$

(C) g achieves a maximum, with slope 0, at $(e, e^{1/e})$

(D) $\text{range}(g) = (0, e^{1/e}]$.

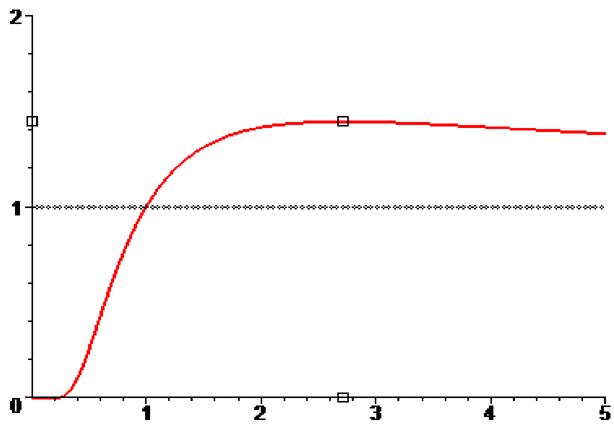

Figure 7. The graph of $y = x^{\frac{1}{x}}$, having max at $(e, e^{\frac{1}{e}})$.

Therefore, since the range of g has maximum value $e^{1/e}$ and $B \subseteq \text{range}(g)$, we have the following, which answers our question:

Proposition 6. $B \subseteq (0, e^{1/e}]$.■

Proposition 6 raises the natural question:

Question. Is it true that $B^* = (1, e^{1/e}]$?

The proof that the answer is "yes" is like our original proof that $b = e^{1/e}$ belongs to B: We must show that any given b in $(1, e^{1/e}]$ lies in B^*. Since $b \in \text{range}(g)$, there is q such that $b = q^{1/q}$. But now clearly

$$\log_b q = q,$$

as required.

This final observation provides us with a nice characerization of the logarithm bases that lead to fixed points:

Theorem 7. *Suppose $b > 0$ and $b \neq 1$. Then the following are equivalent:*

(A) $\log_b$ *has a fixed point; that is, $b \in B$*

(B) *there is $q > 0, q \neq 1$ such that $b = q^{1/q}$.*

(C) $0 < b \leq e^{\frac{1}{e}}$.

There is one remaining piece of the puzzle that should be answered. Our work showed that when $b = e^{1/e}$, $\log_b$ has a *tangent* fixed point. We also observed at the beginning that for all $b \in (0, 1)$, $\log_b$ has a unique fixed point, but not a tangent fixed point. Because of the shape of the graph of $\log_b$ for $b > 1$, it is clear that $\log_b$ can have at most two fixed points, and that $\log_{e^{1/e}}$ must have exactly one. We are led to the following final question:

Question. Which $b \in B^*$ are tangent fixed points? And for which $b \in B^*$ does $\log_b$ have *two* fixed points, and for which does it have just one?

The question is most easily answered by examining the graph of $g(x) = x^{\frac{1}{x}}$ more closely. Notice that if $b \in \text{range}(g)$ and $1 < b < e^{1/e}$, we can read from the graph two fixed points by obtaining the preimages of g at b. For each such b, there is a $q \in (0, e)$ and an $r \in (e, \infty)$ such that $b = g(q) = q^{1/q}$ and $b = g(r) = r^{1/r}$; and certainly $\log_b q = q, \log_b r = r$. For such b, $f_b' < 0$ on $(0, \log_b e)$ (which contains q), and $f_b' > 0$ on $(\log_b e, +\infty)$ (which contains r). Therefore, neither q nor r is a tangent fixed point.

Proposition 8. *Suppose* $b \in B$. *Then,*

(A) *if* $0 < b < 1$, $\log_b$ *has a unique fixed point, which is not a tangent fixed point*

(B) *if* $1 < b < e^{1/e}$, $\log_b$ *has exactly two fixed points, neither of which is a tangent fixed point*

(C) *if* $b = e^{1/e}$, $\log_b$ *has a unique fixed point at* $x = e$, *and it is a tangent fixed point.*

Proposition 8 leads to a new characterization of the number e:

Proposition 9. *The number e is the only real number that is a tangent fixed point for a logarithmic function of the form* $y = \log_b x$. ■