Forcing With Non-wellfounded Models
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ABSTRACT: We develop the machinery for performing forcing over an arbitrary (possibly non-wellfounded) model of set theory. For consistency results, this machinery is unnecessary since such results can always be legitimately obtained by assuming that the ground model is (countable) transitive. However, for establishing properties of a given (possibly non-wellfounded) model, the fully developed machinery of forcing as a means to produce new related models can be useful. We develop forcing through iterated forcing, paralleling the standard steps of presentation found in [19] and [14].

In this paper, we develop the basic theory of forcing in the context of arbitrary (rather than transitive) models of ZFC. For the purpose of establishing relative consistency results, it is always possible to use a (countable) transitive ground model, and the forcing machinery in this setting has already been well developed (see for example [19]). There are occasions, however, in which the objective is of a more model-theoretic nature; for instance, in studying various types of extensions of a given, possibly non-wellfounded model $\mathcal{M}$ of set theory, one may wish to consider forcing extensions of the model as a source of examples. In the literature, the usual way of addressing this need is to work with the Boolean-valued model $\mathcal{M}^B$, for some complete Boolean algebra B, or to construct a Boolean ultrapower of $\mathcal{M}$, again relative to some complete Boolean algebra B; these techniques are discussed in [11]. In many such cases, it could be useful to have on hand the fully developed machinery of forcing for arbitrary models. The purpose of this paper is to fill this need.

A folklore insight about the matter is that roughly the same theorems ought to hold true in the non-wellfounded case as for the transitive case (see for example [20, p. 2]). But if one attempts to formulate the results for the general case precisely, many questions arise. For example, one would not expect the forcing extension M_G of a non-wellfounded model M to be the "smallest" model including M and containing G (a result we call the *Minimality Theorem*), though this assertion is true if M is transitive. One might instead expect that the many forcing results of this kind, in the context of possibly ill-founded models, would now be true "up to isomorphism," in an appropriate sense. But then, how would the standard fact, that, if P is a nontrivial partial order in M, $G \notin M$, be translated in the ill-founded context, "up to isomorphism"?

To answer these and other natural questions once and for all, we develop in this paper the machinery of forcing for arbitrary models of ZFC. Many of the differences from the transitive case are only minor modifications of the usual results. There are some more significant variations, however, that stem from the fact that, in the ill-founded context, it is no longer possible to define the forcing extension as a transitive collapse. This means that elements of the forcing extension end up being equivalence classes of names, and as a result, many convenient methods of proof become unavailable. This fact most significantly affects the proofs of the Minimality Theorem, just discussed, and the Two-Step Iteration Theorem (which asserts that a two-step iteration is equivalent to a certain one-step forcing). Our new statement and proof of the Minimality Theorem makes use of the fact that even a non-wellfounded forcing extension "believes" itself to be obtained by a collection of coherent transitive collapsing functions; this lets us use the standard argument as a guideline, though more bookkeeping is required. Verification that $(M_G)_H$ is canonically isomorphic to $M_{G\otimes H}$ in the Two-Step Iteration Theorem turns out to be more difficult, again because collapsing functions are not available here. In this case, a careful examination of names is required to obtain the result.

The paper is organized as follows: In Section 1, we review basic facts about partial orders, Boolean algebras, and models of set theory that have a possibly non-wellfounded membership relation. In Section 2, we review the necessary results on Boolean-valued models. In Section 3, we develop the analogues to the usual theorems for one-step forcing and in Section 4, for two-step iterations. Finally in Section 5, we make some remarks about general iterations; as we will see, little work beyond that of Section 4 is needed to establish the expected results for general iterations.

This paper is not the first to discuss the forcing machinery for arbitrary models of set theory; in [21] forcing is introduced in the more general context of *semisets*. However, the work in [21] was developed before the modern approach to forcing had been standardized, and model theorists might find this approach inconvenient and impractical. The present paper has the advantage of paralleling the familiar approaches to forcing found in [15] and [19] and may therefore be more suitable as a ready reference.

Another related area, which we do not pursue here, is the relationship between the forcing methodology and *nonstandard universes*, in the sense of nonstandard mathematics. Nonstandard mathematics is the attempt to incorporate the objects and tools of nonstandard analysis into a ZFC-like foundation for mathematics. The work in [9] and [16] survey the developments in this area of research. Typically, a nonstandard set theory postulates three types of objects: standard sets, internal sets, and external sets. Standard sets are meant to correspond to the usual sets of mathematical concern. The class of internal sets represents a (nonstandard) expanded universe consisting of the "ideal" elements of standard sets. The external sets are "everything else". Typically, the applications of nonstandard mathematics exploit the relationship between the standard and internal sets; a desirable goal is to formalize the techniques for studying this relationship in the surrounding universe. One of the most successful theories in this direction, developed in the work of Kanovei and Reeken in [17, 18] is *Hrbaček Set Theory* (*HST*). HST is rich enough to formulate natural questions about the class **S** of standard sets, the class **I** of internal sets, and their relationship. An important example is (roughly stated) the question of whether elementarily equivalent nonstandard extensions are always isomorphic (a more precise statement of this is known as the *Isomorphism Property* or *IP*). The authors of [17] show that IP is not decidable from HST, and they develop a version of forcing over models of HST in order to prove half of this undecidability. The forcing methodology developed for this purpose overlaps to some extent the work we have done here, though in [17], the aim is to establish consistency results rather than to give a full treatment of the topic of forcing in this new context. However, as the referee pointed out to the author, the forcing of [17] generalizes forcing in the nonstandard direction further than we do here: The models we consider here, though possibly non-wellfounded, still satisfy the Axiom of Regularity; they are *internally standard*. By contrast, models of HST are not internally standard; forcing in this context could be described as (in the words of the referee) "essentially nonstandard".

The work in this paper was originally developed as a foundation for another paper in which forcing machinery is developed for the language $\{\in, \mathbf{j}\}$, where $\mathbf{j}$ is a unary function symbol intended to represent an elementary embedding of the universe; see [5]. At present, [5] and [4] are the main applications so far of the material presented here.

1 Preliminaries: Non-wellfounded Models, Partial Orders, and Boolean Algebras

Let $\mathcal{M} = \langle M, E \rangle$ be a (possibly non-wellfounded) model of the language $\{\in\}$—in particular, we assume $\mathcal{M}$ is a model of ZFC. The symbol '$\in$' will be used both for the formal symbol of the language and for the "real" membership relation in the surrounding universe V.

We often need to consider the syntax of the language $\{\in\}$ of set theory as being formalized within set theory, and for this purpose, we follow [10]. In particular, we represent in ZFC $\in$-formulas ϕ by constant terms $\ulcorner\phi\urcorner$ (added to ZFC by definitional extension), having the property that each is an element of V_ω (see [10, pp. 90-91]). We also use, without special mention, simple formulas that describe properties of these sets. One such formula of particular importance is $\mathrm{Sat}(u, M, b)$ which asserts that u encodes the $\in$-formula $\phi(x_1, \ldots, x_m)$ and $\langle M, E(M)\rangle \models \phi(b(1), \ldots, b(m))$, where b is a function defined on ω that specifies set parameters. As in [10], $\mathrm{Sat}(u, M, b)$ is a Δ_1^{ZFC} formula.

Our arguments often require several models with different membership relations. To help avoid confusion about where arguments are taking place at various stages of a proof, we adopt the convention of indicating that $\langle M, E\rangle$ satisfies an atomic formula $x \in y$ at (a, b) by writing

$$\langle M, E\rangle \models a\,E\,b$$

rather than $\langle M, E\rangle \models a \in b$. (Formally, E can be thought of as the binary $\langle M, E\rangle$-class defined by $\langle M, E\rangle \models E(a, b)$ iff $\langle M, E\rangle \models a \in b$.)

For any $X \in M$, we let

$$X_E = \{x \in M : \mathcal{M} \models x\,E\,X\},$$

$$X_{E^2} = \{Y_E : Y \in M \text{ and } \mathcal{M} \models Y\,E\,X\}.$$

The set X_E is the *extension* of X.

We shall assume at the outset that the standard natural numbers (in V) form a (possibly proper) initial segment of the natural numbers of $\mathcal{M}$. Indeed, we will assume from now on that

$$(V_\omega)^V \subseteq (V_\omega)^{\mathcal{M}}_E \text{ and } \forall x \in (V_\omega)^V\, \forall y \in (V_\omega)^{\mathcal{M}}_E\, [(\mathcal{M} \models y\,E\,x) \Longrightarrow y \in x].$$

Using extensions, we can obtain external representatives of the ordered pairs and functions living in $\mathcal{M}$. First we define a pairing function $\mathrm{op} = \mathrm{op}_{\mathcal{M}} : M^2 \to M$:

$$\mathrm{op}(x, y) = \text{unique } u \in M \text{ such that } \mathcal{M} \models u = (x, y). \tag{1.1}$$

For any $X, Y, t \in M$ for which $\mathcal{M} \models$ "$t : X \to Y$ is a function", we define a function $\mathrm{graph}(t) = \mathrm{graph}_{\mathcal{M}}(t)$ having domain X_E by

$$\forall x, y \in M\,(\mathrm{graph}(t)(x) = y \iff \mathcal{M} \models t(x) = y).$$

For any $n \in \omega$ and any $R \in M$ for which $\mathcal{M} \models$ "R is an n-ary relation", we define an n-ary relation $\mathrm{rel}(R) = \mathrm{rel}_{\mathcal{M}}(R)$ as follows:

$$\begin{aligned}\forall (x_1, \ldots, x_n) \in M^n \quad [(x_1, \ldots, x_n) \in \mathrm{rel}(R) \iff \\ \mathcal{M} \models (x_1, \ldots, x_n) \in R].\end{aligned} \tag{1.2}$$

Proposition 1 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC.

(1) *For all $x, y \in M$, $(x, y) = \mathrm{op}(x, y)_{E^2}$.*

(2) *If $\mathcal{M} \models$ "R is a unary relation", then $\mathrm{rel}(R) = R_E$.*

(3) *Suppose $\mathcal{M} \models$ "$t : X \to Y$ is a function".*

   (a) $\mathrm{graph}(t)$ *is one-one if and only if $\mathcal{M} \models$ "t is one-one".*

   (b) $\mathrm{graph}(t)$ *is onto if and only if $\mathcal{M} \models$ "t is onto".*

   (c) *Suppose $n, k \in \omega$ and*

$$\mathcal{M} \models \quad \begin{array}{l}\mathcal{X} = \langle X, R, f\rangle \textit{ and } \mathcal{Y} = \langle Y, S, g\rangle \textit{ are first-order structures of the same type,} \\ R \textit{ and } S \textit{ are } n\textit{-ary, and } t : \mathcal{X} \to \mathcal{Y} \textit{ is structure-preserving.}\end{array}$$

   Then $\mathcal{X}' = \langle X_E, \mathrm{rel}(R), \mathrm{graph}(f)\rangle$ and $\mathcal{Y}' = \langle Y_E, \mathrm{rel}(S), \mathrm{graph}(g)\rangle$ are first-order structures of the same type and $\mathrm{graph}(t) : \mathcal{X}' \to \mathcal{Y}'$ is structure-preserving.

Proof. The proofs of (2) and (3) are easy. For (1), let $u = \mathrm{op}(x, y)$. If $z \in M$ and $\mathcal{M} \models z\,E\,u$, then $\mathcal{M} \models [z = \{x\} \vee z = \{x, y\}]$. Therefore, there are $v, w \in M$ such that

(a) $M \models v = \{x\} \wedge w = \{x, y\}$

(b) $v_E = \{x\}$ and $w_E = \{x, y\}$

(c) $\mathcal{M} \models u = \{v, w\}$, and

(d) $u_E = \{v, w\}$.

We have

$$\begin{aligned} u_{E^2} &= \{z_E : z \in M \text{ and } \mathcal{M} \models z \, E \, u\} \\ &= \{v_E, w_E\} \\ &= \{\{x\}, \{x, y\}\} \\ &= (x, y). \end{aligned}$$

Typically, we will be interested in forcing with a partial order, and to do so we will embed it into its Boolean algebra completion. All partial orders $(P, \leq_P)$, denoted simply by P usually, will be assumed to have a largest element, denoted 1_P or simply 1. A Boolean algebra B can be specified by providing an order relation $\leq$ on B that makes B a complemented distributive lattice, or by providing operations $\vee, \wedge, ^*$ and constants $0, 1$ satisfying the usual axioms of a Boolean algebra (see [3, Section 4]). We also define auxiliary operations $\rightarrow, \leftrightarrow, -$ by

$$b \rightarrow c = b^* \vee c$$

$$b \leftrightarrow c = b \rightarrow c \, \wedge \, c \rightarrow b$$

$$b - c = b \, \wedge \, c^*.$$

A *complete* Boolean algebra is a Boolean algebra B for which $\bigvee X$ exists for every $X \subseteq B$.

If P and Q are partial orders, a function $i : P \rightarrow Q$ is called a *complete embedding* if the following hold (see [19, VII]):

(a) $\forall p_1, p_2 \in P \, (p_1 \leq p_2 \Longrightarrow i(p_1) \leq i(p_2))$

(b) $\forall p_1, p_2 \in P \, (p_1 \perp p_2 \Longleftrightarrow i(p_1) \perp i(p_2))$

(c) $\forall q \in Q \exists p \in P \forall r \in P \, (r \leq p \Longrightarrow (i(r)$ and q are compatible in $Q))$.

A map $e : P \rightarrow Q$ is called a *dense embedding* if the following hold:

(a) $\forall p_1, p_2 \in P \, (p_1 \leq p_2 \Longrightarrow e(p_1) \leq e(p_2))$

(b) $\forall p_1, p_2 \in P \, (p_1 \perp p_2 \Longrightarrow i(p_1) \perp i(p_2))$

(c) $i''P$ is dense in Q.

Suppose B, C are complete Boolean algebras and $i : B \to C$ is a homomorphism. Then i is said to be *complete* if, for all $X \subseteq B$, $i(\bigvee X) = \bigvee(i''X)$. In particular, if B is a subalgebra of C, then B is a *complete subalgebra* if the inclusion map is a complete homomorphism. Typically, if $i : B \to C$ is a one-one complete homomorphism, we will identify B with its image under i (which is a complete subalgebra of C).

The next theorem lists several standard results about partial orders and Boolean algebras that we will need; proofs can be found in [15, Section 17], [3], or [19, VII].

PROPOSITION 2

(1) *Every partial order P has a unique (up to isomorphism) Boolean algebra completion. That is, for each P, there exist a complete Boolean algebra* $\mathrm{ro}(P)$ *(the regular open algebra of P), unique up to isomorphism, and a dense embedding $e : P \to \mathrm{ro}(P) \setminus \{0\}$.*

(2) *If B and C are complete Boolean algebras and $i : B \to C$ is a function, then i is a complete injective homormorphism if and only if $i \restriction B \setminus \{0\} : B \setminus \{0\} \to C \setminus \{0\}$ is complete in the sense of partial orders.*

(3) *Suppose P, Q are partial orders and $B = \mathrm{ro}(P)$ and $C = \mathrm{ro}(Q)$. If $i : P \to Q$ is a complete embedding of partial orders and $e_P : P \to B$, $e_Q : Q \to C$ are dense embeddings, then i lifts to a complete injective homomorphism $\bar{i} : B \to C$.*

(4) *(Rasiowa-Sikorski) Suppose B is a Boolean algebra, $a \in B$, $a \neq 0$, and $\{X_0, X_1, \ldots, X_n, \ldots\}$ is a countable family of subsets of B such that for each n, there is $b \in B$ such that $b = \bigvee X_n$. Then there is an ultrafilter $U \subseteq B$ such that $a \in U$ and for each n,*

$$\bigvee X_n \in U \text{ implies } X_n \cap U \neq \emptyset. \tag{1.3}$$

If e is a dense embedding that witnesses the fact that $B = \mathrm{ro}(P)$, we will often write $e : P \to B$ for convenience, rather than $e : P \to B \setminus \{0\}$.

Suppose $\mathcal{M} = \langle M, E \rangle$ is a model of ZFC and $B \in M$ is such that $\mathcal{M} \models$ "B is a Boolean algebra". It is easy to verify that B_E, with the ordering $b \leq c$ iff $\mathcal{M} \models b \leq c$, is a Boolean algebra (note the external $\leq$ is actually $\mathrm{rel}(\leq)$). We say that B is *M-complete* if, for each $X \in M$, if $\mathcal{M} \models X \subseteq B$, then there is $b \in B_E$ such that $b = \bigwedge X_E$ (where the meet is taken in B_E).

The next proposition says that the extension of a complete Boolean algebra in M is always an M-complete Boolean algebra under the natural ordering.

