The Magical Origin of the Set of Natural Numbers {1, 2, 3, . . . }

Significance of the Natural Numbers in Mathematics

- Traditional View: They are the source of mathematics itself
- Leopold Kronecker (19^{th} century mathematician):

  God made natural numbers; all else is the work of man.
- Peano developed a standard set of axioms for deriving the natural numbers – his axioms are now denoted PA (Peano Arithmetic)

Significance (continued)

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- *From the natural numbers and the knowledge that the natural numbers form an infinite set, all of mathematics can be derived.*

The Perspective from MVS

In 1994-5, Maharishi remarked (mostly can be found in *Absolute Theory of Defence*)

> *Creation arises from the natural numbers 1, 2, 3, . . .*

and

> *Viewing natural numbers as unconnected to their source is the beginning of ignorance.*

MVS (continued)

On the Source of the Natural Numbers: (ATD pp. 614-15)

The ever-expanding value of the universe, in terms of an infinity of numbers, is the natural characteristic feature of the Absolute Number, which enables all numbers to function from their common basis. It is this effect of the Absolute Number on all numbers that actually initiates and maintains order in the ever-evolving infinite diversity of the universe.

How Natural Numbers Are Defined in Mathematics

Were mathematicians the creators of ignorance?

- The purpose of the Axiom of Infinity in ZFC is to assert that the natural numbers 1,2 ,3… can be collected together into a single completed set: {1,2,3,…}
- Since everything in set theory is a set, the "natural numbers" must also be rendered as sets

Natural Numbers in Mathematics (continued)

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Axiom of Infinity leads to definitions of the natural numbers:

$0 = \emptyset$ (empty set)
$1 = \{\emptyset\} = \{0\}$
$2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$
$3 = \{0, 1, 2\}$
etc.

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The ***successor function s*** specifies how to get the next natural number from the previous:

$s(n) = n + 1 = n \cup \{n\}$

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Thesis: We will see the impact whenever mathematics seeks a deeper understanding of what the "infinite" really is. When that happens, set theory will have little to offer.

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Example: The Problem of Large Cardinals

Toward a New Axiom of Infinity: Dedekind-Infinite Sets

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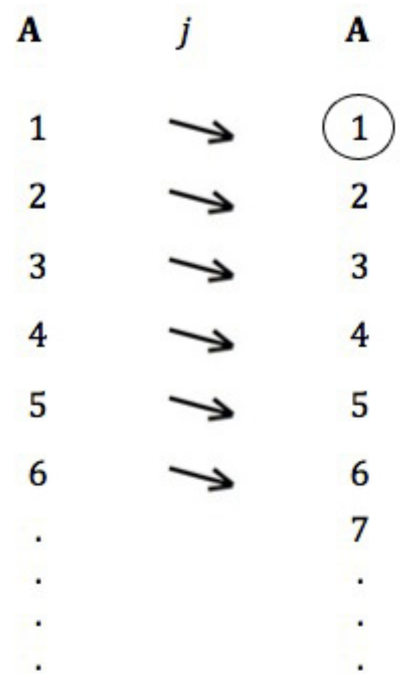
- A set is **Dedekind infinite** if it can be put in 1-1 correspondence with a proper subset of itself.

  $f: A \to B$ where $B \subsetneq A$

- A **Dedekind self-map** $j: A \to A$ with critical point **a** is a matching of elements of A with a subset B of A so that **a** does not belong to B and therefore is not matched with any element of A.

A Dedekind Self-Map with Critical Point = 1



A Dedekind Self-Map $j: A \rightarrow A$:
$j(n) = n + 1$, with critical point =1

NOTE: A = {1 ,2, 3, . . .} and B = {2, 3, 4, . . . }, 1 is not in B and is not matched with any element of A

