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RAMSEY SETS, THE RAMSEY IDEAL, AND OTHER CLASSES OVER R

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Abstract. We improve results of Marczewski, Frankiewicz, Brown, and others comparing the σ-ideals of measure zero, meager, Marczewski measure zero, and completely Ramsey null sets; in particular, we remove CH from the hypothesis of many of Brown's constructions of sets lying in some of these ideals but not in others. We improve upon work of Marczewski by constructing, without CH, a nonmeasurable Marczewski measure zero set lacking the property of Baire. We extend our analysis of σ-ideals to include the completely Ramsey null sets relative to a Ramsey ultrafilter and obtain all 32 possible examples of sets in some ideals and not others, some under the assumption of MA, but most in ZFC alone. We also improve upon the known constructions of a Marczewski measure zero set which is not Ramsey by using a set theoretic hypothesis which is weaker than those used by other authors. We give several consistency proofs: one concerning the relative sizes of the covering numbers for the meager sets and the completely Ramsey null sets; another concerning the size of non($\mathrm{CR}_0^{\mathscr{U}}$); and a third concerning the size of add($\mathrm{CR}_0^{\mathscr{U}}$). We also study those classes of perfect sets which are bases for the class of always first category sets.

§0. Introduction. This paper extends the work of Frankiewicz, et al. [AFP] and J. Brown [B] comparing the σ-ideals of measure zero ($\mathscr{L}_0$), meager ($\mathscr{K}$), Marczewski measure zero ($(s)_0$), and completely Ramsey null (CR_0) sets of reals. The fact that $\mathscr{L}_0$ and $\mathscr{K}$ provide very different notions of "smallness" (there are meager nonmeasurable sets and measure zero sets lacking the propety of Baire; see [O]) is an observation made in a first course in measure theory. Similarly, in this paper we will show that the four notions of smallness given by these four σ-ideals are all quite different; in particular, we will construct numerous examples of sets lying in some of these σ-ideals but not in any of the σ-algebras corresponding to the remaining σ-ideals. We will denote the four corresponding σ-algebras by $\mathscr{L}$ (Lebesgue measurable sets), $\mathrm{B_W}$ (sets having the property of Baire), (s) (Marczewski measurable sets), and CR (completely Ramsey sets).

In [Ma2], Marczewski introduces the classes (s) and $(s)_0$ and gives examples of sets in $\mathscr{L}_0\backslash(s)$ and $\mathscr{K}\backslash(s)$. Moreover, he observes that, assuming CH, $(s)_0\backslash\mathscr{L}\cup \mathrm{B_W} \neq \varnothing$, using the union of a Luzin set and a Sierpinski set as his example. In this paper, we construct[2] in ZFC sets in $(s)_0\backslash\mathscr{L}$ and in $(s)_0\backslash\mathrm{B_W}$ (see §3).

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[1]The results of §3 of this paper were presented at the 845th meeting of the American Mathematical Society, October 28–29, 1988, in Lawrence, KS.

[2]Walsh [W2] has also observed this; see §3. For an application of this result, see [BP].

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In [AFP], Frankiewicz, et al. show that $\mathrm{CR}_0\backslash(s) \neq \varnothing$, and, assuming an axiom weaker than MA, that $(s)_0\backslash\mathrm{CR} \neq \varnothing$. In [B], Brown shows that for each ideal $\mathscr{I}$ in $\mathscr{B} = \{\mathscr{L}_0, \mathscr{K}, (s)_0, \mathrm{CR}_0\}$, there is a set $X \in \mathscr{I}$ which is not in any of the three σ-algebras corresponding to the other three σ-ideals in $\mathscr{B}$; and that for every group of three σ-ideals in $\mathscr{B}$, there is a set in their intersection which is not in the σ-algebra corresponding to the fourth σ-ideal in $\mathscr{B}$. Many of his constructions, however, involve CH; in particular, those examples which require the existence of a set in $(s)_0\backslash\mathrm{CR}$, $(s)_0\backslash\mathscr{L}$, or $(s)_0\backslash\mathrm{B_W}$. Using our examples of sets in $(s)_0\backslash\mathscr{L}$ and $(s)_0\backslash\mathrm{B_W}$, we eliminate CH from the hypothesis in several of Brown's constructions. Of the 16 possible examples of sets in the intersection of a subcollection of $\mathscr{B}$ not in the other σ-algebras, we produce 12 in ZFC alone and the other four under the assumption that $(s)_0\backslash\mathrm{CR} \neq \varnothing$. We do not have a ZFC example[3] of a set in $(s)_0\backslash\mathrm{CR}$; however, we improve upon the known constructions of such sets by using as an extra hypothesis an axiom which is weaker than those used by other authors (namely, Brown in [B] and Reclaw in [AFP]).

In [Ms], Mathias introduced the σ-ideal (σ-algebra) of completely Ramsey null (completely Ramsey) sets relative to a Ramsey ultrafilter $\mathscr{U}$, to be denoted by $\mathrm{CR}_0^{\mathscr{U}}$ ($\mathrm{CR}^{\mathscr{U}}$). In many ways these classes are better behaved than CR_0 and CR (see §4); thus it is natural to carry out an analysis similar to that described above, replacing CR and CR_0 with $\mathrm{CR}^{\mathscr{U}}$ and $\mathrm{CR}_0^{\mathscr{U}}$. Compelled by curiosity, however, we were led to such an analysis involving all five of the σ-ideals $\mathscr{L}_0$, $\mathrm{B_W}$, $(s)_0$, CR_0, and $\mathrm{CR}_0^{\mathscr{U}}$; we obtain examples for all 32 cases, although four of these are constructed assuming MA.

The paper is organized as follows: In §1 we state basic definitions and facts; we also present Miller's proof of the fact[4] that adding a Cohen real forces the set of ground model functions in ω^ω to have strong measure zero (we use this in §2). In §2, we prove basic facts about the classes CR and CR_0; we formulate the axiom we use to construct (in §3) a set in $(s)_0\backslash\mathrm{CR}$; this axiom leads us into a brief study of several ideals related to CR_0, a couple of consistency results, and an analysis of bases for always first category sets. In §3, we give our constructions of sets in $(s)_0\backslash\mathscr{L}$, $(s)_0\backslash\mathrm{B_W}$, and, assuming the axiom described in §2, $(s)_0\backslash\mathrm{CR}$, along with several generalizations; we conclude with a chart giving brief descriptions of each of the 16 constructions. Finally, §4 is devoted to carrying out this kind of analysis using all five σ-ideals; we give several consistency results to highlight differences between CR_0 and $\mathrm{CR}_0^{\mathscr{U}}$; we conclude with a chart giving brief descriptions of the 32 constructions. Ten open problems are stated in the course of the paper.

Let me close this introduction by gratefully acknowledging A. Blass, J. Brown, H. Judah, and A. W. Miller for several helpful discussions on the topics of this paper, and the referee for calling attention to a couple of incorrect proofs in the original draft.

[3] In [C1], I claimed to have such an example; the construction, however, was erroneous.

[4] See p. 155 of [L] where a result of this kind is mentioned.

§1. Preliminaries. In this section we state the basic definitions and facts required for the rest of the paper. Although most of our work will take place in the space $[\omega]^{\omega}$, some results will hold in arbitrary perfect Polish spaces; we begin the section in this more general setting.

Perfect Polish spaces. A topological space is a *perfect Polish space* if it is a complete separable metric space without isolated points; see [M]. Any two perfect Polish spaces are Borel isomorphic [M, 1G.4]. Moreover, if $\mathbf{X}$ and $\mathbf{Y}$ are perfect Polish spaces, there is a Borel isomorphism $f\colon \mathbf{X} \to \mathbf{Y}$ such that f maps the class of meager sets in $\mathbf{X}$ onto those of $\mathbf{Y}$ (see [G]); such an f will be called *category-preserving*.

A subset of perfect Polish space is called *Bernstein dense* (or simply a *Bernstein set*) if both it and its complement meet every perfect set. More generally, if $\mathbf{X}$ is a perfect Polish space and $S \subseteq \mathbf{X}$, a set $B \subseteq S$ is called a *Bernstein set relative to S* (or simply a *Bernstein subset of S*) if both B and $S \setminus B$ meet every subset of S which is perfect in $\mathbf{X}$. Note that a Bernstein set must be nonmeasurable with respect to any finite continuous Borel measure and must lack the property of Baire (defined below).

Let $\mathbf{X}$ be a perfect Polish space. For each $\kappa, \omega_1 \leq \kappa \leq \mathbf{c}$, a set $L \subseteq \mathbf{X}$ is *κ-Luzin* if $|L| \geq \kappa$ and for all meager sets $M \subseteq \mathbf{X}$, $|L \cap M| < \kappa$; an ω_1-Luzin set is called simply a *Luzin set*. Note that κ-Luzin sets are nonmeager and have measure zero relative to every finite continuous Borel measure (see [Mi2]). Assuming CH, a Luzin set can be constructed in any perfect Polish space; assuming MA, a $\mathbf{c}$-Luzin set can be constructed; see [Mi2]. The following result will be used in §2.

1.0. PROPOSITION. *Suppose $\mathbf{X}$ is a perfect Polish space containing a κ-Luzin set L. Then for every perfect Polish space $\mathbf{Y}$, every perfect $P \subseteq \mathbf{Y}$, and every $U \subseteq P$ open in P, there is a subset of U which is κ-Luzin relative to P.*

PROOF. Suppose $U \subseteq P \subseteq \mathbf{Y}$, as in the hypothesis. Let $C \subseteq U$ be the closure of a basic open set relative to U. As C is itself a perfect Polish space, there is a Borel isomorphism $f\colon \mathbf{X} \to C$ which induces a one-one correspondence between the meager subsets of $\mathbf{X}$ and those of C. It is easy to see that $f(L)$ is κ-Luzin in C, and hence also in P. □

Finally, suppose $\mathbf{X}$ is a perfect Polish space equipped with a finite continuous Borel measure μ; a *κ-Sierpinski set* is a set of cardinality $\geq \kappa$ whose intersection with every μ-measure zero set has cardinality $< \kappa$.

σ-algebras and σ-ideals. Apart from notions related to Ramsey sets, the main σ-algebras and σ-ideals that will concern us in this paper are given below; we reserve the definitions of CR, CR_0, $\mathrm{CR}^{\mathscr{U}}$, and $\mathrm{CR}_0^{\mathscr{U}}$ for later sections.

σ-algebras:

$\mathscr{L} = \{$Lebesgue measurable sets$\}$,

$\mathrm{B_W} = \{S\colon$ for some open U and some meager $M, S = U \,\Delta\, M\}$,

$(s) = \{S\colon$ for each perfect set P there is a perfect $Q \subseteq P$ such that either $Q \subseteq S$ or $Q \cap S = \varnothing\}$.

σ-ideals:

$\mathscr{L}_0 = \{$Lebesgue measure zero sets$\}$,

$\mathscr{K} = \{$meager sets$\}$,

$(s)_0 = \{S\colon$ for each perfect set P there is a perfect $Q \subseteq P$ such that $Q \cap S = \varnothing\}$,

$\mathrm{AFC} = \{S\colon$ for each perfect set P, $S \cap P$ is meager in $P\}$.

Except for $\mathscr{L}$ and $\mathscr{L}_0$, the underlying space for each of these classes can be any perfect Polish space; while this is also true for $\mathscr{L}$ and $\mathscr{L}_0$, replacing Lebesgue measure with an arbitrary finite continuous Borel measure, we do not pursue this here. Instead, we adopt the convention that "Lebesgue measure" refers either to the usual Lebesgue measure on $[0,1]$, the product measure on 2^ω, or the measure on $[\omega]^\omega$ inherited from 2^ω (identifying subsets of ω with their characteristic functions). The apparent ambiguity is justified by the fact that the canonical continuous map from 2^ω onto $[0,1]$ is measure preserving; see [Mi2].

Suppose $\mathscr{A}$ is a σ-algebra over a set X. The *hereditary σ-ideal for $\mathscr{A}$* is the collection $\mathscr{I}$ of sets $M \subseteq X$ satisfying

$$M \in \mathscr{I} \quad \text{if and only if} \quad P(M) \subseteq \mathscr{A}$$

(where $P(M)$ denotes the power set of M). It is easy to verify that $\mathscr{L}_0$, $\mathscr{K}$, and $(s)_0$ are the hereditary σ-ideals for $\mathscr{L}$, $\mathrm{B_W}$, and (s), respectively.

The σ-ideals $\mathscr{L}_0$ and $\mathscr{K}$ have the additional property that they are *Borel supported*, i.e., every member of $\mathscr{L}$ (of $\mathrm{B_W}$) is the symmetric difference of a Borel set and a set in $\mathscr{L}_0$ (in $\mathscr{K}$). Walsh [W1] showed that (s) is not Borel-supported.

Associated with any σ-ideal $\mathscr{I}$ on a set X are the following cardinals (see [F1]):

$$\operatorname{add}(\mathscr{I}) = \min\{|\mathscr{A}|: \mathscr{A} \subseteq \mathscr{I} \text{ and } \bigcup\mathscr{A} \notin \mathscr{I}\},$$
$$\operatorname{cov}(\mathscr{I}) = \min\{|\mathscr{A}|: \mathscr{A} \subseteq \mathscr{I} \text{ and } \bigcup\mathscr{A} = X\},$$
$$\operatorname{non}(\mathscr{I}) = \min\{|S|: S \subseteq X \text{ and } S \notin \mathscr{I}\}.$$

Using the fact that there is a category preserving Borel isomorphism between any two perfect Polish spaces, one can show that the values of $\operatorname{add}(\mathscr{K})$, $\operatorname{cov}(\mathscr{K})$, and $\operatorname{non}(\mathscr{K})$ do not depend on the choice of underlying space.

The space $[\omega]^\omega$. Because the classes CR and CR_0 are defined only for the space of subsets of ω, this space will be our main concern. The topology on $[\omega]^\omega$ is that inherited from the product topology on 2^ω, identifying subsets of ω with their characteristic functions. Equivalently, the topology on $[\omega]^\omega$ can be obtained by taking as basic open sets those sets of the form $[F,[n,\omega)]$, where F is finite, $n > \max(F)$, and

$$[F,[n,\omega)] = \{A \in [\omega]^\omega: F \subseteq A \subseteq F \cup [n,\omega) \text{ and } A \cap n = F\}.$$

If $A, B \in [\omega]^\omega$, we write $B \subseteq^* A$ if there is a finite set F such that $B \backslash F \subseteq A$. We adopt the notation $\langle A \rangle$ for the set $\{B \in [\omega]^\omega: B \subseteq^* A\}$. A *tower in* $[A]^\omega$ is a sequence $\langle A_\alpha: \alpha < \kappa\rangle$ of subsets of A such that for all $\beta < \alpha < \kappa$, $A_\alpha \subseteq^* A_\beta$. The axiom $\mathbf{p} = \mathbf{c}$ is the assertion that for every collection $\mathscr{A} \subseteq [\omega]^\omega$ with the properties that

(a) each finite subcollection of $\mathscr{A}$ has infinite intersection, and
(b) $|\mathscr{A}| < \mathbf{c}$,

there is a set $B \in [\omega]^\omega$ such that for all $A \in \mathscr{A}$, $B \subseteq^* A$. Note that towers of length $< \mathbf{c}$ satisfy (a) and (b). Martin's Axiom implies $\mathbf{p} = \mathbf{c}$ (see [F2]).

