THE SPECTRUM OF ELEMENTARY EMBEDDINGS $j : V \to V$

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Abstract. In 1970, K. Kunen, working in the context of Kelley-Morse set theory, showed that the existence of a nontrivial elementary embedding $j : V \to V$ is inconsistent. In this paper, we give a finer analysis of the implications of his result for embeddings $V \to V$ relative to models of ZFC. We do this by working in the extended language $\{\in, \mathbf{j}\}$, using as axioms all the usual axioms of ZFC (for $\in$-formulas), along with an axiom schema that asserts that $\mathbf{j}$ is a nontrivial elementary embedding. Without additional axiomatic assumptions on $\mathbf{j}$, we show that that the resulting theory (denoted $\mathrm{ZFC} + \mathrm{BTEE}$) is weaker than an ω-Erdös cardinal, but stronger than n-ineffables. We show that natural models of $\mathrm{ZFC} + \mathrm{BTEE}$ give rise to Schindler's remarkable cardinals. The approach to inconsistency from $\mathrm{ZFC} + \mathrm{BTEE}$ forks into two paths: extensions of $\mathrm{ZFC} + \mathrm{BTEE} +$ Cofinal Axiom and $\mathrm{ZFC} + \mathrm{BTEE} + \neg$Cofinal Axiom, where Cofinal Axiom asserts that the critical sequence $\kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots$ is cofinal in the ordinals. We describe near-minimal inconsistent extensions of each of these theories. The path toward inconsistency from $\mathrm{ZFC} + \mathrm{BTEE} + \neg$Cofinal Axiom is paved with a sequence of theories of increasing large cardinal strength. Indeed, the extensions of the theory ZFC+"$\mathbf{j}$ is a nontrivial elementary embedding" form a hierarchy of axioms, ranging in strength from Con(ZFC) to the existence of a cardinal that is super-n-huge for every n, to inconsistency. This hierarchy is parallel to the usual hierarchy of large cardinal axioms, and can be used in the same way. We also isolate several intermediate-strength axioms which, when added to $\mathrm{ZFC} + \mathrm{BTEE}$, produce theories having strengths in the vicinity of a measurable cardinal of high Mitchell order, a strong cardinal, ω Woodin cardinals, and n-huge cardinals. We also determine precisely which combinations of axioms, of the form

$$\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_m\text{-Separation}_{\mathbf{j}} + \Sigma_n\text{-Replacement}_{\mathbf{j}}$$

result in inconsistency.

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§1. Introduction.

This paper is a study of elementary embeddings $j : M \to M$, where M is a model of ZFC. Examples of such embeddings abound in the literature. Two familiar examples are $j : \mathbf{L} \to \mathbf{L}$ assuming the existence of $0^{\#}$, and $j : V_\lambda \to V_\lambda$, where λ is a limit above the critical point, known as an I_3 embedding. However, Kunen showed, under reasonable assumptions (such as the Axiom of Choice), that there is no $j : V \to V$. We wish to investigate the difference between these kinds of embeddings — what is it that makes one kind of embedding inconsistent and other kinds consistent with large cardinals?

A common, though coarse, intuition about this question, derived from Kunen's result, tells us that "*external* embeddings from M to M are typically ok, but *internal* embeddings lead to inconsistency." Since any embedding from V to V is necessarily "internal", we expect inconsistency in this case. But how should "internal" and "external" be made precise? A first try is to equate "internal" with *definable (with parameters)*. Though Kunen's Theorem was originally formulated in Kelley-Morse (KM) set theory (since it cannot be formulated in ZFC alone because quantification over classes is necessary[1]), his proof can be carried out in ZFC if one formulates the theorem as follows: *No elementary embedding from V to V is definable (with parameters).* In fact though, Kunen's result forbids more than just the definable embeddings; a proof that definable embeddings are inconsistent can be established with a more direct proof, as f has shown in [Su]. Treating a putative $j : V \to V$ as a KM-class gives the added freedom of defining j using *class parameters*, and such constructions can produce KM classes that are not definable from set parameters alone (see [Le] and [Kr]). Indeed, even in ZFC, Kunen's result forbids more than the definable embeddings. In [Co3], we attempted to come closer to a characterization of the embeddings $j : M \to M$ that are forbidden in the context of ZFC by introducing the concept of *weak definability*: Suppose $M \models \mathrm{ZFC}$. We will say that a subcollection A of M is *weakly definable in M* if the model $\langle M, \in, A\rangle$ satisfies Strong Replacement[2] in the language of set theory extended by a unary predicate. It is

[1] As the referee points out, the result can also be formalized in Gödel-Bernays set theory where such quantification is also allowed; this approach has the advantage that it commits the set theorist to a set theory that is no stronger than ZFC. However, since classes can occur as parameters in KM-definitions, stating the theorem in KM-set theory has the effect of "forbidding" a wider range of embeddings than could be accomplished in GB-set theory.

[2] By Strong Replacement for **j** formulas, we mean all instances of sentences of the form

$$\forall A\,\forall\vec{a}\,\big(\forall x \in A\,\exists^* y\,\psi(x,y,\vec{a}) \Longrightarrow \exists Y\,\forall z\,[z \in Y \Longleftrightarrow (\exists x \in A\,\psi(x,z,\vec{a}))]\big),$$

where $\psi(x,y,\vec{u})$ is a **j**-formula, and $\exists^*$ is short for "there exists at most one". In [Co3], in our definition of weak definability, we mistakenly used the weaker form of Replacement in which $\exists^*$ is straightforward to show that if A is definable in M, it must be weakly definable in M (see [Co3, Theorem 3.8]). Also, there are examples in the literature that show that weak definability is genuinely weaker than definability (see [E] and [Y]). Now, it is straightforward to show, using Kunen's argument, that no elementary embedding $j : M \to M$ can be weakly definable in M. Thus, a somewhat bigger class of embeddings than those definable with parameters are ruled out by Kunen's argument.

Even with these observations, one may still ask, What is it about the definability or weak definability of j that leads to inconsistency? An obvious approach would be to examine closely the exact instances of Strong Replacement for **j** formulas that are used in Kunen's proof. The approach taken in this paper is a somewhat easier variation of this naive approach. However, we have found that any approach to this question will be facilitated by working in a more suitable formal context; namely, we work in the extended language $\{\in, \mathbf{j}\}$, where **j** is a function symbol intended to represent the elementary embedding. Axiomatically, our starting point is ZFC for $\in$-formulas. We then wish to gradually extend ZFC with axioms that regulate the behavior of **j**. We do not automatically assume that the axioms of Separation and Replacement hold for **j**-formulas (of course, if we were working in KM-set theory instead, Separation and Replacement for j-formulas would necessarily hold for any KM-class j). We first add the axiom schema Elementarity which asserts, for each $\in$-formula $\phi(x_1, \dots, x_n)$, that for all $y_1, \dots, y_n$, $\phi(y_1, \dots, y_n) \iff \phi(j(y_1), \dots, j(y_n)))$; in other words, Elementarity asserts that **j** is an $\in$-elementary embedding. We also add an axiom Critical Point, which asserts that there is a least ordinal moved by **j**. We call the axioms Elementarity + Critical Point the *Basic Theory of Elementary Embeddings*, or BTEE. As we will show, ZFC + BTEE is already strong enough to establish that the critical point κ of **j** is n-ineffable for each particular n. Now, in this new context, the question of how inconsistency arises becomes the question, How much Separation and Replacement for **j**-formulas can we consistently add to the theory ZFC+BTEE, and, by contrast, Which combinations of such axioms result in an inconsistent theory?

This issue points to a natural dichotomy, which shows itself in two of the most familiar models of ZFC + BTEE: the models $\langle \mathbf{L}, \in, j\rangle$, where $j : \mathbf{L} \to \mathbf{L}$ is elementary, and $\langle V_\lambda, \in, j\rangle$, where j is an I_3 embedding. In the second model, the critical sequence $\kappa, j(\kappa), j^2(\kappa), \dots$ is cofinal in the ordinals of the model, whereas in the first model, the critical sequence is bounded. We introduce the axiom Cofinal Axiom which asserts that the critical sequence is cofinal, that is, that for every α there are $n \in \omega$ and $\beta > \alpha$ such that $\beta = \mathbf{j}^n(\kappa)$ (we show in Section 2 how to state the axiom more formally). As we show, very little additional large cardinal strength is required to obtain the consistency of

replaced by $\exists!$ — we correct this error here. If one replaces Replacement with Strong Replacement in that paper, all the theorems and proofs (with the obvious modifications) continue to be valid.

either of the theories ZFC + BTEE + Cofinal Axiom or ZFC + BTEE + ¬Cofinal Axiom. Therefore, we consider each of these theories as a starting point for studying how inconsistency arises.

The theme that emerges, as we consider each of these theories, is that extensions of ZFC + BTEE+Cofinal Axiom become inconsistent by introducing "too much" Replacement for **j** formulas, whereas extensions of ZFC+BTEE+¬Cofinal Axiom become inconsistent when "too much" Separation for **j** formulas is added. In the first case, there is a single instance of Replacement for **j**-formulas, which we denote CI (short for "Critical Instance"), which renders ZFC + BTEE + Cofinal Axiom inconsistent. In this case, inconsistency arises because this instance of Replacement for **j** formulas implies that the critical sequence exists (as a *set*), contradicting Cofinal Axiom. In the second case, we observe that a significant consequence of Separation for **j** formulas is Amenability, which asserts the existence of $\mathbf{j} \upharpoonright x$ for any set x. Restricting Amenability to any upper bound on the critical sequence — that is, asserting the existence of $\mathbf{j} \upharpoonright \lambda$ where λ is above all $\mathbf{j}^n(\kappa)$ — yields an axiom that renders ZFC + BTEE + ¬Cofinal Axiom inconsistent. In this case, inconsistency arises because, as we will see, the large cardinal strength that arises from adding axioms of the form "$\mathbf{j} \upharpoonright \beta$ exists", as β increases from κ^+ to $\mathbf{j}(\kappa)$ to $2^{2^{\mathbf{j}^n(\kappa)}}$ to λ (where λ bounds the critical sequence) results in large cardinal strengths that eventually exceed the bounds of consistency, as demonstrated by Kunen's argument. Our analysis provides near-minimal extensions of the theories ZFC + BTEE + Cofinal Axiom and ZFC + BTEE + ¬Cofinal Axiom that are inconsistent.

Another way to state precisely how inconsistency arises is to isolate minimal combinations of Separation and Replacement for **j** formulas in the Levy hierarchy that suffice to carry out Kunen's proof. An example of a result of this kind, from Section 10, is the following: The extension of ZFC + BTEE obtained by adding all Σ_1 instances of Separation for **j** formulas, and all Σ_0 instances of Replacement for **j** formulas, is inconsistent.

An important philosophical conclusion that follows from our analysis of inconsistency is that, relative to ZFC, the assertion "there is no nontrivial embedding $j : V \to V$" is at best imprecise, and is in fact inaccurate. Certainly if V is to be the universe for KM-set theory, this conclusion is warranted, because every KM-class j must satisfy every instance of Separation and Replacement for j-formulas. But our work here shows that no such requirement is present when the underlying theory is ZFC. For example, there is no known proof that forbids the existence of a $j : V \to V$ for which $\langle V, \in, j\rangle \models$ ZFC + BTEE — indeed, as we show in Section 3, the existence of an ω-Erdös cardinal is enough to establish the consistency of this statement[3] . Indeed, our analysis here shows

[3]On the other hand, there are certainly tenable *philosophical* reasons for insisting that any new predicate added to the language of ZFC (such as **j**) should be required to satisfy all instances of Separation and Replacement, at least if we wish to view the theory with the extra predicate as a foundation for mathematics. The philosophical point here is the same as the point raised by the founders of set theory: Separation and Replacement are axiom schema we take to be "true"

that there are many possible ways in which an embedding $M \to M$ can be considered "internal", and these different possibilities span the full spectrum of consistency strengths, with inconsistency arising as an important special case.