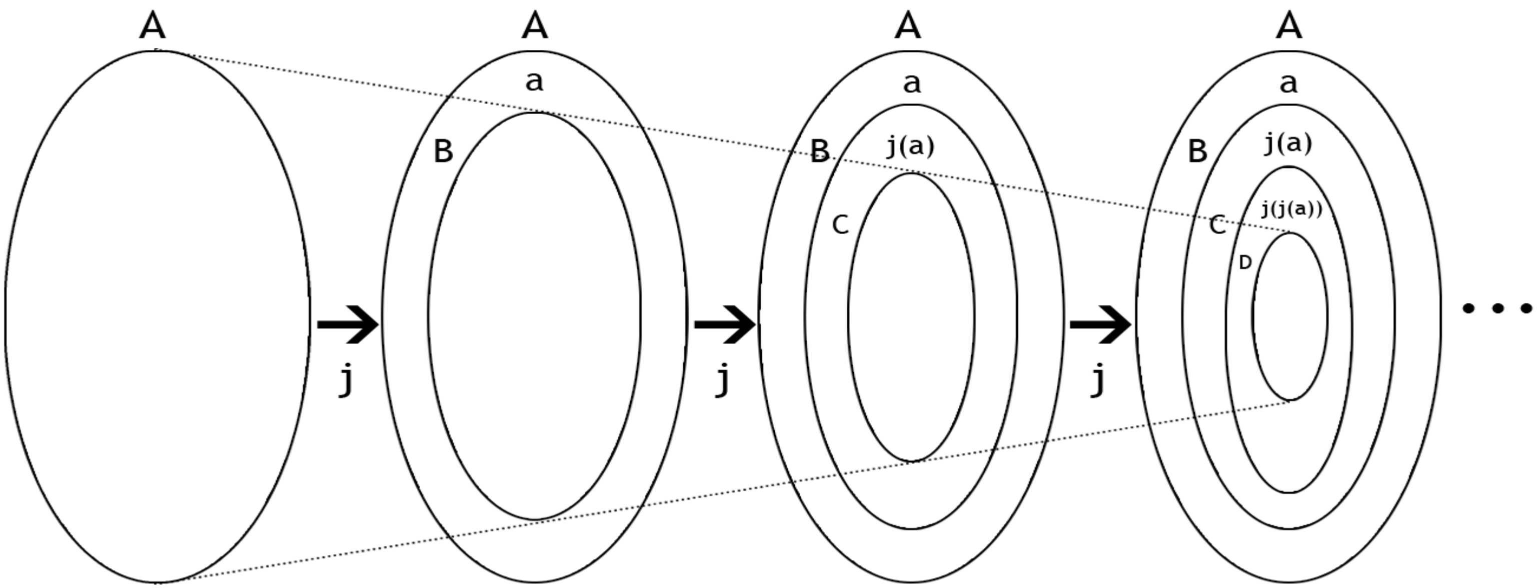
Characteristics of a Dedekind Self Map

Consider a Dedekind self-map j: A -> A with critical point **a**.

- j *preserves* the essential character of A, that of being an *infinite set* (j transforms A into B and B is an infinite subset of A)
- j *transforms A – j* is not just the identity function; values of A are moved around
- j *has a critical point* **a** – the point **a** becomes a focal point for further self-transformation:

  Consider how j acts on B: Its image is another set C, which is a subset of B. Viewed as j: B -> B with critical point j(**a**), we have another Dedekind self-map within j: A -> A.

Infinity of Transformations Within a Dedekind Self-Map j: A -> A



Within this flow by j, there emerges a sequence a, j(a), j(j(a)) . . . This is the *blueprint* of the natural numbers.

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NOTE: The values a, j(a), j(j(a)) . . . cannot yet be specified precisely -- so far, we have only a *blueprint.*

Transformations Within the Blueprint

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(This can be formulated without use of natural numbers.)
- **Step 4:** Verify E is a well-founded partial order. (It is the *blueprint* for the usual "less than" relation on the natural numbers)

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The set that is produced by this collapsing function is

$$M = \{\psi(x) \mid x \text{ is in } W\}$$

M is the "manifestation" of our blueprint. We can now study its contents.

Contents of the Object M: The Origin of 0

Definition of the map ψ: W -> M:

$$\psi(x) = \{\psi(y) \mid y \mathrel{E} x\}$$

Recall **a** is the element of W that is the precursor to 0.
We see what **a** corresponds to in our object M:

$$\psi(\mathbf{a}) = \{\psi(y) \mid y \mathrel{E} \mathbf{a}\} = \{\psi(y) \mid \mathbf{a} = j(y) \text{ or } \mathbf{a} = j(j(y)) \text{ or } \ldots\}$$
$$= \varnothing = 0$$

Therefore, ψ maps the critical point **a** to 0.

Contents of the Object M: The Origin of 1

We expect that j(a) will be "realized" as the number 1. We have:

$$\psi(j(a)) = \{\psi(x) \mid xEj(a)\} = \{\psi(a)\} = \{\emptyset\} = 1$$

Contents of the Object M: The Origin of 2

We expect j(j(a)) will be "realized" as the number 2. We have:

$$\psi(j(j(a))) = \{\psi(x) \mid xEj(j(a))\} = \{\psi(a), \psi(j(a))\} = \{\emptyset, \{\emptyset\}\} = \{0,1\} = 2.$$