A family $\mathscr{A}$ of subsets of $[\omega]^\omega$ is said to be *almost disjoint*, or *a.d.*, if for all distinct pairs $A, B \in \mathscr{A}$, $A \cap B$ is finite; $\mathscr{A}$ is called a *maximal a.d. family* if $\mathscr{A}$ is a.d. and for all infinite B, there is an $A \in \mathscr{A}$ for which $|B \cap A| = \aleph_0$. A *filter* is a nonempty collection $\mathscr{F} \subseteq [\omega]^\omega$ which is upward closed and closed under finite intersections; a filter $\mathscr{F}$ is an *ultrafilter* if one piece of every partition of ω into two pieces is in $\mathscr{F}$. See [F2], [J], and [K1] for more background.

Forcing. Our forcing notation follows [Ba]; see [K2, VII] for an introduction to this subject. We introduce here the forcing notions we will use in this paper.

A set $T \subseteq 2^{<\omega}$ is a *tree* if for all $s \in T$ and all $n \in \omega$, $s | n \in T$. $[T] = \{f \in 2^\omega : \forall n \in \omega\ f | n \in T\}$. Members of $[T]$ are called *branches* of T. T is a *perfect tree* if every $s \in T$ has incomparable extensions in T. For any $X \subseteq 2^\omega$, let $T_X = \{f | n : f \in X\}$. (Note that T_X is a tree.) The proof of the following can be found in [J] or [M].

1.1. PROPOSITION (folklore). *T is perfect if and only if $[T]$ is a perfect subset of 2^ω; for every perfect set $P \subseteq 2^\omega$, there is a perfect tree $T \subseteq 2^{<\omega}$ such that $P = [T]$; for all $X \subseteq 2^\omega$, $[T_X]$ is the topological closure of X.*

Sacks forcing is the set of all perfect trees ordered by $\subseteq$; see [BL]. *Mathias forcing* is the set of all pairs $(F, A) \in [\omega]^{<\omega} \times [\omega]^\omega$ such that $\min(A) > \max(F)$ with the following ordering: $(F, A) < (G, B)$ iff $F \supseteq G$, $A \subseteq B$, and $F \backslash G \subseteq B$; see [Ba, §9] and [Mi1]. Finally, *Cohen forcing* can be defined to be any partial order whose Boolean algebra completion is atomless and has a countable dense set; see [J, p. 542]. Two such orders which concern us here are $Fn(\omega, 2) = \{p: p$ is a finite partial function from ω to $\{0,1\}\}$, and $Fn(\omega, \omega) = \{p: p$ is a finite partial function from ω into $\omega\}$. More generally, we write $Fn(I, J) = \{p: p$ is a finite partial function from I to $J\}$. For work in §2, we will need to know that adding a Cohen real forces the ground model Baire space $(\omega^\omega)^M$ to have strong measure zero. This result was announced by Laver in [L]; we present a slightly modified version of an unpublished proof of this result due to A. W. Miller [Mi3] which the author has kindly agreed to let us include here. We use the usual metric on ω^ω given by

$$d(x, y) = 1/(n+1) \quad \text{iff} \quad n \text{ is least such that } x(n) \neq y(n).$$

We write $N(x, \varepsilon) = \{y: d(x, y) < \varepsilon\}$ for the ε-ball about x. We use a modification of the following definition of strong measure zero, given by:

> $X \in \mathbf{C}$ iff for all $h: \omega \to \omega$ there is a $g: \omega \to \omega^\omega$ such that $X \subseteq \bigcup_n N(g(n), 1/(h(n)+1))$, i.e., such that for all $x \in X$ there is an n such that $x | h(n) = g(n) | h(n)$.

(This definition appears in [BC] and in [B].) For ω^ω, this definition is equivalent to the following—as the reader may easily verify—and more useful for our purposes.

1.2. DEFINITION. A subset X of ω^ω has *strong measure zero* iff for all $h: \omega \to \omega$, there is a $g: \omega \to \omega^{<\omega}$ such that for all $x \in X$ there is $n \in \omega$ such that $x | h(n) = g(n)$.

A real x is a *Cohen real over a model M* of ZFC* (a sufficiently large fragment of ZFC) if $x = \bigcup G$ where G is $\mathscr{P}$-generic over M and $\mathscr{P}$ is the Cohen order; the version of the Cohen order that we will use here is $\mathscr{P} = Fn(\omega, \omega)$.

1.3. THEOREM. *Suppose $x: \omega \to \omega$ is a Cohen real over M. Then $M \cap \omega^\omega$ has strong measure zero in $M[x]$.*

PROOF. Write $\mathscr{P} = \{p_n : n \in \omega\}$. Let $\dot{h}$ be a $\mathscr{P}$-name for a function from ω to ω, and let $k \in M \cap \omega^\omega$ be defined so that for all $n \in \omega$, there is a $q_n \leq p_n$ such that

$$q_n \Vdash k(n) > \dot{h}(n).$$

Let $\dot{x}$ be a canonical name for the Cohen real x. Let $\dot{g}$ be a name for a function from ω to $\omega^{<\omega}$ so that

$$(*) \qquad \Vdash_{\mathscr{P}} \forall n \in \omega\, \dot{g}(n) = \langle \dot{x}(|q_n|), \dot{x}(|q_n| + 1), \ldots, \dot{x}(|q_n| + k(n) - 1)\rangle.$$

Claim. $\Vdash_{\mathscr{P}} \forall y \in M \cap \omega^{\omega}\, \exists n\, y \restriction h(n) = g(n)$.

Proof. Suppose not. Let $p = p_n \in \mathscr{P}$ and $y \in M \cap \omega^{\omega}$ be such that

$$p_n \Vdash y \restriction h(n) \neq g(n).$$

Let $r = q_n{}^\frown y \restriction k(n)$. Then

$$r \Vdash \forall i < \dot{h}(n)\, \dot{x}(|q_n| + i) = y(i).$$

Now by $(*)$,

$$r \Vdash g(n) = y \restriction \dot{h}(n),$$

and we have a contradiction. □

1.4. COROLLARY. *Suppose $\langle \mathscr{P}_\alpha : \alpha \leq \omega_1 \rangle$ is an ω_1-stage finite support iteration of the Cohen order $Fn(\omega_1, 2)$. Then for all $\alpha < \omega_1$, $M[G_{\omega_1}] \models$ "$M[G_\alpha] \cap \omega^{\omega}$ has strong measure zero."*

PROOF. First note that

$$M[G_\omega] \models \text{“}M \cap \omega^{\omega} \text{ has strong measure zero;”}$$

this is because forcing with $\mathscr{P}_\omega$ is the same as forcing with $Fn(\omega \times \omega, 2)$ and hence with the forcing $\mathscr{P}$, as defined in the previous theorem. It follows that for each $\alpha < \omega_1$ and each γ, $\alpha < \gamma < \omega_1$, $M[G_\gamma] \models$ "$M[G_\alpha] \cap \omega^{\omega}$ has strong measure zero." Hence, suppose $h \in M[G_{\omega_1}] \cap \omega^{\omega}$; WLOG, assume $h \in M[G_\gamma] \cap \omega^{\omega}$, $\alpha < \gamma < \omega_1$. In $M[G_\gamma]$ we obtain a $g\colon \omega \to \omega^{<\omega}$ satisfying the property of Definition 1.2; as this property is absolute, it holds in $M[G_{\omega_1}]$, and we are done. □

§2. Basic results. In this section, we define the Ramsey sets, algebras and ideals, discuss a variety of basic results about them and introduce some refinements; some of the basic results are known but have not appeared in the literature. We also give a couple of consistency proofs to show that ZFC does not decide which of the covering numbers of the completely Ramsey ideal and the meager ideal is larger; this result motivates the introduction of a new cardinal invariant, to be used in the next section. We conclude with a discussion of bases for AFC sets and a list of open problems.

2.0. DEFINITION. A set $X \subseteq [\omega]^{\omega}$ is *Ramsey* (*Ramsey null*) and we write $X \in \mathrm{R}$ ($X \in \mathrm{R}_0$) if there is a set $A \in [\omega]^{\omega}$ such that

$$\text{either} \quad [A]^{\omega} \subseteq X \quad \text{or} \quad [A]^{\omega} \cap X = \varnothing \; ([A]^{\omega} \cap X = \varnothing).$$

Any set $A \in [\omega]^{\omega}$ satisfying this property is called *homogeneous for X*. X is *uniformly Ramsey* (*uniformly Ramsey null*) and we write $X \in \mathrm{UR}$ ($X \in \mathrm{UR}_0$) if for all $A \in [\omega]^{\omega}$, there is a $B \in [A]^{\omega}$ which is homogeneous for X.

Often, results about Ramsey sets carry over to uniformly Ramsey sets (see [Mi2, p. 217]); this is a useful fact since the uniformly Ramsey sets form an algebra while the Ramsey sets do not (see below).

2.1. DEFINITION. Suppose $F \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$, and $\max(F) < \min(A)$. Then

$$[F, A] = \{B: F \subseteq B \subseteq F \cup A \text{ and } \max(F) < \min(B \setminus F)\}.$$

Sets of the form $[F, A]$ are called *Ellentuck sets*, or *E-sets.*

Note that E-sets $[F, A]$ are either nowhere dense and of measure zero (if A is coinfinite) or clopen (if A contains an interval $[n, \omega)$).

2.2. DEFINITION. A set $X \subseteq [\omega]^\omega$ is *completely Ramsey*[5] (*completely Ramsey null*) and we write $X \in \mathrm{CR}$ ($X \in \mathrm{CR}_0$) if for each E-set $[F, A]$, there exists a $B \in [A]^\omega$ such that

$$(*) \qquad \text{either} \quad [F, B] \subseteq X \quad \text{or} \quad [F, B] \cap X = \varnothing \; ([F, B] \cap X = \varnothing).$$

Equivalently (see [B]), for each $[F, A]$ there is a $[G, B]$ such that $F \subseteq G$ and $B \subseteq A$ and $(*)$ holds for $[G, B]$.

Obviously,

$$\mathrm{CR} \subseteq \mathrm{UR} \subseteq \mathrm{R} \quad \text{and} \quad \mathrm{CR}_0 \subseteq \mathrm{UR}_0 \subseteq \mathrm{R}_0.$$

The next proposition shows, among other things, that these inclusions are proper.

2.3. PROPOSITION. (1) UR_0 *is an ideal and* UR *is an algebra.*

(2) R_0 *is not closed under finite unions; hence* $\mathrm{UR}_0 \neq \mathrm{R}_0$.

(3) CR_0 *is a* σ*-ideal;* CR *is a* σ*-algebra including the Borel sets.*

(4) UR_0 *is not closed under countable unions; hence* $\mathrm{CR}_0 \neq \mathrm{UR}_0$.

(5) *Martin's Axiom implies that the union of* $< \mathfrak{c}$ CR *sets is* CR, *and that* $\mathrm{add}(\mathrm{CR}_0) = \mathfrak{c}$.

PROOF OF (1). Clearly UR is closed under complementation. For finite unions, suppose $X, Y \in \mathrm{UR}$ and $A \in [\omega]^\omega$. Let $B \in [A]^\omega$ be homogeneous for X and let $C \in [B]^\omega$ be homogeneous for Y. If $[B]^\omega \subseteq X$ or $[C]^\omega \subseteq Y$, then $[C]^\omega \subseteq X \cup Y$; if not, then $[C]^\omega$ misses $X \cup Y$. Hence, C is homogeneous for $X \cup Y$. The proof that UR_0 is an ideal is similar. □

PROOF OF (2). Let $A \in [\omega]^\omega$ be coinfinite and let $B = \omega \setminus A$. Let $\{A_\alpha: \alpha < \mathfrak{c}\}$ enumerate $[A]^\omega$, and let $\{B_\alpha: \alpha < \mathfrak{c}\}$ enumerate $[B]^\omega$. For each α, pick $C_\alpha \neq C'_\alpha$ and $D_\alpha \neq D'_\alpha$ so that

$$\{C_\alpha, C'_\alpha\} \subseteq [A_\alpha]^\omega \setminus [\{C_\beta: \beta < \alpha\} \cup \{C'_\beta: \beta < \alpha\}],$$
$$\{D_\alpha, D'_\alpha\} \subseteq [B_\alpha]^\omega \setminus [\{D_\beta: \beta < \alpha\} \cup \{D'_\beta: \beta < \alpha\}].$$

Set $X = \{C_\alpha: \alpha < \mathfrak{c}\}$ and $Y = \{D_\alpha: \alpha < \mathfrak{c}\}$. As X avoids $[B]^\omega$ and Y avoids $[A]^\omega$, X and Y are Ramsey null; however $X \cup Y$ is not even Ramsey: Given $E \in [\omega]^\omega$, WLOG assume $|E \cap A| = \omega$. Then for some α, $E \cap A = A_\alpha$. Then $[E]^\omega \nsubseteq X \cup Y$ since $C'_\alpha \in [E]^\omega \setminus (X \cup Y)$, and $[E]^\omega \cap (X \cup Y) \neq \varnothing$ since $C_\alpha \in [E]^\omega \cap X$. □

PROOF OF (3). This has been proven by several authors; see for example [GP]. □

[5]In Morgan's unifying theory of category bases [Mo], ($[\omega]^\omega$, $\{E\text{-sets}\}$) is a Baire, point-meager, perfect category base; the CR_0 sets are both singular and meager relative to this base; and the CR sets are the Baire sets with respect to this base.

Proof of (4). Note that sets of the form $[F,[n,\omega)]$, with $F \neq \emptyset$, are in UR_0. Let $\mathscr{A} = \{[F,[\max(F)+1,\omega)]\colon F \neq \emptyset\}$. Since each $A \in [\omega]^\omega$ is in $[\{\min(A)\}, [\min(A)+1,\omega)]$, it follows that $\bigcup\mathscr{A} = [\omega]^\omega$; hence UR_0 is not closed under countable unions. □

Proof of (5). Additivity of CR was proven by Silver [Si]; additivity of CR_0 has been observed by many people; see for example [F2, 23 Na (ii)]. □

The next proposition shows that CR_0 is the hereditary ideal for CR.

2.4. Proposition. *$X \in \mathrm{CR}_0$ if and only if $P(X) \subseteq \mathrm{CR}$.*

The following are useful examples of CR_0 sets.

2.5. Proposition. (1) *Every a.d. family is in* CR_0.

(2) *Every filter is in* CR_0.

(3) *For all $\kappa, \omega_1 \leq \kappa \leq \mathfrak{c}$, all κ-Luzin sets and all κ-Sierpinski sets are in* CR_0.

Clearly, E-sets are perfect in $[\omega]^\omega$. The next proposition shows that there are perfect sets which contain no E-set; Proposition 2.8 shows that every perfect set contains such a perfect set. These results are already known (see [B]) but our techniques are quite different. We will treat $[\omega]^\omega$ as a subspace of 2^ω so that we can add members of $[\omega]^\omega$ together; the fact that such sums may not always lie in $[\omega]^\omega$ will not interfere with our arguments. Note that for any perfect $P \subseteq 2^\omega$ and any $D \in [\omega]^\omega$, $D + P$ is also perfect.