As we will show, extensions of ZFC in the language $\{\in, \mathbf{j}\}$ have consistency strengths that range from Con(ZFC) to $\mathrm{ZFC} + \exists\kappa\, \mathrm{I}_3(\kappa)$, to inconsistency. This observation leads to a second main topic of the paper. We pursue the idea that these extensions provide a ladder of theories that are parallel in consistency strengths to the usual large cardinal axioms and can be used in the same way — for example, as a measure for the consistency strengths of other theories. A program of study that we initiate here is to determine how fine-grained this ladder of axioms is. In this paper, we introduce natural axioms which, when added to ZFC + BTEE, produce theories having consistency strengths in the vicinities of $0^{\#}$, of a measurable cardinal having Mitchell order $> \mathrm{o}(\kappa)$, and of an n-huge cardinal (for each n, a different theory). We also provide lower bounds in the vicinity of a strong cardinal for one theory, and for another, ω Woodin cardinals. For these latter axioms, though, we have only a crude upper bound (a 2-huge cardinal). These results represent a first attempt to solve the following general problem:

The Hierarchy Problem. *For each classical large cardinal axiom $A(x)$ expressible in the language $\{\in\}$, find an extension of* ZFC *in the language $\{\in, \mathbf{j}\}$ whose consistency strength is near $A(x)$.*

The paper is organized as follows: In Section 2, we develop the Basic Theory of Elementary Embeddings (BTEE). We show that the large cardinal strength of ZFC + BTEE is somewhat beyond that of a cardinal that is n-ineffable for every particular n and that of a totally indescribable cardinal. In Section 3, we show that the existence of an ω-Erdös cardinal is sufficient to obtain a transitive model of ZFC + BTEE. We also define the notion of a *good* transitive model of ZFC + BTEE, showing that such models are also derivable from an ω-Erdös cardinal, but also showing that these models give rise to transitive models of Schindler's remarkable cardinals. As a result, good transitive models of ZFC + BTEE are naturally linked to recent results about forcing absoluteness. In Section 4 we introduce induction axioms for $\mathbf{j}$-formulas. We show that Σ_1-Induction$_{\mathbf{j}}$ suffices to establish that the formula $\Psi(n, \beta)$ that defines the critical sequence $\langle \kappa, \mathbf{j}(\kappa), \ldots \rangle$ is a (total) class function, as is the formula Φ that defines the relation $\mathbf{j}^n(x) = y$. These observations allow us to improve results from Section 2 of the form "for each particular $n \ldots$" to results of the form "for *all*

of the universe; Separation is the natural local restriciton of full Comprehension, and Replacement prevents short sequences from being cofinal in the universe. On this view, then, if we wish to supplement ZFC with an elementary embedding j of the universe, the embedding should be required to satisfy all such instances — and therefore, by Kunen's results, we are led to inconsistency.

Our point here is that, though this view is quite reasonable, it is nothing more than a point of view. There is no *logical necessity* for requiring a j to satisfy all instances of Separation and Replacement.

$n \ldots$". In Section 5, we take a closer look at the fairly weak theory $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Induction$_{\mathbf{j}}$. In this theory, it is not possible to prove that $\mathbf{j}^n(x)$ exists for every n, x; the main result describes the conclusions that can be drawn, and yields a number of corollaries — one of these states that Π_1-Induction$_{\mathbf{j}}$ suffices to show that Φ and Ψ are (total) class functions. In Section 6, we introduce the Least Ordinal Principle$_{\mathbf{j}}$, which asserts that for each formula $\phi(x, \vec{y})$, whenever $\phi(\alpha, \vec{b})$ holds for an ordinal α, then $\phi(\beta, \vec{b})$ holds for a least β. This axiom gives us a number of simple consequences, like the fact that $\mathbf{j}(\alpha) \geq \alpha$ for any ordinal α, which are needed in later sections.

In Section 7, after showing that an $\omega + \omega$-Erdös cardinal is sufficient to obtain models of each of the theories ZFC + BTEE + Cofinal Axiom and ZFC + BTEE + $\neg$Cofinal Axiom, we describe near-minimal inconsistent extensions of each theory. In studying extensions of ZFC + BTEE + $\neg$Cofinal Axiom, we pause to examine the logical relationships between statements such as "the critical sequence is a set", "the critical sequence is bounded", "if the critical sequence has a supremum δ, then $\mathbf{j}(\delta) = \delta$". The model that we present in this section that is obtained by performing the iterated ultrapower construction starting from a normal measure on a measurable cardinal κ, producing the model $\langle M_\omega, \in, j \restriction M_\omega \rangle$, provides a rich source of insight into the possibilities of elementary embeddings $M \to M$, and will be used in later sections for other purposes. We then show that adding to ZFC + BTEE the combination of $\neg$Cofinal Axiom and $\exists z\, z = \mathbf{j} \restriction \lambda$, whenever λ bounds the critical sequence of $\mathbf{j}$, produces an inconsistent theory, and suggests that axioms of this kind have significant large cardinal strength. We pursue this point further in Section 8, where we study Amenability$_{\mathbf{j}}$, which asserts that for *every* z, $\mathbf{j} \restriction z$ is a set. For each n, let WA_n denote the axioms of BTEE together with all Σ_n instances of Separation of $\mathbf{j}$-formulas, and let WA denote the union, over all $n \in \omega$, of these sets of statements. After showing that, in ZFC + BTEE, Amenability$_{\mathbf{j}}$ is equivalent to Σ_0-Separation for $\mathbf{j}$-formulas, we show that $\mathrm{ZFC} + \mathrm{WA}_0$ suffices to prove Cofinal Axiom, that $V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec \ldots \prec V$ forms an elementary chain, where κ is the critical point of $\mathbf{j}$, and that κ is super-n-huge for every n. In particular, this latter result shows that all known large cardinal consequences of the theory ZFC + WA, established in [Co3], also hold for $\mathrm{ZFC} + \mathrm{WA}_0$. The consequences we mention here of $\mathrm{ZFC} + \mathrm{WA}_0$ are considerably easier to prove in ZFC + WA (or even $\mathrm{ZFC} + \mathrm{WA}_1$). We have gone to the extra trouble of proving the results from $\mathrm{ZFC} + \mathrm{WA}_0$ for the following reason: The work of Hamkins [Ha1] shows that $\mathrm{ZFC} + \mathrm{WA}_0$ has some extraordinary properties that are not (apparently) shared by $\mathrm{ZFC} + \mathrm{WA}_n$ for $n \geq 1$, or by weaker extensions of ZFC + BTEE. For instance, Hamkins has shown that $\mathrm{ZFC} + \mathrm{WA}_0$ is finitely axiomatizable (this is not known to be true for the other theories mentioned). He also has developed a forcing methodology by which one obtains relative consistency results of the form

$$\mathrm{Con}(\mathrm{ZFC} + \mathrm{WA}_0) \Longrightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{WA}_0 + \sigma),$$

where σ is a statement like GCH or $V = \mathrm{HOD}$. His approach to preserving the embedding does not work for weaker extensions of ZFC + BTEE; and his technique does not preserve WA_n for $n \geq 1$

starting from the theory $\mathrm{ZFC} + \mathrm{WA}_n$.

In Section 9 we turn to a study of axioms of intermediate strength that can be added to $\mathrm{ZFC} + \mathrm{BTEE}$. We examine several statements and in some cases establish fairly tight bounds on their consistency strengths. These axioms range in strength between that of a strong cardinal to that of an n-huge cardinal.

In Section 10, we isolate other combinations of Separation and Replacement axioms that render $\mathrm{ZFC} + \mathrm{BTEE}$ inconsistent; here the interest is in determining how high in the Levy hierarchy of formulas one needs to climb in order to produce inconsistency. In this context, we also study the impact of varying the version of Replacement that is used — we consider Strong Replacement, Replacement, and Collection, for $\mathbf{j}$-formulas. Finally, in Section 11 we list a number of problems left open by our work here.

The reader will find in this paper many familiar theorems about elementary embeddings. However, because we are working primarily in a new context — ZFC and its extensions in the language $\{\in, \mathbf{j}\}$ — details of familiar proofs have had to be re-examined. Since Replacement for $\mathbf{j}$-formulas is generally forbidden in the theories we consider, the resulting set theory often has a different flavor. Two notable differences are:

(A) Definition by transfinite recursion (when the recursion depends on a formula having an occurrence of $\mathbf{j}$) is almost never allowed;
(B) Bounded quantifiers increase the complexity of a $\mathbf{j}$-formula

Much of the work here consists in determining which axioms about the embedding $\mathbf{j}$ are needed to obtain standard theorems. When the extension of $\mathrm{ZFC}_{\mathbf{j}}$ under consideration is too weak to carry out standard proofs, other proofs have been devised or weaker theorems are proved. The result of our efforts, we hope, has been to provide a framework for studying the natural axiomatic extensions of $\mathrm{ZFC}+\mathrm{BTEE}$—all of which formalize the notion "ZFC plus an elementary embedding of the universe to itself". Certain natural questions about embeddings $j : M \to M$ — such as determining the precise axiomatic assumptions about such an embedding that would render it inconsistent with ZFC (via Kunen's argument) — seem to be easier to understand and address within the framework provided here. Our hope is that our framework can be used by others to approach the many other natural questions about embeddings $j : M \to M$ that remain.

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§2. The Basic Theory of Elementary Embeddings

In this section, we work in the language $\mathcal{L} = \{\in, \mathbf{j}\}$, where $\mathbf{j}$ is a unary function symbol; and we introduce the *Basic Theory of Elementary Embeddings*, or BTEE, which consists of the axioms that are needed to assert that $\mathbf{j}$ is a nontrivial elementary embedding. We develop the basic machinery and show that the critical point of the embedding is n-ineffable for each particular n, and totally indescribable.

Formulas in which $\mathbf{j}$ does not occur will be called $\in$-formulas whereas formulas having at least one occurrence of $\mathbf{j}$ will be called $\mathbf{j}$-formulas. Including the function symbol $\mathbf{j}$ means that we need to consider $\mathcal{L}$-terms (which we will call $\mathbf{j}$-terms from now on). As usual, terms are defined by the clauses: (a) a variable is a term, and (b) if t is a term, so is $\mathbf{j}(t)$. The terms are of the form $\mathbf{j}^n(x)$ for variables x (assuming $\mathbf{j}^0(x)$ is taken to be x).

Our basic theory is ZFC, now in the context of the language, and first order logic of, $\mathcal{L}$. In this context, we will have occasion to prove $\mathbf{j}$-sentences from ZFC using the logic of $\mathcal{L}$; we will wish to add $\mathbf{j}$-axioms to ZFC; and we will be considering models of ZFC that are $\mathcal{L}$-models. Since derivations from ZFC, extensions of ZFC, and models of ZFC normally pertain to the language $\{\in\}$, we use the notation $\mathrm{ZFC}_{\mathbf{j}}$ to signify ZFC in the context of $\mathcal{L}$ — so that we may unambiguously refer to derivations from $\mathrm{ZFC}_{\mathbf{j}}$, extensions of $\mathrm{ZFC}_{\mathbf{j}}$, and models of $\mathrm{ZFC}_{\mathbf{j}}$.

$\mathcal{L}$-formulas can be classified by complexity in the usual way, though some of the usual theorems about the Lévy hierarchy of ZFC formulas do not hold here, as we discuss below. An *atomic* formula is any formula of the form $s = t$ or $s \in t$, where s and t are $\mathbf{j}$-terms. A *bounded* formula is one in which all quantifiers are bound. The collection of bounded formulas is denoted Σ_0 (or, equivalently, Π_0 or Δ_0). Continuing the inductive definition in the metatheory, Σ_{n+1} is the set of $\mathcal{L}$-formulas ϕ of the form $\exists x\, \psi$ where ψ is in Π_n, and similarly for Π_{n+1}. If T is an extension of $\mathrm{ZFC}_{\mathbf{j}}$ and ϕ is an $\mathcal{L}$-formula, we say that ϕ is Σ_n^T if for some Σ_n $\mathcal{L}$-formula ψ, $T \vdash \phi \Longleftrightarrow \psi$, and similarly for Π_n^T. A formula is Δ_n^T if and only if it is both Σ_n^T and Π_n^T. In the special case $T = \mathrm{ZFC}_{\mathbf{j}}$, we will often assert that a particular formula is Σ_n (Π_n) when we really mean that the formula is Σ_n^T (Π_n^T); for proper extensions T of $\mathrm{ZFC}_{\mathbf{j}}$, we will not suppress the superscript T.