Proposition 3 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC *and in $\mathcal{M}$ B is a complete Boolean algebra. Then $\langle B_E, \leq\rangle$ is an M-complete Boolean algebra.*

Proof. Suppose $X \in M$ and $\mathcal{M} \models X \subseteq B$. Let $b \in B_E$ be unique such that $\mathcal{M} \models b = \bigwedge X$. Clearly, for each $x \in X_E$, $\mathcal{M} \models b \leq x$, and so $b \leq x$; thus b is a lower bound of X. Suppose $c \in B_E$ and, for each $x \in X_E$, $c \leq x$. Then $\mathcal{M} \models \forall x \in X\, (c \leq x)$, whence $\mathcal{M} \models c \leq b$. Hence $c \leq b$, and we have shown that $b = \bigwedge X$. ∎

Likewise, one can show that each of the X_E as in Proposition 3 has a *join* in B_E. For each $X \subseteq B_E$ let $X^* = \{x^* : x \in X\}$. It is easy to show that if $Y \subseteq B_E$ has a join and a meet, so does Y^*.

The obvious similarity between the structures $(B_E, \leq)$ and $(B, \leq)^{\mathcal{M}}$ derives from the fact that these structures actually have the same first-order properties. This in turn follows from a more general observation that will be useful: Suppose $n, k \in \omega$ and

$$\mathcal{M} \models \quad \text{“}\mathcal{X} = \langle X, R, f\rangle \text{ is a first-order structure, } R \text{ is an } n\text{-ary relation, and } f \text{ is a } k\text{-ary function”.}$$

Let $\mathcal{X}' = \langle X_E, \mathrm{rel}(R), \mathrm{graph}(f)\rangle$. Let $\phi(x_1, \ldots, x_m)$ be a first-order formula in the language of $\mathcal{X}'$. Then for all $b \in M$ for which

$$\mathcal{M} \models \text{“}b : \mathrm{rank}(\ulcorner\phi\urcorner) \to X \text{ is a function”},$$

we have

$$\mathcal{X}' \models \phi[b_0, \ldots, b_{m-1}] \iff \mathcal{M} \models \mathrm{Sat}(\ulcorner\phi\urcorner, X, b), \tag{1.4}$$

where, for each i, $\mathcal{M} \models b_i = b(i)$. The proof is by a straightforward induction on the complexity of ϕ and makes use of the fact that M end-extends the real V_ω. (This convenient observation was pointed out to me by D. Hatch.)

Some easily proven consequences (1.4) are listed in the next proposition:

Proposition 4

(1) *Suppose $\mathcal{M} \models$ “P is a partial order”. Then $\mathcal{M} \models$ “P is separative” if and only if P_E is separative.*

(2) *Suppose* $\mathcal{M} \models$ *"P is a partial order". Then for all* $D \in M$, $\mathcal{M} \models$ *"D is a dense subset of P" if and only if* D_E *is a dense subset of* P_E*. The same holds if "dense subset of" is replaced by "(maximal) antichain in".*

(3) *Suppose* $\mathcal{M} \models$ *"B is a Boolean algebra and* $b, c\, E\, B$*". Then* $\mathcal{M} \models b = c^*$ *if and only if, in* B_E, $b = c^*$*. Analogous statements hold for the operations* $\wedge, \vee$ *and for the constants* $0, 1$.

(4) *Suppose* $\mathcal{M} \models$ *"B is a Boolean algebra and* X, Y *are subsets of B". Then* $\mathcal{M} \models Y = X^*$ *if and only if* $Y_E = X^*_E$.

(5) *Suppose that in* $\mathcal{M}$*, P is a partial order,* $B = \mathrm{ro}(P)$*, and* $e : P \to B$ *is a function. Then* $\mathcal{M} \models$ *"e is a dense embedding" if and only if* $\mathrm{graph}(e) : P_E \to B_E$ *is a dense embedding.*

Proof. We outline the proof of (5): Consider in $\mathcal{M}$ the first-order structure $\langle B, \wedge, \vee, *, 0, 1, P, B, e\rangle$, where e is treated as a binary relation. Clearly, the property of being a dense embedding is first-order relative to this structure, and so (1.4) applies. ■

2 Boolean-valued Models

Given a model $\mathcal{M} = \langle M, E\rangle$ of ZFC and a $B \in M$ such that $\mathcal{M} \models$ "B is a complete Boolean algebra", we build the Boolean valued model $\mathcal{M}^B$ in $\mathcal{M}$ in the usual way: $\mathcal{M}^B = \bigcup_{\alpha \in \mathrm{ON}} \mathcal{M}^B_\alpha$, where $\mathcal{M}^B_0 = \emptyset$, $\mathcal{M}^B_{\alpha+1}$ is the set of all functions $f \in M$ such that $\mathrm{dom}\, f \subseteq \mathcal{M}^B_\alpha$ and $\mathrm{ran}\, f \subseteq B$; and $\mathcal{M}^B_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}^B_\alpha$, when λ is a limit. In $\mathcal{M}$, we also define sets $M_{B,\gamma}$, γ an ordinal in $\mathcal{M}$, as follows:

$$\mathcal{M} \models M_{B,\gamma} = M^B \cap V_\gamma. \tag{2.1}$$

As usual, define a first-order language $\mathcal{L}^B = \mathcal{L}^{\mathcal{M},B}$ consisting of $\in$ together with a constant for each member of $(M^B)_E$. Formulas of $\mathcal{L}^B$ are coded in $\mathcal{M}$ so that the formulas form a definable class in $\mathcal{M}$. We refer to the formulas of $\mathcal{L}^B$ as B-formulas. As usual, there is a Boolean truth value map $[\![\cdot]\!] = [\![\cdot]\!]^{\mathcal{M}}_B$, depending on B and $\mathcal{M}$ and defined within $\mathcal{M}$ by recursion on a well-founded relation, that assigns a value in B to each B-formula. For completeness, we give this definition here.

$$
\begin{aligned}
[\![\sigma \in \tau]\!]_B &= \bigvee\nolimits_{t\,E\,\mathrm{dom}\,(\tau)} (\tau(t) \wedge [\![\sigma = t]\!]_B) \\
[\![\sigma = \tau]\!]_B &= \bigwedge\nolimits_{s\,E\,\mathrm{dom}\,(\sigma)} (\sigma(s) \to [\![s \in \tau]\!]_B) \wedge \bigwedge\nolimits_{t\,E\,\mathrm{dom}\,(\tau)} (\tau(t) \to [\![t \in \sigma]\!]_B) \\
[\![\psi \wedge \phi]\!]_B &= [\![\psi]\!]_B \wedge [\![\phi]\!]_B \\
[\![\neg\psi]\!]_B &= ([\![\psi]\!]_B)^* \\
[\![\exists x\, \psi(x)]\!]_B &= \bigvee\nolimits_{t\,E\,M^B} [\![\psi(t)]\!]_B.
\end{aligned}
$$

In the definition, σ and τ are B-names and ψ, ϕ are B-sentences. Formally, $[\![\psi]\!]_B$ is defined for atomic ψ within $\mathcal{M}$ by recursion on pairs of name-ranks (see [15]). Then the definition proceeds, *by induction in* V, on the complexity of formulas. The definition up to any finite stage is formalizable in $\mathcal{M}$, but, by Tarski's result on undefinability of truth, there is no class function $\mathbf{F}$ defined in $\mathcal{M}$ such that $\mathbf{F}(\ulcorner\psi\urcorner) = [\![\psi]\!]_B^{\mathcal{M}}$ for every B-sentence ψ.

For $b \in B_E$, we express the fact that a formula $\phi(x)$ at τ has Boolean value b in $\mathcal{M}$ with the notation $\mathcal{M} \models [\![\phi(\tau)]\!]_B = b$ or $[\![\phi(\tau)]\!]_B^{\mathcal{M}} = b$; when the underlying Boolean algebra is clear from the context, we shall suppress the subscript "B" in this notation. In $\mathcal{M}$, we say $M^B \models \phi$ if $[\![\phi]\!]_B = 1$. Still in $\mathcal{M}$, for each $x \in M$, we let $\check{x} = \{(\check{y}, 1) : y \in x\} \in M^B$; $\check{x}$ is called the *canonical name for* x. Let $\mathbf{u} = \mathbf{u}_B \in (M^B)_E$ be defined by letting $\mathrm{dom}\,\mathbf{u} = \{\check{b} : b \in B\}$ and defining $\mathbf{u}(\check{b}) = b$ for all $b \in B$. $\mathbf{u}$ is called the *canonical name for a generic ultrafilter in* B. (We will define *generic ultrafilter* for the present context in the next subsection where we deal with two-valued models.) The usual forcing relation $\Vdash$ is defined in $\mathcal{M}$ by

$$b \Vdash \phi \text{ iff } b \leq [\![\phi]\!]_B.$$

Next we state two theorems that outline useful properties of M^B. The first of these is a result about B-names; proofs of parts (1)–(4), (6) can be found in [2]. We will sketch a proof of part (5) using Theorem 10 in the next subsection.

Theorem 5 *Suppose* $\mathcal{M} = \langle M, E\rangle \models \mathrm{ZFC}$ *and, in* $\mathcal{M}$*,* B *is a complete Boolean algebra.*

(1) *(Names of Unions) In* $\mathcal{M}$*: Suppose* $\sigma\, E\, M^B$*. Define* $\tau\, E\, M^B$ *by*

$$\mathrm{dom}\,\tau = \bigcup\{\mathrm{dom}\,\nu : \nu\, E\, \mathrm{dom}\,\sigma\}$$

and

$$\tau(t) = [\![\exists x \in \sigma\, (t \in x)]\!].$$

Then $[\![\tau = \bigcup \sigma]\!]_B = 1$.

(2) *(Names of Subsets) In* $\mathcal{M}$*: Suppose* $\sigma\, E\, M^B$*. Then for every* $\tau_1\, E\, M^B$ *there is* $\tau_2\, E\, M^B$ *such that* $\mathrm{dom}\ \tau_2 = \mathrm{dom}\ \sigma$ *and* $[\![\tau_1 \subseteq \sigma \to \tau_1 = \tau_2]\!]_B = 1$.

(3) *(Names Of Power Sets) In* $\mathcal{M}$*: Suppose* $\sigma\, E\, M^B$*. Let* $\mathbf{p}_B(\sigma)$ *be the* B*-name defined as follows:* $\mathrm{dom}\ \mathbf{p}_B(\sigma) = {}^{\mathrm{dom}\ (\sigma)}B$ *and for all* $t\, E\, \mathrm{dom}\ \mathbf{p}_B(\sigma)$, $\mathbf{p}_B(\sigma)(t) = [\![t \subseteq \sigma]\!]$*. Then* $[\![\mathbf{p}_B(\sigma) = P(\sigma)]\!]_B = 1$.

(4) *(Mixing Lemma) In* $\mathcal{M}$*: Suppose* $A \subseteq B$ *is an antichain, and we have* B*-names* $\{\sigma_a : a\, E\, A\}$*. Then there is* $\sigma\, E\, M^B$ *such that for all* $a\, E\, A$, $a \leq [\![\sigma = \sigma_a]\!]_B$.

(5) *(Unmixing Lemma) In* $\mathcal{M}$*: Suppose* $\sigma, \pi\, E\, M^B$*. Then there is an antichain* A *of elements of* B *below* $[\![\sigma \in \pi]\!]_B$ *such that* $\bigvee A = [\![\sigma \in \pi]\!]_B$ *and for each* $a\, E\, A$ *there is* $\sigma_a\, E\, \mathrm{dom}\ \pi$ *such that* $a \leq [\![\sigma = \sigma_a]\!]_B$.

(6) *In* $\mathcal{M}$*,* M^B *is full; that is, for each* B*-formula* $\phi(x, x_1, \ldots, x_n)$ *and all* $\tau_1, \ldots, \tau_n \in (M^B)_E$*, there is* $\tau \in (M^B)_E$ *such that*

$$\mathcal{M} \models [\![\phi(\tau, \tau_1, \ldots, \tau_n)]\!] = [\![\exists x\, \phi(x, \tau_1, \ldots, \tau_n)]\!].$$

The name $\mathbf{p}_B(\sigma)$ in part (3) will be called the *canonical* B*-name for the power set of* σ. Part (5) asserts that every Boolean-valued element σ of a B-name π is a mixture, in the sense of (4), of the elements of the domain of π by a maximal antichain below $[\![\sigma \in \pi]\!]_B$.

In working with names, it is handy to have a canonical subcollection of names that are relatively small in size and low in rank. For this purpose, we define canonical names for ranks V_α, give bounds on the sizes and ranks of these names, and use these tools to describe a relationship, definable in $\mathcal{M}$, between the rank of a set in a forcing extension and the rank of one of its well-chosen names. The bounds we describe below are convenient for this paper but are not optimal; see [20] and [13] for sharper results in the case of partial-order-based names.

We begin by recursively defining in $\mathcal{M}$ a class sequence $\langle \dot{\mathbf{r}}_\alpha : \alpha \in \mathrm{ON}\rangle$ of names for the ranks V_α: Let $\dot{\mathbf{r}}_0 = \check{0}$. For the inductive step, given $\dot{\mathbf{r}}_\alpha$,

$$\begin{aligned} \mathrm{dom}\ \dot{\mathbf{r}}_{\alpha+1} &= B^{\mathrm{dom}\ (\dot{\mathbf{r}}_\alpha)}; \\ \dot{\mathbf{r}}_{\alpha+1}(t) &= [\![t \subseteq \dot{\mathbf{r}}_\alpha]\!]_B. \end{aligned}$$

For λ a limit:

$$\begin{aligned} \mathrm{dom}\ \dot{\mathbf{r}}_\lambda &= \bigcup\{\mathrm{dom}\ \dot{\mathbf{r}}_\alpha : \alpha < \lambda\}; \\ \dot{\mathbf{r}}_\lambda(t) &= \textstyle\bigvee_{\alpha<\lambda}[\![t \in \dot{\mathbf{r}}_\alpha]\!]_B. \end{aligned}$$

Recall that for an infinite cardinal κ, $\beth_\alpha(\kappa)$ is defined recursively as follows: $\beth_0(\kappa) = \kappa$; $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$; $\beth_\lambda(\kappa) = \bigcup_{\alpha<\lambda} \beth_\alpha(\kappa)$ for limit λ. Also, for any ordinals α, β, we define $\mathrm{reg}(\beta, \alpha)$ to be the least regular cardinal $> \max\{\alpha, \beta\}$.

THEOREM 6 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC *and, in $\mathcal{M}$, B is a complete Boolean algebra.*

(1) $\mathcal{M} \models \forall\alpha \in \mathrm{ON}\, [\![\dot{\mathbf{r}}_\alpha = V_\alpha]\!]_B = 1$.

(2) $\mathcal{M} \models \forall\alpha \in \mathrm{ON}\, |\dot{\mathbf{r}}_\alpha| \leq \beth_\alpha(|B|)$.

(3) *In $\mathcal{M}$: Whenever σ is a B-name with domain* $\mathrm{dom}\, \dot{\mathbf{r}}_\alpha$, *then $\sigma \in V_\rho$ where* $\rho = \mathrm{reg}(\mathrm{rank}(B), \alpha)$.

(4) $\mathcal{M} \models \forall\alpha < \lambda\, E\, \mathrm{ON}\, [(\lambda$ *a strong limit and* $B\, E\, V_\lambda) \implies \dot{\mathbf{r}}_\alpha\, E\, V_\lambda]$.

(5) *In $\mathcal{M}$: If λ is a strong limit, $B\, E\, V_\lambda$, and $\sigma\, E\, M^B$, then there is $\tau\, E\, M_{B,\lambda}$ such that* $[\![\sigma \in V_\lambda \longrightarrow \sigma = \tau]\!]_B = 1$.

(6) *There is an $\mathcal{M}$-class function $T = T_B$ with $\mathcal{M} \models T : \mathrm{ON} \to \mathrm{ON}$ having the following property in $\mathcal{M}$:*

$$\forall\alpha\, E\, \mathrm{ON}\, \forall\sigma\, E\, M^B\, ([\![\sigma \in V_\alpha]\!]_B = 1 \implies \exists\tau\, E\, M_{B,T(\alpha)}\ [\![\sigma = \tau]\!]_B = 1). \tag{2.2}$$

In particular, if T is defined in $\mathcal{M}$ by $\mathcal{M} \models T(\alpha) = \mathrm{reg}(\mathrm{rank}B, \alpha)$, then T satisfies (2.2).