2.6. Proposition. *Given any coinfinite $A \in [\omega]^\omega$, there is a $D \in [\omega]^\omega$ such that $D + [A]^\omega$ contains no E-set. Hence there is a perfect set in $[\omega]^\omega$ containing no E-set.*

Proof. Let $D = \omega\backslash A$. Notice that if $K \in D + [A]^\omega$, then $D \subseteq K$. Thus, for any E-set $[F,B]$, let $n \in D\backslash(\max(F)+1)$; then since $D \nsubseteq F \cup B\backslash\{n\} \in [F,B]$, it follows that $[F,B] \nsubseteq D + [A]^\omega$. Now $(D + [A]^\omega) \cap [\omega]^\omega$ is the required perfect set in $[\omega]^\omega$. □

2.7. Lemma. *Every E-set includes a perfect set which contains no E-set.*

Proof. Note that for each $A \in [\omega]^\omega$, the function $f_A\colon [A]^\omega \to [\omega]^\omega$ induced by a bijection from A onto ω is a homeomorphism. For any E-set $[F,A]$, define the function $g_{[F,A]}\colon [F,A] \to [A]^\omega$ by $g_{[F,A]}(C) = C\backslash F$. (Our f_A and $g_{[F,A]}$ are essentially the inverses of the f and g of [GP, Lemma 7].) Observe that $f_A \circ g_{[F,A]}$ takes E-sets to E-sets, and, in particular, relative basic open sets to relative basic open sets (recall that basic open sets relative to $[F,A]$ are of the form $[G,[n,\omega)] \cap [F,A]$). As $g_{[F,A]}$ is open, continuous, and finite-one, preimages of perfect sets are perfect. Thus, if P is the perfect set of the last proposition, $(f_A \circ g_{[F,A]})^{-1}(P)$ is perfect in $[F,A]$ and contains no E-set. □

2.8. Proposition. *For every perfect $P \subseteq [\omega]^\omega$, there is a perfect $Q \subseteq P$ which contains no E-set. Moreover, such a Q is* CR_0.

Proof. If P contains no E-set, let $Q = P$; if P does contain an E-set $[F,A]$, use the previous lemma to obtain $Q \subseteq [F,A]$ which contains no E-set. Note that Q, being Borel, is in CR; since it contains no E-set, $Q \in \mathrm{CR}_0$. □

Perfect subsets of $[\omega]^\omega$ can be viewed as trees in $2^{<\omega}$; we now characterize the E-sets using trees and, as an application, prove a variation of the last proposition which will be used in the next section.

2.9. Definition. Suppose $T \subseteq 2^{<\omega}$ is a tree. A node $s \in T \cap 2^n$ is called a *splitting node* if there are $t_1, t_2 \in T \cap 2^{n+1}$ which extend s. The *stem* of a tree that has a splitting node is its splitting node of shortest length. The stem of a tree T is denoted stem(T).

Consider the following property of trees:

$E(T)$: (a) For each $n \geq$ length(stem(T)), either every $s \in T \cap 2^n$ is a splitting node or every $s \in T \cap 2^n$ is extended uniquely in $T \cap 2^{n+1}$ by $s\hat{\ }0$.

(b) T has infinitely many splitting nodes.

We show that the E-sets in $[\omega]^\omega$ correspond canonically to the trees $T \subseteq 2^{<\omega}$ satisfying $E(T)$. For simplicity, we identify each $A \in P(\omega)$ with its characteristic function. Let $\Delta = \{F \in P(\omega)\colon F \text{ is finite}\}$.

Observe that if T is a tree for which $E(T)$ holds, we may obtain an E-set $[F, A] = [F_T, A_T]$ as follows: Let C be the rightmost branch of T, let F be the stem of T, and let $A = C \backslash F$. Now note that $[T] \cap [\omega]^\omega = [F, A]$. We also observe here that $[T] \cap \Delta$ is dense in $[T]$.

Conversely, if $[F, A]$ is an E-set, let $T = T_{[F,A]}$ (see Proposition 1.1). Let $m = \min(A)$. Note that stem$(T) = F \,|\, m$; thus, m is the length of stem(T). We verify that if $n \geq m$ and $n \in A$, then each $s \in T \cap 2^n$ is a splitting node: Each $s \in T \cap 2^n$ is the characteristic function of a finite set $G \subseteq n$ where $F \subseteq G$. Let $D_1 = G \cup (A \backslash n)$ and $D_0 = D_1 \backslash \{n\}$. As $n \in A$, $D_0 \,|\, n+1 \neq D_1 \,|\, n+1$ are incomparable extensions of s in $T \cap 2^{n+1}$. On the other hand, if $n \notin A$ and for some $s \in T \cap 2^n$, $s\hat{\ }1 \in T \cap 2^{n+1}$, let $D \in [F, A]$ be such that $D \,|\, n+1 = s\hat{\ }1$. Then $D(n) = 1$, and we have a contradiction. Finally, note that because A is infinite, T has infinitely many splitting nodes.

In addition, the correspondence described above is one-one and onto: If $T = T_{[F,A]}$, then $[F_T, A_T] = [F, A]$. Likewise, given a tree T such that $E(T)$ holds, $T_{[F_T, A_T]} = T$.

We now apply trees to construct a variation of the example in Corollary 2.8 that will be used in §3. We begin with a definition.

2.10. Definition. A set $X \subseteq [\omega]^\omega$ is *E-nowhere dense* if $X \cap [F, A]$ is nowhere dense in $[F, A]$ for every E-set $[F, A]$.

2.11. Theorem. *For every perfect set $P \subseteq [\omega]^\omega$, there is a perfect $Q \subseteq P$ which is E-nowhere dense; moreover, there is such a $Q \in \mathscr{L}_0$.*

Proof. The main insight for the construction—which was observed in [AFP]—is that any open set in 2^ω which includes Δ relativizes to an open set in $[\omega]^\omega$ which is open dense in every E-set.

Suppose P is perfect in $[\omega]^\omega$. Then $[T_P]$ is the closure of P in 2^ω. Let P' be a measure zero perfect subset of $[T_P]$, and let Q—as a subspace of 2^ω—be a perfect nowhere dense subset of $P' \backslash \Delta$. Let $W = 2^\omega \backslash Q$. We show that Q is E-nowhere dense. Given an E-set $[F, A]$, let $T = T_{[F,A]}$; recall that in 2^ω, $[T] \cap \Delta$ is dense in $[T]$. Hence, if $U = V \cap [F, A]$ is open in $[F, A]$, where V is open in 2^ω, $V \cap W$ meets Δ. Thus $V \cap W \cap [T]$ is a nonempty set open in $[T]$ which misses Q; it follows that $V \cap W \cap [F, A] \subseteq U$ is open in $[F, A]$ and misses Q. □

It will be useful for later work to make a few more distinctions related to E-sets:

2.12. Definition. A set $X \subseteq [\omega]^\omega$ is *E-meager* if for all E-sets $[F, A]$, $X \cap [F, A]$ is meager in $[F, A]$. X is *σ-E-nowhere dense* if X is a countable union of E-nowhere dense sets. X is *E-thin* if X contains no E-set.

Note that

$$E\text{-nowhere dense} \to \sigma\text{-}E\text{-nowhere dense} \to E\text{-meager}$$

and that

$$\mathrm{CR}_0 \to E\text{-thin}.$$

The only other implications are that E-meager implies E-thin and that σ-E-nowhere dense implies CR_0. To prove the latter, it suffices to show that every E-nowhere dense set is CR_0. Suppose X is E-nowhere dense and $[F, A]$ is an E-set. Recall that any open set contains a basic open set of the form $[G, [n, \omega)]$; it follows that if U is open in $[F, A]$, there are $G \supseteq F$ and $B \subseteq A$ such that $[G, B] \subseteq U$ and $[G, B] \cap X = \varnothing$. Following Proposition 2.13, we give counterexamples demonstrating that no other implications hold.

The σ-E-nowhere dense sets will be used in the next section to improve upon results of Brown [B] and Reclaw [AFP, Theorem 4] concerning Ramsey sets and $(s)_0$ sets. The E-meager sets provide a natural weakening of the notion of always first category sets; we will discuss AFC sets at greater length at the end of this section. The next proposition shows that the classes described in the diagrams above collapse to CR_0 on closed sets.

2.13. PROPOSITION. *If P is a closed subset of $[\omega]^\omega$, the following are equivalent:*

(1) *P is E-nowhere dense.*

(2) *P is σ-E-nowhere dense.*

(3) *P is E-meager.*

(4) *P is CR_0.*

(5) *P is E-thin.*

PROOF. From earlier remarks, the following implications hold: (1) $\to$ (2) $\to$ (3) $\to$ (5) and (4) $\to$ (5). Thus, it suffices to show (5) $\to$ (4) and (5) $\to$ (1). We show that each closed E-thin set is both CR_0 and E-nowhere dense. Suppose P is closed and E-thin. Given an E-set $[F, A]$, $P \cap [F, A]$ being closed either is nowhere dense in $[F, A]$ or contains a relative open set; but if it contained a relative open set, it would contain the intersection of a basic open set—of the form $[G, B]$—with $[F, A]$ and hence would contain an E-set. To see P is CR_0, note that $P \in \mathrm{CR}$ (since every Borel set is in CR); if $P \in \mathrm{CR}\setminus\mathrm{CR}_0$, P must contain an E-set; thus $P \in \mathrm{CR}_0$. □

Properties (2)–(5) remain equivalent for F_σ sets; it is easy to find an F_σ set which is σ-E-nowhere dense but not E-nowhere dense. Properties (4) and (5) remain equivalent for G_δ sets (they are in fact equivalent for all CR sets, hence for all Borel sets; note that any Bernstein set is E-thin but not CR_0). Brown [B] gives an example of a dense G_δ CR_0 set (this is our set N in Proposition 2.16). In the same paper, he uses CH to give an example of an AFC set which is not CR_0 (his set has other stronger properties as well); since AFC $\to$ E-meager, we have a proof that, under CH, E-meager does not imply CR_0. I do not know if there is such an example without CH nor if, under CH, there is a Borel example.

OPEN PROBLEM #1. Assuming CH, is there a Borel example of an E-meager set which is not in CR_0? In ZFC, is there any example at all of an E-meager set not in CR_0?

a splitting node is its splitting node of shortest length. The stem of a tree T is denoted stem(T).

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Note that

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and that

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The only other implications are that E-meager implies E-thin and that σ-E-nowhere dense implies CR_0. To prove the latter, it suffices to show that every E-nowhere dense set is CR_0. Suppose X is E-nowhere dense and $[F, A]$ is an E-set. Recall that any open set contains a basic open set of the form $[G, [n, \omega)]$; it follows that if U is open in $[F, A]$, there are $G \supseteq F$ and $B \subseteq A$ such that $[G, B] \subseteq U$ and $[G, B] \cap X = \varnothing$. Following Proposition 2.13, we give counterexamples demonstrating that no other implications hold.

The σ-E-nowhere dense sets will be used in the next section to improve upon results of Brown [B] and Reclaw [AFP, Theorem 4] concerning Ramsey sets and $(s)_0$ sets. The E-meager sets provide a natural weakening of the notion of always first category sets; we will discuss AFC sets at greater length at the end of this section. The next proposition shows that the classes described in the diagrams above collapse to CR_0 on closed sets.

2.13. Proposition. *If P is a closed subset of $[\omega]^\omega$, the following are equivalent*:

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(3) *P is E-meager.*

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Proof. From earlier remarks, the following implications hold: (1) → (2) → (3) → (5) and (4) → (5). Thus, it suffices to show (5) → (4) and (5) → (1). We show that each closed E-thin set is both CR_0 and E-nowhere dense. Suppose P is closed and E-thin. Given an E-set $[F, A]$, $P \cap [F, A]$ being closed either is nowhere dense in $[F, A]$ or contains a relative open set; but if it contained a relative open set, it would contain the intersection of a basic open set—of the form $[G, B]$—with $[F, A]$ and hence would contain an E-set. To see P is CR_0, note that $P \in \mathrm{CR}$ (since every Borel set is in CR); if $P \in \mathrm{CR}\backslash\mathrm{CR}_0$, P must contain an E-set; thus $P \in \mathrm{CR}_0$. □

Properties (2)–(5) remain equivalent for F_σ sets; it is easy to find an F_σ set which is σ-E-nowhere dense but not E-nowhere dense. Properties (4) and (5) remain equivalent for G_δ sets (they are in fact equivalent for all CR sets, hence for all Borel sets; note that any Bernstein set is E-thin but not CR_0). Brown [B] gives an example of a dense G_δ CR_0 set (this is our set N in Proposition 2.16). In the same paper, he uses CH to give an example of an AFC set which is not CR_0 (his set has other stronger properties as well); since AFC → E-meager, we have a proof that, under CH, E-meager does not imply CR_0. I do not know if there is such an example without CH nor if, under CH, there is a Borel example.

Open problem #1. Assuming CH, is there a Borel example of an E-meager set which is not in CR_0? In ZFC, is there any example at all of an E-meager set not in CR_0?

The term 'E-meager' has been used by Brown [B] to mean "meager with respect to the Ellentuck topology on $[\omega]^\omega$;" the Ellentuck topology is that obtained by taking the E-sets as a base. Ellentuck showed (see [B]) that the sets which are meager in this topology are precisely the CR_0 sets. Thus, as our remarks above indicate, the class we have termed 'E-meager' differs from Brown's.

We now consider a cardinal which will be used as a bound in the next section. As we mentioned in §0, Reclaw in [AFP, Theorem 4] constructed a set $X \in (s)_0\backslash \mathrm{R}$ assuming $\mathrm{cov}(\mathscr{K}) = \mathfrak{c}$; Brown in [B] did the same assuming CH; his arguments can be carried out under the weaker assumption[6] $\mathrm{cov}(\mathrm{CR}_0) = \mathfrak{c}$. We wish to improve these results slightly by weakening the hypothesis to:

$$\mathrm{cov}(\sigma\text{-END}) = \mathfrak{c},$$

where σ-END is the class of σ-E-nowhere dense sets.

The next theorem indicates why the cardinal bounds of Reclaw and Brown need improvement: ZFC does not decide which of $\mathrm{cov}(\mathscr{K})$ and $\mathrm{cov}(\mathrm{CR}_0)$ is larger.

2.14. THEOREM. *Assuming* ZFC *is consistent, so is each of the following*:

(1) ZFC + $\mathrm{cov}(\mathscr{K}) < \mathrm{cov}(\mathrm{CR}_0)$,

(2) ZFC + $\mathrm{cov}(\mathrm{CR}_0) < \mathrm{cov}(\mathscr{K})$.

