Fixed Points of Logs and a Characterization of $e$

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Abstract. We describe the set of all $b$ for which the equation $x = \log_b x$ has a solution. A consequence of this computation is a new characterization of the transcendental number $e$.

Introduction

Is there some real $b$ such that the graph of $y = \log_b x$ is tangent to $y = x$, as in the following diagram?

Figure 1. $y = x$ and $y = \log_b x$ for some value of $b$.

In working with logarithms, we are more accustomed to graphs like the following, where the log function lies strictly below $y = x$.

Figure 2. $y = x$ and $y = \log_e x$. 
For a real-valued function $f$ defined on the reals, we say $x$ is a fixed point of $f$ if $f(x) = x$. In this paper, we study the set of all $b$ for which $\log_b$ has a fixed point; that is, we study the set $B$ where

$$B = \{ b > 0 : \log_b x = x \text{ for some } x \}.$$ 

Is $B$ non-empty? Infinite? Do the elements of $B$ admit a simple characterization? We will show the answer is “yes” in every case. One surprising consequence of our results is a new characterization of the number $e$, the base of natural logarithms.

I developed the material for this paper when, in preparing for a class, I found myself unable to prove that $x > \log_b x$ for every $x$ and every $b > 1$. After working out the correct results, I discovered that they could be organized into an interesting student project (the math pre-requisite is just a first course in calculus). To support this project format, I have included a number of exercises in the paper.

**The Fixed Points of Logarithms**

To study the set $B = \{ b > 0 : \log_b$ has a fixed point $\}$, we begin by dispensing with some trivial cases. First, notice that $\log_1$ is not defined anywhere; even $\log_1 1$ is ambiguous, since $1^0 = 1 = 1^1$. We will therefore omit $b = 1$ from the range of possible elements of $B$. (However, as a curiosity, if we were to define $\log_1 1 = 1$, then $x = 1$ would be the only solution to $\log_1 x = x$.)

Another easy case occurs when $0 < b < 1$.

![Figure 3. $y = x$ and $y = \log_{\frac{1}{2}} x$.](image)

Using properties of logarithms, one verifies that for any $c > 1$,

$$\log_{\frac{1}{c}} x = -\log_c x.$$ 

Therefore, when $0 < b < 1$, $\log_b$ is the reflection in the $x$-axis of $\log_c$, where $c = 1/b$. Therefore, by a simple inspection of the graph (or by checking limits at $0$ and $\infty$ and noticing that the derivative of $\log_b$ is always negative), one sees that $\log_b$ is strictly decreasing, extending from $+\infty$ as $x \to 0^+$ to $-\infty$ as $x \to +\infty$. One can then use the Intermediate Value Theorem (applied to the function $x - \log_b x$) to conclude that, for such $b$, $\log_b$ must always have a unique fixed point; moreover, the fixed point must lie in the interval $(0, 1)$. 

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Proposition 1. \((0, 1) \subseteq B. \) ■

Therefore, the case of interest is \(b > 1\). We let \(B^* = \{b \in B : b > 1\}\). The interesting answers to the questions raised earlier pertain, as we shall see, to \(B^*\) rather than \(B\).

We study the function \(f(x) = f_b(x) = x - \log_b x\); a fixed point of \(\log_b\) is then a zero of \(f\). We begin with several straightforward computations:

\[
\lim_{x \to 0^+} f(x) = +\infty \\
\lim_{x \to +\infty} f(x) = +\infty \\
f'(x) = 1 - (\log_b e) \frac{1}{x} \\
f''(x) = (\log_b e) \frac{1}{x^2}
\]

**Exercise.** Prove the second of these limits. **Hint.** Use the identity

\[
x - \log_b x = (\log_b x)\left(\frac{x}{\log_b x} - 1\right)
\]

and apply L'Hôpital’s Rule.

These observations about limits, together with the observation that \(f''(x) > 0\) on \((0, +\infty)\), show that the graph of \(f\) may be one of three types, always having a minimum at \(x = \log_b e\):

![Figure 4. A Type 1 variant of \(y = x - \log_b x\), \(b > 1\).](image)
Figure 5. A Type 2 variant of $y = x - \log_b x$, $b > 1$.

Figure 6. A Type 3 variant of $y = x - \log_b x$, $b > 1$.

Type I graphs arise when the usual bases for logarithms are used, such as 2, e, 10. For instance, when $b = e$, the minimum is achieved at

$$x = \log_b e = \log_e e = 1.$$  

Since $f_e(1) = 1 > 0$, it follows that the natural log has no fixed points, as expected.