Proof. The proof of (1) is by induction in $\mathcal{M}$ on the ordinals and uses Theorem 5(3) at successor stages. For the limit stage, working in $\mathcal{M}$, notice that if λ is a limit, we can let σ be the name having domain $\{\dot{\mathbf{r}}_\alpha : \alpha < \lambda\}$ and constant value 1. Then for all $t\, E\, \sigma$,

$$\begin{aligned} \dot{\mathbf{r}}_\lambda(t) &= \textstyle\bigvee_{\alpha<\lambda} [\![t \in \dot{\mathbf{r}}_\alpha]\!]_B \\ &= [\![\exists x \in \sigma\, t \in x]\!]_B. \end{aligned}$$

It follows from Theorem 5(1) and the induction hypothesis that

$$[\![\dot{\mathbf{r}}_\lambda = \bigcup\{\dot{\mathbf{r}}_\alpha : \alpha < \lambda\} = \bigcup\{V_\alpha : \alpha < \lambda\} = V_\lambda]\!]_B = 1.$$

The proofs of (2) and (4) are also straightforward inductions (in $\mathcal{M}$). To prove (3), we proceed by induction, in $\mathcal{M}$, on the ordinals. The basis step is trivial. For the successor step, suppose dom $\sigma = $ dom $\dot{\mathbf{r}}_{\alpha+1}$. Then dom $\sigma = B^{\text{dom }(\dot{\mathbf{r}}_\alpha)}$. Let $\rho = \text{reg}(\text{rank}(B), \alpha)$. Clearly $\rho = \text{reg}(\text{rank}(B), \alpha+1)$. By induction hypothesis, we have easily that $\{B, \text{dom } \dot{\mathbf{r}}_\alpha, \dot{\mathbf{r}}_\alpha\} \in V_\rho$. It follows easily that $\sigma \in V_\rho$, as required. For the limit step, suppose λ is a limit ordinal and dom $\sigma = $ dom $\dot{\mathbf{r}}_\lambda = \bigcup\{\text{dom } \dot{\mathbf{r}}_\alpha : \alpha < \lambda\}$. For each $\alpha < \lambda$, let $\beta_\alpha = \text{rank}(\dot{\mathbf{r}}_\alpha)$. By the induction hypothesis, $\beta_\alpha < \text{reg}(\text{rank}B, \alpha)$. Let $\beta = \sup\{\beta_\alpha : \alpha < \lambda\}$. Then $\bigcup_{\alpha<\lambda} \text{dom } \dot{\mathbf{r}}_\alpha \subseteq V_\beta$. Let $\rho = \text{reg}(\text{rank}B, \lambda)$. Since $\lambda < \rho$ and each $\beta_\alpha < \rho$, by regularity of ρ we have $\beta < \rho$. Thus dom $\dot{\mathbf{r}}_\lambda \in V_\rho$. Since $B \in V_\rho$, it follows that $\sigma \in V_\rho$.

To prove (5), suppose $\mathcal{M} \models \lambda$ is a strong limit, $B \, E \, V_\lambda$, and $\sigma \, E \, M^B$. Arguing in $\mathcal{M}$, since $\text{sat}(B) < \lambda$, there is $\alpha < \lambda$ such that $[\![\sigma \in V_\lambda]\!]_B \leq [\![\sigma \subseteq V_\alpha]\!]_B$. Now by Theorem 5(2), we obtain a B-name τ such that

$$\text{dom } \tau = \text{dom } \dot{\mathbf{r}}_\alpha \quad \text{and} \quad [\![\sigma \subseteq V_\alpha]\!]_B \leq [\![\sigma = \tau]\!]_B.$$

The result follows.

For (6), we define T by $\mathcal{M} \models T(\alpha) = \text{reg}(\text{rank}B, \alpha)$. Suppose α and σ are such that $[\![\sigma \in V_\alpha]\!]_B^{\mathcal{M}} = 1$. Using Theorem 5(2), we obtain in $\mathcal{M}$ a τ having domain dom $\dot{\mathbf{r}}_\alpha$ such that $[\![\sigma = \tau]\!]_B^{\mathcal{M}} = 1$. By (3), $\mathcal{M} \models \tau \, E \, V_{T(\alpha)}$. ■

The next theorem is a list of results about Boolean-valued set theory that we will need in our exposition; again, proofs can be found in [Be].

Theorem 7 *Suppose $\mathcal{M} = \langle M, E \rangle \models$ ZFC and, in $\mathcal{M}$, B is a complete Boolean algebra.*

(1) *For each axiom ψ of* ZFC, $[\![\psi]\!]_B^{\mathcal{M}} = 1$.

(2) *For each $\tau \in (M^B)_E$,*

$$[\![\tau \in \mathbf{u}]\!]_B^{\mathcal{M}} = [\bigvee_{c \, E \, B} (c \wedge [\![\tau = \check{c}]\!])]_B^{\mathcal{M}} = \bigvee_{c \in B_E} (c \wedge [\![\tau = \check{c}]\!]_B^{\mathcal{M}}).$$

For each $b \in B$,

$$\mathcal{M} \models [\![\check{b} \in \mathbf{u}]\!]_B = b.$$

(3) *For each $x \in M$ and $\tau \in (M^B)_E$,*

$$[\![\tau \in \check{x}]\!]_B^{\mathcal{M}} = \Big(\bigvee_{y \, E \, x} [\![\tau = \check{y}]\!]_B\Big)^{\mathcal{M}} = \bigvee_{y \in x_E} [\![\tau = \check{y}]\!]_B^{\mathcal{M}}.$$

(4) *For each* $x, y \in M$

$$\begin{aligned} x\,E\,y &\iff (M^B \models \check{x} \in \check{y})^{\mathcal{M}} \\ x = y &\iff (M^B \models \check{x} = \check{y})^{\mathcal{M}} \end{aligned}$$

(5) *For any* Σ_0 *formula* $\phi(x_1, \ldots, x_n)$ *and any* $y_1, \ldots, y_n \in M$

$$M \models \phi(y_1, \ldots, y_n) \iff (M^B \models \phi(\check{y_1}, \ldots, \check{y_n}))^{\mathcal{M}}.$$

(6) *For all* $\tau \in (M^B)_E$,

$$[\![\tau \text{ is an ordinal}]\!]_B^{\mathcal{M}} = \Big(\bigvee_{\alpha\, E\, ON} [\![\tau = \check{\alpha}]\!]\Big)^{\mathcal{M}} = \bigvee_{\alpha \in ON_E^{\mathcal{M}}} [\![\tau = \check{\alpha}]\!]_B^{\mathcal{M}}.$$

(7) *Suppose that in* $\mathcal{M}$*,* C *is a complete Boolean algebra and* B *is a complete subalgebra of* C*. Then for any* Σ_0 *formula* $\phi(x_1, \ldots, x_n)$ *and any* $\tau_1, \ldots, \tau_n \in (M^B)_E$,

$$[\![\phi(\tau_1, \ldots, \tau_n)]\!]_B^{\mathcal{M}} = [\![\phi(\tau_1, \ldots, \tau_n)]\!]_C^{\mathcal{M}}.$$

We remark here that the basic results concerning λ-cc forcing and λ-closed forcing hold in the present context of non-wellfounded ground models because they hold in the Boolean-valued model — namely, λ-cc forcing preserves cardinals and cofinalities $\geq \lambda$ and λ-closed forcing adds no new functions on sets of size $< \lambda$. After stating relevant definitions, we record these results below in the language of Boolean-valued models; see [2] and [15] for proofs.

Still working in a model $\langle M, E\rangle$ of ZFC, suppose λ is an infinite cardinal. Recall that a partially ordered set P is $< \lambda$-Baire if the intersection of less than λ open dense subsets of P is dense. If P is $< \lambda$-Baire, so is $\mathrm{ro}(P) \setminus \{0\}$. Moreover, we say that a complete Boolean algebra B is $< \lambda$-Baire iff $B \setminus \{0\}$ is $< \lambda$-Baire in the sense of partial orders. If $x, y \in M$ and $\mathcal{M} \models$ "B is $< \lambda$-Baire and $|x| < \lambda$ and $F = y^x$", then $[\![\check{y}^{\check{x}} = \check{F}]\!]_B^{\mathcal{M}} = 1$.

Still in $\mathcal{M}$ recall that if P has the λ-cc then $B = \mathrm{ro}(P)$ does too, and in either case, whenever $\theta \geq \lambda$ is a cardinal of cofinality γ, then $[\![\check{\theta}$ is a cardinal and $\mathrm{cf}(\check{\theta}) = \check{\gamma}]\!]_B = 1$. We record these facts:

Proposition 8 *Suppose* $\mathcal{M} = \langle M, E\rangle$ *is a model of* ZFC *and, in* $\mathcal{M}$*,* P *is a partial order and* $B = \mathrm{ro}(P)$*, and* λ *is an infinite cardinal.*

(1) *If, in* $\mathcal{M}$*,* P *is* λ*-closed (or even* $<\lambda$*-Baire), then for all* $x, y, F \in M$ *with*

$$\mathcal{M} \models |x| < \lambda \text{ and } F = y^x,$$

we have

$$[\![\check{y}^{\check{x}} = \check{F}]\!]_B^{\mathcal{M}} = 1.$$

(2) *Suppose in* $\mathcal{M}$ P *is* λ*-cc,* $\theta \geq \lambda$ *is a cardinal, and* $\mathrm{cf}(\theta) = \gamma$*. Then*

$$[\![\check{\theta} \text{ is a cardinal and } \mathrm{cf}(\check{\theta}) = \check{\gamma}]\!]_B^{\mathcal{M}} = 1. \blacksquare$$

We shall write $\mathrm{sat}(P)$ (or $\mathrm{sat}(B)$) for the least κ such that P (or B) has the κ-cc.

We conclude this subsection with some facts about the canonical name for generic filters in the context of Boolean-valued models. (Again, we postpone the actual definition of a *generic filter* to the next subsection.) In $\mathcal{M}$, suppose P is a partial order, $B = \mathrm{ro}(P)$, and $e : P \to B$ is a dense embedding. We define $\mathbf{g} = \mathbf{g}_{P,e} \in (M^B)_E$ as follows: Let dom $\mathbf{g} = \{\check{p} : p \in P\}$ and define $\mathbf{g}(\check{p}) = e(p)$. The name $\mathbf{g}$ is called the *canonical name for a generic filter in* P *with respect to* e. The following theorem is an easy corollary to Theorem 7:

THEOREM 9 *Suppose* $\mathcal{M} = \langle M, E\rangle \models \mathrm{ZFC}$ *and, in* $\mathcal{M}$*,* P *is a partial order,* $B = \mathrm{ro}(P)$*, and* $e : P \to B$ *is a dense embedding.*

(1) *For each* $\tau \in M^B$*,*

$$[\![\tau \in \mathbf{g}]\!]^{\mathcal{M}} = [\bigvee_{p\,E\,P} (e(p) \wedge [\![\tau = \check{p}]\!]_B)]^{\mathcal{M}} = \bigvee_{p \in P_E} (e(p)^{\mathcal{M}} \wedge [\![\tau = \check{p}]\!]_B^{\mathcal{M}}).$$

(2) *For each* $p \in P$*,*

$$[\![\check{p} \in \mathbf{g}]\!]_B^{\mathcal{M}} = (e(p))^{\mathcal{M}}.$$

(3) *For each* $p \in P$*,*

$$[\![\check{p} \in \mathbf{g} \longleftrightarrow \check{e}(\check{p}) \in \mathbf{u}]\!]_B^{\mathcal{M}} = 1.$$

3 Forcing Over Arbitrary Models

The properties given in the Theorem 7 are internal to $\mathcal{M}$; consistency results in the context of Boolean-valued models take the form

$$\mathcal{M} \models S \Rightarrow M^B \models S + \sigma,$$

where S is an extension of ZFC. Here, however, we are interested in casting our results in terms of two-valued models. To obtain such a model from M^B, we collapse M^B with an ultrafilter U that is "contained in" B. When M is transitive, we can use an ultrafilter $U \subseteq B$, but when M is arbitrary, we need to take $U \subseteq B_E$. Even in the transitive case, M^B/U is a poor substitute for the usual generic extension $M[G]$, unless U is endowed with genericity. In the transitive case, we can define U to be generic if $\bigwedge X \in U$ whenever $X \in M$ and $X \subseteq U$, but this definition has to be modified for arbitrary M. In the transitive case, using a generic U gives us that M^B/U is well-founded with transitive collapse precisely equal to $M[U]$. For arbitrary M, using a generic U gives us a new model M_U that closely resembles its transitive analogue; Lemma 14 and Theorems 15 and 16 list the relevant properties. Before proving these results, we establish a few additional preliminaries:

Definition 1 Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of ZFC and, in $\mathcal{M}$, B is a complete Boolean algebra.

(1) (*S-Genericity*) Suppose $\mathcal{M} \models S \subseteq P(B)$. We will call an ultrafilter $U \subseteq B_E$ *S-generic over $\mathcal{M}$* if, whenever $X \in M$, $X \in S_E$, and $X_E \subseteq U$, we have $\bigwedge X_E \in U$.

(2) (*Genericity*) An ultrafilter $U \subseteq B_E$ is *B-generic over $\mathcal{M}$* if U is $(P(B))^{\mathcal{M}}$-generic over $\mathcal{M}$.

(3) (*Internal Genericity*) Suppose $\Gamma, S \in M$ and

$$\mathcal{M} \models \text{“}\Gamma \subseteq B \text{ is an ultrafilter and } S \subseteq P(B)\text{”}.$$

Then Γ is *internally S-generic (for B) in $\mathcal{M}$* if

$$\mathcal{M} \models \forall X \in S\, (X \subseteq \Gamma \implies \bigwedge X\, E\, \Gamma). \tag{3.1}$$

(4) (*Genericity in a Model*) Suppose $\Gamma, S \in M$. Then we say $\mathcal{M} \models$ "Γ is S-generic in B" if $\mathcal{M} \models$ "$\Gamma \subseteq B$ is an ultrafilter and $S \subseteq P(B)$" and (3.1) holds.

Parts (3) and (4) are different ways of saying the same thing; indeed, Γ is internally S-generic in $\mathcal{M}$ if and only if $\mathcal{M} \models$ "Γ is S-generic in B". Parts (3) and (4) are different from part (1) because we may be dealing with non-wellfounded models. An example of internal genericity is $\mathbf{u}_U$: In $\mathcal{M}$ let $P = P(B)$. Then $\mathbf{u}_U$ is internally P-generic in $\mathcal{M}_U$. The next theorem is the analogue of the usual result that generics over countable transitive models always exist:

THEOREM 10 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a countable model of* ZFC *and $\mathcal{M} \models$ "B is a complete Boolean algebra". Then, for each nonzero $b \in B_E$, there is an ultrafilter $U_b \subseteq B_E$ such that $b \in U_b$ and U_b is B-generic over $\mathcal{M}$.*

Proof. Let $\mathcal{P}_M = \{X \in M : \mathcal{M} \models X \subseteq B\}$ and let $b \in B_E$. Since M is countable, so is $\mathcal{P} = \{X_E : X \in \mathcal{P}_M\}$ and we can write $\mathcal{P} = \{X_E^{(0)}, X_E^{(1)}, \ldots, X_E^{(n)}, \ldots\}$. Since B is M-complete, each $X_E^{(n)}$ has a join and a meet in B_E. By the Rasiowa-Sikorski Theorem applied to B_E and the family $\mathcal{P}$, we obtain an ultrafilter $U_b \subseteq B_E$ such that $b \in U_b$ and (1.3) holds. Assume that for some n, $X_E^{(n)} \subseteq U_b$ but $\bigwedge X_E^{(n)} \notin U_b$. Then $\bigvee (X_E^{(n)})^* \in U_b$. By (1.3), some $x^* \in (X_E^{(n)})^*$ must be in U_b. But this is impossible since x is also in U_b. The result follows. ∎

As promised in the last subsection, we can use Theorem 10 to prove Theorem 5(5): Work in $\mathcal{M}$: Let $b = [\![\sigma \in \pi]\!]_B$ and let $B_b = \{c \in B : c \leq b\}$. Let $K = \{(\tau, c)\, E \operatorname{dom} \pi \times B_b : c \leq [\![\tau = \sigma]\!]\}$. Let K_0 be a subset of K that is maximal with respect to the property that for all $(\tau_1, c_1), (\tau_2, c_2)\, E\, K_0$, $c_1 \wedge c_2 = 0$. Let $A = \{c\, E\, B_b : \exists \tau\, E \operatorname{dom} \pi (c, \tau)\, E\, K_0\}$. Clearly, A is an antichain below b. We prove that $\bigvee A = b$; it suffices to show that A is a *maximal* antichain below b. Suppose $d\, E\, B_b$ is such that $d \wedge c = 0$ for all $c\, E\, A$. Let U be B-generic over $\mathcal{M}$ with $c \in U$. Since, in $\mathcal{M}$, $c \leq [\![\sigma \in \pi]\!]_B$, there must be, by the definition of Boolean-valued membership and genericity, a $\tau' \in M$ with $\mathcal{M} \models \tau'\, E \operatorname{dom} \pi$ and $[\![\sigma = \tau']\!]^{\mathcal{M}}\, E\, U$. Thus, in $\mathcal{M}$, we can find d' below both d and $[\![\sigma = \tau']\!]^{\mathcal{M}}$. Now $(\tau', d')\, E\, K$ satisfies the property that for any $(\tau, c)\, E\, K_0$, $d' \wedge c = 0$, contradicting the maximality property of K_0. Therefore, as claimed, $\bigvee A = b$. To complete the proof, arguing in $\mathcal{M}$, for each $a\, E\, A$, we let σ_a be such that $(\sigma_a, a)\, E\, K_0$; these σ_a have the required property.