PROOF OF (1). It suffices to produce a model of "$\omega_1 = \mathrm{cov}(\mathscr{K}) < \mathrm{cov}(\mathrm{CR}_0) = \omega_2 = \mathfrak{c}$." In a model of GCH, let $\langle \mathscr{Q}_\alpha \colon \alpha \le \omega_2\rangle$ be an ω_2-stage countable support iteration of the Mathias order; no cardinals are collapsed and $\mathfrak{c} = \omega_2$ in the extension. Miller [Mi1] shows that in this model, $\mathrm{cov}(\mathscr{K}) = \omega_1$. To see that "$\mathrm{cov}(\mathrm{CR}_0) = \mathfrak{c}$" also holds in the model, we argue as in [JMS] nearly verbatim, replacing the $(s)_0$ ideal by CR_0. First note that Mathias forcing is isomorphic to the poset $\mathscr{Q}$ of all E-sets ordered by inclusion; we use the latter order in our iteration. Suppose $\langle X_\alpha \colon \alpha < \omega_1\rangle$ are CR_0 sets in $M[G_{\omega_2}]$. In $M[G_{\omega_2}]$ let $f_\alpha \colon \mathscr{Q} \to \mathscr{Q}$ be defined so that for each $[F, A] \in \mathscr{Q}$ $f_\alpha([F, A]) \cap X_\alpha = \varnothing$. Using the ω_2-cc and a Löwenheim-Skolem type argument, one can find a $\gamma < \omega_2$ such that

$$\langle f_\alpha \restriction Q^{M[G_\gamma]} \colon \alpha < \omega_1\rangle \in M[G_\gamma].$$

But now the γth Mathias real x_γ is not in $\bigcup_{\alpha<\omega_1} X_\alpha$: If it were, then for some $p \in Q_{[\gamma,\omega_2]}$ and some $\alpha < \omega_1$, we would have:

$$p \Vdash x_\gamma \in X_\alpha.$$

But then let $[F, A] = p(\gamma) \in Q^{M[G_\gamma]}$, $r(\gamma) = f_\alpha([F, A])$ and $r(\beta) = p(\beta)$ for $\beta > \gamma$; we are left with the contradiction that $r \Vdash x_\gamma \notin X_\alpha$. □

PROOF OF (2). The model is obtained by forcing with an ω_1-stage finite support iteration of the Cohen order starting from a model of MA + $\neg$CH. We first prove that $\mathfrak{c}$-Luzin sets are preserved by such a forcing; our argument is a modification of an argument suggested to us by Miller and Judah showing that adding one Cohen real does not destroy a Luzin set.

[6] The reader is warned not to make the same mistake the author did and suppose that Brown's arguments can be carried out under the weaker hypothesis $\mathrm{cov}(\mathrm{CR}_0 \cap \mathscr{L}_0) = \mathfrak{c}$; under this hypothesis, there is no guarantee that Brown's inductive construction of a set in $(s)_0\backslash\mathrm{R}$ can be completed.

Claim. If $\mathfrak{c} = \omega_2$ and there is a $\mathfrak{c}$-Luzin set L, then in the model obtained by forcing with $Fn(\omega_1, 2)$, L continues to be $\mathfrak{c}$-Luzin.

Proof of Claim. We show that every set L in the ground model M which has size $\mathfrak{c}$ intersection with a meager set X in the extension must contain a ground model meager set of size $\mathfrak{c}$; this will show that such sets L are *not* $\mathfrak{c}$-Luzin, completing the proof.

Let

$$\mathbf{1} \Vdash \dot{X} \text{ is meager} \quad \text{and} \quad L \cap \dot{X} = \{\dot{x}_\alpha : \alpha < \omega_2\}.$$

For each $\alpha < \omega_2$, let $x_\alpha \in M$ and $p_\alpha \in Fn(\omega_1, 2)$ be such that

$$p_\alpha \Vdash \dot{x}_\alpha = x_\alpha.$$

Since $|Fn(\omega_1, 2)| = \aleph_1$, there is a condition p and a set A of cardinality ω_2 such that

$$p \Vdash \text{“}\dot{x}_\alpha = x_\alpha\text{''} \quad \text{for all } \alpha \in A.$$

Let $Y = \{x_\alpha : \alpha \in A\}$. Since $p \Vdash Y \subseteq \dot{X}$, it follows that $p \Vdash Y$ is meager. □

Now to prove part (2) of the theorem, we force with an ω_1-stage finite-support iteration $\langle \mathscr{P}_\alpha : \alpha \leq \omega_1 \rangle$ of the Cohen order from a model M of MA + $\neg$CH. In the extension, $\mathfrak{c} = \omega_2$; also, by the claim, the extension contains a $\mathfrak{c}$-Luzin set. It follows that $\mathrm{cov}(\mathscr{K}) = \mathfrak{c}$. It follows from Corollary 1.4 that, in the extension, each $M[G_\alpha] \cap [\omega]^\omega$ has strong measure zero relative to the metric induced by ω^ω. In [B], Brown shows that such strong measure zero sets are in fact CR_0. It follows from the ccc property that

$$M[G_{\omega_1}] \cap [\omega]^\omega = \bigcup_\alpha M[G_\alpha] \cap [\omega]^\omega.$$

Hence, $\mathrm{cov}(\mathrm{CR}_0) = \omega_1$ in this model. □

The next proposition shows that the hypothesis “$\mathrm{cov}(\sigma\text{-END}) = \mathfrak{c}$” is weaker than either of those used by Reclaw or Brown.

2.15. PROPOSITION. (1) $\mathrm{cov}(\mathscr{K}) \leq \mathrm{cov}(\sigma\text{-END})$.

(2) $\mathrm{cov}(\mathrm{CR}_0) \leq \mathrm{cov}(\sigma\text{-END})$.

PROOF. This follows because $\sigma\text{-END} \subseteq \mathscr{K}$ and $\sigma\text{-END} \subseteq \mathrm{CR}_0$. □

I do not know if there is a model in which $\leq$ can be replaced by $<$ in the above proposition; however, we show in the next proposition that σ-END is a proper subcollection of $\mathscr{K} \cap \mathrm{CR}_0$.

2.16. PROPOSITION. *There is a G_δ set $X \in \mathscr{K} \cap \mathrm{CR}_0$ which is not E-meager (and hence not σ-END).*

PROOF. We expand upon an example given in Brown [B]. Let $E_0 = E = \{2n: n \in \omega\}$ and $O_0 = O = \{2n + 1: n \in \omega\}$. Let $E_n = E_0 \cup n$ and $O_n = O_0 \cup n$. Let $H = \bigcup_{n\in\omega}[E_n]^\omega \cup \bigcup_{n\in\omega}[O_n]^\omega$ and let $N = [\omega]^\omega \backslash H$. As Brown observes, N is CR_0, comeager and of measure 1; note also that N is G_δ. Let $K = \{3n: n \in \omega\}$. The required set is $X = [K]^\omega \cap N$; since X is clearly meager and CR_0, it suffices to show that $[K]^\omega \cap N$ is a dense G_δ relative to $[K]^\omega$. For each n, let

$$R_n^{\text{even}} = \{A \subseteq K: \text{every } x \in A \cap (n, \omega) \text{ is even}\},$$
$$R_n^{\text{odd}} = \{A \subseteq K: \text{every } x \in A \cap (n, \omega) \text{ is odd}\};$$

each is closed nowhere dense relative to $[K]^\omega$. But then

$$[K]^\omega \cap H = \bigcup_{n\in\omega} R_n^{\text{even}} \cup \bigcup_{n\in\omega} R_n^{\text{odd}}$$

is a relative meager F_σ; the result follows. □

OPEN PROBLEM #2. Construct a model of ZFC in which $\text{cov}(\mathscr{K}) + \text{cov}(\text{CR}_0) = \omega_1$ and $\text{cov}(\sigma\text{-END}) = \omega_2 = \mathbf{c}$.

2.17. REMARK. The existence of a set which is a CR_0 measure 1 dense G_δ (like the set N in the last theorem) implies the following curious fact: Suppose $\mathscr{I} \neq \mathscr{J} \in \{\mathscr{L}_0, \mathscr{K}, \text{CR}_0\}$; then there are $X \in \mathscr{I}$ and $Y \in \mathscr{J}$ such that $X \cup Y = [\omega]^\omega$. The same fact does not hold if CR_0 is replaced by $(s)_0$.

We conclude this section with a discussion of AFC sets. Note that the definition of E-meager sets resembles that of AFC sets in that one tests a set X for membership in the ideal of E-meager sets by intersecting X with each member of a large class of perfect sets, namely, the E-sets. Proposition 2.11 shows, however, that the class of E-sets is not adequate to test for membership in AFC since AFC contains no perfect set. When *is* a class of perfect sets adequate? Following the referee's suggestion, we restrict ourselves to a brief outline of results. We begin with a definition.

DEFINITION. Let $\mathscr{P}$ denote the class of perfect sets in a perfect Polish space $\mathbf{X}$ and $\mathscr{R} \subseteq \mathscr{P}$. We call the set $\{X: \forall P \in \mathscr{R}\, X \cap P \text{ is meager in } P\}$ the *collection of $\mathscr{R}$-based sets*. The set $\mathscr{R}$ is called a *base for* AFC if

$$X \in \text{AFC} \quad \text{iff} \quad X \text{ is an } \mathscr{R}\text{-based set.}$$

The following easily verified facts are suggestive:

1. If there is a Luzin set, the class of all nowhere dense perfect sets is *not* a base for AFC.

2. A base for AFC need not be dense in the partial order of all perfect sets (ordered by $\subseteq$). For instance, if $\mathbf{X} = [0,1]$ and if $\mathscr{R} = \{P\colon P \text{ is perfect and } P \not\subseteq [0,1/2]\}$, then $\mathscr{R}$ is a (nondense) base for AFC.

3. The set of all closures of basic open sets together with the class of nowhere dense perfect sets forms a base for AFC (see [C2]).

LEMMA. *Suppose $P \subseteq Q$ are perfect. Suppose that for all $N \subseteq Q$ for which $N \cap P \neq \varnothing$, we have that N contains a nonempty set which is open relative to P iff N contains a nonempty set which is open relative to Q. Then for all sets X, $X \cap P$ is nowhere dense relative to P iff $X \cap Q$ is nowhere dense relative to Q.*

We will say that if $P \subseteq Q$ are perfect sets satisfying the hypothesis of the lemma, then (P, Q) has the *nowhere dense property*.

PROPOSITION. *Suppose $\mathscr{R}$ is a collection of perfect sets satisfying the following property:*

If $P \notin \mathscr{R}$ is perfect, then there is a countable collection $\{Q_n\colon n \in \omega\} \subseteq \mathscr{R}$ such that

(a) *$\bigcup_n \text{int}_P(P \cap Q_n)$ is dense in P;*

(b) *for each n, $(P \cap Q_n, Q_n)$ has the nowhere dense property.*

Then $\mathscr{R}$ is a base for AFC.

PROPOSITION. *Assume* MA (*in fact, assume that there is a cardinal κ, $\omega_1 \leq \kappa \leq \mathbf{c}$, such that* $\text{non}(\mathscr{K}) = \kappa$ *and some perfect Polish space contains a κ-Luzin set*). *Suppose*

$\mathscr{R}$ is a base for AFC. *Then $\mathscr{R}$ satisfies the following property:*

If $P \notin \mathscr{R}$ is perfect, then there is a countable collection $\{Q_n : n \in \omega\} \subseteq \mathscr{R}$ such that

(a) $\bigcup_n \operatorname{int}_P(P \cap Q_n)$ *is dense in* P;

(b) *for each* n, $\operatorname{int}_{Q_n}(P \cap Q_n) \neq \varnothing$.

SKETCH OF PROOF. Given $P \notin \mathscr{R}$ perfect, use 1.0 to show that

$$\bigcup\{\operatorname{int}_P(P \cap Q) : Q \in \mathscr{R}_0\} \text{ is dense in } P,$$

where

$$\mathscr{R}_0 = \{Q \in \mathscr{R} : \operatorname{int}_P(P \cap Q) \neq \text{ and } \operatorname{int}_Q(P \cap Q) \neq \varnothing\}. \qquad \square$$

Let base(AFC) $= \min\{|\mathscr{R}| : \mathscr{R}$ is a base for AFC$\}$.

PROPOSITION. $\operatorname{cov}(\mathscr{K}) \le \operatorname{cov}(\text{AFC}) \le \operatorname{base}(\text{AFC})$. *Thus, in particular,* AFC *has no countable base and, assuming* MA, base(AFC) $= \mathbf{c}$.

We are left with the following open problems.

OPEN PROBLEM #3. Is it provable in ZFC that base(AFC) $= \mathbf{c}$? If not, what values can cf(base(AFC)) have? Is base(AFC) regular? Does the value of base(AFC) depend on the underlying perfect Polish space?

OPEN PROBLEM #4. Is there a model in which the class of nowhere dense perfect sets is a base for AFC?

OPEN PROBLEM #5. Find necessary and sufficient conditions *in* ZFC for a subclass of the perfect sets to be a base for AFC.

§3. Some examples. As we mentioned in the Introduction, Brown in [B], using CH in many cases, showed that for each ideal $\mathscr{I}$ in $\mathscr{B} = \{\mathscr{L}_0, \mathscr{K}, (s)_0, \mathrm{CR}_0\}$, there is a set $X \in \mathscr{I}$ which is not in any of the three σ-algebras corresponding to the other three σ-ideals in $\mathscr{B}$; and that for every group of three σ-ideals in $\mathscr{B}$, there is a set in their intersection which is not in the σ-algebra corresponding to the fourth σ-ideal in $\mathscr{B}$. Our goal in this section is to eliminate the use of CH in the construction of many of these examples. As mentioned before, we have been unable to accomplish this goal for those constructions which involve the existence of a set in $(s)_0 \backslash \mathrm{CR}$. Reclaw [AFP] and Brown [B] give constructions of such sets using various hypotheses; we will do a similar construction with a set-theoretic hypothesis (namely, "$\operatorname{cov}(\sigma\text{-END}) = \mathbf{c}$," introduced in §2) which is weaker than both of theirs. Although we have no ZFC construction of such sets, we are able to prove in ZFC alone that if any set in $(s)_0 \backslash \mathrm{CR}$ *does* exist, then this fact is enough to obtain all the other related examples we wish to construct. Thus, we have several results like the following:

$$\text{ZFC} \vdash \text{“If } (s)_0 \backslash \mathrm{CR} \neq \varnothing, \text{ then } \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \backslash \mathrm{CR} \neq \varnothing.\text{”}$$

The various constructions we will do are fairly elementary once certain results are in place. Hence, we will present the necessary results first and then organize all the relevant examples into a chart with brief descriptions of the techniques of construction. Among these examples are those which show that for any *two* ideals in $\mathscr{B}$, there is a set in their intersection which is not in the σ-algebras corresponding to the other two ideals in $\mathscr{B}$. Brown informed the author that he had also observed this fact, though some of his constructions used CH. One additional feature of the constructions given in the chart is that they are all of cardinality $\mathbf{c}$; we indicate why this is so and mention several other points of interest immediately after our presentation of the chart.

We begin with several constructions of $(s)_0$ sets.

3.0. THEOREM. *There is a set in $\mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0$ of cardinality* $\mathfrak{c}$.

For the proof, we need the following lemma which essentially shows that if Walsh's construction [W1] of an $(s)_0$ set of power $\mathfrak{c}$ is carried out with disjoint E-sets, the resulting set is in $(s)_0 \cap \mathrm{CR}_0$.