Some arguments will require formalization of syntax; to the extent this formalization will be needed, we follow [Dr]. In particular, we represent in ZF $\mathbf{j}$-terms t and $\mathcal{L}$-formulas ϕ by constant terms $\ulcorner t \urcorner$ and $\ulcorner \phi \urcorner$, respectively (added to ZF by definitional extension), using absolute formulas, and having the property that each is an element of V_ω (see [Dr, pp. 90-91]). We also have available the usual simple formulas that describe properties of these sets, such as "x is a variable" and "u represents a $\mathbf{j}$-formula". Two such formulas of particular importance are those that formalize the satisfaction relation, for both of the languages $\{\in\}$ and $\{\in, \mathbf{j}\}$:

(A) $\mathrm{Sat}(u, M, b)$: "u is an $\in$-formula $\phi(x_1, \ldots, x_m)$ and $\langle M, E(M)\rangle \models \phi(b(1), \ldots, b(m))$"

(B) $\mathrm{Sat}(u, M, i, b)$: "u is an $\mathcal{L}$-formula $\phi(x_1, \ldots, x_m)$ and $\langle M, E(M), i\rangle \models \phi(b(1), \ldots, b(m))$"

As in [Dr], $\mathrm{Sat}(u, M, b)$ and $\mathrm{Sat}(u, M, i, b)$ are Δ_1^{ZF} formulas. Also, we have the following standard result:

Theorem 2.1. *For each $\mathcal{L}$-formula $\phi(x_1, \dots, x_m)$,*

$$\forall M\, \forall b, i\, \Big[(\text{“}b \text{ and } i \text{ are functions”} \wedge\, b : \mathrm{rank}(\ulcorner \phi \urcorner) \to M\, \wedge\, i : M \to M)$$
$$\implies \big[\phi^{\langle M, E(M), i\rangle}(b(1), \dots, b(m)) \iff \mathrm{Sat}(\ulcorner \phi \urcorner, M, i, b)\big]\Big]. \blacksquare$$

Different kinds of models of $\mathrm{ZFC}_{\mathbf{j}}$ are possible, depending on one's assumptions about the surrounding universe. In this paper, all models will live in a ZFC universe $\langle V, \in\rangle$, fixed once and for all, and in particular, if $\langle M, E, i\rangle$ is a model of $\mathrm{ZFC}_{\mathbf{j}}$, we assume i is definable in V. We call such models *sharp-like* because they fit the familiar pattern of an elementary embedding $j : \mathbf{L} \to \mathbf{L}$ given by the axiom "$0^{\#}$ exists." An alternative approach, which we explore only briefly in this paper, would be to consider models $\langle M, E, i\rangle$ living in a $\mathrm{ZFC}_{\mathbf{j}}$ universe $\langle V, \in, j\rangle$. In this approach, i would be definable in $\langle V, \in, j\rangle$, but possibly not in $\langle V, \in\rangle$. Such models which, in addition, are not sharp-like, will be called *strictly* **j**-*definable*. An important subclass of these will be called **j**-*inherited* — models of the form $\langle M, \in, i\rangle$ for which $i = j \restriction M$. These are the submodels of $\langle V, \in, j\rangle$. Sharp-like models have interesting properties that strictly **j**-definable models often do not have; often, we can see what goes wrong in the latter case by considering a **j**-inherited example. Our plan, then, is to work with sharp-like models (referring to them simply as "models"), but occasionally mention variations that arise when strictly **j**-definable models are used. Our philosophical reason for considering these sometimes strange variants of the background theory originates with our work in [Co3], where we suggested that $\mathrm{ZFC} + \mathrm{WA}$ could provide a reasonably natural extension of ZFC in which all (or virtually all) large cardinals are derivable. Our brief observations about **j**-definable and **j**-inherited models serve as further explorations along these lines.

We note here that, given a model $\mathcal{M} = \langle M, E, j\rangle$ of an extension of $\mathrm{ZFC}_{\mathbf{j}}$, it is often useful to consider another model $\mathcal{M}_0 = \langle M_0, E, j \restriction M_0\rangle$, where $M_0 \subseteq M$. In such cases, $\mathcal{M}_0$ will not typically be sharp-like with respect to $\mathcal{M}$; this fact is not a violation of our convention (of restricting ourselves to sharp-like models), because $\mathcal{M}_0$ will be sharp-like with respect to V as long as $\mathcal{M}$ is. This situation arises in forcing arguments, where $\mathcal{M}_0$ is the ground model and $\mathcal{M}$ is the forcing extension; see [Co1].

Another convention we will adopt is that *every well-founded proper class model will be assumed to be set-like*; that is, for every element of the model $\langle M, E\rangle$, its class of E-predecessors is a set.

We observe next that familiar absoluteness results for Σ_0 and Δ_1 formulas hold in the present context; these will be useful in forcing arguments. Given a model $\langle M, E\rangle$ of the language $\{\in\}$, we shall call A a transitive subset of M if $A \subset M$ and for all $x \in A$ and all $y \in M$, if $y\, E\, x$ then $y \in A$.

Proposition 2.2. *Suppose $\mathcal{M} = \langle M, E, j\rangle$ is a model of $T = \mathrm{ZFC}_{\mathbf{j}}$. Suppose A is a transitive subset of M and $j \restriction A : A \to A$. Let $\mathcal{A} = \langle A, E, j \restriction A\rangle$.*

(1) *Suppose* $\phi(x_1, \ldots, x_n)$ *is a* Σ_0 $\mathcal{L}$*-formula. Then for all* $a_1, \ldots, a_n \in A$,

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \iff \mathcal{A} \models \phi[a_1, \ldots, a_n].$$

(2) *Suppose* $\mathcal{A} \models T$ *and* $\phi(x_1, \ldots, x_n)$ *is a* Δ_1^T $\mathcal{L}$*-formula. Then*

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \iff \mathcal{A} \models \phi[a_1, \ldots, a_n].$$

Proof. For (1), notice that the result easily holds for quantifier-free formulas, possibly involving $\mathbf{j}$-terms; and for formulas with bounded quantifiers, since the bounds always lie in A, the result follows as in the usual ZFC setting. For (2), the proof is essentially the same as the standard result in ZFC. To emphasize that the possible non-wellfoundedness of E does not affect the proof, we give the details for one direction. Let $\gamma(x_1, \ldots, x_n)$ be Σ_1 and $\psi(x_1, \ldots, x_n)$ be Π_1 $\mathcal{L}$-formulas such that

$$\mathrm{ZFC}_{\mathbf{j}} \vdash \forall a_1, \ldots, a_n \left[\phi(a_1, \ldots, a_n) \longleftrightarrow \phi(a_1, \ldots, a_n) \longleftrightarrow \gamma(a_1, \ldots, a_n).\right.$$

Also, write

$$\gamma(x_1, \ldots, x_n) \equiv \exists y\, \gamma'(y, x_1, \ldots, x_n)$$
$$\psi(x_1, \ldots, x_n) \equiv \forall z\, \psi'(z, x_1, \ldots, x_n),$$

where γ' and ψ' are Σ_0. Suppose $a_1, \ldots, a_n \in A$. Then

$$\begin{aligned}
\mathcal{M} \models \phi[a_1, \ldots, a_n] &\implies \mathcal{M} \models \forall z\, \psi'[z, a_1, \ldots, a_n] \\
&\implies \forall z \in M\, \big(\mathcal{M} \models \psi'[z, a_1, \ldots, a_n] \\
&\implies \forall z \in A\, \big(\mathcal{A} \models \psi'[z, a_1, \ldots, a_n] \\
&\implies \mathcal{A} \models \forall z\, \psi'[z, a_1, \ldots, a_n] \\
&\implies \mathcal{A} \models \phi[a_1, \ldots, a_n].
\end{aligned}$$

A similar argument, using γ and γ' in place of ψ and ψ', establishes upward absoluteness. ∎

One important difference between the hierarchy of $\mathcal{L}$ formulas and the usual Lévy hierarchy of $\in$-formulas is that it is not generally the case that $\exists x \in y\, \phi$ is equivalent to a Π_n formula if ϕ is Π_n, nor that $\forall x \in y\, \psi$ is equivalent to a Σ_n formula if ψ is Σ_n. The reason is that the usual proof of this equivalence involves some form of Replacement (for example, see [Je2, Lemma 14.2] or [Dr, 3.2.7]); as was discussed in Section 1, the extensions of $\mathrm{ZFC}_{\mathbf{j}}$ that will concern us primarily in this paper will *not* satisfy even Σ_1-Replacement for $\mathbf{j}$-formulas. Therefore, we issue the following caveat, to which we will refer from time to time:

(2.1)
Π_n *is not generally closed under bounded existential quantification*; and
Σ_n *is not generally closed under bounded universal quantification*

We turn to the task of explicitly introducing the axioms that govern the behavior of $\mathbf{j}$; we do not assume that the usual axioms of Separation or Replacement hold for $\mathbf{j}$-formulas. In order to make an initial observation, we introduce the following standard terminology: Suppose T is a theory that extends some sufficiently large fragment of ZFC. We will say that a formula $\Gamma(x, y)$ *defines a class function in T* if $T \vdash \forall x \exists! y\, \Gamma(x, y)$. In particular, *functions* and *class functions* are always assumed to be *total*. We observe that the formula $\mathbf{j}(x) = y$ defines a class function in all extensions of $\mathrm{ZFC}_{\mathbf{j}}$: Since $\mathbf{j}$ is a unary function symbol, it follows that for any model $\mathcal{M} = \langle M, E, j\rangle$ of $\mathrm{ZFC}_{\mathbf{j}}$, $\forall x \in M\, \exists! y \in M\, \mathcal{M} \models j(x) = y$. Thus $\mathcal{M} \models \forall x\, \exists! y\, j(x) = y$. By the Completeness Theorem, $\mathrm{ZFC}_{\mathbf{j}} \vdash \forall x\, \exists! y\, \mathbf{j}(x) = y$.

As we introduce $\mathcal{L}$-sentences to axiomatize the behavior of $\mathbf{j}$, we adopt the following convention: If σ is an $\mathcal{L}$-sentence having an occurrence of $\mathbf{j}$, then we shall denote the theory $\mathrm{ZFC}_{\mathbf{j}} + \sigma$ by simply $\mathrm{ZFC} + \sigma$, with the understanding that our language is $\mathcal{L}$ and we are using the first order logic for $\mathcal{L}$.

To capture the idea that $\mathbf{j}$ is a nontrivial elementary embedding from the universe to itself, we supplement the theory with the following axioms:

Elementarity. Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula:

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right).$$

Nontriviality. $\exists x\, \mathbf{j}(x) \neq x$.

Note that Elementarity is an axiom schema, whereas Nontriviality is a single axiom. Elementarity and Nontriviality impose the minimal conditions on $\mathbf{j}$ to guarantee that each interpretation of $\mathbf{j}$ is a nontrivial elementary embedding of the universe. Note that "elementarity" is with respect to $\in$-formulas only. We cannot derive Kunen's inconsistency result from this theory — indeed, as R. Holmes reminded the author, the mere consistency of ZFC is enough to get a model (see [CK, Theorems 3.3.10, 3.3.11(d)]):

Proposition 2.3. *Con(*ZFC*) implies that there is a model $\langle M, E, j\rangle$ of* ZFC + Elementarity + Nontriviality. ∎

For the proof, assuming ZFC is consistent, one begins by extending the language with countably many constants corresponding to some infinite ordered set $\langle I, <\rangle$, and extending ZFC with axioms that assert that these constants are indiscernibles. Using Ramsey's Theorem and the Compactness Theorem, and the fact that ZFC has a model, one shows that the extended theory is consistent. Let $\mathcal{N} = \langle N, E\rangle$ be the reduct of a model of this theory. Now $I \subset N$ is a set of indiscernibles for $\mathcal{N}$. Assuming, without loss of generality, that $\mathcal{N}$ has built-in Skolem functions, one then may extend any order-preserving $f : I \to I$ to an elementary embedding $j : \mathfrak{H}^{\mathcal{N}}(I) \to \mathfrak{H}^{\mathcal{N}}(I)$ by defining $j(t[i_1, \ldots, i_m]) = t[f(i_1, \ldots, f(i_m)]$. The final model is therefore $\langle M, E, j\rangle$ where $M = \mathfrak{H}^{\mathcal{N}}(I)$.

In contrast to Proposition 2.3, large cardinal assumptions are needed in order to obtain a *well-founded* model. This is shown in Lemma 2.7 below.