Emergence of the Successor Function

We see the successor function $s : M \to M$ on the natural numbers emerging:

$$\begin{array}{ccc} W & \xrightarrow{j} & W \\ \downarrow{\psi} & & \downarrow{\psi} \\ M & \xrightarrow{s} & M \end{array}$$

$$s(0) = s(\emptyset) = s(\psi(a)) = \psi(j(a)) = \{\emptyset\} = 1$$

$$s(1) = s(\{\emptyset\}) = s(\psi(j(a))) = \psi(j(a)) = \{\emptyset, \{\emptyset\}\} = 2$$

Moreover, the unique solution s that makes the diagram commutative is

$$s(z) = z \cup \{z\}$$

as seen by

$$\begin{aligned} \psi(j(x)) &= \{\psi(y) \mid yEj(x)\} \\ &= \{\psi(y) \mid yEx\} \cup \{\psi(x)\} \\ &= \psi(x) \cup \{\psi(x)\}. \end{aligned}$$

M Is the Set of Natural Numbers

Using j-induction (induction within the "blueprint"), one shows that M, together with its successor function, is an inductive set – indeed the *smallest* inductive set.

So

$$M = \{0, 1, 2, 3, \ldots\}$$

and

s: M -> M is the successor function:

$$s(n) = n + 1$$

The Natural Numbers Derived

Conclusion: The set M of natural numbers is the concrete manifestation of self-referral dynamics generated by a Dedekind self-map.

j: A -> A $\Rightarrow$ **j : W -> W** (blueprint)

$\Rightarrow$ **s: M -> M** (concrete natural numbers)

Application to the Problem of Large Cardinals

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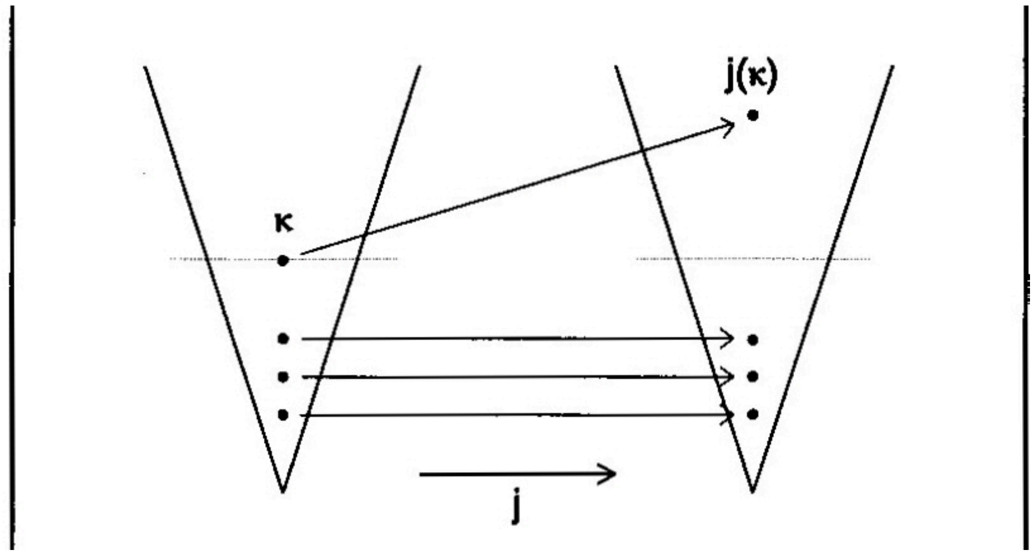
- Second Try: Use the new Axiom of Infinity ("there is a Dedekind self-map j: A -> A with critical point a") to get a hint about how to solve the problem
- Maybe large cardinals arise in the interaction of a Dedekind self-map *of the universe* with its critical point?

A Solution to the Problem of Large Cardinals



A solution is given by a Dedekind self-map j: V -> V with critical point κ (the least cardinal moved by j is a critical point of j), with the added feature that j is an *elementary embedding.*

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- This examination of the Axiom of Infinity has resulted in an acceptable justification for just a a few of the small to middle-range large cardinals.

Q & A: A Solution to the Problem of Large Cardinals

- The **Wholeness Axiom** (WA) asserts that there is a nontrivial elementary embedding $j : V \to V$ with the property that $j \mid X$ belongs to V whenever X belongs to V.
- **Theorem.** Assume WA and let $j : V \to V$ be the WA-embedding. Then the first cardinal κ moved by j (so that $j(\kappa) \neq \kappa$) is super-n-huge for every n (and more).
- The cardinal κ is obtained as a collapse of V to a point, to the first hint of transformation, analogous to the collapse of A to K.
- This is a solution to the Problem of Large Cardinals

Q & A
j: V -> V According to WA