A familiar equivalent form of genericity is given in the next proposition. The proof is an easy variant of the usual one in the context of transitive models (see, for instance, [15, 17.4]).

PROPOSITION 11 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC*, B is, in $\mathcal{M}$, a complete Boolean algebra, and $U \subseteq B_E$ is an ultrafilter. Then U is B-generic over $\mathcal{M}$ if and only if, for each $D \in M$, $D_E \cap U \neq \emptyset$ whenever $\mathcal{M} \models$ "D is dense in $B \setminus \{0\}$".* ∎

We proceed to a description of the model $M_U = (M^B)_E/U$, where U is some B-generic ultrafilter[1] over $\mathcal{M}$. Given such a U, define an equivalence relation $\sim_U$ on $(M^B)_E$ by

$$\tau_1 \sim_U \tau_2 \text{ iff } [\![\tau_1 = \tau_2]\!]^{\mathcal{M}}_B \in U.$$

We denote by $\tau_U = \tau_U^{\mathcal{M}}$ the $\sim_U$-equivalence class containing τ. We let $M_U = \{\tau_U : \tau \in (M^B)_E\}$. Define a membership relation E_U on M_U by

$$\sigma_U \, E_U \, \tau_U \text{ iff } [\![\sigma \in \tau]\!]^{\mathcal{M}}_B \in U.$$

As usual, E_U respects equivalence classes. We have the following:

THEOREM 12 *Suppose $\phi(x_1, \ldots, x_n)$ is a formula and $\tau_1, \ldots, \tau_n \in M^B$. Then*

$$\mathcal{M}_U \models \phi((\tau_1)_U, \ldots, (\tau_n)_U) \text{ iff } [\![\phi(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}}_B \in U.$$

In particular, $\mathcal{M}_U \models$ ZFC.

Proof. The last part follows from the first. The proof of the first part is by induction on the complexity of ϕ. The only nontrivial case is the existential quantifier case where fullness of M^B is used. Suppose $\phi(x_1, \ldots, x_n) \equiv \exists x\, \psi(x, x_1, \ldots, x_n)$. Then for any $\tau_1, \ldots, \tau_n \in M^B$,

$$\begin{aligned} \mathcal{M}_U \models \phi((\tau_1)_U, \ldots, (\tau_n)_U) &\iff \exists \tau \in M^B\ \mathcal{M}_U \models \psi(\tau_U, (\tau_1)_U, \ldots, (\tau_n)_U) \\ &\iff \exists \tau \in M^B\ [\![\psi(\tau, \tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}} \in U \\ &\iff [\![\exists x\, \psi(x, \tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}} \in U \\ &\iff [\![\phi(\tau_1, \ldots, \tau_n)]\!]^{\mathcal{M}} \in U. \blacksquare \end{aligned}$$

The analogues to the usual Forcing Theorems now follow as a corollary:

THEOREM 13 (FORCING THEOREMS) *Let ψ be a sentence of the B-language for $\mathcal{M}$.*

(1) *Suppose $b \in B_E$. Then $\mathcal{M} \models b \Vdash \psi$ if and only if, for every U that contains b and is B-generic over $\mathcal{M}$, we have $\mathcal{M}_U \models \psi$.*

(2) *$\mathcal{M}_U \models \psi$ if and only if there is $b \in U$ such that $\mathcal{M} \models b \Vdash \psi$.*

[1]Though we do not pursue this direction here, interesting things can be said about M_U for an arbitrary (not necessarily generic) ultrafilter. See for example [11].

Proof. For (2), both directions follow immediately from Theorem 12. For (1), if $\mathcal{M} \models b \Vdash \psi$ and $b \in U$, where U is B-generic over $\mathcal{M}$, then $\mathcal{M}_U \models \psi$ by Theorem 12. For the converse, if $\mathcal{M} \not\models b \Vdash \psi$, there is $c \in B_E$, $c \leq b$ such that $c \neq 0$ and $c \wedge [\![\psi]\!] = 0$. Then $\mathcal{M} \models c \leq [\![\neg\psi]\!]_B$. Let U be B-generic over $\mathcal{M}$ such that $c \in U$. But now $b \in U$ and, by Theorem 12 again, $\mathcal{M}_U \models \neg\psi$, and this suffices to complete the proof. ■

Next, we describe properties of the natural embedding of M into M_U. Since we are working with possibly non-wellfounded models, it will be helpful to review the usual mappings that are used when M is transitive, and then indicate the difference in the present context. When forcing over a countable transitive ground model M with a generic ultrafilter U in B, one has:

$$M \quad \stackrel{\check{}}{\longrightarrow} \quad M^B \quad \stackrel{\eta_U}{\longrightarrow} \quad M^B/U \quad \stackrel{m}{\longrightarrow} \quad M[U],$$

and $m \circ \eta_U$ is often denoted i_U. In the present context, the map m, which is the Mostowski collapsing function, is not generally an isomorphism since E_U is typically non-wellfounded, but all the other maps are defined and used in the usual way. (Technically, the definition of η_U must be changed to $\eta_U : (M^B)_E \to (M^B)_E/U$, and the check function is to be thought of as defined within $\mathcal{M}$.) Without the transitive collapsing function, it will not generally be true that M is a subset of the forcing extension. We therefore define the insertion map that gives the canonical isomorphism: $s_U = \eta_U \circ \check{}$; in other words, for all $x \in M$,

$$s_U(x) = \check{x}_U.$$

The next theorem lists the properties of s_U. We need some definitions. We follow [2] in defining an element $y \in M_U$ to be a *standard ordinal of* $\mathcal{M}_U$ if $M_U \models$ "y is an ordinal" and for some $\alpha \in M$ for which $\mathcal{M} \models$ "α is an ordinal" we have $M_U \models y = \check{\alpha}_U$. Also, given models $\langle A, E\rangle$ and $\langle B, F\rangle$ of $\{\in\}$ with $A \subseteq B$, we shall say that A is *transitive in* B if for all $x \in A$, $y \in B$, if $y\,F\,x$, then we have $y \in A$ and $y\,E\,x$. Given models $\mathcal{C} = \langle C, E\rangle$ and $\mathcal{D} = \langle D, F\rangle$ of $\{\in\}$ and a function $f : C \to D$, we will say that f is a *transitive embedding*, and that $\mathcal{C}$ *is transitively embedded in* $\mathcal{D}$ *by* f, if $f : \mathcal{C} \to \langle f''C, F\rangle$ is an (E, F)-isomorphism and $f''C$ is transitive in D. (A warning is in order here. Typically, in this paper, when we speak of a model A being a *transitive subset* of another model B, the intended meaning will be as in the above definition, and *not* in the more familiar sense that A is in fact a transitive set that is a subset of B.)

Lemma 14 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC*. Suppose that, in $\mathcal{M}$, B is a complete Boolean algebra, and that U is an ultrafilter in B_E, which is B-generic over $\mathcal{M}$.*

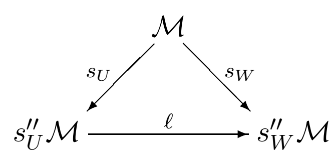
(1) *The map $s_U : M \to M_U$ is a transitive embedding; that is,*

   (a) *$s_U : M \to s_U'' M$ is an (E, E_U)-isomorphism*

   (b) *$s_U'' M \subseteq M_U$ is transitive in $\mathcal{M}_U$.*

(2) *M and M_U have the "same" ordinals. That is, for every $\alpha \in M$, if α is an ordinal in $\mathcal{M}$, then $\check{\alpha}_U$ is a standard ordinal of $\mathcal{M}_U$, and every ordinal of $\mathcal{M}_U$ is standard.*

(3) *Suppose $\mathcal{M} \models$ "C is a complete Boolean algebra" and W is C-generic over $\mathcal{M}$. Then the map $\ell : s_U''\mathcal{M} \to s_W''\mathcal{M}$ defined by $\ell(\check{x}_U) = \check{x}_W$ is an isomorphism satisfying $s_W = \ell \circ s_U$.*



Proof of (1). If $x \neq y$ are elements of M, then by Theorem 7(4), $[\![\check{x} \neq \check{y}]\!]^{\mathcal{M}} = 1 \in U$. By Theorem 12, $\mathcal{M}_U \models \check{x}_U \neq \check{y}_U$. Thus, s_U is one-one. Replacing $=$ with appropriate forms of the membership relation in the above argument leads to the conclusion that s_U is in fact an isomorphism.

To see that $M' = s''M$ is transitive in $\mathcal{M}_U$, suppose $\check{w}_U \in M'$ and $\mathcal{M}_U \models z_U\, E_U\, \check{w}_U$; we show that $z_U \in M'$ by showing that, for some $y \in M$, $[\![z = \check{y}]\!]^{\mathcal{M}} \in U$. Now $\mathcal{M}_U \models z_U\, E_U\, \check{w}_U$ implies $[\![z \in \check{w}]\!]^{\mathcal{M}} \in U$. By Theorem 7(3),

$$[\![z \in \check{w}]\!]^{\mathcal{M}} = \bigvee_{y \in w_E} [\![z = \check{y}]\!]^{\mathcal{M}}.$$

By genericity of U, there is $y \in w_E$ such that $[\![z = \check{y}]\!]^{\mathcal{M}} \in U$, as required. This completes the proof of (1). ■

Proof of (2). To see that each ordinal in $\mathcal{M}$ is mapped to a standard ordinal, suppose $\mathcal{M} \models$ "α is an ordinal". By Theorem 7(5), $[\![$"$\check{\alpha}$ is an ordinal"$]\!]^{\mathcal{M}} = 1 \in U$. By Theorem 12, $\mathcal{M}_U \models$ "$\check{\alpha}_U$ is an ordinal". Therefore α is mapped to a standard ordinal. Conversely, to see that every ordinal of M_U is standard, we show that each ordinal τ_U in M_U is equivalent to a standard ordinal

$\check{\alpha}_U$:

$$\begin{aligned}
\mathcal{M}_U \models \text{“}\tau_U \text{ is an ordinal”} &\iff [\![\text{“}\tau \text{ is an ordinal”}]\!]^{\mathcal{M}} \in U \\
&\iff \bigvee_{\alpha\in \mathrm{ON}_E^M} [\![\tau = \check{\alpha}]\!]^{\mathcal{M}} && \text{(by Theorem 7(6))} \\
&\iff \exists \alpha \in \mathrm{ON}_E^M\, [\![\tau = \check{\alpha}]\!]^{\mathcal{M}} \in U && \text{(by genericity)} \\
&\iff \exists \alpha \in \mathrm{ON}_E^M\, \tau_U = \check{\alpha}_U && \text{(by Theorem 12)}
\end{aligned}$$

as required. ∎

Proof of (3). Immediate. ∎

Notice that by transitivity, as in (1), for any $x \in M$, the members of $s_U(x)$ are of the form $s_U(y)$ for $y \in M$. Intuitively, this says that $s_U(x) = s_U''(x)$, but this notation is incorrect. The intuition can be made precise with the formula:

$$[s_U(x)]_{E_U} = s_U''(x_E). \tag{3.2}$$

By (2), the ordinals of $\mathcal{M}_U$ must be standard. Therefore, we will use the same notation — Greek letters α, β, etc. — to denote the ordinals in both $\mathcal{M}$ and $\mathcal{M}_U$. This identification makes s_U the identity on ON^M; that is, for all $\alpha \in \mathrm{ON}_E$,

$$s_U(\alpha) = \alpha.$$

Let ω^V denote the set of standard integers and $(V_\omega)^V$ the set of standard hereditarily finite sets. Our convention of identifying the standard elements of ω^M with the elements of ω^V, and the standard elements of $(V_\omega)^M$ with the elements of $(V_\omega)^V$ leads to the following further identification:

$$\forall x \in (V_\omega)^M\ s_U(x) = x.$$

We also wish to identify B with its image under s_U. It is easy to see that s_U induces the isomorphism

$$\langle B_E, \leq\rangle \cong \langle [s_U(B)]_{E_U}, \mathrm{rel}_{\mathcal{M}_U}(s_U(\leq))\rangle;$$

in other words, B and its image are isomorphic under s_U. We therefore make the identification:

$$\text{for all } b \in B_E,\ s(b) = b.$$

This identification implies that

$$s_U''U = U.$$

It is important for later work not to identify M with $s''_U M$, though in some circumstances the identification is warranted. The problem is that there will be times when we need to know whether one forcing extension is truly a subset of another; to make use of this identification in such circumstances would be incorrect. However, for arguments that are strictly "up to isomorphism" (and so do not, for example, make claims about one model being a subset of another), the identification is justified and will be used sometimes for the sake of readability.

THEOREM 15 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC. *Suppose $\mathcal{M} \models$ "B is a complete Boolean algebra" and U is an ultrafilter in B_E that is B-generic over $\mathcal{M}$. Then the model $\mathcal{M}_U = \langle M_U, E_U\rangle$ has the following properties:*

(1) *If M is countable, then M_U is also countable.*

(2) $(\mathbf{u}_U)_{E_U} = U$ *(where $\mathbf{u}_U$ is the U-equivalence class containing* $\mathbf{u}$ *and $(\mathbf{u}_U)_{E_U}$ is its extension).*

(3) *Suppose $\mathcal{N} = \langle N, F\rangle$ is another model of* ZFC *and M is transitive subset of N that is definable with parameters in $\mathcal{N}$. Suppose that for some $\Gamma \in N, \Gamma_F = U$. Then there is a one-one map $f : M_U \to N$ satisfying, for all $x, y \in M_U$,*

$$x\, E_U\, y \iff f(x)\, F\, f(y).$$

Proof of (1). Assume M is countable. Note that $(M^B)_E$ is a subset of M, so $(M^B)_E$ is countable. The map $\eta_U : (M^B)_E \to M_U : \tau \mapsto \tau_U$ is onto; therefore M_U is also countable. ∎

Proof of (2). We first observe that, by genericity and Theorem 7(2), for all $\tau \in M^B$,

$$\begin{aligned}\llbracket \tau \in \mathbf{u} \rrbracket^{\mathcal{M}} \in U \quad &\iff \exists b \in B_E\, (b \wedge \llbracket \tau = \check{b} \rrbracket^{\mathcal{M}}) \in U \\ &\iff \exists b \in U\, \llbracket \tau = \check{b} \rrbracket^{\mathcal{M}} \in U.\end{aligned}$$

Thus (making use of the identification $s_U \restriction B_E : b \mapsto b$),

$$\begin{aligned}(\mathbf{u}_U)_{E_U} \quad &= \{\tau_U \in M_U : M_U \models \tau_U\, E_U\, \mathbf{u}_U\} \\ &= \{\tau_U \in M_U : \llbracket \tau \in \mathbf{u} \rrbracket^{\mathcal{M}} \in U\} \\ &= \{\tau_U \in M_U : \exists b \in U\, \llbracket \tau = \check{b} \rrbracket^{\mathcal{M}} \in U\} \\ &= \{\check{b}_U : b \in U\} \\ &= s''_U U \\ &= U.\end{aligned}$$

Proof of (3). The Boolean-valued model M^B is definable in $\mathcal{N}$; we claim that

$$\mathcal{N} \models \text{“}\Gamma \text{ is } B\text{-generic over } M\text{”} :$$

Suppose $D \in M$ and $\mathcal{M} \models$ “D is dense in $B \setminus \{0\}$”. By transitivity of M in N, $D_E = D_F$. Thus, there is $d \in M$ such that $d \in D_E \cap U = D_F \cap \Gamma_F$. It follows that $\mathcal{N} \models d\, F\, D \cap \Gamma$, as required. Thus, we can define in $\mathcal{N}$ the class $M_\Gamma = \{\sigma_\Gamma : \sigma\, F\, M^B\}$. (To do this properly, we must use Scott's trick in the definition of the equivalence classes since, without this restriction, each equivalence class τ_Γ would be a proper class in $\mathcal{M}$.) Now if we define $f : M_U \to N$ by $f(\tau_U) = (\tau_\Gamma)^{\mathcal{N}}$, f is easily seen to have the required properties. ■

The result described in (3) above is not optimal since we have required that M be a class in $\mathcal{N}$. The reason that the usual proof—which does not rely on this assumption—fails here is that it relies on the existence of the usual collapsing map from M^B to the forcing extension, defined recursively by $i_U(\tau) = \{i_U(\sigma) : \tau(\sigma) \in U\}$; when such a map exists (and the models involved are transitive), one can argue that the range of the restriction of this map to each M^B_α is included in N, whence the entire forcing extension lies in N. In the present context, although we do not have such a collapsing map, once $\mathcal{M}_U$ has been built, $\mathcal{M}_U$ believes that it is the range of such a collapsing map, or at least of a coherent collection of set maps that collapse names in the same way. This is true because if one builds the forcing extension entirely within M^B using the canonical name for a generic ultrafilter, a collapsing map is definable. In the next paragraph, we develop these ideas, and use them to improve Theorem 15(3). We shall call a collection $\mathcal{F}$ of functions *coherent* if its elements are pairwise compatible (relative to the usual inclusion relation).