In standard set-theoretic practice, one studies elementary embeddings having a least ordinal moved. However, an instance of Separation for $\mathbf{j}$-formulas is required to prove the existence of such an ordinal. As it turns out, adding such an axiom to the theory greatly increases its large cardinal strength. We call this new axiom *Critical Point*:

Critical Point: There is a least ordinal moved by $\mathbf{j}$.

Certainly Critical Point implies Nontriviality. Let $\text{Separation}_{\mathbf{j}}$ denote Separation for $\mathbf{j}$-formulas. Before showing that Critical Point is derivable from a Σ_0 instance of $\text{Separation}_{\mathbf{j}}$, we make precise the notion of an *instance* of $\text{Separation}_{\mathbf{j}}$, make some general remarks about such instances, and then prove a useful lemma.

Formally, an *instance* of $\text{Separation}_{\mathbf{j}}$ is a sentence

$$\forall A\,\forall\vec{a}\,\exists z\,\forall u\,[u \in z \longleftrightarrow u \in A \wedge \phi(u, A, \vec{a})],$$

where ϕ is a $\mathbf{j}$-formula; in particular, this is the instance of $\text{Separation}_{\mathbf{j}}$ that is *determined by* ϕ.
2.4 Remark. Many instances of Separation for $\mathbf{j}$-formulas can be proved directly from the theory ZFC+BTEE. For example, for any $\in$-formula $\phi(x, y, z)$ and any sets A, Y, Z, $\{u \in A : \phi(u, Y, j(Z))\}$ is a set because ZFC proves that for all sets W, $\{u \in A : \phi(u, Y, W)\}$ is a set. However, it is not necessarily true that $\{u \in A : \phi(j(u), Y, Z)\}$ is a set. A familiar counter-example is the attempt to construct a measurable ultrafilter from j: Let $U = \{X \in P(\kappa) : \kappa \in j(X)\}$. U fails to be a set in the model $\langle \mathbf{L}, \in, j\rangle$, where $j : \mathbf{L} \to \mathbf{L}$ is any embedding obtained from Silver indiscernibles (assuming $0^{\#}$ exists).

Lemma 2.5. *The theory* ZFC + Elementarity + Nontriviality *proves that if x and $\mathbf{j}(x)$ have the same rank, and for all sets y for which* $\text{rank}(y) < \text{rank}(x)$, $\mathbf{j}(y) = y$, *then* $\mathbf{j}(x) = x$.

Proof. If $y \in x$, then by elementarity, $\mathbf{j}(y) \in \mathbf{j}(x)$. Also, since $\mathbf{j}(y) = y$, $y \in \mathbf{j}(x)$, and we have shown that $x \subseteq \mathbf{j}(x)$. Conversely, if $y \in \mathbf{j}(x)$, we have $\mathbf{j}(y) = y \in \mathbf{j}(x)$, whence $y \in x$. The result follows. ∎

Now we show that Critical Point is derivable from ZFC + Elementarity + Nontriviality together with the instance of Σ_0-$\text{Separation}_{\mathbf{j}}$ determined by the formula $\mathbf{j}(x) \neq x$. Seeking a contradiction, assume Critical Point fails. There are two cases: The first case (which was brought to the attention of the author by the referee) is that some ordinal α is moved by $\mathbf{j}$, but there is no least such. In that case, by an application of the instance of Σ_0-$\text{Separation}_{\mathbf{j}}$ determined by "$\mathbf{j}(x) \neq x$", the following is a set:

$$S = \{\beta < \alpha + 1 \mid \mathbf{j}(\beta) \neq \beta\}.$$

Since $\alpha \in S$, S has a least element (arguing in ZFC alone), yielding a contradiction. The second case is that for all α, $\mathbf{j}(\alpha) = \alpha$. Using Nontriviality, let x be such that $\mathbf{j}(x) \neq x$. Let $\alpha = \text{rank}(x) + \omega$

and let $X = V_\alpha$. Let $M = \{x \in X : \mathbf{j}(x) \neq x\}$; the fact that M is a set follows from an application of the instance of Σ_0-Separation$_\mathbf{j}$ determined by "$\mathbf{j}(x) \neq x$". Let $B = \{\text{rank}(x) : x \in M\}$; B is a set by Replacement for $\in$-formulas. Also, $B \neq \emptyset$ since $M \neq \emptyset$. Let $\alpha = \inf B$ and let $y \in M$ be such that $\text{rank}(y) = \alpha$. By the lemma and the leastness of $\text{rank}(y)$, we must have $\mathbf{j}(y) = y$, and we have a contradiction.

When Critical Point holds, we will denote the critical point of $\mathbf{j}$ (and of any of its interpretations) with the letter κ, and also with the notation $\text{cp}\,\mathbf{j}$ or $\text{cp}\,j$. We think of κ as a constant added by definitional extension. Note that the $\mathbf{j}$-formula "x is the critical point of $\mathbf{j}$" is $\Sigma_0^{\mathrm{ZFC}_\mathbf{j}}$. By elementarity, as usual, $\mathbf{j}(\kappa) > \kappa$:

Proposition 2.6. ZFC + Elementarity + Critical Point $\vdash \mathbf{j}(\kappa) > \kappa$. ■

The axioms Elementarity + Critical Point capture the basic features of elementary embeddings, as they are used in practice; so we give this collection of axioms the name *Basic Theory of Elementary Embeddings*, or BTEE. Although this theory is not strong enough to obtain inconsistency either, the critical point of the embedding must be a large cardinal. For the moment, we prove that κ must be inaccessible, and prove more after setting up some preliminaries.

We begin by observing that κ is an infinite ordinal $> \omega$: First, since each standard integer is definable, we have $\mathbf{j}(0) = 0$ and $\mathbf{j}(1) = 1$. It follows that no finite ordinal is the critical point of $\mathbf{j}$, for if $\kappa = n + 1$, we would have $\mathbf{j}(n + 1) = \mathbf{j}(n) + 1 = n + 1$. (This argument actually shows that (ZFC $-$ Infinity) + BTEE $\vdash$ Infinity, which shows that each axiom of the form "there exists $j : V \to V$ having a critical point" can be viewed as a generalized Axiom of Infinity.) Finally, $\kappa > \omega$ since, by definability of ω, $\mathbf{j}(\omega) = \omega$.

To see κ is a regular uncountable cardinal, we can argue as follows (in ZFC+BTEE): Whenever $f : \alpha \to \kappa$, where $\alpha < \kappa$, we have, by elementarity and leastness of κ, that $j(f) = f$; thus it would be impossible for such an f to be a bijection or even cofinal.

To see that κ is inaccessible, first observe that for any bounded subset A of κ, $\mathbf{j}(A) = A$: If $\alpha \in A$, then $\alpha = \mathbf{j}(\alpha) \in \mathbf{j}(A)$; conversely, if $\beta < \kappa$ is such that $A \subset \beta$ then $\mathbf{j}(A) \subseteq \mathbf{j}(\beta) = \beta$, and so $\alpha \in \mathbf{j}(A)$ implies $\mathbf{j}(\alpha) = \alpha$, whence $\alpha \in A$. For the proof of inaccessibilty, assume there is some $\alpha < \kappa$ for which there is a surjection $g : P(\alpha) \to \kappa$. Then $\mathbf{j}(g) : P(\alpha) \to \mathbf{j}(\kappa)$ is also a surjection. Now for each $A \subset \alpha$, by our previous observation and the fact that ran $(g) = \kappa$,

$$\mathbf{j}(g)(A) = \mathbf{j}(g)(\mathbf{j}(A)) = \mathbf{j}(g(A)) = g(A),$$

whence $\mathbf{j}(g) = g$. But since $\mathbf{j}(\kappa) > \kappa$, this is impossible.

We can now give the reason that well-founded set models of ZFC + Elementarity + Nontriviality have large cardinal strength; we begin with a useful lemma:

Lemma 2.7. *Suppose $\langle M, E, i\rangle$ is a well-founded model of* ZFC$_\mathbf{j}$. *Let $\pi : \langle M, E\rangle \to \langle N, \in\rangle$ be the Mostowski collapsing isomorphism. If $j = \pi \circ i \circ \pi^{-1}$, then $\langle N, \in, j\rangle \models$* ZFC$_\mathbf{j}$. *Moreover, j is the* *unique function $N \to N$ satisfying*

(2.2) $\quad$ *π is an isomorphism between the structures $\langle M, E, i\rangle$ and $\langle N, \in, j\rangle$.*

In addition, if $\langle M, E, i\rangle \models$ ZFC + Elementarity + Nontriviality, *then* $\langle N, \in, j\rangle \models$ ZFC+ Elementarity + Nontriviality.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ N \arrow[r, "j"] & N \end{tikzcd}$$

Proof. The fact that $\langle N, \in, j\rangle \models \mathrm{ZFC}_{\mathbf{j}}$, where $j = \pi \circ i \circ \pi^{-1}$, is obvious. To see that π is an isomorphism between the structures $\mathcal{M}$ and $\mathcal{N}$, it suffices to observe that π respects j. But this follows from the equation

$$\pi \circ i = j \circ \pi, \tag{2.3}$$

which is easily derived from the definition of j. Finally, notice that *any* function $j : N \to N$ satisfying (2.2) must satisfy the equation (2.3), and hence we must have $j = \pi \circ i \circ \pi^{-1}$. If $\langle M, E, i\rangle \models$ ZFC + Elementarity + Nontriviality, elementarity and nontriviality of j in $\langle N, \in, j\rangle$ follow immediately from its definition. ■

Proposition 2.8. *Any well-founded model of* ZFC + Elementarity + Nontriviality *also satisfies* Critical Point, *and hence* BTEE.

Proof. Since Critical Point is preserved by isomorphisms between models it suffices, by Lemma 2.7, to prove the result for any *transitive* model of ZFC + Elementarity + Nontriviality. Thus, suppose $\langle N, \in, j\rangle$ is such a model. In V, we can form the set $S = \{y \in N : j(y) \neq y\}$ (since j is definable in V); by absoluteness, $S \neq \emptyset$. Let x be a set in S of least rank (obtained in V), and let $\kappa = \mathrm{rank}(x)$. Now in N, x also has a rank, and by absoluteness of the rank function, this rank must be κ. We show that, in V, $j(\kappa) \neq \kappa$: If $j(\kappa) = \kappa$, then

$$\mathrm{rank}(j(x)) = j(\mathrm{rank}(x)) = j(\kappa) = \kappa = \mathrm{rank}(x).$$

Now one can argue as in Lemma 2.5 to conclude that $j(x) = x$, which is impossible. Thus V, and hence also $\langle N, \in, j\rangle$ by absoluteness, satisfies

$$j(\kappa) > \kappa \wedge \forall \alpha < \kappa\, (j(\alpha) = \alpha).$$

It follows that $\langle N, \in, j\rangle \models$ Critical Point. ■

The referee suggests the following alternative proof for Proposition 2.8: First observe that Nontriviality is equivalent to the assertion that there is an ordinal α with $\mathbf{j}(\alpha) \neq \alpha$. This is true because by AC any set is coded with a set of ordinals. If all ordinals were fixed by $\mathbf{j}$, then any set of ordinals would also be fixed by $\mathbf{j}$, and so every set would be fixed by $\mathbf{j}$. Proposition 2.8 now follows, since if a model is well-founded and has an ordinal moved, it has a least ordinal moved.

With additional hypotheses, we get comparable results for $\mathbf{j}$-inherited models; these additional hypotheses appear to be necessary:

Proposition 2.9. *Suppose* $\langle M, \in, i\rangle$ *is a transitive* $\mathbf{j}$*-inherited model of* ZFC + Elementarity + Nontriviality. *Assume that* $\langle V, \in, j\rangle \models$ ZFC + BTEE. *Then* $\langle M, \in, i\rangle \models$ Critical Point.

Proof. Suppose κ is the critical point of j in V. Since Elementarity + Nontriviality holds in M, some set x is moved by i, and hence also by j. But then $\mathrm{rank}(x) \geq \kappa$. Since rank is computed the same way in both models, $\kappa \in M$. By absoluteness, κ is the critical point of i in M.■

Assuming that $\langle V, \in, j\rangle$ satisfies slightly more than ZFC + BTEE, we prove in Proposition 6.7 that there is no *countable* transitive $\mathbf{j}$-inherited model of ZFC + Elementarity + Nontriviality.