3.1. LEMMA. *Suppose $\{B_\alpha : \alpha < \mathfrak{c}\}$ are disjoint Borel sets each of which contains an E-set. Then there is a set $X \in (s)_0 \cap \mathrm{CR}_0$ such that $|X \cap B_\alpha| = 1$ for all $\alpha < \mathfrak{c}$.*

PROOF OF LEMMA. Let $\langle P_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of the perfect sets in $[\omega]^\omega$ and for each $\alpha < \mathfrak{c}$, let $[F_\alpha, A_\alpha]$ denote an E-set included in B_α. Define $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ inductively as follows:

$$x_\alpha \in [F_\alpha, A_\alpha] \backslash \bigcup \{P_\beta : |P_\beta \cap [F_\alpha, A_\alpha]| \leq \omega, \beta < \alpha\}.$$

As in [W1], $X \in (s)_0$. To see $X \in \mathrm{CR}_0$ as well, suppose $[F, A]$ is an E-set and β is such that $[F, A] = P_\beta$. There are two cases: If $[F, A]$ has uncountable intersection with some $[F_\alpha, A_\alpha]$, then (since the E-sets form a basis for a T_1 topology; see [E]) this intersection includes an E-set which avoids x_α and hence all of X. The other case is that $[F, A]$ has countable intersection with each $[F_\alpha, A_\alpha]$. Then $[F, A]$ misses $\{x_\alpha : \alpha > \beta\}$, and since E-sets can be partitioned into $\mathfrak{c}$ disjoint pieces, each an E-set, one such piece misses X entirely. □

PROOF OF THEOREM 3.0. Apply the lemma with each B_α a subset of a fixed E-set $[G, B]$, where B is coinfinite (so that $[G, B]$ is meager and of measure zero); the set X in the conclusion of the lemma is now also meager and of measure zero. □

The next theorem provides, among other things, an example of an $(s)_0$ set which neither has the property of Baire nor is Lebesgue measurable; this answers an old problem implicit in Marczewski's original work [Ma2] on (s) and $(s)_0$. Independently, and with different methods, Walsh [W2] obtained a generalization of the same result shortly before we obtained ours. Our techniques also generalize, and so we state and prove Walsh's result in Corollary 3.4. For 3.2–3.4, we fix perfect Polish spaces $\mathbf{X}$ and $\mathbf{Y}$ and continuous finite Borel measures μ on $\mathbf{X}$ and ν on $\mathbf{Y}$.

3.2. THEOREM. (1) *There is an $(s)_0$ subset of $\mathbf{X} \times \mathbf{Y}$ which meets every full $\mu \times \nu$-measure F_σ set in $\mathbf{X} \times \mathbf{Y}$.*

(2) *There is an $(s)_0$ subset of $\mathbf{X} \times \mathbf{Y}$ which meets every dense G_δ set in $\mathbf{X} \times \mathbf{Y}$.*

To prove the theorem, we need the following lemma:

3.3. LEMMA. *Suppose $\langle G_\alpha : \alpha < \mathfrak{c}\rangle$ is an enumeration of the full $\mu \times \nu$-measure F_σ (dense G_δ) subsets of $\mathbf{X} \times \mathbf{Y}$. Then there is a set $Q = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbf{X}$ such that for all α, the αth vertical section $G_\alpha^{x_\alpha}$ of G_α is a ν-full-measure (dense G_δ) subset of $\mathbf{Y}$.*

PROOF OF LEMMA. Assume that for $\alpha < \mathfrak{c}$, x_β $(\beta < \alpha)$ have been defined. Since $Q_\alpha = \{x_\beta : \beta < \alpha\}$ is not a μ-full-measure (dense G_δ) subset of $\mathbf{X}$, we can apply Fubini's Theorem (the Ulam-Kuratowski Theorem; see [O]) to obtain $x_\alpha \notin Q_\alpha$ for which $G_\alpha^{x_\alpha}$ is a ν-full-measure (dense G_δ) subset of $\mathbf{Y}$. □

PROOF OF (1). We modify the proof of Theorem 5.10 in [Mi2]. Let $\langle G_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of the full $\mu \times \nu$-measure F_σ's in $\mathbf{X} \times \mathbf{Y}$ and $Q = \{x_\alpha : \alpha < \mathfrak{c}\}$ as in the lemma. Let $\langle P_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of the perfect sets in $\mathbf{X} \times \mathbf{Y}$. We inductively pick points $y_\alpha \in \mathbf{Y}$ so that the set $S = \{(x_\alpha, y_\alpha) : \alpha < \mathfrak{c}\}$ has the required properties. Choose y_α so that

$$y_\alpha \in G_\alpha^{x_\alpha} \setminus \bigcup \{P_\beta^{x_\alpha} : \beta < \alpha \text{ and } P_\beta^{x_\alpha} \text{ is countable}\}.$$

Such a y_α exists since $|\bigcup\{P_\beta^{x_\alpha} : \beta < \alpha \text{ and } P_\beta^{x_\alpha} \text{ is countable}\}| < \mathbf{c}$.

As in the proof of Theorem 5.10 in [Mi2], $S \in (s)_0$. As S meets every full $\mu \times \nu$-measure F_σ, S is not $\mu \times \nu$-null. Since S contains no perfect set, it is not $\mu \times \nu$-measurable either. □

PROOF OF (2). Proceed as in (1), replacing the notions "full $\mu \times \nu$-measure F_σ," "$\mu \times \nu$-null," and "is $\mu \times \nu$-measurable" with "dense G_δ," "meager," and "has the property of Baire," respectively. □

3.4. COROLLARY. *There is an $(s)_0$ subset $Z \subseteq \mathbf{X} \times \mathbf{Y}$ which does not have the property of Baire and which is nonmeasurable with respect to every product measure $\hat{\mu} \times \hat{\nu}$, where $\hat{\mu}$ and $\hat{\nu}$ are continuous finite Borel measures on $\mathbf{X}$ and $\mathbf{Y}$, respectively.*

PROOF. Note that there are only $\mathbf{c}$ many pairs of finite Borel measures on $\mathbf{X}$ and $\mathbf{Y}$. Thus, in the lemma, if we assume that $\langle G_\alpha : \alpha < \mathbf{c}\rangle$ is an enumeration of the set

$$\{K \subseteq \mathbf{X} \times \mathbf{Y} : K \text{ is } F_\sigma \text{ and there exist continuous finite Borel measures } \mu \text{ on } \mathbf{X} \text{ and } \nu \text{ on } \mathbf{Y} \text{ such that } K \text{ has full } \mu \times \nu\text{-measure}\}$$

then a set $Q' = \{x_\alpha : \alpha < \mathbf{c}\}$ may be obtained so that for each α, the x_αth vertical section has full measure with respect to some $\hat{\nu}$ on $\mathbf{Y}$. Now, if we carry out the construction in the proof of 3.2(1), the resulting $(s)_0$ set S meets every F_σ which has full measure with respect to some continuous finite Borel product measure and is therefore nonmeasurable with respect to all such measures.

Next, let T be constructed as in 3.2(2). Then $Z = S \cup T$ is the required set. □

3.5. COROLLARY. (1) *There is an $(s)_0$ subset of $[\omega]^\omega$ whose intersection with every measure 1 F_σ has cardinality* $\mathbf{c}$.

(2) *There is an $(s)_0$ subset of $[\omega]^\omega$ whose intersection with every dense G_δ has cardinality* $\mathbf{c}$.

PROOF. In the proof of 3.2(1), start with an enumeration $\langle G_\alpha : \alpha < \mathbf{c}\rangle$ of measure 1 F_σ's (in $2^\omega \times 2^\omega$) such that each set is enumerated $\mathbf{c}$ times; in building the set S, make sure that $y_\alpha \notin \{y_\beta : \beta < \mathbf{c}\}$. The intersection of S with each measure 1 F_σ subset of $2^\omega \times 2^\omega$ has power $\mathbf{c}$; there is a homeomorphism $\varphi: 2^\omega \times 2^\omega \to 2^\omega$ such that $\varphi(S)$ has the same property in 2^ω (see [R, Theorem 15.3.9]). Thus, $\varphi(S) \cap [\omega]^\omega$ is the required set for part (1). For part (2), use dense G_δ's instead of measure 1 F_σ's and a category preserving homeomorphism in place of φ. □

We proceed to several equivalent forms of the statement "$(s)_0 \setminus \mathrm{CR} \neq \varnothing$;" these may be of some assistance in devising a ZFC proof of this statement.

3.6. THEOREM. *The following are equivalent*:

(1) $(s)_0 \setminus \mathrm{CR}_0 \neq \varnothing$.

(2) $(s)_0 \setminus \mathrm{CR} \neq \varnothing$.

(3) $(s)_0 \setminus \mathrm{R} \neq \varnothing$.

(4) *There is a dense subclass $\mathscr{R}$ of the poset $\mathscr{P}$ of perfect sets such that for all $\mathscr{R}_0 \subseteq \mathscr{R}$ for which $|\mathscr{R}_0| < \mathbf{c}$, $\bigcup \mathscr{R}_0 \not\supseteq [B]^\omega$ for any $B \in [\omega]^\omega$.*

PROOF. Clearly, (3) → (2) → (1). We show (1) → (3) and (3) ↔ (4).

(1) → (3): Suppose $X \in (s)_0 \setminus \mathrm{CR}_0$ and let $[F, A]$ be such that for all $B \in [A]^\omega$, $X \cap [F, B] \neq \varnothing$. Let $Y = X \cap [F, A]$. Define the functions $f_A: [A]^\omega \to [\omega]^\omega$ and

$g_{[F,A]}\colon [F,A] \to [A]^\omega$ as in the proof of Lemma 2.7. We claim that $f_A(g_{[F,A]}(Y)) \in (s)_0\backslash\mathrm{R}$. Given $C \in [\omega]^\omega$, it's easy to check that $(f_A \circ g_{[F,A]})^{-1}([C]^\omega)$ includes an E-set $[F,D]$. Thus, if $W \in Y \cap [F,D]$, $f_A(g_{[F,A]}(W)) \in f_A(g_{[F,A]}(Y)) \cap [C]^\omega$, as required.

(3) $\leftrightarrow$ (4): Assume (3) $\wedge \neg$(4). Let $X \in (s)_0\backslash\mathrm{R}$ and let $\mathscr{R}$ denote the class of all perfect sets which miss X; clearly, $\mathscr{R}$ is dense in $\mathscr{P}$. By $\neg$(4), there is a class $\mathscr{R}_0 \subseteq \mathscr{R}$ of power $<\mathbf{c}$ and a set $B \in [\omega]^\omega$ for which $\bigcup\mathscr{R}_0 \supseteq [B]^\omega$, contradicting the fact that X meets $[B]^\omega$.

For the converse, assume (4) holds and let $\mathscr{R}$ be the dense subclass of $\mathscr{P}$ given in (4). As $|\mathscr{R}| = \mathbf{c}$ (since perfect sets contain $\mathbf{c}$ disjoint perfect subsets), write $\mathscr{R} = \{P_\alpha\colon \alpha < \mathbf{c}\}$. Let $\langle B_\alpha\colon \alpha < \mathbf{c}\rangle$ be an enumeration of $[\omega]^\omega$. Obtain $X = \{x_\alpha\colon \alpha < \mathbf{c}\}$ inductively by picking

$$x_\alpha \in [B_\alpha]^\omega \left(\bigcup_{\beta<\alpha} P_\beta \cup \{x_\beta\colon \beta < \alpha\}\right).$$

$X \in (s)_0$ since for each perfect P, we can find β such that $P \supseteq P_\beta$; and as $|P_\beta \cap X| < \mathbf{c}$, P_β has a perfect subset which misses X. $X \notin \mathrm{R}_0$ since X meets every $[B]^\omega$; thus $X \notin \mathrm{R}$ (since X contains no perfect set). □

It follows from the theorem that (2) is equivalent to each of the statements "$(s)_0\backslash\mathrm{UR}_0 \neq \varnothing$," "$(s)_0\backslash\mathrm{UR} \neq \varnothing$," and "$(s)_0\backslash\mathrm{R}_0 \neq \varnothing$."

We now prove $(s)_0\backslash\mathrm{R} \neq \varnothing$ assuming that cov(σ-END) $= \mathbf{c}$; recall from §2 that this hypothesis is a natural weakening of those used by Reclaw [AFP] and Brown [B] to obtain the same result.

3.7. THEOREM. *Assume* cov(σ-END) $= \mathbf{c}$. *Then* $(s)_0\backslash\mathrm{R} \neq \varnothing$.

PROOF. We use criterion (4) from the last theorem. Let $\mathscr{R}$ denote the class of all σ-E-nowhere dense perfect sets; by Theorem 2.11(1), $\mathscr{R}$ is dense in the perfect sets. Now if $\mathscr{R}_0 \subseteq \mathscr{R}$ is a subcollection of size $< \mathbf{c}$ and $\bigcup\mathscr{R}_0 \supseteq [F,A]$, for some $[F,A]$, we show how to obtain another subcollection $\mathscr{R}_1 \subseteq \mathscr{R}$ of size $< \mathbf{c}$ which covers $[\omega]^\omega$; since such a collection violates the hypothesis cov(σ-END) $= \mathbf{c}$, the proof will be complete once the collection $\mathscr{R}_1$ has been exhibited. Assume each $Z \in \mathscr{R}_0$ is a subset of $[F,A]$. Let $\mathscr{R}_1 = \{f_A(g_{[F,A]}(Z))\colon Z \in \mathscr{R}_0\}$ (where f_A and $g_{[F,A]}$ are defined as in the proof of the previous theorem). Because $f_A \circ g_{[F,A]}$ is onto, $\bigcup\mathscr{R}_1 = [\omega]^\omega$; and because this map preserves E-nowhere denseness, $\mathscr{R}_1 \subseteq \mathscr{R}$, and we are done. □

We now proceed to a chart which summarizes our constructions of sets which lie in some of the σ-ideals and not in other σ-algebras, as discussed at the beginning of this section. We begin by fixing notation for certain sets and functions described earlier:

Symbol	Definition
E	The set of even numbers
O	The set of odd numbers
$\varphi_A\colon [\omega]^\omega \to [A]^\omega$	The homeomorphism induced by the increasing enumeration $\omega \to A$ (see Lemma 2.7)
G	A measure zero dense G_δ

Symbol	Definition
N	A CR_0 dense G_δ having measure 1 (see the proof of Proposition 2.16 or [B])
U	An $(s)_0$ set whose intersection with every measure 1 F_σ has cardinality $\mathbf{c}$ (see Corollary 3.5(1))
V	An $(s)_0$ set whose intersection with every dense G_δ has cardinality $\mathbf{c}$ (see Corollary 3.5(2))
W	A member of $(s)_0\backslash\mathrm{R}$ (see Theorem 3.7)

Note that we have assumed there is a set W in $(s)_0\backslash\mathrm{R}$; no hypotheses beyond ZFC other than this one will be needed to carry out the constructions below. Moreover, all examples other than (b), (g), (i), and (o) are constructed in ZFC alone.