We begin with some facts that are provable in M^B. Recall that we may add a constant symbol $\check{V}$ to our forcing language $\mathcal{L}^B$ that represents the ground model in the sense that, in $\mathcal{M}$

$$[\![\tau \in \check{V}]\!]_B = \bigvee_{x\, E\, V} [\![\tau = \check{x}]\!]_B.$$

One shows (see [2]) that, in $\mathcal{M}$, the following statements have B-value 1:

- “$\check{V}$ is a transitive model of ZFC containing all the ordinals”;
- $(\text{“}\check{B} \text{ is a complete Boolean algebra”})^{\check{V}}$;
- “$\mathbf{u}$ is $\check{B}$-generic over $\check{V}$”.

Defining B-names and the collapsing map within M^B, one also proves that

$$[\![\check{V}[\mathbf{u}] \text{ is a transitive model of } \ulcorner \mathrm{ZFC} \urcorner,\ \check{V} \subseteq \check{V}[\mathbf{u}], \text{ and } \mathbf{u} \in \check{V}[\mathbf{u}]]\!]_B = 1.$$

Finally, one can show in $\mathcal{M}$ that

$$[\![\forall x\,(x \in \check{V}[\mathbf{u}])]\!]_B = 1. \tag{3.3}$$

Formula (3.3) says that, when the forcing machinery is developed inside M^B, every element of the real forcing extension is realized by a B-valued term defined in M^B.

We can restate Theorem 6(5) in M^B as follows:

$$\begin{aligned}[\![\forall \alpha, X, \sigma\, \exists z\, [(\quad & (\alpha \text{ a strong limit})^{\check{V}} \wedge B \in \check{V}_\alpha \wedge \sigma \in V_\alpha \wedge \\ & z \in \check{M}_{B,\alpha}) \longrightarrow \sigma = z]]\!]_B = 1.\end{aligned}$$

In other words, if α is a strong limit in the ground model and σ is forced to be an element of V_α, there is a name τ in the V_α of the ground model that is forced to equal σ. It follows that

$$[\![\forall \alpha\,((\alpha \text{ a strong limit})^{\check{V}} \longrightarrow V_\alpha = \check{V}_\alpha[\mathbf{u}])]\!]_B^{\mathcal{M}} = 1. \tag{3.4}$$

Putting together (3.3) and (3.4), we obtain

$$[\![\forall x\, \exists \alpha\,(x \in \check{V}_\alpha[\mathbf{u}])]\!]_B^{\mathcal{M}} = 1. \tag{3.5}$$

The consequence of (3.5) and (3.4) after collapsing by U is that we have

$$\mathcal{M}_U \models \forall x\, \exists \alpha\,(x \in s_U(V_\alpha^{\mathcal{M}})[\mathbf{u}_U]), \tag{3.6}$$

and

$$\mathcal{M} \models \text{“}\alpha \text{ is a strong limit”} \Longrightarrow \mathcal{M}_U \models V_\alpha = s_U(V_\alpha^{\mathcal{M}})[\mathbf{u}_U]. \tag{3.7}$$

Now we can define our coherent collection of collapsing maps inside $\mathcal{M}_U$: For each γ, recursively define $i_{\gamma,\mathbf{u}_U} = i_\gamma$ on $s_U(M_{B,\gamma})$ by

$$i_\gamma(s_U(\tau)) = \{i_\gamma(s_U(\sigma)) : s_U(\sigma)\, E_U \operatorname{dom} s_U(\tau) \wedge s_U(\tau)(s_U(\sigma)) \in \mathbf{u}_U\}. \tag{3.8}$$

To verify coherence, one shows that

$$\mathcal{M}_U \models \forall \alpha, \beta\,(\alpha < \beta \Longrightarrow i_\alpha = i_\beta \restriction s_U(M_{B,\alpha})). \tag{3.9}$$

To do this, fix an ordinal β and prove by $\in$-induction in $\mathcal{M}_U$ that whenever $s_U(\tau)\, E_U\, s_U(M_{B,\beta})$ then $i_\beta(s_U(\tau)) = i_\alpha(s_U(\tau))$ for all α for which $s_U(\tau)\, E_U\, s_U(M_{B,\alpha})$.

The fact that every element of $\mathcal{M}_U$ is in the range of some i_α follows from (3.6) since, for each $\alpha \in \mathrm{ON}_E$ there is a $\gamma \in \mathrm{ON}_E$ and a name μ_γ for i_γ such that

$$[\![V_\alpha \subseteq \check{V}_\gamma[\mathbf{u}] = \{\mu_\gamma(\check{\sigma}) : \check{\sigma} \in \check{V}_\gamma\}]\!]_B = 1.$$

(In fact $\gamma = T(\alpha)$ works, where T is defined in $\mathcal{M}$ as in Theorem 6(6).)

Note that the i_α's need not form a class sequence in $\mathcal{M}_U$ since M (and M^B) need not be definable in $\mathcal{M}_U$. Moreover, though it would seem reasonable that for each $\tau \in (M_{B,\alpha})_E$, we should have $i_\alpha(s_U(\tau))$ equal to τ_U, the recursion one might hope to perform in order to prove this inside $\mathcal{M}_U$ cannot be carried out since $\mathcal{M}_U$ does not know how τ_U is constructed from τ. Nonetheless, the result can be proven by resorting again to the model M^B. Assuming that in $\mathcal{M}$, $\gamma\, E\, \mathrm{ON}$ is such that $\tau\, E\, M_{B,\gamma}$, and letting μ_γ be as above, we can reason by recursion in M^B to obtain:

$$[\![\mu_\gamma(\check{\tau}) = \{\mu_\gamma(\check{\sigma}) : \check{\sigma} \in \mathrm{dom}\ \check{\tau}\} = \{\sigma : \sigma \in \tau\} = \tau]\!]_B = 1.$$

Collapsing to $\mathcal{M}_U$ gives us that

$$\mathcal{M}_U \models i_\gamma(s_U(\tau)) = \tau_U. \tag{3.10}$$

We can now provide an improved version of Theorem 15(3):

Theorem 16 (*Minimality Theorem*) *Suppose* $\mathcal{M} = \langle M, E\rangle$ *and* $\mathcal{N} = \langle N, F\rangle$ *are models of* ZFC. *Suppose, in* $\mathcal{M}$, B *is a complete Boolean algebra. Suppose that* U *is* B*-generic over* $\mathcal{M}$*. Suppose also that:*

(A) *There is a transitive embedding* $f : M \to N$.

(B) *There is* $\Gamma \in N$ *such that* $\Gamma_F = U$.

Then there is a transitive embedding $g : \mathcal{M}_U \to \mathcal{N}$ *for which* $g \circ s_U = s_U \circ f$.

Proof. For the proof, since results are correct only "up to isomorphism," we identify both s_U and the embedding f mentioned in part (A) with the corresponding identity maps. This means that we are assuming M is a transitive subset of both M_U and N, and that we must prove that g is a transitive embedding which is the identity on M.

Since for each $\gamma \in \mathrm{ON}^{\mathcal{M}}$, $M_{B,\gamma} \in N$, we can define define the maps $i_{\gamma,\Gamma}$ in $\mathcal{N}$ in the same way we defined the $i_{\gamma,\mathbf{u}_U}$ in $\mathcal{M}_U$. Before defining g, we make several observations. Let $\gamma \in \mathrm{ON}^{\mathcal{M}}$.

(1) For all $x \in M$ for which $\mathcal{M} \models \check{x} \in M_{B,\gamma}$,

$$\mathcal{N} \models i_{\gamma,\Gamma}(\check{x}) = x.$$

(2) $\mathcal{N} \models i_{\gamma,\Gamma}(\mathbf{u}) = \Gamma$.

(3) For all $\tau \in (M^B)_E$:

$$\mathcal{N} \models \forall t\, F \operatorname{dom} \tau\, [i_{\gamma,\Gamma}(t)\, F\, i_{\gamma,\Gamma}(\tau) \iff \tau(t)\, F\, \Gamma]$$

(4) If $\mathcal{M} \models \sigma, \tau\, E\, M_{B,\gamma}$,

$$\begin{aligned} \mathcal{M}_U \models i_{\gamma,\mathbf{u}_U}(\sigma)\, E_U\, i_{\gamma,\mathbf{u}_U}(\tau) \quad &\iff [\![\sigma \in \tau]\!]^{\mathcal{M}}_B \in U \\ &\iff \mathcal{N} \models i_{\gamma,\Gamma}(\sigma)\, F\, i_{\gamma,\Gamma}(\tau). \end{aligned}$$

Likewise,

$$\begin{aligned} \mathcal{M}_U \models i_{\gamma,\mathbf{u}_U}(\sigma) = i_{\gamma,\mathbf{u}_U}(\tau) \quad &\iff [\![\sigma = \tau]\!]^{\mathcal{M}}_B \in U \\ &\iff \mathcal{N} \models i_{\gamma,\Gamma}(\sigma) = i_{\gamma,\Gamma}(\tau). \end{aligned}$$

The analogues of (1)–(3) for $\mathcal{M}_U$, as well as the first parts of (4), follow from (3.10). For (1), proceed by $\in$-induction inside $\mathcal{N}$ as follows: Assuming the result holds for all σ for which $\mathcal{N} \models \sigma\, F \operatorname{dom} \check{x}$, we have in $\mathcal{N}$:

$$\begin{aligned} i_{\gamma,\Gamma}(\check{x}) \quad &= \{i_{\gamma,\Gamma}(\check{y}) : \check{y}\, F\, \check{x} \text{ and } \check{x}(\check{y})\, F\, \Gamma\} \\ &= \{y : y\, F\, x\} = x. \end{aligned}$$

We have used here the fact that M is a *transitive* subset of N.

For (2), we have in $\mathcal{N}$:

$$\begin{aligned} i_{\gamma,\Gamma}(\mathbf{u}) \quad &= \{i_{\gamma,\Gamma}(\check{b}) : b\, F\, B \text{ and } \mathbf{u}(\check{b})\, F\, \Gamma\} \\ &= \{b\, F\, B : b\, F\, \Gamma\} = \Gamma. \end{aligned}$$

Observation (3) follows immediately from the definition of $i_{\gamma,\Gamma}$. For (4), it suffices to prove the result for each infinite cardinal γ. In order to perform an induction involving pairs of names, we define in $\mathcal{M}$ a class function ρ on M^B by

$$\rho(\sigma) = \text{least } \alpha \text{ such that } \sigma\, E\, M_{B,\alpha+1}.$$

In $\mathcal{M}$, let $\rho_\gamma = \rho \restriction M_{B,\gamma}$. Clearly, $\rho_\gamma \in N$. We prove both parts of (4) simultaneously by induction in $\mathcal{N}$ on pairs $(\rho_\gamma(\sigma), \rho_\gamma(\tau))$, well-ordered in the canonical way. We have

$$\begin{aligned}
[\![\sigma \in \tau]\!]^{\mathcal{M}}_B \in U &\iff (\textstyle\bigvee_{t\, E\, \mathrm{dom}\ \tau} \tau(t) \wedge [\![\sigma = t]\!]_B)^{\mathcal{M}} \in U \\
&\iff \text{for some } t \in (\mathrm{dom}\ \tau)^{\mathcal{M}}_E,\ [\tau(t)^{\mathcal{M}} \in U \text{ and } [\![\sigma = \tau]\!]^{\mathcal{M}}_B \in U] \\
&\iff \text{for some } t \in (\mathrm{dom}\ \tau)^{\mathcal{M}}_E,\ [\tau(t)^{\mathcal{M}} \in U \text{ and } \mathcal{N} \models i_{\gamma,\Gamma}(\sigma) = i_{\gamma,\Gamma}(t)] \\
&\iff \mathcal{N} \models \exists t\, F\, \mathrm{dom}\ \tau\, [\tau(t)\, F\, \Gamma \text{ and } i_{\gamma,\Gamma}(\sigma) = i_{\gamma,\Gamma}(\tau)] \\
&\iff \mathcal{N} \models \exists t\, F\, \mathrm{dom}\ \tau\, [i_{\gamma,\Gamma}(t)\, F\, i_{\gamma,\Gamma}(\tau) \text{ and } i_{\gamma,\Gamma}(\sigma) = i_{\gamma,\Gamma}(\tau)] \\
&\iff \mathcal{N} \models i_{\gamma,\Gamma}(\sigma)\, F\, i_{\gamma,\Gamma}(\tau).
\end{aligned}$$

For the equality case, it suffices to prove the following:

$$\mathcal{N} \models i_{\gamma,\Gamma}(\sigma) \subseteq i_{\gamma,\Gamma}(\tau) \iff [\![\sigma \subseteq \tau]\!]^{\mathcal{M}}_B \in U. \tag{3.11}$$

We have:

$$\begin{aligned}
[\![\sigma \subseteq \tau]\!]^{\mathcal{M}}_B \in U &\iff (\textstyle\bigwedge_{s\, E\, \mathrm{dom}\ \sigma} \sigma(s) \to [\![s \in \tau]\!]_B)^{\mathcal{M}} \in U \\
&\iff \forall s \in (\mathrm{dom}\ \sigma)^{\mathcal{M}}_E\ (\sigma(s)^{\mathcal{M}} \in U \implies [\![s \in \tau]\!]^{\mathcal{M}}_B \in U) \\
&\iff \mathcal{N} \models \forall s\, F\, \mathrm{dom}\ \sigma\, (\sigma(s)\, F\, \Gamma \implies i_{\gamma,\Gamma}(s)\, F\, i_{\gamma,\Gamma}(\tau)) \\
&\iff \mathcal{N} \models \forall s\, ([s\, F\, \mathrm{dom}\ \sigma\ \wedge\ \sigma(s)\, F\, \Gamma] \implies i_{\gamma,\Gamma}(s)\, F\, i_{\gamma,\Gamma}(\tau)) \\
&\iff \mathcal{N} \models \forall s\, ([i_{\gamma,\Gamma}(s)\, F\, i_{\gamma,\Gamma}(\sigma)] \implies i_{\gamma,\Gamma}(s)\, F\, i_{\gamma,\Gamma}(\tau)) \\
&\iff \mathcal{N} \models i_{\gamma,\Gamma}(\sigma) \subseteq i_{\gamma,\Gamma}(\tau).
\end{aligned}$$

This completes the proof of Observations (1)-(4). We now define g by

$$g((i_{\gamma,\mathbf{u}_U}(\sigma))^{\mathcal{M}_U}) = (i_{\gamma,\Gamma}(\sigma))^{\mathcal{N}}.$$

By (3.9), g does not depend upon the choice of γ. Moreover, g is well-defined and one-one because

$$\begin{aligned}
g((i_{\gamma,\mathbf{u}_U}(\sigma))^{\mathcal{M}_U}) = g((i_{\gamma,\mathbf{u}_U}(\tau))^{\mathcal{M}_U}) &\iff i_{\gamma,\Gamma}(\sigma)^{\mathcal{N}} = i_{\gamma,\Gamma}(\tau)^{\mathcal{N}} \\
&\iff [\![\sigma = \tau]\!]^{\mathcal{M}}_B \in U \\
&\iff (i_{\gamma,\mathbf{u}_U}(\sigma))^{\mathcal{M}_U} = (i_{\gamma,\mathbf{u}_U}(\tau))^{\mathcal{M}_U}.
\end{aligned}$$

We can establish the isomorphism property of g by replacing equality with the appropriate membership relations in the above argument. The proof that $g''M_U$ is a transitive subset of N follows immediately from the definition of g and of the i_γ's. The proof that g is the identity on M follows from Observation (1) and its analogue for M_U. ∎

Typically, if U is B-generic over $\mathcal{M}$, then $U \notin M$; unfortunately, $U \notin M_U$ either, typically. The correct formulation is a minor variation of the the usual result.

PROPOSITION 17 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC. *Suppose $\mathcal{M} \models$ "B is an
atomless complete Boolean algebra".*

(1) *If U is B-generic over $\mathcal{M}$ and U has a meet in B_E, then $\bigwedge U \notin U$.*

(2) *For any U that is B-generic over $\mathcal{M}$, $\mathbf{u}_U \notin s''_U M$.*

(3) *Suppose $\mathcal{M} \models$ "$\Gamma \subseteq B$ and $P = P(B)$". Then Γ is not internally P-generic in $\mathcal{M}$.*

Proof of (1). Suppose U has a meet in B_E and $\bigwedge U \in U$. First we show that $\bigwedge U$ is an atom of B_E: Suppose there exists $b \in B_E$ for which $0 < b < \bigwedge U$. Let $D = \{c \in B_E : 0 < c < \bigwedge U\}$. By considering the dense set $\{d \in B_E : d < \bigwedge U$ or $d \wedge \bigwedge U = 0\}$, one shows that there is $d \in U \cap D$. But now d is an element of U below the meet of U; since this is impossible, $\bigwedge U$ must be an atom of B_E.