We now show that the theory ZFC + BTEE implies that the critical point of $\mathbf{j}$ is *n-ineffable* for every $n \geq 1$. A cardinal λ is n-ineffable if every partition $f : [\lambda]^{n+1} \to 2$ has a stationary homogeneous set (that is, a stationary set $H \subset \lambda$ such that $f \restriction [H]^{n+1}$ is constant). λ is said to be *ineffable* if λ is 1-ineffable. It is known that the n-ineffables below an $n+1$-ineffable λ form a stationary set in λ. Every measurable cardinal is ineffable; assuming $0^{\#}$ exists, every Silver indiscernible is ineffable in $\mathbf{L}$; and an ineffable cardinal λ is the λth weakly compact (see [Je2] and [KM]). Also, it is known that λ is ineffable iff for each sequence $\langle A_\alpha : \alpha < \lambda\rangle$ satisfying $A_\alpha \subseteq \alpha$ for all $\alpha < \lambda$, there is a set $A \subseteq \lambda$ such that $\{\alpha < \lambda : A \cap \alpha = A_\alpha\}$ is stationary in λ.

We need the following standard lemma:

Proposition 2.10. ZFC + BTEE *proves the following:*

(1) $\forall A \in P(\kappa)\, \mathbf{j}(A) \cap \kappa = A$.

(2) $\forall A \in P(\kappa)\, \big(\kappa \in \mathbf{j}(A) \implies$ *"A is stationary"*$\big)$.

(3) *Suppose* $\alpha < \kappa, S \subseteq \kappa, \kappa \in \mathbf{j}(S)$, *and* $S = \bigcup_{\beta<\alpha} S_\beta$ *Then there is* $\beta < \kappa$ *such that* $\kappa \in \mathbf{j}(S_\beta)$.

Proof of (1). By Remark 2.4, $\mathbf{j}(A) \cap \kappa$ is a set. If $\alpha \in A$, then $\alpha = \mathbf{j}(\alpha) \in \mathbf{j}(A) \cap \kappa$. Conversely, if $\alpha \in \mathbf{j}(A) \cap \kappa$ then $\alpha = \mathbf{j}(\alpha)$, whence $\alpha \in A$. ■

Proof of (2). Suppose $C \subseteq \kappa$ is closed and unbounded. κ is a limit point of $C = \mathbf{j}(C) \cap \kappa$, whence $\kappa \in \mathbf{j}(C)$. Since $\mathbf{j}(A) \cap \mathbf{j}(C) \neq \emptyset$, we have $A \cap C \neq \emptyset$, by elementarity. Thus, A is stationary. ■

Proof of (3). Define $f : \alpha \to P(\kappa)$ by $f(\beta) = S_\beta$. By elementarity, $\mathbf{j}(f) = \langle \mathbf{j}(S_\beta) : \beta < \alpha\rangle$. Then since

$$\kappa \in \mathbf{j}(S) = \mathbf{j}\big(\bigcup \mathrm{ran}\ f\big) = \bigcup \mathrm{ran}\ \mathbf{j}(f) = \bigcup_{\beta<\alpha} \mathbf{j}(S_\beta),$$

it follows that $\kappa \in \mathbf{j}(S_\beta)$ for some $\beta < \kappa$. ■

We also need a lemma that gives us information about slightly stronger versions of the equivalent definitions of ineffable given above. Only one of the possible implications is needed for our work here:

Lemma 2.11. *Consider the following two statements, expressed in the language $\mathcal{L}$:*

(2.4) *For each sequence $\langle A_\alpha : \alpha < \kappa\rangle$ satisfying $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, there exist A, S such that $A \subseteq \kappa$, $S = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$, and $\kappa \in \mathbf{j}(S)$,*

and

(2.5) *Each $f : [\kappa]^2 \to 2$ has a homogeneous set H for which $\kappa \in \mathbf{j}(H)$.*

Then it is provable in $\mathrm{ZFC} + \mathrm{BTEE}$ *that (2.4) implies (2.5).*

Proof. The proof is like [Je2, Lemma 32.7(a)], except that we must verify that no axioms beyond $\mathrm{ZFC} + \mathrm{BTEE}$ are used when working with $\mathbf{j}$. Assume (2.4), and let $f : [\kappa]^2 \to 2$ be a partition of $[\kappa]^2$. Using ZFC, define $A_\alpha \subseteq \alpha$, $\alpha < \kappa$, by

$$A_\alpha = \{\beta < \alpha : f(\beta, \alpha) = 1\}.$$

Using (2.4), let A, S be such that $A \subseteq \kappa$, $S = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$, and $\kappa \in \mathbf{j}(S)$. Using Proposition 2.10(3), we define H to be the element of $\{S \cap A, S \setminus A\}$ for which $\kappa \in \mathbf{j}(H)$. If $H = S \cap A$ and $\beta < \alpha$ are in H, then, since $\beta \in A_\alpha$, $f(\beta, \alpha) = 1$. If $H = S \setminus A$ and $\beta < \alpha$ are in H, then, since $\beta \notin A_\alpha$, $f(\beta, \alpha) = 0$. Either way, f is constant on H, so H is homogeneous for f. ■

Theorem 2.12. *For each particular (methatheoretic) natural number $n \geq 1$,* $\mathrm{ZFC} + \mathrm{BTEE} \vdash$ *κ is n-ineffable.*

Proof. By induction in the metatheory, we prove the following slightly stronger result, for each particular $n \geq 1$:

(2.6) Each $f : [\kappa]^{n+1} \to 2$ has a homogeneous set H such that $\kappa \in \mathbf{j}(H)$.

(This statement implies n-ineffability by Proposition 2.10(2).)

For the basis step $n = 1$, we prove (2.4), which suffices by Lemma 2.11. We follow the argument in [Je2, Lemma 32.7]. Suppose $f = \langle A_\alpha : \alpha < \kappa\rangle$, where, for each $\alpha < \kappa$, $A_\alpha \subseteq \alpha$. Let $A = \mathbf{j}(f)(\kappa)$;

$A \subseteq \kappa$. Note by Separation for $\in$-formulas, using the Remark 2.4, $S = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is a set. Now, because $\mathbf{j}(A) \cap \kappa = A$ (by Proposition 2.10(1)), $\kappa \in \mathbf{j}(S)$, as required.

For the induction step, assume (2.6) holds for $n \geq 1$, and let $f : [\kappa]^{n+2} \to 2$ be a partition. For each $\alpha < \kappa$, define $f_\alpha : [\kappa]^{n+1} \to 2$ by

$$f_\alpha(\xi_0, \ldots, \xi_n) = f(\xi_0, \ldots, \xi_n, \alpha).$$

By the induction hypothesis, for each α there is a set $H_\alpha \subseteq \kappa$ such that H_α is homogeneous for f and $\kappa \in \mathbf{j}(H_\alpha)$. Using ZFC only, we form the sets $K_\varepsilon = \{\alpha < \kappa : f''_\alpha H_\alpha = \varepsilon\}$, for $\varepsilon \in \{0, 1\}$. Using Proposition 2.10(3), we let K denote the element of $\{K_0, K_1\}$ for which $\kappa \in \mathbf{j}(K)$. Without loss of generality, we assume $K = K_1$. Let $A_\alpha = H_\alpha \cap \alpha$. By the basis step, there exist subsets A, S of κ such that $S = \{\alpha < \kappa : A_\alpha = A \cap \alpha\}$ and $\kappa \in \mathbf{j}(S)$. By elementarity, $\kappa \in \mathbf{j}(S \cap K)$. We define H to be $(S \cap K) \cap A$ if $\kappa \in \mathbf{j}\big((S \cap K) \cap A\big)$ or $(S \cap K) \setminus A$ if $\kappa \in \mathbf{j}\big((S \cap K) \setminus A\big)$. By Proposition 2.10(3), H is well-defined. We prove that H is homogeneous in the case in which $H = (S \cap K) \cap A$; the other case is handled similarly. Let $\xi_0 < \ldots < \xi_n < \alpha$ be elements of H. Since $\alpha \in S$, it follows that $A \cap \alpha = A_\alpha = H_\alpha \cap \alpha$. Since $\xi_0, \ldots, \xi_n \in H_\alpha \cap \alpha$ and $\alpha \in K$ (and using the assumption that $K = K_1$),

$$1 = f_\alpha(\xi_0, \ldots, \xi_n) = f(\xi_0, \ldots, \xi_n, \alpha).$$

Thus, H is homogeneous for f.∎

The slightly stronger versions of ineffability that we have introduced here, replacing stationarity of a set S with the condition "$\kappa \in \mathbf{j}(S)$," have allowed us to "step around" the well-known obstacle to proving the equivalence of n-ineffability and m-ineffability for all m and n.

Corollary 2.13. *Every well-founded set model of* ZFC+Elementarity+Nontriviality, *also satisfies "there is a cardinal that is n-ineffable for every n".*

Proof. This follows immediately from Propositions 2.8 and 2.12. ∎

Once we know that κ is n-ineffable for every n, we can show that the cardinals below κ that also have this property form a stationary set, and that there are more than κ many such cardinals above κ. This follows from a more general fact:

Theorem 2.14. *Suppose $A(x)$ is a large cardinal property, expressible in the language $\{\in\}$, and* $\mathrm{ZFC} + \mathrm{BTEE} \vdash A(\kappa)$. *Then,*

(1) $\mathrm{ZFC} + \mathrm{BTEE} \vdash$ "$\{\alpha < \kappa : A(\alpha)\}$ *is stationary*".

(2) *For each particular (metatheoretic) natural number n,*

$$\mathrm{ZFC} + \mathrm{BTEE} \vdash |\{\lambda : A(\lambda)\}| > \mathbf{j}^n(\kappa).$$

In order to prove (2), we will need to be able to talk about iterates $\mathbf{j} \circ \mathbf{j}, \mathbf{j} \circ \mathbf{j} \circ \mathbf{j}, \ldots$ of $\mathbf{j}$ in the formal theory. We do this in the usual way, by adding, for each particular $n > 0$, a function symbol $\mathbf{j}^n$ by definitional extension, where

$$\mathbf{j}^1(x) = \mathbf{j}(x)$$
$$\mathbf{j}^{n+1}(x) = (\mathbf{j}^n \circ \mathbf{j})(x) = \mathbf{j}^n(\mathbf{j}(x))$$

It is straightforward to show that for each particular n, the formula $\mathbf{j}^n(x) = y$ defines a class function and that $\mathbf{j}^n$ satisfies the Elementarity schema. We can now prove Theorem 2.14:

Proof of Theorem 2.14(1). Let $B = \{\alpha < \kappa : A(\alpha)\}$. To see that B is unbounded in κ, assume $B \subset \beta < \kappa$. Applying $\mathbf{j}$ to the true formula $\forall \gamma\, (\beta < \gamma < \kappa \Longrightarrow \neg A(\gamma))$ yields

$$\forall \gamma\, (\beta < \gamma < \mathbf{j}(\kappa) \Longrightarrow \neg A(\gamma)),$$

which contradicts the fact that $A(\kappa)$ is true. Finally, notice that, since $\mathbf{j}(B)$ contains all the cardinals $\lambda < \mathbf{j}(\kappa)$ for which $A(\lambda)$ holds, $\kappa \in \mathbf{j}(B)$. Therefore, by Proposition 2.10(2), B is stationary. ∎

Proof of Theorem 2.14(2). By elementarity of each $\mathbf{j}^n$ and by induction in the metatheory, $A(\mathbf{j}^n(\kappa))$ is true for each particular natural number n. Elementarity of $\mathbf{j}^n$ also shows that the set $\{\lambda < \mathbf{j}^n(\kappa) : A(\lambda)\}$ is unbounded in $\mathbf{j}^n(\kappa)$. The result follows. ∎