Where X lives	How X is constructed
(a) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0$	Theorem 3.0
(b) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0\backslash\mathrm{R}$	$X = \varphi_E(W) \cup \varphi_O(W)$
(c) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap \mathrm{CR}_0\backslash(s)$	(Brown [B]) X is a Bernstein subset of a measure zero perfect subset of N
(d) $X \in \mathscr{L}_0 \cap (s)_0 \cap \mathrm{CR}_0\backslash\mathrm{B_W}$	$X = G \cap N \cap V$
(e) $X \in \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0\backslash\mathscr{L}$	$X = N \cap U\backslash G$
(f) $X \in \mathscr{L}_0 \cap \mathscr{K}\backslash(s) \cup \mathrm{R}$	X is a Bernstein subset of $[E]^\omega \cup [O]^\omega$
(g) $X \in \mathscr{L}_0 \cap (s)_0\backslash\mathrm{B_W} \cup \mathrm{R}$	$X = \varphi_E(W) \cup \varphi_O(W) \cup (G \cap V)$
(h) $X \in \mathscr{L}_0 \cap \mathrm{CR}_0\backslash\mathrm{B_W} \cup (s)$	X is a Bernstein subset of $G \cap N$
(i) $X \in \mathscr{K} \cap (s)_0\backslash\mathscr{L} \cup \mathrm{R}$	$X = \varphi_E(W) \cup \varphi_O(W) \cup (U\backslash G)$
(j) $X \in \mathscr{K} \cap \mathrm{CR}_0\backslash\mathscr{L} \cup (s)$	X is a Bernstein subset of $N\backslash G$
(k) $X \in (s)_0 \cap \mathrm{CR}_0\backslash\mathscr{L} \cup \mathrm{B_W}$	$X = N \cap (U \cup V)$
(m) $X \in \mathscr{L}_0\backslash\mathrm{B_W} \cup (s) \cup \mathrm{R}$	(Brown [B]) X is a Bernstein subset of $[E]^\omega \cup [O]^\omega \cup G$
(n) $X \in \mathscr{K}\backslash\mathscr{L} \cup (s) \cup \mathrm{R}$	X is a Bernstein subset of $[E]^\omega \cup [O]^\omega \cup ([\omega]^\omega\backslash G)$
(o) $X \in (s)_0\backslash\mathscr{L} \cup \mathrm{B_W} \cup \mathrm{R}$	$X = U \cup V \cup W$
(p) $X \in \mathrm{CR}_0\backslash\mathscr{L} \cup \mathrm{B_W} \cup (s)$	(Brown [B]) X is a Bernstein subset of N
(q) $X \notin \mathscr{L} \cup \mathrm{B_W} \cup (s) \cup \mathrm{R}$	X is any Bernstein set

3.8. REMARKS. (1) In [B], Brown gives CH examples of (b), (d), (e), and (o), and ZFC examples of (c), (m), and (p). His version of example (n) is actually in $\mathscr{K} \backslash \mathscr{L} \cup (s) \cup \mathrm{UR}$; we have improved this result slightly here. Also, our example (m) is slightly different from Brown's.

(2) The reader may notice that many of our examples (b, f, g, i, m, n, o, and q) are constructed so that they lie outside of R even though we promised only to give examples lying outside of CR; consequently, some of our constructions are somewhat more complicated than they would have been otherwise. For example, in part (b), if we are content to find a set in $\mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \backslash \mathrm{CR}$, $\varphi_E(W)$ suffices. The main observation which allows us to construct stronger examples (i.e., lying outside of R) is that the intersection of the set $[E]^\omega \cup [O]^\omega$ with any $[A]^\omega$ includes a $[B]^\omega$; thus if X meets every $[C]^\omega$ for $C \in [E]^\omega \cup [O]^\omega$, then X must meet every set of this form, and this fact is often sufficient to establish that $X \notin R$. Arguing in a similar fashion, replacing $[E]^\omega \cup [O]^\omega$ by the set H of Proposition 2.16, it is possible to further improve each of these examples X so that both X and its complement actually meet every E-set.

(3) Another feature of the above examples is that each has cardinality $\mathbf{c}$. This is clear for example (a) (by Theorem 3.0) and for those sets which are not in $(s) \cap \mathrm{CR}$ (since $(s)_0 \cap \mathrm{CR}_0$ contains all sets of cardinality $< \mathbf{c}$). We are left with (d), (e), and (k); in each of these three cases, the set X includes (or is equal to) the intersection either of U with a measure 1 F_σ or of V with a dense G_δ, and hence has cardinality $\mathbf{c}$.

(4) At the request of the referee, we verify one line of the chart: we show that the construction in line (k) produces the desired set X. Being a subset of $U \cup V$, $X \in (s)_0$, and being a subset of N, $X \in \mathrm{CR}_0$. Note that X meets every measure 1 F_σ. Given an F_σ F, since $F \cap N$ has measure 1 (and hence includes a measure 1 F_σ), we have

$$X \cap F = (U \cup V) \cap (F \cap N) \neq \varnothing$$

(in fact, $X \cap F$ has cardinality $\mathbf{c}$). Thus $X \notin \mathscr{L}_0$. Since $X \in (s)_0$, $X \notin \mathscr{L}$. A similar argument shows that $X \notin \mathrm{B_W}$.

OPEN PROBLEM #6. Is there a ZFC example of a set in $(s)_0 \backslash \mathrm{CR}$?

§4. Examples with a Ramsey ultrafilter. Our original plan for this section was to study examples as in §3 with CR_0 and CR replaced by $\mathrm{CR}_0^{\mathscr{U}}$ and $\mathrm{CR}^{\mathscr{U}}$ where $\mathscr{U}$ is a Ramsey ultrafilter (defined below); however, in working with these classes, it seemed natural to expand the range of examples somewhat. Our motivation for this study is that $\mathrm{CR}_0^{\mathscr{U}}$ and $\mathrm{CR}^{\mathscr{U}}$ are in many ways better behaved that CR_0 and CR:

(A) $\mathrm{CR}_0^{\mathscr{U}}$ has the ccc property (every collection of subsets of $\mathrm{CR}^{\mathscr{U}} \backslash \mathrm{CR}_0^{\mathscr{U}}$ all of whose pairwise intersections are in $\mathrm{CR}_0^{\mathscr{U}}$ is countable) [Ms, Proposition 1.11], while CR_0 does not.

(B) $\mathrm{CR}^{\mathscr{U}}$ is *Borel-supported*, i.e., each $X \in \mathrm{CR}^{\mathscr{U}}$ is the symmetric difference of a Borel set and a $\mathrm{CR}_0^{\mathscr{U}}$ set ([Ms, Proposition 1.9]); CR does not have this property (Proposition 4.0).

Once it is observed that both $\mathrm{CR}_0 \backslash \mathrm{CR}^{\mathscr{U}}$ and $\mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{CR}$ are nonempty (Proposition 4.4), it is natural to proceed as in §3 with the *five* σ-ideals

$\{\mathscr{L}_0, \mathscr{K}, (s)_0, \mathrm{CR}_0, \mathrm{CR}_0^{\mathscr{U}}\}$ and their corresponding σ-algebras $\{\mathscr{L}, \mathrm{B_W}, (s), \mathrm{CR}, \mathrm{CR}^{\mathscr{U}}\}$. Thus, in this section we shall show that for any nonempty subcollection $\mathscr{H}$ of these σ-ideals, there is a set in their intersection which is not in the union of the σ-algebras corresponding to the σ-ideals lying outside of $\mathscr{H}$.

After proving Proposition 4.0 and stating the basic definitions and facts concerning $\mathrm{CR}_0^{\mathscr{U}}$ and $\mathrm{CR}^{\mathscr{U}}$, we proceed to several central observations from which all the required examples can easily be constructed. As in the last section, most of our examples can be constructed in ZFC alone (although Ramsey ultrafilters cannot be proven to exist in ZFC [K2], our constructions require only that the filter $\mathscr{U}$ be an ultrafilter; Ramsey-ness of $\mathscr{U}$ is needed as background to establish that $\mathrm{CR}^{\mathscr{U}}$ is a σ-algebra including the Borel sets, and that $\mathrm{CR}_0^{\mathscr{U}}$ is a σ-ideal [Ms, §1]). Eight of the other examples are constructed assuming, as in §3, that $(s)_0 \backslash \mathrm{R} \neq \varnothing$. Finally, there are four "exceptional" examples which we construct assuming Martin's Axiom; in the remarks following Proposition 4.8, we give a plausibility argument for why the axiom "$(s)_0 \backslash \mathrm{R} \neq \varnothing$" is not enough to construct these examples.

Our first proposition asserts that unlike $\mathscr{L}$ and $\mathrm{B_W}$, CR is not Borel supported. Walsh [W1] observed that (s) also lacks this property, and Brown [B] proved the result for CR assuming CH.

4.0. THEOREM. CR *is not Borel-supported.*

PROOF. The proof is a modification of a classical proof due to Sierpinski [S]; see also [B]. If the theorem were false, then for each $X \in \mathrm{CR}$ there would be a Borel set B such that $X \Delta B \in \mathrm{CR}_0$; moreover, it follows that if there were a collection $\mathscr{G} \subseteq \mathrm{CR}$ having $> \mathbf{c}$ elements, there would be a Borel set B such that for $> \mathbf{c}$ many X in $\mathscr{G}$, $X \Delta B \in \mathrm{CR}_0$. Thus, we will be done when we exhibit a collection $\mathscr{G} \subseteq \mathrm{CR}$ of cardinality $2^{\mathbf{c}}$ such that for all $X \neq Y \in \mathscr{G}$, $X \Delta Y \notin \mathrm{CR}_0$. To obtain $\mathscr{G}$, let $\{A_\alpha : \alpha < \mathbf{c}\}$ enumerate a maximal a.d. family of subsets of ω (see §1). Recall that for each $\alpha < \mathbf{c}$, $\langle A_\alpha \rangle = \{B : B \subseteq^* A_\alpha\}$. Note that for $\alpha \neq \beta$, $\langle A_\alpha \rangle \cap \langle A_\beta \rangle = \varnothing$. For each $s : \mathbf{c} \to 2$, let

$$E(s) = \bigcup\{\langle A_\alpha \rangle : s(\alpha) = 1\},$$

and let $\mathscr{G} = \{E(s) : s \in {}^{\mathbf{c}}2\}$. Each $E(s)$ is in CR since if $[F, B]$ is such that $|B \cap A_\alpha| = \omega$ for some α for which $s(\alpha) = 1$, $[F, B \cap A_\alpha] \subseteq \langle A_\alpha \rangle$, while if $|B \cap A_\alpha| = \omega$ for some α for which $s(\alpha) = 0$, $[F, B \cap A_\alpha] \cap \langle A_\alpha \rangle = \varnothing$; moreover, one of these cases obtains for each $B \in [\omega]^\omega$ by maximality of the a.d. family. Finally, note that if s is less than t in the lexicographic ordering of ${}^{\mathbf{c}}2$, then there is an E-set in $E(t) \backslash E(s)$; hence, their symmetric difference is not in CR_0, and we are done. □

We come to the definitions of the new concepts of this section.

4.1. DEFINITION. An ultrafilter $\mathscr{U}$ is *Ramsey* if for every infinite partition $\{A_n : n \in \omega\}$ of ω, either there is an n such that $A_n \in \mathscr{U}$ or there is $B \in \mathscr{U}$ such that for all n, $|B \cap A_n| \leq 1$. If $\mathscr{U}$ is a Ramsey ultrafilter, an E-set $[F, A]$ is called a *$\mathscr{U}$-E-set* if $A \in \mathscr{U}$. A set $X \subseteq [\omega]^\omega$ is said to be *completely Ramsey* (*completely Ramsey null*) *relative to* $\mathscr{U}$, and we write $X \in \mathrm{CR}^{\mathscr{U}}$ ($X \in \mathrm{CR}_0^{\mathscr{U}}$), if for every $\mathscr{U}$-E-set $[F, A]$, there is a $\mathscr{U}$-E-set $[F, B] \subseteq [F, A]$ either contained in or missing (missing) X.

The next proposition collects some important facts about $\mathrm{CR}^{\mathscr{U}}$ and $\mathrm{CR}_0^{\mathscr{U}}$; most of the proofs can be found in [Ms, §1].

4.2. PROPOSITION. *Suppose $\mathcal{U}$ is a Ramsey ultrafilter.*
(1) $\mathrm{CR}^{\mathcal{U}}$ *is a σ-algebra and* $\mathrm{CR}_0^{\mathcal{U}}$ *is a σ-ideal.*
(2) $\mathrm{CR}_0^{\mathcal{U}}$ *is a ccc ideal.*
(3) $\mathrm{CR}^{\mathcal{U}}$ *is Borel supported.*
(4) $\mathrm{CR}_0^{\mathcal{U}}$ *is the hereditary ideal for* $\mathrm{CR}^{\mathcal{U}}$.
(5) $\mathrm{CR}_0^{\mathcal{U}} \subseteq \mathrm{R}_0$ *and* $\mathrm{CR}^{\mathcal{U}} \subseteq \mathrm{R}$.

Notice that part (5) suggests that it ought to be easier to find an example of a set in $(s)_0 \setminus \mathrm{CR}^{\mathcal{U}}$ than a set in $(s)_0 \setminus \mathrm{R}$, but whether such a set exists (in ZFC + "there is a Ramsey ultrafilter") is still open:

OPEN PROBLEM #7. Is there a proof of "$(s)_0 \setminus \mathrm{CR}^{\mathcal{U}} \neq \varnothing$" from ZFC + "there is a Ramsey ultrafilter"?

One important difference between CR_0 and $\mathrm{CR}_0^{\mathcal{U}}$ is that all sets of size $< \mathbf{c}$ are in CR_0, while this need not be the case for $\mathrm{CR}_0^{\mathcal{U}}$, as the next theorem shows. We need the following notion: If $\mathcal{U}$ is an ultrafilter, a set $X \subseteq \mathcal{U}$ *generates* $\mathcal{U}$ if $\mathcal{U}$ is the unique ultrafilter extending the set X.

4.3. THEOREM. *The following statement is independent of* ZFC:

$$(*) \qquad \mathrm{non}(\mathrm{CR}_0^{\mathcal{U}}) = \mathbf{c}.$$

In particular, assuming MA, *there is a Ramsey ultrafilter $\mathcal{U}$ for which $(*)$ is true; and, in the iterated Sacks model, there is a Ramsey ultrafilter $\mathcal{U}$ for which $(*)$ fails.*

PROOF. To see that $(*)$ is consistent, we use the following two consequences of MA (see §1 and [K1, Chapter II]):

(1) $\mathbf{p} = \mathbf{c}$

and

(2) $\forall \alpha < \mathbf{c}\; 2^{\alpha} = \mathbf{c}$.

We first observe that:

(3) $\forall X\, \forall A\, [|X| < \mathbf{c}$ and $A \in [\omega]^{\omega} \to \exists B \in [A]^{\omega}(\langle B \rangle \cap X = \varnothing)]$.

To see this, simply recall from the proof of 4.0 that an a.d. family of subsets of A of size $\mathbf{c}$ gives rise to a disjoint collection of subsets of $[A]^{\omega}$ of the form $\langle B \rangle$.