To complete the proof, let $b = \bigwedge U$. By (1.4), $\mathcal{M} \models$ "b is an atom of B." ■

Proof of (2). Suppose U is B-generic over $\mathcal{M}$ and $\mathbf{u}_U \in s''_U M$. Let $\Gamma \in M$ be such that $\mathbf{u}_U = s_U(\Gamma)$. We show that U has a meet in B_E and $\bigwedge U \in U$, contradicting (1). Using (3.2) and Proposition 15, we have

$$s''_U(\Gamma_E) = s_U(\Gamma)_{E_U} = (\mathbf{u}_U)_{E_U} = U = s''_U U,$$

and it follows that $\Gamma_E = U$. Thus Γ is a set $X \in M$ for which $X_E \subseteq U$; thus $\Gamma_E = U$ has a meet in U. ■

Proof of (3). Suppose Γ is internally P-generic in $\mathcal{M}$ (recall Definition 1(2)). Let $U = \Gamma_E$. By (1.4), U is an ultrafilter in B_E; we show it is B-generic over $\mathcal{M}$: Suppose $\mathcal{M} \models X \subseteq B$ and $X_E \subseteq U$. By (1.4) again, $\mathcal{M} \models X \subseteq \Gamma$. By genericity of Γ in $\mathcal{M}$, $\mathcal{M} \models b = \bigwedge X \in \Gamma$. By (1.4), $b \in U$ and b is the meet of X_E in B_E. We have shown $\bigwedge(X_E) \in U$, and hence that U is B-generic over $\mathcal{M}$. But now again notice that Γ itself is an $X \in M$ for which $X_E \subseteq U$, and so $\bigwedge U = \bigwedge \Gamma_E \in U$, contradicting (1).■

If b is an atom of B in $\mathcal{M}$, the usual proof shows that the filter Γ generated by b is an ultrafilter that is internally $P(B)$-generic in $\mathcal{M}$. Letting $U = \Gamma_E$, we have that

$$[s_U(\Gamma)]_{E_U} = s''_U(\Gamma_E) = s''_U U = (\mathbf{u}_U)_{E_U},$$

from which it follows that $\mathbf{u}_U \in s''_U M$.

In the present context of possibly non-wellfounded models, since isomorphism is not the same as equality (as it is in the transitive case), it might seem possible that forcing over $\mathcal{M}$ with an atomless complete Boolean algebra always produces a model $\mathcal{M}_U$ that is *not isomorphic* to $\mathcal{M}$. This is not true, though. If, for example, $\mathcal{M}$ is itself a forcing extension $(\mathcal{M}_0)_{U_0}$ obtained by adding a single Cohen real, and $\mathcal{M}_U$ is obtained from $\mathcal{M}$ again by adding a single Cohen real, then it is well-known that $\mathcal{M} \cong \mathcal{M}_U$. (To work out the proof of this in the present context, use Proposition 18(1) and Theorem 21.)

Next we show that forcing with isomorphic complete Boolean algebras produces isomorphic forcing extensions.

PROPOSITION 18 . *Suppose* $\mathcal{M} = \langle M, E\rangle$ *is a model of* ZFC.

(1) *Suppose that, in* $\mathcal{M}$*,* B *and* C *are complete Boolean algebras and* $i : B \to C$ *is an isomorphism. Then for any ultrafilter* U *that is* B*-generic over* $\mathcal{M}$*,* $\mathrm{graph}(i)''U$ *is* C*-generic over* $\mathcal{M}$ *and* i *induces an isomorphism* $i^U : \mathcal{M}_U \to \mathcal{M}_{U'}$*, where* $U' = \mathrm{graph}(i)''U$*. Moreover* $i^U \circ s_U = s_{U'}$*.*

(2) *In* $\mathcal{M}$*, suppose* B *is a complete Boolean algebra. Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are both* B*-valued models of* ZFC *and that there is an isomorphism (a structure-preserving bijection)* $j : \mathcal{A} \to \mathcal{B}$*, all defined in* $\mathcal{M}$*. Suppose* U *is* B*-generic over* $\mathcal{M}$*. Let* $\mathcal{M}_{\mathcal{A},U}, \mathcal{M}_{\mathcal{B},U}$ *denote the respective collapses of* $\mathcal{A}, \mathcal{B}$ *by* U*. Then* $\mathcal{M}_{\mathcal{A},U} \cong \mathcal{M}_{\mathcal{B},U}$

Proof of (1). Using the fact that i induces an isomorphism $j : B_E \to C_E$, it is easy to verify that $U' = \mathrm{graph}(i)''U$ is C-generic over $\mathcal{M}$. The usual argument [2, 3.12], shows that, in $\mathcal{M}$, i induces a Boolean-valued isomorphism $\bar{i} : M^B \to M^C$; in particular, for all $\sigma, \tau \, E \, M^B$ and $b \, E \, B$,

$$\begin{array}{ll} [\![\sigma = \tau]\!]_B = b & \iff [\![\bar{i}(\sigma) = \bar{i}(\tau)]\!]_C = i(b) \\ [\![\sigma \in \tau]\!]_B = b & \iff [\![\bar{i}(\sigma) \in \bar{i}(\tau)]\!]_C = i(b). \end{array}$$

Define (in V) $i^U : \mathcal{M}_U \to \mathcal{M}_{U'}$ by

$$i^U(\tau_U) = \text{unique } \sigma_{U'} \in \mathcal{M}_{U'} \text{ such that } \mathcal{M} \models \bar{i}(\tau) = \sigma.$$

Verification that i^U is a well-defined isomorphism makes use of the properties of $\bar{i}$; the proofs are routine so we omit them. To see that $i^U \circ s_U = s_{U'}$, use the fact that, in $\mathcal{M}$ $\bar{i}(\check{x}) = \check{x}$ for all x.∎

Proof of (2). Define $f : \mathcal{M}_{\mathcal{A},U} \to \mathcal{M}_{\mathcal{B},U}$ by

$$f(\sigma_U^{\mathcal{A}}) = (j(\sigma))_U^{\mathcal{B}}.$$

Now the fact that f is a well-defined isomorphism follows from onto-ness of j and the following two equations (which hold for all $\sigma, \tau \in \mathcal{A}$):

$$\begin{aligned} [\![\sigma = \tau]\!]_{\mathcal{A}} &= [\![j(\sigma) = j(\tau)]\!]_{\mathcal{B}} \\ [\![\sigma \in \tau]\!]_{\mathcal{A}} &= [\![j(\sigma) \in j(\tau)]\!]_{\mathcal{B}}. \blacksquare \end{aligned}$$

Suppose $i : B \to C$ in $\mathcal{M}$ is an isomorphism and U is a B-generic ultrafilter over $\mathcal{M}$. Let $U' = i''U$. Then we will say that U and U' are *canonically isomorphic generic ultrafilters.*

To conclude this subsection, we develop some of the ideas needed for doing forcing with partial orders in $\mathcal{M}$. We let $\mathcal{M}, P, B$ be defined as above. Let $e : P \to B$ be a dense embedding. Let G be a filter in P_E. We will say that G is P generic over $\mathcal{M}$ if, for every $D \in \mathcal{M}$ for which $\mathcal{M} \models$ "D is dense in P" we have $G \cap D_E \neq \emptyset$.

Proposition 19 *Let $\mathcal{M} = \langle M, E\rangle$ be a model of* ZFC *such that, in $\mathcal{M}$, P is a partial order, B is a complete Boolean algebra, and $e : P \to B$ is a dense embedding.*

(1) *Suppose U is B-generic over $\mathcal{M}$. Define G by*

$$G = \{p \in P_E : e(p)^{\mathcal{M}} \in U\}. \tag{3.12}$$

Then G is P-generic over $\mathcal{M}$.

(2) *Suppose G is P-generic over $\mathcal{M}$. Define U by*

$$U = \{b \in B_E : \exists p \in G\, \mathcal{M} \models e(p) \leq b\}. \tag{3.13}$$

Then U is B-generic over $\mathcal{M}$.

Proof. The proof is very much like the usual one (see [15, Lemma 17.4]), using Proposition 4 to weave in and out of $\mathcal{M}$ as needed. We prove the genericity part of (1) and leave the rest to the reader.

Suppose $\mathcal{M} \models$ "D is dense in P". Then, in $\mathcal{M}$, $D_e = e''D$ is dense in $B \setminus \{0\}$. So $(D_e)_E = \text{graph}(e)''(D_E)$ is dense in $B_E \setminus \{0\}$, and we can find $p \in D_E$ such that $e(p) \in (D_e)_E \cap U$. It follows that $p \in D_E \cap G$. $\blacksquare$

Whenever we are given G as above, we will call U, as defined in (3.13), the *B-generic ultrafilter over $\mathcal{M}$ derived from G and e.* Likewise, if we are given U, we call G, as defined in (3.12), the *P-generic filter over $\mathcal{M}$ derived from U and e.* We suppress mention of e if it is clear from the context. It is easy to verify that

$$\begin{aligned} &U \text{ is the } B\text{-generic ultrafilter derived from } G, e \iff \\ &\qquad G \text{ is the } P\text{-generic filter derived from } U, e. \end{aligned} \tag{3.14}$$

Whenever we are given $\mathcal{M}, P, B, e$ as above, and G is P-generic over $\mathcal{M}$, we evaluate terms $\sigma \in (M^B)_E$ by putting $\sigma_G = \sigma_U$ and we let $\mathcal{M}_G$ be simply $\mathcal{M}_U$, where U is the B-generic ultrafilter over $\mathcal{M}$ derived from G.

Whenever P and Q are partial orders (in $\mathcal{M}$) having isomorphic completions, we say that P and Q are *forcing equivalent* and write $P \sim Q$. Clearly, forcing with forcing equivalent partial orders produces isomorphic extensions. We also make the following definition:

Suppose in $\mathcal{M}$, $i : \text{ro}(P) \to \text{ro}(Q)$ is an isomorphism, $e_P : P \to \text{ro}(P)$ and $e_Q : Q \to \text{ro}(Q)$ are dense embeddings, G is P-generic over $\mathcal{M}$, H is Q-generic over $\mathcal{M}$, and $\text{graph}(i)''[\text{graph}(e_P)''G] = \text{graph}(e_Q)''H$. Then G and H are said to be *canonically equivalent generic filters.*

The next corollary gives more information about the canonical name for a generic filter in P:

Corollary 20 Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of ZFC and, in $\mathcal{M}$, P is a partial order, $B = \text{ro}(P)$, and $e : P \to B$ is a dense embedding.

(1) $[\![\mathbf{g}$ is the generic filter in $\check{P}$ derived from $\mathbf{u}$ and $\check{e}]\!]^{\mathcal{M}}_B = 1$.

(2) $[\![\mathbf{u}$ is the generic ultrafilter in $\check{B}$ derived from $\mathbf{g}$ and $\check{e}]\!]^{\mathcal{M}}_B = 1$.

(3) Suppose G is P-generic over $\mathcal{M}$ and let U be the B-generic ultrafilter derived from G. Then $G = (\mathbf{g}_U)_{E_U}$ (where $(\mathbf{g}_U)_{E_U}$ denotes the extension of $\mathbf{g}_U \in \mathcal{M}_U$).

Proof. Parts (1) and (2) follow easily from Theorem 9(3). For (3), we have the following chain of equivalences for a given $p \in P$:

$$\begin{aligned} p \in G &\iff (e(p))^{\mathcal{M}} \in U \\ &\iff (e(p))^{\mathcal{M}} \in (\mathbf{u}_U)_{E_U} \\ &\iff \mathcal{M}_U \models e(p)\, E_U\, \mathbf{u}_U \\ &\iff \mathcal{M}_U \models p\, E_U\, \mathbf{g}_U \\ &\iff p \in (\mathbf{g}_U)_{E_U}. \end{aligned}$$

4 Two-step Iterations

Our objective in this section is to show that if, in $\mathcal{M}$, B is a complete Boolean algebra and, still in $\mathcal{M}$, $[\![\chi$ is a complete Boolean algebra $]\!] = 1$, then there is a complete Boolean algebra $C = B * \chi$ defined in $\mathcal{M}$ such that forcing with C is "the same as" forcing with B and then with χ. The proof requires maneuvers among the internal worlds of several (possibly) non-wellfounded models, and these steps require some care. The usual proof for transitive models makes substantial use of the transitive collapsing function $\eta_U : M^B \to M[U]$; our proof requires that we work with the equivalence classes by U directly. This leads only to an *isomorphism* (rather than equality) between the model obtained via a two-step iteration and that obtained via its canonical one-step analogue.

We begin by fixing the following notation: $\mathcal{M} = \langle M, E\rangle$ is a model of ZFC, and $P, B, \pi, \chi \in M$ are such that, in $\mathcal{M}$ P is a partial order and $B = \mathrm{ro}(P)$, and

$$([\![\pi \text{ is a partial order and } \chi = \mathrm{ro}(\pi)]\!]_B = 1)^{\mathcal{M}}.$$

In $\mathcal{M}$ we define an equivalence relation $\sim$ on the $\mathcal{M}$-class

$$\{\sigma : \sigma\, E\, M^B \text{ and } [\![\sigma \in \chi]\!]_B = 1\}$$

by putting $\sigma \sim \tau$ if and only if $[\![\sigma = \tau]\!]_B = 1$. In $\mathcal{M}$, let $B * \chi$ denote a set of representatives from the $\sim$-equivalence classes (C is a set by Theorem 5(5) since each member of $B * \chi$ is determined by a pair (A, W) where A is a maximal antichain in B and $W \subseteq \mathrm{dom}\, \chi$. In $\mathcal{M}$, let $C = B * \chi$. In $\mathcal{M}$, define a meet operation $\wedge = \wedge_C$ on C by

$$\sigma \wedge \tau = \mu \qquad \text{iff} \qquad [\![\sigma \wedge \tau = \mu]\!]_B = 1.$$

In a similar fashion, define the operations $\vee_C, *_C$. Still in $\mathcal{M}$, define a map $u = u_{B,\chi} : B \to B * \chi$ as follows: For each $b \in B$, let σ_b be the unique element of C such that $[\![\sigma_b = 1_C]\!]_B = b$ and $[\![\sigma_b = 0_C]\!] = b^*$. The map is well-defined by Theorem 5(4).

In $\mathcal{M}$, let e_P and $\dot{e}_\pi$ witness that the completions of P and π are B and χ, respectively; that is, $e_P : P \to B$ is a dense embedding and $[\![\dot{e}_\pi : \pi \to \chi$ is a dense embedding $]\!]_B = 1$. Let $P_e = e_P'' P$ and let π_e be a B-name such that $[\![e''\pi = \pi_e]\!]_B = 1$.

Define $P_e * \pi_e$ to be the following suborder of C: Put $\sigma \in P_e * \pi_e$ if and only if there exist $p \in P_e$ and $\mu \in C$ such that

$$[\![\mu \in \pi_e]\!]_B = 1 \qquad \text{and} \qquad \sigma = u(p) \wedge_C \mu.$$

An alternative definition of two-step iteration for partial orders is useful. In $\mathcal{M}$, we define $P \otimes \pi$ as follows: Let $\bar{\pi}$ be a set of representatives of equivalence classes determined by the equivalence relation $[\![\sigma = \tau]\!]_B = 1$, defined on the $\mathcal{M}$-class $\{\sigma : [\![\sigma \in \pi]\!]_B = 1\}$. (Theorem 5(5) can be used to show that $\bar{\pi}$ is a set.) Then the underlying set for $P \otimes \pi$ is $P \times \bar{\pi}$. (This is a way of ensuring that "full names" are used in iterations, in the sense of [19, Chapter VIII].) Identify elements $(p, \sigma), (q, \tau) \in P \otimes \pi$ whenever $p = q$ and $p \Vdash \sigma = \tau$. Define an order relation on $P \otimes \pi$ by putting $(p, \sigma) \leq (q, \tau)$ if and only if $p \leq q$ and $p \Vdash \sigma \leq \tau$.