A result related to Theorem 2.14 is the fact that κ must be totally indescribable. Recall that a cardinal λ is Π^n_m-indescribable if, whenever $U \subseteq V_\lambda$ and σ is a Π^n_m sentence such that $\langle V_\lambda, \in, U\rangle \models \sigma$, then for some $\alpha < \lambda$, $\langle V_\alpha, \in, U \cap V_\alpha\rangle \models \sigma$ (treating U as a unary predicate). λ is totally indescribable if it is Π^n_m-indescribable for every m, n. It is known (see [Je2, Exercise 32.13]) that if $\langle M, \in, j\rangle$ is a transitive model of ZFC + BTEE, then $\mathrm{cp}(j)$ is totally indescribable in M. Showing that ZFC + BTEE $\vdash$ "κ is totally indescribable" represents a slight improvement of this result, though, in fact, essentially the same proof works: Let $\mathrm{Sent}(X, n, p)$ assert that $n \in \omega$ and p codes an $(n+1)$th order sentence in the language $\{\in, X\}$, where X denotes a first-order unary predicate symbol. Let $\mathrm{Sat}(n, p, M, U)$ be a formula asserting $\mathrm{Sent}(X, n, p)$ and that $\langle M, E(M), U\rangle \models \sigma$ (using $(n+1)$th-order satisfaction). Let $\Pi(n, m, p)$ say that p codes a Π^n_m formula. Working in ZFC + BTEE, fix positive integers m, n and a p such that $\mathrm{Sent}(X, n, p)$ and $\Pi(p, n, m)$. Let $U \subset V_\kappa$. Note that $U = \mathbf{j}(U) \cap V_\kappa$. If $\mathrm{Sat}(n, p, V_\kappa, U)$, then $\exists \alpha < \mathbf{j}(\kappa)\ \mathrm{Sat}(n, p, V_\alpha, \mathbf{j}(U) \cap V_\alpha)$. By elementarity, $\exists \alpha < \kappa\, \mathrm{Sat}(n, p, V_\alpha, U \cap V_\alpha)$, as required. We record this observation here:

Proposition 2.15. ZFC + BTEE $\vdash$ "*κ is totally indescribable*". ∎

To close this section, we give the definition of the 3-parameter formula $\Phi(n, x, y)$, uniformizing the formulas $\mathbf{j}^n(x) = y$ mentioned earlier. Assuming enough Induction axioms for $\mathbf{j}$-formulas, this formula defines the functional relation $y = \mathbf{j}^n(x)$; in Section 4, we will introduce these additional induction axioms. Here, we use the formula to prove some basic results that do not require these extra axioms.

Define

$$\Phi(n,x,y) \equiv n \in \omega \Longrightarrow \exists f\, \Theta(f,n,x,y), \tag{2.7}$$

where

$$\begin{aligned} \Theta(f,n,x,y) \equiv &\text{“}f \text{ is a function”} \wedge \operatorname{dom} f = n+1 \wedge f(0) = x \wedge \\ &\forall i \left(0 < i \leq n \Longrightarrow f(i) = \mathbf{j}(f(i-1))\right) \wedge f(n) = y. \end{aligned} \tag{2.8}$$

An important variant of $\Phi(n,x,y)$ is given by the Σ_1 formula

$$\Psi(n,y) \equiv \exists x \in y \left[\Phi(n,x,y) \wedge x = \kappa\right]. \tag{2.9}$$

Without extra induction axioms, it is easy to verify that, whenever $\Theta(f,n,x,y)$ holds, so must $\Theta(f \restriction m+1, m, x, f(m))$ for any $m < n$. We prove next that whenever $\Theta(f,n,\alpha,y)$ and α is an ordinal, then f is a sequence of ordinals. We shall say that $\mathbf{j}^n(x)$ *exists* or *is defined* just in case there is some y for which $\Phi(n,x,y)$.

Lemma 2.16. $\mathrm{ZFC} + \mathrm{BTEE} \vdash \forall f,n,x,y \left[\Theta(f,n,x,y) \wedge \text{“}x \text{ is an ordinal”} \Longrightarrow \text{“}y \text{ is an ordinal”}\right]$.

Proof. Let f,n,x,y be such that $n \in \omega$, x is an ordinal, and $\Theta(f,n,x,y)$. One then shows by a straightforward (ordinary) bounded induction that

$$\forall m \leq n\, f(m) \text{ is an ordinal.} \blacksquare$$

Proposition 2.17. *In* ZFC + BTEE*, suppose α is an ordinal.*

1. *If $\mathbf{j}(\alpha) > \alpha$ and f,n,β satisfy $\Theta(f,n,\alpha,\beta)$, then f is a strictly increasing sequence of ordinals.*
2. *If $\mathbf{j}(\alpha) \geq \alpha$ and f,n,β satisfy $\Theta(f,n,\alpha,\beta)$, then f is a nondecreasing sequence of ordinals.*
3. *The sequence $\langle \mathbf{j}^n(\kappa) : n \in \omega$ and $\mathbf{j}^n(\kappa)$ exists$\rangle$ is strictly increasing.*

Proof. Part (3) follows from (1). The proofs for parts (1) and (2) are nearly identical, so we just prove (1). Let f,n,α,β be such that $\Theta(f,n,\alpha,\beta)$ and α is an ordinal. By Lemma 2.16, β is also an ordinal. It suffices to prove by (ordinary) bounded induction

$$\forall m \leq n\, \forall i < m\, f(i) < f(m).$$

The basis step is true vacuously. For the induction step, let $m < n$; we prove $\forall i < m+1\, f(i) < f(m+1)$. This formula holds when $m = 0$ by hypothesis, so assume $m > 0$. By induction hypothesis, $f(m-1) < f(m)$. Applying $\mathbf{j}$ to the latter formula, we have

$$f(m) = \mathbf{j}(f(m-1)) < \mathbf{j}(f(m)) = f(m+1);$$

this completes the induction and the proof of (1).■

The hypotheses "$\mathbf{j}(\alpha) > \alpha$" and "$\mathbf{j}(\alpha) \geq \alpha$" in Proposition 2.17(1),(2), respectively, cannot be eliminated. We show in Proposition 6.4 that even the theory ZFC + BTEE + Σ_0-Induction$_{\mathbf{j}}$ does not suffice to prove that for all $\alpha, \mathbf{j}(\alpha) \geq \alpha$.

The next lemma establishes the simple ZFC+BTEE fact that the sets y for which $\Phi(n, \mathbf{j}(x), y)$ are the same as the sets y for which $\Phi(n+1, x, y)$. Intuitively, this says that $\mathbf{j}^n(\mathbf{j}(x)) = \mathbf{j}^{n+1}(x)$. This result is used in the proof of Proposition 5.3.

Lemma 2.18. $\mathrm{ZFC} + \mathrm{BTEE} \vdash \forall n \in \omega\, \forall x\, \forall y \left[\Phi(n, \mathbf{j}(x), y) \iff \Phi(n+1, x, y)\right]$.

Proof. We define the following formulas:

$$Q_0(n, f) \equiv n \in \omega \,\wedge\, \text{“}f \text{ is a function”} \,\wedge\, \mathrm{dom}\,(f) = n + 1,$$
$$Q_1(n, x, g) \equiv n \in \omega \,\wedge\, \text{“}g \text{ is a function”} \,\wedge\, \mathrm{dom}\,(g) = n + 2 \,\wedge\, g(0) = x.$$

$Q_0(n, f)$ says that f is a function with domain $n + 1$ and $Q_1(n, x, g)$ says g is a function with domain $n + 2$ for which $g(0) = x$. It is easy to see that

$$\text{(2.10)} \qquad \mathrm{ZFC} \vdash \forall n, x, f\, \exists! g \left[Q_0(n, f) \Longrightarrow Q_1(n, x, g) \,\wedge\, \forall i \left(0 \leq i \leq n \Longrightarrow g(i+1) = f(i)\right)\right],$$

and

$$\text{(2.11)} \qquad \mathrm{ZFC} \vdash \forall n, g\, \exists! f \left[Q_1(n, g(0), g) \Longrightarrow Q_0(n, f) \,\wedge\, \forall i \left(0 \leq i \leq n \Longrightarrow f(i) = g(i+1)\right)\right].$$

Now, suppose $n \in \omega, x, y, f$ are such that $\Theta(f, n, x, y)$ holds. Since $Q_0(n, f)$ holds, we obtain from (2.10) a unique g for which $Q_1(n, \kappa, g)$ is true and $g(i + 1) = f(i)$ whenever $0 \leq i \leq n$. For each i, with $0 < i \leq n$, we have

$$g(i+1) = f(i) = \mathbf{j}(f(i-1)) = \mathbf{j}(g(i)).$$

It follows that $\Theta(g, n + 1, x, y)$ holds, whence $\Phi(n + 1, x, y)$. A similar argument demonstrates the reverse implication. ■

We also show here that $\mathbf{j}(\mathbf{j}^n(x)) = \mathbf{j}^n(\mathbf{j}(x))$:

Lemma 2.19. $\mathrm{ZFC} + \mathrm{BTEE} \vdash \forall n \in \omega\, \forall x\, \forall y \left[\Phi(n, x, y) \implies \Phi(n, \mathbf{j}(x), \mathbf{j}(y))\right]$.

Proof. From ZFC alone it follows that for any function f defined on $n + 1$ there is a g defined on n such that for all $i < n$, $g(i) = f(i + 1)$. Now, assume $\Theta(f, n, x, y)$. Define g as above, and define $\hat{g}$ on $n + 1$ by $\hat{g} = g \cup \{(n, \mathbf{j}(y))\}$. Clearly, $\Theta(g, n, \mathbf{j}(x), \mathbf{j}(y))$, and the result follows. ■

Assuming Σ_0-Induction, one may extend these results to show that, if there is y such that $y = \mathbf{j}^n(x)$, then for each particular integer k, $\mathbf{j}^n(\mathbf{j}^k(x)) = \mathbf{j}^{n+k}(x) = \mathbf{j}^k(\mathbf{j}^n(x))$; see Proposition 5.3(1).

§3. Transitive Models of BTEE and Remarkable Cardinals

In this section, we show that only a rather weak large cardinal hypothesis (namely, the existence of an ω-Erdös cardinal) is necessary to obtain models of ZFC + BTEE. After describing a canonical procedure for obtaining such models from a set of indiscernibles, we discuss a particularly nice class of models that are rich enough to prove the consistency of Schindler's *remarkable* cardinals.

Whenever we have a set I of ordinal indiscernibles of type ω for a transitive set model of ZFC having built-in or definable Skolem functions, we can obtain a transitive model of ZFC + BTEE, and we can do so in a canonical way:

3.1 Remark (*Canonical Construction of Models of* ZFC + BTEE). Given a transitive $M \models$ ZFC with built-in or definable Skolem functions and $I \subset \mathrm{ON}^M$ of indiscernibles for M having ordertype ω. Define $B = \mathfrak{H}^M(I) \prec M$. Let $\pi : B \to N$ be the transitive collapsing map, and let $e : N \to M$ denote the induced elementary embedding ($e = \pi^{-1}$). Define $i_0 : I \to I$ so that i_0 takes each element α of I to the next element $s_I(\alpha)$ of I above α. Define $i : B \to B$ by

$$i(t^M[\alpha_1, \ldots, \alpha_k]) = t^M[i_0(\alpha_1), \ldots, i_0(\alpha_k)]$$

where $t(x_1, \ldots, x_k)$ is any Skolem term and $\alpha_1 < \ldots < \alpha_k$ are in I; as usual, i is well-defined and is an elementary embedding. Letting $j = \pi \circ i \circ \pi^{-1}$, we have, by Lemma 2.7, that $\langle N, \in, j\rangle \models$ ZFC + BTEE. Note that if $J = \pi''I$, then J is a set of indiscernibles for N, and j acts on J by sending each $\beta \in J$ to $s_J(\beta)$. We call the model $\langle N, \in, j\rangle$ the *canoncial transitive model of* ZFC + BTEE *derived from* M, I. (Of course, N also depends on the choice of Skolem functions, but this dependency will not need to be made explicit in any of our arguments here. In particular, when we work in models of type $\langle L_\gamma, \in, i\rangle$, we will always use the definable Skolem functions already available in the model.)

The only way known (so far) for building models of ZFC + BTEE under mild large cardinal hypotheses ($0^\#$ or weaker) is by using a set of indiscernibles that is a subset of some ordinal. If $0^\#$ exists, indiscernibles for $\mathbf{L}$ and L_λ for cardinals λ are always available. For our purposes, though, it usually suffices to assume the existence of an α-Erdös cardinal for some countably infinite limit ordinal α. We pause here to review a central theorem about $0^\#$ and the main properties of α-Erdös cardinals. (We assume familiarity with the development of $0^\#$ as in [Je2] or [Dr].) The proof of Theorem 3.2 can be found in [De]; proofs of (1)-(5) of Theorem 3.3 can be found in [Je2, Chapter 32]; and the proof of part (6) of that theorem is a special case of Theorem 8.2.4 of [Dr].