Now $\mathcal{U}$ is constructed as in the usual CH argument (see [J, §38]) as the upward closure of a tower over ω, using (3) to avoid each set of size $< \mathbf{c}$ along the way. Here are the details: Let $\langle \mathscr{E}_{\alpha} : \alpha < \mathbf{c} \rangle$ be an enumeration of the infinite partitions of ω, and $\langle X_{\alpha} : \alpha < \mathbf{c} \rangle$ an enumeration of $[[\omega]^{\omega}]^{<\mathbf{c}}$ (using (2)). Build a tower $A_0 \,{}^*{\supseteq}\, A_1 \,{}^*{\supseteq} \cdots {}^*{\supseteq}\, A_{\alpha} \,{}^*{\supseteq} \cdots (\alpha < \mathbf{c})$ of infinite sets inductively as follows: $A_0 \in [\omega]^{\omega}$. For each α, let $A_{\alpha+1} \subseteq A_{\alpha}$ be an infinite subset of a member of $\mathscr{E}_{\alpha}$ if A_{α} has infinite intersection with a member of $\mathscr{E}_{\alpha}$; otherwise obtain $A_{\alpha+1}$ by thinning out A_{α} so that it meets each member of $\mathscr{E}_{\alpha}$ at most once. In either case, use (3) to make sure that $\langle A_{\alpha+1} \rangle \cap X_{\alpha+1} = \varnothing$. For limit α, use (1) to obtain A_{α} such that $A_{\alpha} \subseteq^* A_{\beta}$ for all $\beta < \alpha$; by taking an infinite subset if necessary, use (3) to make sure $\langle A_{\alpha} \rangle \cap X_{\alpha} = \varnothing$. Now let $\mathcal{U} = \{B : \exists \alpha\, B \supseteq A_{\alpha}\}$. As in the usual proof, $\mathcal{U}$ is a Ramsey ultrafilter. To see that $(*)$ holds for $\mathrm{CR}_0^{\mathcal{U}}$, suppose $X_{\alpha} \in [[\omega]^{\omega}]^{<\mathbf{c}}$ and suppose $[F, A]$ is a $\mathcal{U}$-E-set. Let β be such that $A \supseteq A_{\beta}$. Then

$$[F, A_{\alpha} \cap A_{\beta}] \subseteq \langle A_{\alpha} \rangle \subseteq [\omega]^{\omega} \setminus X_{\alpha},$$

as required.

For $\neg(*)$, we use the iterated Sacks model; in that model, $\mathbf{c} = \omega_2$. In [BL], Baumgartner and Laver show that each Ramsey ultrafilter in the ground model can be extended to a Ramsey ultrafilter in the extension, and that all Ramsey ultrafilters in the extension are ω_1-generated. Thus, given such an ultrafilter $\mathscr{U}$ in the extension, let $Y \subseteq \mathscr{U}$ be a set of cardinality $\aleph_1$ which generates $\mathscr{U}$. We assume Y is closed under finite intersections; let

$$X = \{F \cup B: F \in [\omega]^{<\omega} \text{ and } B \in Y\}.$$

To see that $X \notin \mathrm{CR}_0^{\mathscr{U}}$, for each $[F, A]$ with $A \in \mathscr{U}$, let $B \subseteq A$ and $B \in Y$; then $F \cup B \in [F, A] \cap X$. □

4.4. REMARK. Later on in this section, we will make use of the fact that the Ramsey ultrafilter $\mathscr{U}$ constructed above under the assumption of MA is actually a $p(\mathbf{c})$-point,[7] i.e., if $\{B_\beta: \beta < \kappa\} \subseteq \mathscr{U}$, $\kappa < \mathbf{c}$ and for all $\gamma < \beta < \kappa$, $B_\gamma \subseteq^* B_\beta$, then there is a set $B \in \mathscr{U}$ such that for all β, $B \subseteq^* B_\beta$. To see this, given $B_0 \,{}^*\!\supseteq B_1 \,{}^*\!\supseteq \cdots {}^*\!\supseteq B_\beta \,{}^*\!\supseteq \cdots (\beta < \kappa)$, inductively construct $A_{\alpha_\beta} \subseteq^* B_\beta$ so that $\beta < \gamma$ implies $\alpha_\beta < \alpha_\gamma$ (where $\mathscr{A} = \{A_\alpha: \alpha < \mathbf{c}\}$ is defined as in the above proof). Use regularity of $\mathbf{c}$ to obtain $B \in \mathscr{A}$ such that for all β, $B \subseteq^* A_{\alpha_\beta}$; B is the required set.

OPEN PROBLEM #9. Does MA imply that for all Ramsey ultrafilters $\mathscr{U}$, $\mathrm{non}(\mathrm{CR}_0^{\mathscr{U}}) = \mathbf{c}$?

The demanding reader will insist that the results of the last theorem ought to be consistent with the statement (**) $(s)_0 \backslash \mathrm{CR} \neq \varnothing$. Since MA implies $\mathrm{cov}(\mathscr{K}) = \mathbf{c}$, (*) and (**) are consistent. However, I do not know if (**) holds in the Sacks model, nor if $\neg(*) + (**)$ is consistent at all.

OPEN PROBLEM #10. Assuming ZFC is consistent, is there a model of ZFC in which there are X, Y, and $\mathscr{U}$ such that $\mathscr{U}$ is a Ramsey ultrafilter, $|X| < \mathbf{c}$ and $X \notin \mathrm{CR}_0^{\mathscr{U}}$, and $Y \in (s)_0 \backslash \mathrm{CR}$?

The next proposition collects together a few simple examples; we leave the proofs to the reader. For each $A \in [\omega]^\omega$, let $\mathscr{F}_A$ be the filter generated by A, that is, $\mathscr{F}_A = \{B \subseteq \omega: B \supseteq A\}$. Note that $\mathscr{F}_A$ is perfect and of measure zero.

4.5. PROPOSITION. *Suppose $\mathscr{U}$ is a Ramsey ultrafilter.*

(1) *Each $\mathscr{F}_A$ is in $\mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}}$.*

(2) *For all $A \in [\omega]^\omega$, $[A]^\omega \in \mathrm{CR}_0^{\mathscr{U}}$ if and only if $A \notin \mathscr{U}$. Hence, if $A \in [\omega]^\omega \backslash \mathscr{U}$, any Bernstein subset of $[A]^\omega$ is in $\mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{CR}$.*

Notice that (2) implies that if $\mathscr{U}$ and $\mathscr{V}$ are distinct Ramsey ultrafilters, then both $\mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{CR}^{\mathscr{V}}$ and $\mathrm{CR}_0^{\mathscr{V}} \backslash \mathrm{CR}^{\mathscr{U}}$ are nonempty. The next proposition provides us with analogues to the sets H and N of Proposition 2.16.

4.6. PROPOSITION. *Suppose $\mathscr{U}$ is a Ramsey ultrafilter. Then there is a set $\hat{H}$ with the following properties:*

(1) *$\hat{H}$ is meager and has measure zero.*

(2) *For every $\mathscr{U}$-E-set $[G, B]$, there is $C \in \mathscr{U} \cap [B]^\omega$ such that $[G, C] \subseteq \hat{H}$.*

(3) *There is a set $B \in [\omega]^\omega$ such that $\hat{H} \cap [B]^\omega = \varnothing$.*

PROOF. Pick a coinfinite set $A \in \mathscr{U}$ and let $\hat{H} = \langle A \rangle$. To verify property (3), let $B = \omega \backslash A$. □

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[7] Note that this is not generally true of Ramsey ultrafilters under MA: In [So], Solomon shows that MA + $\neg$CH implies that there is a Ramsey ultrafilter which is not a $p(\omega_2)$-point.

4.7. REMARKS. (1) Note that the construction of $\hat{H}$ depends on our choice of A; when A is used in the construction, we shall say that $\hat{H}$ is *built from A*.

(2) We define $\hat{N}$ to be the complement of $\hat{H}$. Notice that $\hat{N} \in \mathrm{CR}_0^{\mathscr{U}}$, is a measure 1 dense G_δ, and is a member of $\mathrm{CR} \backslash \mathrm{CR}_0$.

4.8. PROPOSITION. *Suppose $\mathscr{U}$ is a Ramsey ultrafilter.*

(1) *There is a Bernstein set S relative to $\mathscr{U}$ which meets every measure* 1 *set in* $[\omega]^\omega$.

(2) *There is a Bernstein set T relative to $\mathscr{U}$ which meets every comeager subset of* $[\omega]^\omega$.

In addition, such S and T must meet every set of the form $[A]^\omega$ *for* $A \in \mathscr{U}$.

PROOF. To begin, assume one of the sets S, T has been constructed—call it R. Suppose $A \in \mathscr{U}$: let $B \subseteq A$ with $|B| = |A \backslash B|$ and $B \in \mathscr{U}$. Then since $\{C: B \subseteq C \subseteq A\} \subseteq \mathscr{U}$ is perfect, R meets $[A]^\omega$. This proves the last part of the proposition.

We conclude with a proof of (1); (2) is similar. Let $\{P_\alpha: \alpha \text{ a successor} < \mathbf{c}\}$, $\{K_\alpha: \alpha \text{ a limit} < \mathbf{c}\}$ be enumerations of the perfect sets lying in $\mathscr{U}$ and the measure 1 F_σ's, respectively. Build $S = \{x_\alpha: \alpha < \mathbf{c}\}$ and $\{y_\alpha: \alpha \text{ a successor} < \mathbf{c}\}$ inductively as follows: For successor α, let $x_\alpha \neq y_\alpha \in P_\alpha \backslash \{x_\beta, y_\beta: \beta < \alpha\}$. For limit α, pick $x_\alpha \in K_\alpha \cap \mathscr{U}$. S is the required set. □

The next two theorems are concerned with the construction of the four exceptional examples, mentioned earlier; these appear in the chart below as cases (a2), (d2), (e2), and (k2). The basic problem is to find an example of a set in $(s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{CR}^{\mathscr{U}}$. Since we know of no methods in ZFC alone to construct a set in $(s)_0 \backslash \mathrm{CR}^{\mathscr{U}}$, our construction must rely on the set $X \in (s)_0 \backslash \mathrm{R}$, given as hypothesis. Since $X \notin \mathrm{CR}$, we need to intersect it with a CR_0 set which is "large" in the sense of $\mathrm{CR}_0^{\mathscr{U}}$. The "largest" candidate we know of is $\mathscr{U}$ itself. As we show in the next proposition, however, it is quite possible that $X \cap \mathscr{U} = \varnothing$. Moreover, three of the four examples must lie in either $\mathscr{L}_0$ or $\mathscr{K}$; to ensure this, it is natural to consider $\varphi_A(X)$ (using the notation of the last section) in place of X, where A is a coinfinite member of $\mathscr{U}$; $\varphi_A(X)$ is a meager measure zero member of $(s)_0 \backslash \mathrm{CR}$. Again, we would like to try intersecting with $\mathscr{U}$ to obtain the desired example. But as we show below, it is consistent that for some $\mathscr{U}$, $\varphi_A(X) \cap \mathscr{U} \in \mathrm{CR}_0^{\mathscr{U}}$ for any choice of $A \in \mathscr{U}$. In Theorem 4.10, we use MA to show that, with some care, our intuitively motivated construction will work; in particular, that there exist $\mathscr{U}$, A, and X for which $\varphi_A(X) \cap \mathscr{U} \notin \mathrm{CR}^{\mathscr{U}}$.

4.9. THEOREM. *Assume Martin's Axiom. Then there exist a Ramsey ultrafilter $\mathscr{U}$ and a set $X \in (s)_0 \backslash \mathrm{R}$ such that $X \cap \mathscr{U} = \varnothing$, and for all $B \in \mathscr{U}$, $\varphi_B(X) \cap \mathscr{U} \in \mathrm{CR}_0^{\mathscr{U}}$.*

PROOF. Let $\mathscr{U}$ be the Ramsey ultrafilter constructed under MA in Theorem 4.3. Let $\langle P_\alpha: \alpha < \mathbf{c} \rangle$ enumerate the CR_0 perfect sets and let $\langle A_\alpha: \alpha < \mathbf{c} \rangle$ enumerate $[\omega]^\omega$, with $A_0 = \omega$. Build $X = \{x_\alpha: \alpha < \mathbf{c}\}$ inductively so that for each α,

(1) $x_\alpha \in [A_\alpha]^\omega \backslash \bigcup\{P_\beta: \beta < \alpha\}$, and

(2) for all $\beta < \alpha$, $\varphi_{A_\beta}(x_\alpha) \notin \mathscr{U}$.

We obtain x_α as follows: For $\alpha = 0$, we let x_α be an arbitrary element of the complement of $\mathscr{U}$. Now suppose $\alpha > 0$. Since $\mathrm{add}(\mathrm{CR}_0) = \mathbf{c}$ (Proposition 2.3(5)), there is $B_\alpha \in [A_\alpha]^\omega$ such that $[B_\alpha]^\omega \subseteq [A_\alpha]^\omega \backslash \bigcup\{P_\beta: \beta < \alpha\}$. Build a tower

$$x_{\alpha,0} \,{}^*\!\supseteq x_{\alpha,1} \,{}^*\!\supseteq \cdots {}^*\!\supseteq x_{\alpha,\beta} \,{}^*\!\supseteq \cdots \qquad (\beta < \alpha)$$

in $[B_\alpha]^\omega$: Let $x_{\alpha,0} \in [B_\alpha]^\omega$ be such that $\varphi_{A_0}(x_{\alpha,0}) \notin \mathscr{U}$. Let $x_{\alpha,\beta+1} \subseteq x_{\alpha,\beta}$ be such that $\varphi_{A_{\beta+1}}(x_{\alpha,\beta+1}) \notin \mathscr{U}$. For limit β, use the fact that $\mathbf{p} = \mathbf{c}$ to obtain $x_{\alpha,\beta}$ such that $x_{\alpha,\beta} \in [B_\alpha]^\omega$, $x_{\alpha,\beta} \subseteq^* x_{\alpha,\gamma}$ for all $\gamma < \beta$, and $\varphi_{A_\beta}(x_{\alpha,\beta}) \notin \mathscr{U}$. Finally, use "$\mathbf{p} = \mathbf{c}$" again (if necessary) to obtain $x_\alpha \in [B_\alpha]^\omega$ for which $x_\alpha \subseteq^* x_{\alpha,\beta}$ for all $\beta < \alpha$. Clearly (1) and (2) are satisfied, and the construction of X is complete.

Since the CR_0 perfect sets form a dense suborder of the perfect sets, $X \in (s)_0 \setminus \mathrm{R}$ (see [B]). Because of our care in defining A_0 and x_0, our set X is disjoint from $\mathscr{U}$. Finally, given $B \in \mathscr{U}$, let β be such that $A_\beta = B$. Since for all $\alpha > \beta$, $\varphi_{A_\beta}(x_\alpha) \notin \mathscr{U}$, $|\varphi_{A_\beta}(X) \cap \mathscr{U}| < \mathbf{c}$; by the proof of Theorem 4.3 it follows that $\varphi_{A_\beta}(X) \cap \mathscr{U} \in \mathrm{CR}_0^{\mathscr{U}}$. $\square$

The next theorem will be needed for the construction of the four exceptional examples, (a2), (d2), (e2), and (k2).