Given a B-generic ultrafilter U_1 over $\mathcal{M}$ and a χ_{U_1}-generic ultrafilter U_2 over $\mathcal{M}_{U_1}$, we define

$$U_1 * U_2 = \{\sigma \in (B * \chi)_E : \sigma_{U_1} \in U_2\}.$$

If G_1 is P-generic over $\mathcal{M}$ and G_2 is π_{G_1}-generic over $\mathcal{M}_{G_1}$, we define

$$G_1 \otimes G_2 = \{\mathrm{op}_{\mathcal{M}}(p, \sigma) \in (P \otimes \pi)_E : p \in G_1 \text{ and } \sigma_{G_1} \in G_2\}.$$

Theorem 21 *Suppose* $\mathcal{M} = \langle M, E\rangle$ *is a model of* ZFC *and suppose* $B, \chi, C, P, \pi, e_P, \dot{e}_\pi, P_e, \pi_e, u_{B,\chi}$ *are defined as above.*

(1) $\mathcal{M} \models$ *"C is a complete Boolean algebra under the operations* $\wedge_C, \vee_C, *_C$*".*

(2) *In* $\mathcal{M}$ *: The order relation* $\leq_C$ *induced by the Boolean operations* $\wedge_C, \vee_C, *_C$ *satisfies:*

$$\sigma \leq_C \tau \qquad \textit{iff} \qquad [\![\sigma \leq_\chi \tau]\!]_B = 1.$$

(3) *In* $\mathcal{M}$*, the map* $u_{B,\chi}$ *is a one-one complete homomorphism.*

(4) *In* $\mathcal{M}$*,* $\mathrm{ro}(P_e * \pi_e) \cong B * \chi$*.*

(5) *In* $\mathcal{M}$*,* $P_e \otimes \pi_e \cong P_e * \pi_e$*.*

(6) *In* $\mathcal{M}$*,* $\mathrm{ro}(P \otimes \pi) \cong B * \chi$*. Indeed, the function* $f : P \otimes \pi \to B * \chi$ *defined in* $\mathcal{M}$ *by* $f(p, \sigma) = e_P(p) \wedge_C \sigma_e$ *(where* σ_e *is a B-name for* $e_\pi(\sigma)$*) is a dense embedding with the following property: Suppose that* U_1, U_2 *are as above, and* G_1, G_2 *are the corresponding derived generic filters, or, equivalently, that* G_1, G_2 *are as above and* U_1, U_2 *are the corresponding derived generic ultrafilters. Then*

$$G_1 \otimes G_2 = \{\mathrm{op}_{\mathcal{M}}(p, \sigma) \in (P \otimes \pi)_E : f(p, \sigma)^{\mathcal{M}} \in U_1 * U_2\}.$$

(7) *Suppose* U_1, U_2, G_1, G_2 *are defined as above.*

(a) $U_1 * U_2$ *is* $B * \chi$*-generic over* $\mathcal{M}$.

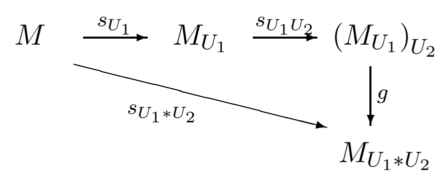
(b) *If* f *is defined as in (6),* $G_1 \otimes G_2$ *is the* $P \otimes \pi$*-generic filter over* $\mathcal{M}$ *that is derived from* $U_1 * U_2$ *and* f.

(c) *There is an isomorphism* $g : (\mathcal{M}_{U_1})_{U_2} \to \mathcal{M}_{U_1 * U_2}$ *with the following property: if* s_{U_1}, $s_{U_1U_2}, s_{U_1*U_2}$ *represent the usual insertion maps, then*

$$g \circ s_{U_1U_2} \circ s_{U_1} = s_{U_1*U_2},$$

and $g \circ s_{U_1U_2}$ *is a transitive embedding. Moreover, treating a* B*-name* σ *as a* $B * \chi$ *name, we have*

$$g(s_{U_1U_2}(\sigma_{U_1})) = \sigma_{U_1*U_2}. \tag{4.1}$$



Remarks

(1) Among the standard proofs that show that two-step iterations are equivalent to canonical one-step iterations, the one that seems most easily adapted to the context of non-wellfounded models is the Boolean-valued model approach. Part (7) of the theorem, along with Theorem 23 below, provides the details of this adaptation. However, many theorems about iterated forcing are most easily stated in terms of the partial order approach. Part (6) of the theorem shows that, as is the case for transitive ground models, the partial order approach can be used in combination with the Boolean algebra approach.

(2) In light of (3), we will treat B as a complete subalgebra of $B * \chi$ in parts (6) and (7), and in the sequel.

(3) By (3.14), one may also conclude in (7b) that $U_1 * U_2$ is the $B * \chi$-generic ultrafilter over $\mathcal{M}$ that is derived from $G_1 \otimes G_2$ and f.

(4) In the case of transitive ground models, one easily proves that $M[G_1][G_2] = M[G_1 \otimes G_2]$ by invoking the standard Minimality Theorem. In the present context, the relevant minimality theorem is Theorem 16, but this only gives us one-one embeddings in either direction between $(\mathcal{M}_{U_1})_{U_2}$ and $\mathcal{M}_{U_1 * U_2}$ — it is not obvious that either embedding is onto; nor is it obvious that the embeddings are inverses of each other. We have taken a simpler approach in our proof that these models are isomorphic by using instead the well-known isomorphism between the Boolean-valued models $(M^B)^C$ and M^{B*C}.

(5) With reference to (7c), it is easy to show that *any* isomorphism $h : (\mathcal{M}_{U_1})_{U_2} \to \mathcal{M}_{U_1 * U_2}$ has the property that $h \circ s_{U_1 U_2}$ is a transitive embedding.

Proof of Theorem. Proofs of (1)–(4) can be found in [15] and [2]. For (5), the map that works is $P_e \otimes \pi_e \to P_e * \pi_e : (p, \sigma) \to p \wedge_C \sigma$ (see [15] for more details). For (6), because, in $\mathcal{M}$, $\mathrm{ro}(P) \cong \mathrm{ro}(P_e)$ and $[\![\mathrm{ro}(\pi) \cong \mathrm{ro}(\pi_e)]\!]_B = 1$, it follows (see [19, VIII.K1]) that

$$\mathrm{ro}(P \otimes \pi) \cong \mathrm{ro}(P_e \otimes \pi_e) \cong \mathrm{ro}(P_e * \pi_e) \cong B * \chi.$$

To obtain the specific results for f, we give an outline:

Argue in $\mathcal{M}$. The fact that $f''P \otimes \pi$ is dense in C follows from (5). To see that $(p, \sigma) \leq (q, \tau)$ implies $f(p, \sigma) \leq f(q, \tau)$, note that, by (5) (and the map given in the proof), it suffices to show that

(a) $e_P(p) \leq e_P(q)$ and

(b) $e_P(p) \Vdash \dot{e}_\pi(\sigma) \leq \dot{e}_\pi(\tau)$.

Part (a) follows because e_P is a dense embedding. For part (b), likewise, since, in $\mathcal{M}^B$, $\dot{e}_\pi$ is a dense embedding, we have

$$p \Vdash \sigma \leq \tau \Longrightarrow e_P(p) \leq [\![\sigma \leq \tau]\!]_B \Longrightarrow e_P(p) \leq [\![\dot{e}_\pi(\sigma) \leq \dot{e}_\pi(\tau)]\!]_B.$$

To see that $(p, \sigma) \perp (q, \tau)$ implies $f(p, \sigma) \perp f(q, \tau)$, assume $f(p, \sigma)$ and $f(q, \tau)$ are compatible. Then for some $r \in P$,

$$e_P(r) \leq [\![\exists x\,(x \leq e_\pi(\sigma) \,\wedge\, x \leq e_\pi(\tau))]\!]_B \leq [\![\exists x\,(x \leq \sigma \,\wedge\, x \leq \tau)]\!]_B.$$

Let μ be such that $e_P(r) \leq [\![\mu \leq \sigma \wedge \mu \leq \tau]\!]$. It is easy to check that r must be compatible with p, and any s below both of these must be compatible with q. Pick t below such an s and q. Then $(t, \mu) \leq (p, \sigma), (q, \tau)$, as required.

To prove the last part of (6), it suffices to prove the following: For each $\mathrm{op}_{\mathcal{M}}(p,\sigma) \in (P \otimes \pi)_E$,

$$\mathrm{op}_{\mathcal{M}}(p,\sigma) \in G_1 \otimes G_2 \iff (e_P(p)\ \wedge_C\ \sigma_e)^{\mathcal{M}} \in U_1 * U_2.$$

The main step in the proof is the following claim:

Claim. $e_P(p)^{\mathcal{M}} \in U_1 * U_2$ if and only if $e_P(p)^{\mathcal{M}} \in U_1$.

Proof of Claim. For the proof, we set $p_e = e_P(p)^{\mathcal{M}}$. Recall that p_e is implicitly embedded in $C = B * \chi$ by identifying p_e with the unique $c_e \in C$ for which $[\![c_e = 1_\chi]\!]_B = p_e$ and $[\![c_e = 0_\chi]\!]_B = p_e^*$. Thus:

$$\begin{aligned} p_e \in U_1 &\iff [\![c_e = 1_\chi]\!]_B \in U_1 \\ &\iff (c_e)_{U_1} \in U_2 \\ &\iff (p_e)_{U_1} \in U_2 \\ &\iff p_e \in U_1 * U_2, \end{aligned}$$

and this proves the claim.

Continuation of the Proof of the Theorem. Notice also that

$$\sigma_e \in U_1 * U_2 \iff ((\dot{e}_\pi)_{U_1}(\sigma_{U_1}))^{\mathcal{M}_{U_1}} \in U_2. \tag{4.2}$$

By the Claim and (4.2), we have

$$\begin{aligned} \mathrm{op}_{\mathcal{M}}(p,\sigma) \in G_1 \otimes G_2 &\iff p \in G_1 \text{ and } \sigma_{G_1} \in G_2 \\ &\iff e_P(p)^{\mathcal{M}} \in U_1 \text{ and } ((\dot{e}_\pi)_{U_1}(\sigma_{U_1}))^{\mathcal{M}_{U_1}} \in U_2 \\ &\iff e_P(p)^{\mathcal{M}} \in U_1 * U_2 \text{ and } \sigma_e \in U_1 * U_2 \\ &\iff (e_P(p)\ \wedge_C\ \sigma_e)^{\mathcal{M}} \in U_1 * U_2, \end{aligned}$$

as required.

We turn to the proof of (7). First notice that (7b) follows immediately from (6) and the genericity of $U_1 * U_2$, by Proposition 19. To prove (7a) — that $U_1 * U_2$ is $B * \chi$-generic (we leave the proof that it is an ultrafilter in C_E to the reader) — begin by setting $C = B * \chi$ in $\mathcal{M}$. Suppose $\mathcal{M} \models$ "D is dense in C".

Claim. $\mathcal{M}_{U_1} \models$ "$\check{D}_{U_1}$ is dense in χ_{U_1}".

Proof of Claim. In $\mathcal{M}_{U_1}$ let $\tau_{U_1}\, E_{U_1}\, \chi_{U_1}$. Then there is, in $\mathcal{M}$, a σ in C such that $[\![\sigma = \tau]\!]_B^{\mathcal{M}} \in U_1$. Since $\mathcal{M} \models$ "D is dense in C", there is a $\delta \in M$ such that $\mathcal{M} \models \delta\, E\, D\, \wedge\, \delta \leq_C \sigma$. Thus,

$$\mathcal{M}_{U_1} \models \delta_{U_1} \leq_{\chi_{U_1}} \tau_{U_1} \text{ and } \delta_{U_1}\, E_{U_1}\, \check{D}_{U_1}.$$

Continuation of the Proof of the Theorem. Let $Q, S \in M_{U_1}$ be such that

$$\mathcal{M}_{U_1} \models S = \check{D}_{U_1} \text{ and } Q = \chi_{U_1}.$$

Since U_2 is Q-generic over $\mathcal{M}_{U_1}$, it follows that there is $\tau_{U_1} \in M_{U_1}$ such that $\tau_{U_1} \in S_{E_{U_1}} \cap U_2$. We can find $\sigma \in (M^B)_E$ such that $[\![\sigma = \tau]\!]_B^{\mathcal{M}} \in U_1$ and $\mathcal{M} \models \sigma \, E \, D$. Thus, $\sigma_{U_1} = \tau_{U_1}$ and $\sigma_{U_1} \in U_2$. It follows that $\sigma \in U_1 * U_2$. Thus, we have shown that $(U_1 * U_2) \cap D_E \neq \emptyset$, as required.

Next, we prove that $\mathcal{M}_{U_1 * U_2} \cong (\mathcal{M}_{U_1})_{U_2}$. As in [Be, Chapter 6], we define in $\mathcal{M}$ the following class of names:

$$J^\chi = \{\sigma \, E \, M^B : [\![\sigma \text{ is a } \chi\text{-name}]\!]_B = 1\}.$$

Bell [Be, Chapter 6] shows that J^χ can be endowed with a $B * \chi$-valued structure with the following definitions:

$$\begin{aligned} [\![\sigma = \tau]\!]_{J^\chi} &= \text{unique } c \in B * \chi \text{ such that } [\![c = [\![\sigma = \tau]\!]_\chi]\!]_B = 1 \\ [\![\sigma \in \tau]\!]_{J^\chi} &= \text{unique } c \in B * \chi \text{ such that } [\![c = [\![\sigma \in \tau]\!]_\chi]\!]_B = 1. \end{aligned}$$

Using this structure, Bell shows that, in $\mathcal{M}$, J^χ is isomorphic (as a $B * \chi$ structure) to $\mathcal{M}^{B*\chi}$, and it is easy to verify that in his proof, canonical names are matched in the following way: For any $x \in M$,

$$\check{\check{x}} \mapsto \check{x}. \tag{4.3}$$

For the rest of the proof, we identify J^χ with $\mathcal{M}^{B*\chi}$, treating $\mathcal{M}_{U_1 * U_2}$ as obtainable by collapsing either of these $B * \chi$-valued models by U (this identification is justified by Bell's result and by Theorem 18(2)). As a notational consequence, we shall rewrite Boolean values $[\![\phi]\!]_{J^\chi}$ as $[\![\phi]\!]_{B*\chi}$.

Define $g : (\mathcal{M}_{U_1})_{U_2} \to \mathcal{M}_{U_1 * U_2}$ as follows: Let $\sigma \in (M^B)_E$ be such that $\mathcal{M}_{U_1} \models$ "σ_{U_1} is a χ_{U_1}-name". Note that every element of $(\mathcal{M}_{U_1})_{U_2}$ is of the form $(\sigma_{U_1})_{U_2}$ for such a σ — we shall call such names *U_1-good names*. Let $\sigma' \in (M^B)_E$ be such that $[\![\sigma' \text{ is a } \chi\text{-name}]\!]_B^{\mathcal{M}} = 1$ and $\sigma_{U_1} = \sigma'_{U_1}$. Note that $\sigma' \in (J^\chi)_E$. We shall call σ' an *auxiliary name associated with* σ. Now, using our identification of J^χ and $M^{B*\chi}$, we define g at $(\sigma_{U_1})_{U_2}$ by

$$g((\sigma_{U_1})_{U_2}) = \sigma'_{U_1 * U_2}.$$

We verify that g is well-defined and one-one as follows: Given U_1-good names σ, τ with associated names $\sigma', \tau' \in (J^\chi)_E$, let $c \in (B * \chi)_E$ be such that

$$c = [\![\sigma' = \tau']\!]_{B*\chi}^{\mathcal{M}}. \tag{4.4}$$

By definition of the $B * \chi$ structure on J^χ, we have in $\mathcal{M}$:

$$[\![c = [\![\sigma' = \tau']\!]_\chi]\!]_B = 1. \tag{4.5}$$

We obtain the following chain of equivalences:

$$\begin{aligned}
(\sigma_{U_1})_{U_2} = (\sigma_{U_1})_{U_2} &\iff [\![\sigma'_{U_1} = \tau'_{U_1}]\!]^{\mathcal{M}_{U_1}}_{\chi_{U_1}} \in U_2 \\
&\iff c_{U_1} \in U_2 \qquad\qquad \text{(by (4.5))} \\
&\iff c \in U_1 * U_2 \\
&\iff [\![\sigma' = \tau']\!]^{\mathcal{M}}_{B*\chi} \in U_1 * U_2 \qquad \text{(by (4.4))} \\
&\iff \sigma'_{U_1*U_2} = \tau'_{U_1*U_2} \\
&\iff g((\sigma_{U_1})_{U_2}) = g((\tau_{U_1})_{U_2}).
\end{aligned}$$

Replacing equality with appropriate forms of the membership relation (E_{U_2} or $E_{U_1*U_2}$) in the above chain of equivalences yields a proof that

$$(\sigma_{U_1})_{U_2}\, E_{U_2}\, (\sigma_{U_1})_{U_2} \iff g((\sigma_{U_1})_{U_2})\, E_{U_1*U_2}\, g((\tau_{U_1})_{U_2}).$$

To complete the proof, we must show that g is onto. If $\sigma'_{U_1*U_2} \in \mathcal{M}_{U_1*U_2}$, where $\sigma' \in (J^\chi)_E$, then clearly σ' is a name associated with itself, and we have easily that $g((\sigma'_{U_1})_{U_2}) = \sigma'_{U_1*U_2}$, as required.

To prove (7c), notice that

$$s_{U_1U_2}(s_{U_1}(x)) = ((\check{\check{x}})_{U_1})_{U_2}.$$

Thus, by (4.3),

$$g(s_{U_1U_2}(s_{U_1}(x))) = g\Big(((\check{\check{x}})_{U_1})_{U_2}\Big) = \check{x}_{U_1*U_2} = s_{U_1*U_2}(x).$$

For the second part of (7c), if $x \in M_{U_1}$ and $z\, E_{U_1*U_2}\, g(s_{U_1U_2}(x))$, there is a $y \in (\mathcal{M}_{U_1})_{U_2}$ such that $z = g(y)$ and so $y\, E_{U_2}\, s_{U_1U_2}(x)$. Since $s_{U_1U_2}$ is a transitive embedding, for some $w \in M_{U_1}$, $y = s_{U_1U_2}(w)$. Therefore $z = g(y) = g(s_{U_1U_2}(w)) \in (g \circ s_{U_1U_2})'' M_{U_1}$, as required.