Exercise. Show that $y = x - \log_b x$ yields a Type I graph whenever $b = 2$ or 10.

We now show that a Type II graph is indeed possible. A Type II graph must contain the point $(\log_b e, 0)$. But this happens if and only if each of the following is true:

$$f(\log_b e) = 0$$
$$\log_b e = -\log_b (\log_b e)$$
$$e = \log_b e$$
$$b^e = e$$
But now, the equation $b^e = e$ has a solution

$$b = e^{1/e}.$$  

Therefore, when $b = e^{1/e}$ and $x = \log_b e$, then $\log_b x = x$. It follows that $e$ itself is also a fixed point of $\log_b$.

In this case, the fixed point represents a point of tangency of $y = x$ to $\log_b x$ — we shall therefore call such a fixed point a tangent fixed point. The fixed points we discovered for $\log_b$ when $0 < b < 1$ are not tangent fixed points.

**Exercise.** Verify that $x = e$ is a tangent fixed point for $y = x^2 - (2e - 1)x + e^2$.

We summarize our findings so far with the following:

**Proposition 2.** When $b = e^{1/e}$, $\log_b$ has a fixed point. Indeed, $e$ is a tangent fixed point of $\log_b$. ■

We have established that $B^*$ is nonempty. This result leads to the next question:

**Question.** Is $B^*$ infinite?

It is reasonable to hope that more candidates for elements of $B^*$ can be found using powers of $e$ in place of $e$. If we try

$$b = (e^2)^{1/e}, x = e^2$$

we again find that $\log_b x = x$. This observation generalizes to:

**Proposition 3.** For each natural number $n \geq 1$, if

$$b = (e^n)^{1/e},$$

$e^n$ is a fixed point of $\log_b$. ■

We wish to conclude from the Proposition that $B^*$ is infinite; however, one has to check whether any of the numbers $(e^n)^{1/e}$ are duplicates. This concern is addressed in the following exercise:

**Exercise.** Prove the following: The sequence

$$e^{1/e}, (e^2)^{1/e^2}, (e^3)^{1/e^3}, \ldots$$

is strictly decreasing and converges to 1.

The Exercise gives us the next Proposition:

**Proposition 4.** There are infinitely many $b > 1$ for which $\log_b$ has a fixed point. That is, $B^*$ is infinite. ■

So far, our only examples of $b \in B^*$ lie in the half-open interval $(1, e^{1/e}]$ (note that $e^{1/e} \approx 1.4$).

**Question.** Must every $b \in B^*$ lie in $(1, e^{1/e}]$?
Reasoning as we did to obtain a Type II graph, it is straightforward to show that, whenever \( b^e > e \), \( x - \log_b x \) yields a Type I graph; therefore the answer to the question is “yes”. However, using a different approach, we can arrive at this answer in a way that reveals more of the structure of \( B^* \). We begin by asking another question: Suppose \( b, c \in B, b \neq c, \) and \( \log_b x = x \). Is it possible that \( \log_c x = x \)? If this were possible, we would have

\[
x = \log_c x = \frac{\log_b x}{\log_b c} = \frac{x}{\log_b c}.
\]

But this would imply \( \log_b c = 1 \), which contradicts the fact that \( b \neq c \). Therefore, \( \log_c x \neq x \).

We call this principle the Changing Base Property of logarithmic fixed points: Different fixed points correspond to different bases.

Now we more closely examine the elements of \( B^* \) that we have found so far. These all have the form \( b = q^{1/q} \) for some \( q \). A reasonable conjecture is that all elements of \( B^* \) are of this form. To test the conjecture, we let \( b \in B^* \); there must be an \( x \) such that

\[
\log_b x = x.
\]

Letting \( q = b^x \) gives us immediately that \( \log_b q = x \), and since \( \log_b \) is one-one, we conclude that \( q = x \). With this choice of \( q \), we have therefore

\[
(*) \quad b^q = q.
\]

We wish to conclude from \( (*) \) that \( b = q^{1/q} \), but it might be possible that some \( b \) different from \( q^{1/q} \) also satisfies equation \( (*) \). However, if such a \( b \) exists, we would have

\[
\log_b q = q, \\
\log_{q^{1/q}} q = q, \quad \text{and} \\
b \neq q^{1/q}
\]

in violation of the Changing Base Property. Therefore, we must have \( b = q^{1/q} \), and we have shown that every \( b \in B^* \) has the desired form. A quick review of the proof shows that, for \( b \in B^* \), the \( q \) that works is \( > 1 \); also, the same proof goes through for \( \text{any } b \in B \), but we may require only that \( q > 0 \). Summarizing,