4.10. THEOREM. *Assume Martin's Axiom. Then there is a Ramsey ultrafilter $\mathscr{U}$ such that for any $A \in \mathscr{U}$, there is an $X \in (s)_0 \setminus \mathrm{R}$ such that $\varphi_A(X) \cap \mathscr{U} \notin \mathrm{CR}^{\mathscr{U}}$; moreover, $|\varphi_A(X) \cap \mathscr{U}| = \mathbf{c}$.*

PROOF. Let $\mathscr{U}$ be the Ramsey ultrafilter constructed in the proof of Theorem 4.3. We begin by making a couple of observations:

Claim 1. Every perfect set in $[\omega]^\omega$ includes a $\mathrm{CR}_0^{\mathscr{U}}$ perfect set.

Proof. Suppose P is perfect. If P is already $\mathrm{CR}_0^{\mathscr{U}}$, we're done; if not, then $P \in \mathrm{CR}^{\mathscr{U}} \setminus \mathrm{CR}_0^{\mathscr{U}}$ and hence must contain a $\mathscr{U}$-E-set $[F, A]$. But now $[F, A]$ contains a $\mathrm{CR}_0^{\mathscr{U}}$ perfect set, namely $[F, B]$ where $B \in [A]^\omega$ and $B \notin \mathscr{U}$. $\square$

Claim 2. $\mathrm{add}(\mathrm{CR}_0^{\mathscr{U}}) = \mathbf{c}$.

Proof. Suppose $\{X_\alpha : \alpha < \kappa\}$, $\kappa < \mathbf{c}$, are $\mathrm{CR}_0^{\mathscr{U}}$ sets. It suffices to consider the collection $\{\hat{X}_\alpha : \alpha < \kappa\}$ where $\hat{X}_\alpha = \{C \setminus G : C \in X_\alpha \text{ and } G \in [\omega]^{<\omega}\}$. Given a $\mathscr{U}$-E-set $[F, A]$, use the fact that $\mathscr{U}$ is a $p(\mathbf{c})$-point (Remark 4.4) to obtain a tower

$$A_0 \,{}^*\!\supseteq A_1 \,{}^*\!\supseteq \cdots {}^*\!\supseteq A_\alpha \,{}^*\!\supseteq \cdots \qquad (\alpha < \kappa)$$

in $\mathscr{U}$ such that for all α, $A_\alpha \subseteq A$, and $[F, A_\alpha] \cap \hat{X}_\beta = \varnothing$ for all $\beta \leq \alpha$, as follows: For each α let $A_\alpha \subseteq A_{\alpha+1}$ be such that $[F, A_{\alpha+1}] \cap \hat{X}_{\alpha+1} = \varnothing$. If α is a limit, let $A_\alpha \in \mathscr{U}$ be such that $A_\alpha \subseteq^* A_\beta$, $\beta < \alpha$, and $[F, A_\alpha] \cap \hat{X}_\alpha = \varnothing$. To see that the induction hypothesis is satisfied in this case, let $\beta < \alpha$ and assume $C \in [F, A_\alpha] \cap \hat{X}_\beta$; then for some finite set G, $C \setminus G \in [F, A_\beta] \cap \hat{X}_\beta$, which is impossible. Having constructed the tower, we can reason in a similar way and conclude that there is a $B \in \mathscr{U}$ such that $B \subseteq A$ and $[F, B] \cap \hat{X}_\alpha = \varnothing$ for all α. $\square$

With these preliminaries in place, we prove the theorem. Suppose $A \in \mathscr{U}$. Let $\langle P_\alpha : \alpha < \mathbf{c} \rangle$ enumerate the perfect sets in $\mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}}$; by Claim 1 and Proposition 2.8, the P_α are dense in the poset of all perfect sets. Let $\langle A_\alpha : \alpha < \mathbf{c} \rangle$ be an enumeration of $[A]^\omega$. Build $Y = \{y_\alpha : \alpha < \mathbf{c}\}$ inductively: Pick $y_\alpha \in [A_\alpha]^\omega \setminus \bigcup \{P_\beta : \beta < \alpha\}$ and if $A_\alpha \in \mathscr{U}$, let $y_\alpha \in \mathscr{U}$ (this is possible by Claim 2). Let $X = \varphi_A^{-1}(Y)$. It is easy to see that both X and Y are $(s)_0$ and that $X \notin \mathrm{R}$. Moreover, for any $B \in \mathscr{U}$, if α is such that $A_\alpha = A \cap B$, then $y_\alpha \in [B]^\omega \cap Y \cap \mathscr{U}$; hence $\varphi_A(X) \cap \mathscr{U} \notin \mathrm{CR}^{\mathscr{U}}$. The fact that $|\varphi_A(X) \cap \mathscr{U}| = \mathbf{c}$ follows from the proof of Theorem 4.3. $\square$

In an earlier version of this paper, we asked whether MA implies that for any Ramsey ultrafilter $\mathscr{U}$, $\mathrm{add}(\mathrm{CR}_0^{\mathscr{U}}) = \mathbf{c}$. A. Louveau sent us a copy of his paper [Lo] in which this problem is solved. In particular, his Theorem 3.7 implies that for each cardinal κ, $\omega_1 \leq \kappa \leq \mathfrak{c}$, if a Ramsey ultrafilter $\mathcal{U}$ is not a $p(\kappa)$-point, then $\mathrm{CR}_0^{\mathcal{U}}$ is not κ-additive.

For our chart of examples, we fix some notation; for readability, we repeat some of the notation given for the chart of §3 as well.

Symbol	Definition
E	The set of even numbers
O	The set of odd numbers
$\varphi_A\colon [\omega]^\omega \to [A]^\omega$	The homeomorphism induced by the increasing enumeration $\omega \to A$
C	A coinfinite set in $\mathcal{U}$
D	A coinfinite set not in $\mathcal{U}$
G	A measure zero dense G_δ
$\hat{H}$	A measure zero meager co-$\mathrm{CR}_0^{\mathcal{U}}$ set built from C (Proposition 4.6)
N	A measure 1 dense G_δ in CR_0
$\hat{N}$	A measure 1 dense G_δ in $\mathrm{CR}_0^{\mathcal{U}}$
B	A coinfinite subset of ω such that $[B]^\omega \subseteq \hat{N}$ (Proposition 4.6, Remark 4.7)
S	A Bernstein set relative to $\mathcal{U}$ which meets every measure 1 set in $[\omega]^\omega$ and meets every set of the form $[A]^\omega$ for $A \in \mathcal{U}$ (Proposition 4.8(1))
T	A Bernstein set relative to $\mathcal{U}$ which meets every comeager subset of $[\omega]^\omega$ and meets every set of the form $[A]^\omega$ for $A \in \mathcal{U}$ (Proposition 4.8(2))
$\mathcal{U}$	A Ramsey ultrafilter
U	An $(s)_0$ set whose intersection with every measure 1 F_σ has cardinality $\mathfrak{c}$
V	An $(s)_0$ set whose intersection with every dense G_δ has cardinality $\mathfrak{c}$
W	A member of $(s)_0 \backslash \mathrm{R}$

The chart is organized as follows: Including $\mathrm{CR}_0^{\mathcal{U}}$ and $\mathrm{CR}^{\mathcal{U}}$ in our analysis gives us a range of 32 examples; these will be organized by splitting each case from the chart in §3 in two. Thus, for instance, Case (a) will split into Case (a1) ($X \in \mathcal{L}_0 \cap \mathcal{K} \cap (s)_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathcal{U}}$) and Case (a2) ($X \in \mathcal{L}_0 \cap \mathcal{K} \cap (s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{CR}_0^{\mathcal{U}}$).

As in §3, the eight examples derived from cases (b), (g), (i), and (o) use the assumption that there is a set W in $(s)_0 \backslash \mathrm{R}$. The four exceptional examples, (a2), (d2), (e2), and (k2), are built assuming MA; in particular, the sets $\mathcal{U}$, C, and W stand for the Ramsey ultrafilter, the coinfinite element of $\mathcal{U}$, and the member of $(s)_0 \backslash \mathrm{R}$, respectively, that were constructed in Theorem 4.10. The remaining 20 examples are constructed in ZFC alone.

Where X lives	How X is constructed
(a1) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}}$	As in the proof of Theorem 3.0 starting with $[D]^\omega$
(a2) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{CR}^{\mathscr{U}}$	**(MA)** $\varphi_C(W) \cap \mathscr{U}$
(b1) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{CR}$	$X = \varphi_D(W)$
(b2) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap (s)_0 \backslash \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (b)
(c1) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash (s)$	X is a Bernstein subset of $\mathscr{F}_C$
(c2) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap \mathrm{CR}_0 \backslash (s) \cup \mathrm{CR}^{\mathscr{U}}$	X is a Bernstein subset of $\hat{H} \cap \mathscr{U}$
(d1) $X \in \mathscr{L}_0 \cap (s)_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{B_W}$	$X = G \cap N \cap \hat{N} \cap V$
(d2) $X \in \mathscr{L}_0 \cap (s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{B_W} \cup \mathrm{CR}^{\mathscr{U}}$	**(MA)** $X = [\varphi_C(W) \cap \mathscr{U}] \cup [G \cap N \cap V]$
(e1) $X \in \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L}$	$X = N \cap \hat{N} \cap U \backslash G$
(e2) $X \in \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0 \backslash \mathscr{L} \cup \mathrm{CR}^{\mathscr{U}}$	**(MA)** $X = [\varphi_C(W) \cap \mathscr{U}] \cup [N \cap U \backslash G]$
(f1) $X \in \mathscr{L}_0 \cap \mathscr{K} \cap \mathrm{CR}_0^{\mathscr{U}} \backslash (s) \cup \mathrm{CR}$	X is a Bernstein subset of $[D]^\omega$
(f2) $X \in \mathscr{L}_0 \cap \mathscr{K} \backslash (s) \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	X is a Bernstein subset of $[C]^\omega$
(g1) $X \in \mathscr{L}_0 \cap (s)_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{B_W} \cup \mathrm{CR}$	$X = \varphi_D(W) \cup (G \cap V \cap \hat{N})$
(g2) $X \in \mathscr{L}_0 \cap (s)_0 \backslash \mathrm{B_W} \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (g)
(h1) $X \in \mathscr{L}_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{B_W} \cup (s)$	X is a Bernstein subset of $G \cap N \cap \hat{N}$
(h2) $X \in \mathscr{L}_0 \cap \mathrm{CR}_0 \backslash \mathrm{B_W} \cup (s) \cup \mathrm{CR}^{\mathscr{U}}$	$X = T \cap (G \cup \hat{H})$
(i1) $X \in \mathscr{K} \cap (s)_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup \mathrm{CR}$	$X = \varphi_D(W) \cup (\hat{N} \cap U \backslash G)$
(i2) $X \in \mathscr{K} \cap (s)_0 \backslash \mathscr{L} \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (i)
(j1) $X \in \mathscr{K} \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup (s)$	X is a Bernstein subset of $\hat{N} \cap N \backslash G$
(j2) $X \in \mathscr{K} \cap \mathrm{CR}_0 \backslash \mathscr{L} \cup (s) \cup \mathrm{CR}^{\mathscr{U}}$	$X = S \cap [([\omega]^\omega \backslash G) \cup \hat{H}]$
(k1) $X \in (s)_0 \cap \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup \mathrm{B_W}$	$X = N \cap \hat{N} \cap (U \cup V)$
(k2) $X \in (s)_0 \cap \mathrm{CR}_0 \backslash \mathscr{L} \cup \mathrm{B_W} \cup \mathrm{CR}^{\mathscr{U}}$	**(MA)** $X = [\varphi_C(W) \cap \mathscr{U}] \cup [N \cap (U \cup V)]$
(m1) $X \in \mathscr{L}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathrm{B_W} \cup (s) \cup \mathrm{CR}$	X is a Bernstein subset of $\hat{N} \cap (G \cup [B]^\omega)$
(m2) $X \in \mathscr{L}_0 \backslash \mathrm{B_W} \cup (s) \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (m)
(n1) $X \in \mathscr{K} \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup (s) \cup \mathrm{CR}$	X is a Bernstein subset of $[B]^\omega \cup (\hat{N} \backslash G)$

Where X lives	How X is constructed
(n2) $X \in \mathscr{K} \backslash \mathscr{L} \cup (s) \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (n)
(o1) $X \in (s)_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup \mathrm{B_W} \cup \mathrm{CR}$	$X = \varphi_D(W) \cup [(U \cup V) \cap \hat{N}]$
(o2) $X \in (s)_0 \backslash \mathscr{L} \cup \mathrm{B_W} \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	Example (o)
(p1) $X \in \mathrm{CR}_0 \cap \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup \mathrm{B_W} \cup (s)$	X is a Bernstein subset of $N \cap \hat{N}$
(p2) $X \in \mathrm{CR}_0 \backslash \mathscr{L} \cup \mathrm{B_W} \cup (s) \cup \mathrm{CR}^{\mathscr{U}}$	$X = S \cup T$
(q1) $X \in \mathrm{CR}_0^{\mathscr{U}} \backslash \mathscr{L} \cup \mathrm{B_W} \cup (s) \cup \mathrm{CR}$	X is any Bernstein set relative to $\hat{N}$
(q2) $X \notin \mathscr{L} \cup \mathrm{B_W} \cup (s) \cup \mathrm{CR} \cup \mathrm{CR}^{\mathscr{U}}$	X is any Bernstein set

4.11. Remarks. (1) Some of the constructions look unpleasant, but even these are easily shown to have the desired properties. As an example of the "right" approach to these consider example (d2): Theorem 4.10 tells us that the left half of the union is in $(s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{CR}^{\mathscr{U}}$; since this left half is a subset of $[C]^\omega$, it also has measure zero. The right half is also in $\mathscr{L}_0 \cap (s)_0 \cap \mathrm{CR}_0$; because $G \cap N$ is a dense G_δ, the right half meets every dense G_δ. Hence, the union lies outside of both $\mathrm{CR}^{\mathscr{U}}$ and $\mathrm{B_W}$.

(2) Each example has cardinality $\mathfrak{c}$: The techniques of §3 can be used to verify this for all examples other than (a2), (d2), (e2), and (k2); these four are of cardinality $\mathfrak{c}$ by Theorem 4.10.

(3) Finally, we remark that because $\mathrm{CR}^{\mathscr{U}} \subseteq \mathrm{R}$, we have made no attempt to create examples lying outside of R.

What remains to be done to complete the work on the charts given in §§3 and 4 can be summarized as follows: First, obtain a ZFC example of a set in $(s)_0 \backslash \mathrm{CR}$; such an example guarantees ZFC constructions for each of the 16 cases in §3 and all but four of the cases in the present section. Next, prove in ZFC + "there is a Ramsey ultrafilter $\mathscr{U}$" (or, if possible, from ZFC alone, like most of the other constructions) that there is a measure zero meager set in $(s)_0 \cap \mathrm{CR}_0 \backslash \mathrm{CR}^{\mathscr{U}}$; from this example the other three can also be constructed using the techniques given above.

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