Finally, we verify equation (4.1). When we view a B-name σ as a $B * \chi$ name, we have automatically that $[\![\sigma \text{ is a } \chi \text{ name}]\!]_B = 1$. Thus, σ is its own auxiliary name, and we have

$$g(s_{U_1U_2}(\sigma_{U_1})) = g\Big(((\sigma_{U_1})\check{\ })_{U_2}\Big) = \sigma_{U_1*U_2}. \blacksquare$$

The following is a useful technical corollary to Theorem 21(7). It says, roughly, that the canonical isomorphism $g : (M_{U_1})_{U_2} \to M_{U_1*U_2}$ respects internal collapsing maps.

COROLLARY 22 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC *and suppose* $B, \chi, C, P, \pi, e_P, \dot{e}_\pi, P_e, \pi_e, U_1$, *and* U_2 *are defined as in Theorem 21. Let* $g : (M_{U_1})_{U_2} \to M_{U_1 * U_2}$ *be the canonical isomorphism and let* $s_{U_1}, s_{U_1U_2}$, *and* $s_{U_1*U_2}$ *be the insertion maps, again as in Theorem 21. Let* $i_{\gamma,\mathbf{u}_{U_1}}$ *and* $i_{\gamma,\mathbf{u}_{U_1*U_2}}$ *be the* γ*th internal collapsing maps for* M_{U_1} *and* $M_{U_1*U_2}$, *respectively, as defined in* (3.8). *Then for all* $\sigma \in M_{B,\gamma}$,

$$g(s_{U_1U_2}(i_{\gamma,\mathbf{u}_{U_1}}(s_{U_1}(\sigma)))) = i_{\gamma,\mathbf{u}_{U_1*U_2}}(s_{U_1*U_2}(\sigma)).$$

Proof. By our remarks preceding Theorem 16,

$$i_{\gamma,\mathbf{u}_{U_1}}(s_{U_1}(\sigma)) = \sigma_{U_1} \text{ and } i_{\gamma,\mathbf{u}_{U_1*U_2}}(s_{U_1*U_2}(\sigma)) = \sigma_{U_1*U_2}.$$

The result now follows from the final clause of Theorem 21(7). ∎

A version of the standard converse to Theorem 21(7) is also true; the proof does not differ much from the usual one. We present it as a separate result because we make slightly different assumptions from those used in Theorem 21.

THEOREM 23 *Suppose $\mathcal{M} = \langle M, E\rangle$ is a model of* ZFC. *Suppose that in $\mathcal{M}$, B and C are complete Boolean algebras and $[\![\chi$ is a complete Boolean algebra$]\!]_B = 1$. Suppose $\mathcal{M} \models$ "$h : C \to B * \chi$ is an isomorphism". Suppose U is a C-generic ultrafilter over $\mathcal{M}$.*

(1) *Let* $U_1 = (\mathrm{graph}(h)''U) \cap B_E$. *Then* U_1 *is* B*-generic over* $\mathcal{M}$.

(2) *Define* $U_2 \subseteq \chi_{U_1}$ *as follows: For each* $\tau \in (M^B)_E$ *for which* $[\![\tau \in \chi]\!]^{\mathcal{M}}_B = 1$, *let* τ'—the name associated with τ—*be the unique element of* $B * \chi$ *for which* $[\![\tau' = \tau]\!]^{\mathcal{M}}_B = 1$. *Put* $\tau_{U_1} \in U_2$ *if and only if* $\mathrm{graph}(h^{-1})(\tau') \in U$. *Then* U_2 *is a* χ_{U_1}*-generic ultrafilter over* $\mathcal{M}_{U_1}$.

(3) $\mathrm{graph}(h)''U = U_1 * U_2$.

(4) $\mathcal{M}_U \cong \mathcal{M}_{U_1*U_2}$.

Proof. For (1), we verify genericity only: Suppose $X \in M$ and $X_E \subseteq U_1$. Suppose $\mathcal{M} \models Y = h^{-1}(X)$. Let c be such that $\mathcal{M} \models c = \bigwedge_C Y$. Let b be such that $\mathcal{M} \models b = h(\bigwedge_C Y) = \bigwedge_{B*\chi} X$. Since $c \in U$ and h is an isomorphism, we have

$$\mathrm{graph}(h)(c) = b \in \mathrm{graph}(h)''U.$$

Since $X_E \subseteq B_E$ and $\mathcal{M} \models$ "B is a complete subalgebra of $B * \chi$", it follows that $b = \bigwedge_{B_E} X_E \in (\mathrm{graph}(h)''U) \cap B_E = U_1$.

For (2), suppose $\mathcal{M}_{U_1} \models$ "D_1 is dense in $\chi_{U_1} \setminus \{0\}$". We show that $(D_1)_{E_{U_1}} \cap U_2 \neq \emptyset$, and leave the verification that U_2 is an ultrafilter to the reader. Let $\dot{D}_1$ be a name for D_1 and let $b \in U_1$ be such that

$$\mathcal{M} \models b \leq [\![\dot{D}_1 \text{ is dense in } \chi]\!]_B.$$

Let D be such that

$$\mathcal{M} \models D = \{c\, E\, (B * \chi) \setminus \{0\} : c \leq [\![c \in \dot{D}_1]\!]_B\}.$$

The usual argument (see [15, Lemma 23.4]) shows that

$$\mathcal{M} \models \text{"}D \text{ is dense in } (B * \chi) \setminus \{0\}\text{"}.$$

Now let c, z be such that

$$z \in \mathrm{graph}(h^{-1})(D_E) \cap U,$$

equivalently,

$$\mathrm{graph}(h)(z) = c \in D_E \cap \mathrm{graph}(h)''U. \tag{4.6}$$

Since $\mathrm{graph}(h^{-1})(c) \in U$, by definition, $c_{U_1} \in U_2$. To complete the proof of (2), it suffices to show $\mathcal{M}_{U_1} \models c_{U_1}\, E_{U_1}\, D_1$. Since $\mathcal{M} \models c\, E\, D$, we have $\mathcal{M} \models c \leq [\![c \in \dot{D}_1]\!]_B \in B$, and we conclude from (4.6) that

$$[\![c \in \dot{D}_1]\!]_B^{\mathcal{M}} \in \mathrm{graph}(h)''U \cap B_E = U_1.$$

Thus, $\mathcal{M}_{U_1} \models c_{U_1}\, E_{U_1}\, D_1$, and we are done.

For (3), it suffices to prove $\mathrm{graph}(h)''U \subseteq U_1 * U_2$. Suppose $c \in U$ and let $\mathrm{graph}(h)(c) = d$. Now by definition, $d_{U_1} \in U_2$; that is, $d \in U_1 * U_2$.

For (4), since we have shown that the graph of the isomorphism h carries U to $U_1 * U_2$, we can invoke Theorem 18(1) to conclude that $\mathcal{M}_U \cong \mathcal{M}_{U_1 * U_2}$. ∎

As usual, a kind of inverse operation for $*$ can be defined as follows: In $\mathcal{M}$, suppose D is a complete Boolean algebra and B is a complete subalgebra of D. Let σ be a B-name satisfying $[\![\sigma \text{ is the filter in } \check{D} \text{ generated by } \mathbf{u}_B]\!]_B=1$. Then D/B is a B-name τ satisfying $[\![\tau = \check{D}/\sigma]\!]_B = 1$. The proof of the next proposition can be found in [15] and [2].

Proposition 24 *Suppose in $\mathcal{M}$ we have that B is a complete subalgebra of a complete Boolean algebra D. Then $D \cong B * (D/B)$.* ∎

5 Iterated Forcing

Since iteration of partial orders takes place entirely within the ground model, there are no concerns about iterated forcing that are unique to the setting of non-wellfounded ground models. A typical application of the usual Factor Lemma (which is proven entirely within the ground model) involves breaking up a model $M[G_\alpha]$ obtained by iterated forcing into a model $M[G_\gamma][G_{\alpha-\gamma}]$ obtained by two-step forcing. In the context of arbitrary ground models, this sort of maneuver is addressed by our Two-Step Iteration Theorem (and so, using the analogous notation of this paper, we would have that $\mathcal{M}_{U_\alpha} \cong (\mathcal{M}_{U_\gamma})_{U_{\alpha-\gamma}}$). Therefore, this section on iterated forcing has been included just for the sake of completeness. Since we are using the Boolean algebra approach to forcing, we follow closely the treatment in [15].

We begin by fixing an arbitrary model $\mathcal{M} = \langle M, E\rangle$ of ZFC. Working in $\mathcal{M}$, an α-stage iterated forcing is an object

$$\{\langle P_\xi : \xi \leq \alpha\rangle, \langle B_\xi : \xi \leq \alpha\rangle, \langle e_\xi : \xi \leq \alpha\rangle, \langle \pi_\xi : \xi < \alpha\rangle, \langle i_{\xi\gamma} : \xi < \gamma \leq \alpha\rangle$$

satisfying

(1) Each P_ξ is a partial order.

(2) Each B_ξ is a complete Boolean algebra and $e_\xi : P_\xi \to B_\xi$ is a dense embedding.

(3) For all $\xi \leq \alpha$, $[\![\pi_\xi \text{ is a partial order}]\!]_{B_\xi} = 1$.

(4) For all $\xi < \gamma \leq \alpha$, $i_{\xi\gamma} : B_\xi \to B_\gamma$ is a one-one complete homomorphism, and $\langle i_{\xi\gamma} : \xi < \gamma \leq \alpha\rangle$ is a commutative system.

(5) For each $\xi < \alpha$, $P_{\xi+1} \cong P_\xi \otimes \pi_\xi$.

(6) If $\beta \leq \alpha$ is a limit, then P_β is either the direct or inverse limit of the P_ξ, $\xi < \beta$, and $i_{\xi\beta}$ are the corresponding embeddings.

As in [15], elements of P_α can be identified with functions $p = \langle p_\xi : \xi < \alpha\rangle$ satisfying

(A) $\forall \xi < \alpha\, (p \restriction \xi \in P_\xi)$;

(B) $\forall \xi < \alpha\, ([\![p_\xi \in \pi_\xi]\!]_{B_\xi} = 1)$;

(C) $\forall q, r \in P_\alpha\, (q \leq_\alpha r \iff \forall \xi < \alpha\, [q \restriction \xi \leq_\xi r \restriction \xi \text{ and } q \restriction \xi \Vdash_\xi q_\xi \leq_{\pi_\xi} r_\xi])$.

Moreover, P_α consists of all functions that satisfy (A)–(C) if α is a limit and P_α is an inverse limit. If P_α is a direct limit, then P_α consists of all functions $p = \langle p_\xi : \xi < \alpha \rangle$ satisfying (A)–(C) and also

$$\exists \xi < \alpha \, \forall \beta \, (\beta \geq \xi \implies p_\xi = 1).$$

We may also assume that

$$\text{the embeddings } e_{\xi\gamma} : B_\xi \to B_\gamma \text{ satisfy } e_{\xi\gamma}(p) = p^\frown 1^\frown 1^\frown \ldots. \tag{5.1}$$

When P_α is a direct limit, it is sometimes useful to identify its elements with functions $p = \langle p_\xi : \xi < \beta \rangle$ for some $\beta < \alpha$ that includes the support of p; see [1].

As usual, one can prove the standard Factor Lemma, which says that an iteration P_α can be factored as $P_\beta \otimes \tau_\beta$, where τ_β is, in M^{B_β}, an $(\alpha - \beta)$-stage iteration. See [15, Lemma 36.6].

Our statement of the Factor Lemma here will make use of simplifications due to Baumgartner [1]. We write G_α to denote a filter that is P_α-generic over $\mathcal{M}$. For $\beta < \alpha$, we assume $G_\beta = \{p \restriction \beta \mid p \in G_\alpha\}$; this assumption is warranted by the fact — which can be proved using the standard argument [1, Theorem 1.2] (carried out inside $\mathcal{M}$) — that the set $\{p \restriction \beta \mid p \in G_\alpha\}$ is in fact P_β-generic over $\mathcal{M}$.

As a further simplification, we may specify the tail τ_γ of the previous paragraph as a P_β-name for the set $P_{\beta\alpha}$, which is defined in $\mathcal{M}$ as follows:

$$P_{\beta\alpha} = \{p^\beta : p \in P_\alpha\} \text{ where } p^\beta = p \restriction \{\gamma : \beta \leq \gamma < \alpha\}.$$

The ordering on $P_{\beta\alpha}$ is defined relative to a generic G_β by setting $f \leq g$ (in $\mathcal{M}$) if and only if for some $p \in G_\beta$, $\mathcal{M} \models p \cup f \leq p \cup g$ in P_α. (Here, we have identified $P_{\beta\alpha}$ with its image $s_{U_\beta}(P_{\beta\alpha})$, where $s_{U_\beta} : \mathcal{M} \to \mathcal{M}_{U_\beta}$ is the insertion map and U_β is the generic ultrafilter derived from G_β.) The standard proof [1, Theorem 5.1], carried out in the ground model, then establishes that P_α can be viewed as a two-step iteration of P_β and τ_β:

THEOREM 25 ([1]) *In $\mathcal{M}$, P_α is isomorphic to a dense subset of $P_\beta \otimes \tau_\beta$.* ∎

Then, the Factor Lemma establishes that τ_β itself is a $\alpha - \beta$-stage iteration, as viewed in $\mathcal{M}_{G_\beta}$:

THEOREM 26 ([1]) *In $\mathcal{M}$,*

$$1 \Vdash_{P_\beta} \tau_\beta \textit{ is isomorphic to an } \alpha - \beta\textit{-stage iteration},$$

where $1 = 1_{P_\beta}$ and $\Vdash_{P_\beta}$ is the forcing relation for P_β, in $\mathcal{M}$.

References

[1] J. Baumgartner. *Iterated forcing* in Surveys in Set Theory (A.R.D. Mathias, ed.). Cambridge Univ. Press, 1983, pp 1–59.

[2] J. L. Bell. *Boolean-Valued Models and Independence Proofs in Set Theory.* Clarendon Press, Oxford, 1985.

[3] J. L. Bell, M. Machover. *A Course In Mathematical Logic* North-Holland Publishing Company, 1977.

[4] P. Corazza. *A detailed proof of Hamkins' consistency result for* WA_0. Unpublished notes

[5] P. Corazza. *The indestructibility of wholeness.* In preparation.

[6] P. Corazza. *The spectrum of elementary embeddings* $j : V \to V$ Annals of Pure and Applied Logic. (2006) 139, pp. 327–399.

[7] P. Corazza. *Consistency of* $V = HOD$ *with the wholeness axiom.* Archive for Mathematical Logic. (2000) 39, pp. 219–226.

[8] P. Corazza. *The wholeness axiom and Laver sequences.* Annals of Pure and Applied Logic. (2000) 105, pp. 157–260.

[9] M. Di Nasso. *On the foundations of nonstandard mathematics.* Math. Japonica (1999) 50, pp. 131–160.

[10] F. Drake. *Set theory: An introduction to large cardinals.* American Elsevier Publishing Co., New York, 1974.

[11] A. Enayat. *Minimal elementary extensions of models of set theory and arithmetic.* Archive for Mathematical Logic (1990) 30, pp. 181–192.

[12] J. Hamkins. *The wholeness axioms and* $V = \mathrm{HOD}$. Archive for Mathematical Logic. (2001) 40, 1, pp. 1-8.

[13] J. Hamkins. *Fragile measurability.* Journal of Symbolic Logic (1994) 59, 1, pp. 262-282.

[14] T. Jech. *Boolean-valued models* Handbook of Boolean Algebras North-Holland, 1989, pp. 1197–1211.

[15] T. Jech. *Set Theory.* Academic Press, New York, 1978.

[16] V. Kanovei and M. Reeken. *Nonstandard analysis, axiomatically* Springer-Verlag, Heidelberg Berlin New York, 2004.

[17] V. Kanovei and M. Reeken. *Isomorphism property in nonstandard extensions of the ZFC universe* Ann. Pure Appl. Logic (1997) 88, pp. 1–25.

[18] V. Kanovei and M. Reeken. *Mathematics in a nonstandard universe* Math. Japonica (1997) 45, Part I, pp. 369–408; Part II, pp. 555–571.

[19] K. Kunen. *Set theory: an introduction to independence proofs.* North Holland, New York, 1980.

[20] S. Shelah. *Proper and Improper Forcing.* Springer-Verlag, 1998.

[21] P. Vopěnka, P., P. Hajek. *The Theory of Semisets.* North-Holland Publ. Co., Amsterdam, 1972.