**Proposition 5.** For every \( b \in B \), there is a \( q > 0 \) such that \( b = q^{1/q} \). Moreover, for every \( b \in B^* \), there is a \( q > 1 \) such that \( b = q^{1/q} \). \( \blacksquare \)

So how does Proposition 5 help us prove that every \( b \in B \) must be \( \leq e^{1/e} \)? Since every \( b \in B \) has the form \( x^{1/x} \) it is helpful to consider the graph of \( y = x^{1/x} \). Using calculus, we can identify the graph’s main features:

**Exercise.** Verify the following properties of the function \( g(x) = x^{1/x} \):

- (A) \( \lim_{x \to 0^+} x^{1/x} = 0 \)
(B) \( \lim_{x \to +\infty} x^{1/x} = 1 \)
(C) \( g \) achieves a maximum, with slope 0, at \((e, e^{1/e})\)
(D) range\((g) = (0, e^{1/e}]\).

**Figure 7.** The graph of \( y = x^{1/x} \), having max at \((e, e^{1/e})\).

Therefore, since the range of \( g \) has maximum value \( e^{1/e} \) and \( B \subseteq \text{range}(g) \), we have the following, which answers our question:

**Proposition 6.** \( B \subseteq (0, e^{1/e}] \).

**Proposition 6** raises the natural question:

**Question.** Is it true that \( B^* = (1, e^{1/e}] \)?

The proof that the answer is “yes” is like our original proof that \( b = e^{1/e} \) belongs to \( B \): We must show that any given \( b \) in \((1, e^{1/e}]\) lies in \( B^* \). Since \( b \in \text{range}(g) \), there is \( q \) such that \( b = q^{1/q} \). But now clearly

\[ \log_b q = q, \]

as required.

This final observation provides us with a nice characterization of the logarithm bases that lead to fixed points:

**Theorem 7.** Suppose \( b > 0 \) and \( b \neq 1 \). Then the following are equivalent:

(A) \( \log_b \) has a fixed point; that is, \( b \in B \)
(B) there is \( q > 0, q \neq 1 \) such that \( b = q^{1/q} \).
(C) \( 0 < b \leq e^{1/e} \).

There is one remaining piece of the puzzle that should be answered. Our work showed that when \( b = e^{1/e} \), \( \log_b \) has a tangent fixed point. We also observed at the beginning that for all \( b \in (0, 1) \), \( \log_b \) has a unique fixed point, but not a tangent fixed point. Because of the shape of the graph of \( \log_b \) for \( b > 1 \), it is clear that \( \log_b \) can have at most two fixed points, and that \( \log_{e^{1/e}} \) must have exactly one. We are led to the following final question:
**Question.** Which $b \in B^*$ are tangent fixed points? And for which $b \in B^*$ does $\log_b$ have two fixed points, and for which does it have just one?

The question is most easily answered by examining the graph of $g(x) = x^{1/b}$ more closely. Notice that if $b \in \text{range}(g)$ and $1 < b < e^{1/e}$, we can read from the graph two fixed points by obtaining the preimages of $g$ at $b$. For each such $b$, there is a $q \in (0, e)$ and an $r \in (e, \infty)$ such that $b = g(q) = q^{1/q}$ and $b = g(r) = r^{1/r}$; and certainly $\log_b q = q, \log_b r = r$. For such $b$, $f'_b < 0$ on $(0, \log_b e)$ (which contains $q$), and $f'_b > 0$ on $(\log_b e, +\infty)$ (which contains $r$). Therefore, neither $q$ nor $r$ is a tangent fixed point.

**Proposition 8.** Suppose $b \in B$. Then,

(A) if $0 < b < 1$, $\log_b$ has a unique fixed point, which is not a tangent fixed point
(B) if $1 < b < e^{1/e}$, $\log_b$ has exactly two fixed points, neither of which is a tangent fixed point
(C) if $b = e^{1/e}$, $\log_b$ has a unique fixed point at $x = e$, and it is a tangent fixed point.

Proposition 8 leads to a new characterization of the number $e$:

**Proposition 9.** The number $e$ is the only real number that is a tangent fixed point for a logarithmic function of the form $y = \log_b x$. □