Given an infinite ordinal α, λ is *α-Erdös* if

$$\lambda \text{ is least such that } \lambda \to (\alpha)^{<\omega}.$$

Theorem 3.2. *The following are equivalent:*

(1) $0^{\#}$ *exists.*

(2) *There is an elementary embedding* $j : L_\alpha \to L_\beta$*, where* α *and* β *are limit ordinals and* $\mathrm{cp}(j) < |\alpha|$.

(3) *For any uncountable cardinal* λ*, there is a nontrivial elementary embedding* $j : L_\lambda \to L_\lambda$.

(4) *There is a nontrivial elementary embedding* $\mathbf{L} \to \mathbf{L}$. ■

Part (4) of this theorem cannot be stated in this form in ZFC. Also, as we show in Example 9.2, (4) is *not* equivalent to (1)-(3) unless it is understood that j is "sufficiently" definable in V; certainly, requiring j to be a class (defined with parameters) in V suffices for the proof (and this is nearly always assumed to be the case in this context) — but much less definability will do.

Theorem 3.3 (α*-Erdös cardinals*)**.** *Assume* α *and* β *are infinite limit ordinals.*

(1) *If* $\alpha < \omega_1$ *and* λ *is* α*-Erdös, then* λ *is* α*-Erdös in* $\mathbf{L}$.

(2) *If there is an* ω_1*-Erdös cardinal, then* $0^{\#}$ *exists.*

(3) *If* $\alpha < \beta$ *and* λ_β *is* β*-Erdös, then the* α*-Erdös cardinal* λ_α *exists and* $\lambda_\alpha < \lambda_\beta$.

(4) *Each* α*-Erdös cardinal is inaccessible.*

(5) *If there is an* α*-Erdös cardinal* λ *and* $\mathfrak{A}$ *is a model whose language has less than* λ *symbols and whose domain contains every element of* λ*, then* $\mathfrak{A}$ *has a set of indiscernibles of ordertype* α.

(6) *Suppose* λ *is* α*-Erdös. Let* $\mathcal{M}$ *denote either* $\langle L_\lambda, \in\rangle$ *or* $\langle V_\lambda, \in, h_\phi^{\mathcal{M}}\rangle_{\phi\in\mathrm{Fmla}_\in}$ *(where each* $h_\phi^{\mathcal{M}}$ *is a Skolem function for* ϕ *in* $\langle L_\lambda, \in\rangle$ *or* $\langle V_\lambda, \in\rangle$*, respectively).*

   (a) *For any set* $I \subseteq \lambda$ *of indiscernibles for* $\mathcal{M}$*, we have that if* $\alpha_1 < \ldots < \alpha_k < \beta$ *are in* I*,* $t(x_1, \ldots, x_k)$ *is a Skolem term, and* $\mathcal{M} \models$ "$t(\alpha_1, \ldots, \alpha_k)$ *is an ordinal", then* $\mathcal{M} \models t(\alpha_1, \ldots, \alpha_k) < \beta$.

   (b) *If* $\alpha > \omega$*, then there is a set* $I \subseteq \lambda$ *of indiscernibles for* $\mathcal{M}$ *such that if* $\alpha_1 < \ldots < \alpha_k < \beta < \gamma_1 < \ldots < \gamma_m$ *are in* I*,* $t(x_1, \ldots, x_k, z_1, \ldots, z_m)$ *is a Skolem term, and*

$$\mathcal{M} \models \text{"}t(\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_m) \text{ is an ordinal and } \beta < t(\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_m)\text{"},$$

   then

$$\mathcal{M} \models \gamma_1 \leq t(\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_m). \blacksquare$$

Part (6) of the theorem is needed for the next corollary; the proof of the corollary is essentially the same as the standard proof used to derive the same properties for Silver indiscernibles from an ω_1-Erdös cardinal (as in [Je2, Chapter 30] for example).

Corollary 3.4. *Suppose* λ *is* α*-Erdös, where* α *is an infinite limit ordinal. Let* $\mathcal{M}$ *denote either* $\langle L_\lambda, \in\rangle$ *or* $\langle V_\lambda, \in, h_\phi^{\mathcal{M}}\rangle_{\phi\in\mathrm{Fmla}_\in}$ *(where each* $h_\phi^{\mathcal{M}}$ *is a Skolem function for* ϕ *in* $\langle L_\lambda, \in\rangle$ *or* $\langle V_\lambda, \in\rangle$*).*

(1) *For any set $I \subseteq \lambda$ of indiscernibles for $\mathcal{M}$, if $B = \mathfrak{H}^{\mathcal{M}}(I)$, then I is unbounded in the ordinals of B.*

(2) *If $\alpha > \omega$, there is a set $I = \{\alpha_\xi : \xi < \alpha\} \subseteq \lambda$ of indiscernibles for $\mathcal{M}$ such that for each infinite limit ordinal $\zeta < \alpha$, $\alpha_\zeta = \sup\{\alpha_\xi : \xi < \zeta\}$.*

Proof. Part (1) follows immediately from Theorem 3.3(6a). For (2), assume ζ and β are such that

$$\zeta \text{ is an infinite limit ordinal, } \beta < \alpha_\zeta, \text{ and for all } \xi < \zeta,\ \alpha_\xi < \beta. \tag{3.1}$$

Let $t(x_1, \ldots, x_k)$ be a Skolem term such that $\beta = t^{\mathcal{M}}(\delta_1, \ldots, \delta_k)$, where $\delta_1 < \ldots < \delta_k$ are in I. Suppose first that $\delta_k < \alpha_\zeta$. Since ζ is a limit, there is $\alpha_\xi \in I$ such that $\delta_k < \alpha_\xi < \alpha_\zeta$. By (3.1), $\alpha_\xi < \beta$, but by (1) we have $\beta < \alpha_\xi$; thus, not all the δ_i are below α_ζ. Thus, we can write

$$\beta = t^{\mathcal{M}}(\gamma_1, \ldots, \gamma_r, \nu_1, \ldots, \nu_s)$$

where $s > 0$ and $\gamma_1 < \ldots \gamma_r < \nu_1 < \ldots \nu_s$ are in I, and $\gamma_r < \alpha_\zeta \le \nu_1$. Using (3.1) and the fact that ζ is a limit, there is α_ξ such that $\gamma_r < \alpha_\xi < \beta < \alpha_\zeta$. Now, applying Theorem 3.3(6b), since $\alpha_\xi < \beta$, we must have $\nu_1 \le \beta$. Since $\beta < \alpha_\zeta \le \nu_1$, we have a contradiction. ∎

We can use the indiscernibles obtained from an ω-Erdös cardinal to get a transitive model of ZFC + BTEE:

Proposition 3.5. *Assume there is an ω-Erdös cardinal. Then there is a transitive set model of* ZFC + BTEE.

Proof. Let λ be an ω-Erdös cardinal. Let $\mathcal{M} = \langle V_\lambda, \in, h_\phi^{\mathcal{M}} \rangle_{\phi \in \mathrm{Fmla}_\in}$, where each $h_\phi^{\mathcal{M}}$ is a Skolem function for ϕ in $\langle V_\lambda, \in \rangle$. By Theorem 3.3, $\mathcal{M}$ has a set of indiscernibles $I \subset \lambda$ of order type ω. Now, the required model can be obtained from $\mathcal{M}, I$ in the canonical way, as in Remark 3.1. ∎

Assuming that $0^\#$ exists, we can get a stronger result than Proposition 3.5 — we show that for each δ, there is a transitive model $\langle M, \in, j \rangle$ of ZFC + BTEE with $\mathrm{cp}(j) > \delta$. To see this, let κ be an uncountable cardinal $> \delta$. Define f on the Silver indiscernibles $S_\kappa = \{\alpha_\xi : \xi < \kappa\}$ below κ by

$$f(\alpha_\xi) = \begin{cases} \alpha_\xi & \text{if } \alpha_\xi \le \delta \\ \alpha_{\xi+1} & \text{otherwise.} \end{cases}$$

Extend f to an elementary embedding $i : L_\kappa \to L_\kappa$ via Skolem terms. Now $\langle L_\kappa, \in, i \rangle$ is the required model.

Returning to the more restrictive hypotheses, we record two of the particularly nice properties of the models of ZFC + BTEE that one gets from an ω-Erdös cardinal.

Proposition 3.6. *Suppose λ is an ω-Erdös cardinal and $I \subset \lambda$ is a set of indiscernibles for $\mathcal{M}$ of ordertype ω, where $\mathcal{M}$ is either $\langle V_\lambda, \in, h_\phi^{\mathcal{M}} \rangle_{\phi \in \mathrm{Fmla}_\in}$ or $\langle L_\lambda, \in \rangle$. Suppose $\mathcal{N} = \langle N, \in, j \rangle$ is the* *canonical model of* ZFC + BTEE *derived from* $\mathcal{M}, I$, *and let* $B, \pi : B \to N, e : \mathcal{N} \to \mathcal{M}$, *and* J *be as in the canonical construction. Then*

(1) j *is an extension of an order-preserving function* $J \to J$.

(2) $J = \{\kappa, j(\kappa), j^2(\kappa), \ldots\}$ *is cofinal in* $ON^{\mathcal{N}}$ *(where* $\kappa = \mathrm{cp}(j)$*).*

Proof. (1) follows immediately from the canonical construction of $\mathcal{N}$. For (2), by Corollary 3.4, I is cofinal in $\mathrm{ON}^{\mathcal{N}}$. It is easy to verify that the map $X \mapsto \pi''X$ preserves cofinal sets; hence J is cofinal in $\mathrm{ON}^{\mathcal{N}}$. ■

We have shown that the consistency strength of the theory ZFC + BTEE is bounded below by the existence of n-ineffable cardinals and of totally indescribable cardinals; and it is bounded above by the existence of an ω-Erdös cardinal. Schindler [Sc] has shown that the consistency strength of *remarkable cardinals* has the same upper and lower bounds. We define a particularly good class of transitive models of ZFC + BTEE and show that whenever one of these models exists, there is a transitive model of a remarkable cardinal. In particular, a transitive model $\langle M, \in, j\rangle$ of ZFC+BTEE will be considered *good* in this sense if $\langle M, \in\rangle$ can be elementarily embedded into $\langle L_\kappa, \in\rangle$, where κ is inaccessible. This fact, by way of Schindler's results, links models of ZFC + BTEE to models of absoluteness of set forcing over $\mathbf{L}(R)$; we will indicate some of these connections below.

For each infinite cardinal θ, $H(\theta)$ denotes the sets hereditarily of cardinality $< \theta$. We begin with the definition of a remarkable cardinal.

3.7 Definition [Sc]. A cardinal α is *remarkable* if for each regular cardinal $\theta > \alpha$, there exist a countable transitive M and an elementary embedding $e : M \to H(\theta)$ with $\alpha \in \mathrm{ran}\,(e)$ and also a countable transitive N and an elementary embedding $\sigma : M \to N$ such that

(1) $\mathrm{cp}(\sigma) = e^{-1}(\alpha)$;

(2) $(\mathrm{ON}^M$ is a regular cardinal$)^N$;

(3) $M = H^N(\mathrm{ON}^M)$;

(4) $\sigma(e^{-1}(\alpha)) > \mathrm{ON}^M$.

We also need the following definition from [Sc]: Say that $\mathbf{L}(R)$ *is absolute under proper forcings* if for each proper forcing P, each formula $\phi(\vec{v})$, and each finite sequence $\vec{x}$ of reals in V, we have:

$$\mathbf{L}(R) \models \phi(\vec{x}) \Longleftrightarrow \Vdash_P \mathbf{L}(\dot{R}) \models \phi(\check{\vec{x}}),$$

where $\dot{R}$ is a P-name for the set of reals in the extension.

Similarly, $\mathbf{L}(R)$ is *absolute with ordinal parameters under proper forcings* if for each proper forcing P, each formula $\phi(\vec{v}, \vec{w})$, each $\vec{x} \subseteq R$, and each $\vec{\alpha} \subseteq \mathrm{ON}$,

$$\mathbf{L}(R) \models \phi(\vec{\alpha}, \vec{x}) \Longleftrightarrow \Vdash_P \mathbf{L}(\dot{R}) \models \phi(\check{\vec{\alpha}}, \check{\vec{x}}).$$

Some of Schindler's results on remarkable cardinals are the following:

Theorem 3.8 [Sc]**.**

(1) *If there is an ω-Erdös cardinal, then there is a transitive model of a remarkable cardinal. In particular, there are $\alpha < \beta < \omega_1$ such that $L_\beta \models \mathrm{ZFC} +$ "α is remarkable".*

(2) *Every remarkable cardinal is n-ineffable for every n, and is totally indescribable.*

(3) *The existence of a remarkable cardinal is equiconsistent with the statement that $\mathbf{L}(R)$ is absolute under proper forcings, and also with the statement that $\mathbf{L}(R)$ is absolute with ordinal parameters under proper forcings.* ∎

We now specify the conditions on a transitive model of ZFC + BTEE that will suffice to establish consistency of remarkable cardinals. We will call a transitive model $\mathcal{M} = \langle M, \in, j\rangle$ of ZFC + BTEE *good* if

(1) M is countable;

(2) the set $\{\kappa, j(\kappa), j^2(\kappa), \ldots\}$ is cofinal in $\mathrm{ON}^{\mathcal{M}}$;

(3) there exist λ, e such that λ is inaccessible and $e : \langle M, \in\rangle \to \langle L_\lambda, \in\rangle$ is an elementary embedding.

We observe that the canonically derived models of ZFC + BTEE one obtains from an ω-Erdös cardinal satisfy these properties:

Theorem 3.9. *Suppose λ is an ω-Erdös cardinal. Then there is a good transitive model of* ZFC + BTEE.

Proof. Let $I \subset \lambda$ be a set of indiscernibles of type ω for L_λ, and let $\mathcal{N} = \langle N, \in, j\rangle$ be the canonical transitive model of ZFC + BTEE derived from L_λ, I, with $e : N \to L_\lambda$ and $J \subset \mathrm{ON}^N$ defined as in the canonical construction. Clearly, N is countable. By Proposition 3.6(2), $J = \{\kappa, j(\kappa), j^2(\kappa), \ldots\}$ is cofinal in $\mathrm{ON}^{\mathcal{N}}$. Finally, e witnesses (3) in the definition of good since λ is inaccessible. ∎

Theorem 3.10. *Suppose there is a good transitive model of* ZFC + BTEE*. Then there is a countable transitive model of a remarkable cardinal.*

Proof. Let $\mathcal{N} = \langle L_\gamma, \in, j\rangle$ be a good transitive model of ZFC + BTEE. Let $\kappa = \mathrm{cp}(j)$ and λ, e be such that λ is inaccessible and $e : \langle L_\gamma, \in\rangle \to \langle L_\lambda, \in\rangle$ is a nontrivial elementary embedding. Let $J = \{\kappa, j(\kappa), j^2(\kappa), \ldots, \}$. By the lemma, J is a generating set of indiscernibles for L_γ.

Let $\alpha = \kappa$ and $\beta = j(\kappa)$. We now show that $L_\beta \models$ "α is remarkable". Let θ be an ordinal such that $\alpha < \theta < \beta$ and (θ is a regular cardinal)$^{L_\beta}$. We claim that the following holds in L_λ:

$$\begin{aligned}(3.2)\qquad \exists M\,\exists e_\theta\,\exists\sigma_\theta\,\exists\bar\theta\,\big[&\text{“}M\text{ is countable and transitive”} \wedge \text{“}e_\theta : M \to L_{e(\theta)}\text{ is elementary”} \wedge\\ &e(\alpha) \in \operatorname{ran}(e_\theta) \wedge \text{“}\sigma_\theta : M \to L_{\bar\theta}\text{ is elementary”} \wedge\\ &\operatorname{cp}(\sigma_\theta) = e_\theta^{-1}(e(\alpha)) \wedge \text{“}\bar\theta\text{ is countable”} \wedge \sigma_\theta(e_\theta^{-1}(e(\alpha))) > \mathrm{ON}^M \wedge\\ &\text{“}\mathrm{ON}^M\text{ is a regular cardinal in }L_{\bar\theta}\text{”} \wedge \big(M = H(\theta)\big)^{L_{\bar\theta}}\big].\end{aligned}$$

Letting $M = L_\theta$, $e_\theta = e \restriction L_\theta$, $\sigma_\theta = j \restriction L_\theta$, and $\bar\theta = j(\theta)$, it is easy to verify that the claim is true. Since (by elementarity) $L_\lambda \models$ “$e(\beta)$ is inaccessible”, the formula (3.2) also holds in $L_{e(\beta)}$. By elementarity of e, pulling back, we have:

$$\begin{aligned}L_\beta \models \exists M\,\exists e_\theta\,\exists\sigma_\theta\,\exists\bar\theta\,\big[&\text{“}M\text{ is countable and transitive”} \wedge \text{“}e_\theta : M \to L_\theta\text{ is elementary”} \wedge\\ &\alpha \in \operatorname{ran}(e_\theta) \wedge \text{“}\sigma_\theta : M \to L_{\bar\theta}\text{ is elementary”} \wedge\\ &\operatorname{cp}(\sigma_\theta) = e_\theta^{-1}(\alpha) \wedge \text{“}\bar\theta\text{ is countable”} \wedge \sigma_\theta(e_\theta^{-1}(\alpha)) > \mathrm{ON}^M \wedge\\ &\text{“}\mathrm{ON}^M\text{ is a regular cardinal in }L_{\bar\theta}\text{”} \wedge \big(M = H(\theta)\big)^{L_{\bar\theta}}\big].\end{aligned}$$

This proves the theorem. ■

Schindler's work now gives us the following:

Corollary 3.11. *If there is a good transitive model of* ZFC + BTEE, *then each of the following is consistent:*

(1) $\mathbf{L}(R)$ *is absolute under proper forcings.*

(2) $\mathbf{L}(R)$ *is absolute with ordinal parameters under proper forcings.*■

§4. Induction Axioms

Because not every model of ZFC + BTEE is an ω-model (see [Ku1,IV.10]), we cannot prove (from ZFC+BTEE) induction on the natural numbers relative to $\mathbf{j}$-formulas. We therefore introduce this property as an axiom schema, which we call *Induction*$_\mathbf{j}$, and study some of its consequences. We will adopt the convention of referring to the natural numbers in the metatheory as *particular* (metatheoretic) natural numbers, and to the natural numbers formalized within the theory at hand (usually some extension of ZFC$_\mathbf{j}$) as *formal* natural numbers.

We show that transitive models always satisfy Induction$_j$, and that the schema Σ_1-Induction$_j$ is sufficient to show that $\Phi(n, x, y)$ and $\Psi(n, y)$ (defined at the end of Section 2) are class functions. Using weak forms of Induction$_j$, we will be able to improve some of our results in Section 2 of the form "for each particular natural number $n \ldots$" to results of the form "for all formal $n \ldots$". One such result is that, by full Induction$_\mathbf{j}$, $\mathbf{j}^n$ is elementary for all *formal* $n \geq 1$.

Induction$_\mathbf{j}$: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\big[\phi(0, \vec{a}) \;\wedge\; \forall n \in \omega\, [\phi(n, \vec{a}) \Longrightarrow \phi(n + 1, \vec{a})]\big] \Longrightarrow \forall n \in \omega\, \phi(n, \vec{a}).$$

We let Σ_n-*Induction*$_\mathbf{j}$ (Π_n-*Induction*$_\mathbf{j}$) denote Induction$_\mathbf{j}$ restricted to Σ_n (Π_n) $\mathbf{j}$-formulas. (We continue to follow our convention of calling a formula Σ_n (Π_n) when it may only be $\Sigma_n^{\mathrm{ZFC}_\mathbf{j}}$ ($\Pi_n^{\mathrm{ZFC}_\mathbf{j}}$).) For each n, Σ_n-Induction$_\mathbf{j}$ follows from Σ_n-Separation for $\mathbf{j}$-formulas: if the hypothesis of the Induction$_\mathbf{j}$ axiom holds for the Σ_n $\mathbf{j}$-formula ϕ, and yet $\exists n\, \neg\phi(n, \vec{a})$, the $\mathbf{j}$-class $\{m \in \omega : \neg\phi(m, \vec{a})\}$ is a set by Σ_n-Separation (since this is equivalent to Π_n-Separation). One can then take the least element of this set to obtain a contradiction as usual. The same proof shows that Π_n-Induction$_\mathbf{j}$ follows from Σ_n-Separation for $\mathbf{j}$-formulas. Finally, Hatch [H] has observed that Σ_n-Induction$_\mathbf{j}$ implies Π_n-Induction$_\mathbf{j}$; it is unknown whether the converse is true.

Hatch [H] has shown that Induction$_j$ need not hold in models of ZFC + BTEE: Given a nonstandard model $\mathcal{M} = \langle M, E, j\rangle$ of ZFC + BTEE, he shows that the model $\mathcal{N}$ whose domain is $N = \{x \in M : \exists n \in \omega\, \mathcal{M} \models \mathrm{rank}(x) < \mathbf{j}^n(\kappa)\}$, and whose embedding is $j \restriction N$, is a model of ZFC + BTEE + $\neg\Sigma_1$-Induction$_j$. (In the sequel, we will refer to this model as *Hatch's model.*) For future reference, we mention here that the Σ_1 formula for which Σ_1-Induction$_j$ fails in Hatch's model is the formula $\Psi(n, \beta)$, defined at the end of Section 2. Hatch also shows that, assuming additional large cardinal hypotheses (weaker than the existence of $0^\#$), it is consistent with ZFC + BTEE for Σ_0-Induction$_\mathbf{j}$ to fail.

The next result shows that, by contrast, Induction$_\mathbf{j}$ always holds in well-founded models of ZFC + BTEE:

Proposition 4.1. *Any well-founded model of* ZFC$_\mathbf{j}$ *is also a model of* Induction$_\mathbf{j}$.

Proof. Since Induction$_\mathbf{j}$ is preserved by isomorphisms between $\mathcal{L}$-structures, it suffices, by Lemma 2.7, to prove the proposition for all *transitive* models of ZFC$_\mathbf{j}$. Given such a model

$\mathcal{N} = \langle N, \in, j\rangle$, suppose

$$\mathcal{N} \models \phi(0, \vec{a}) \wedge \forall x \in \omega \left(\phi(x, \vec{a}) \Longrightarrow \phi(\mathbf{s}(x), \vec{a})\right), \tag{4.1}$$

and also

$$\mathcal{N} \models \exists x \in \omega \, \neg\phi(x, \vec{a}), \tag{4.2}$$

for some formula $\phi(x, \vec{y})$. Since $\omega^{\mathcal{N}} = \omega$, we obtain from (4.2) that there is (in V) a least n for which $\mathcal{N} \models \neg\phi(n, \vec{a})$. But now this choice of n contradicts (4.1) (in the usual way). Thus $\mathcal{N} \models \mathrm{Induction}_{\mathbf{j}}$. ∎

Two familiar variations on the $\mathrm{Induction}_j$ schema are *bounded induction* and *total induction*. We formulate these and state the standard results about them without proof:

Bounded Induction$_{\mathbf{j}}$: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\forall n \in \omega \left(\left[\phi(0, \vec{a}) \wedge \forall m \left(m < n \wedge \phi(m, \vec{a}) \Longrightarrow \phi(m+1, \vec{a}) \right) \right] \Longrightarrow \forall m \leq n \, \phi(m, \vec{a}) \right).$$

We let Σ_n-*Bounded Induction*$_{\mathbf{j}}$ (Π_n-*Bounded Induction*$_{\mathbf{j}}$) denote Bounded $\mathrm{Induction}_{\mathbf{j}}$ restricted to Σ_n (Π_n) $\mathbf{j}$-formulas.

Proposition 4.2. *For each particular (methatheoretic) k, the theory* $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_k$-$\mathrm{Induction}_{\mathbf{j}}$ *proves each instance of* Σ_k-Bounded $\mathrm{Induction}_{\mathbf{j}}$. *In particular* $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{Induction}_{\mathbf{j}}$ *proves each instance of* Bounded $\mathrm{Induction}_{\mathbf{j}}$. ∎

Total Induction: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\left(\forall n \in \omega \left[\phi(0, \vec{a}) \wedge \forall m < n \, \phi(m, \vec{a}) \implies \phi(n, \vec{a}) \right] \right) \implies \forall n \in \omega \, \phi(n, \vec{a}).$$

We let Σ_n-*Total Induction*$_{\mathbf{j}}$ (Π_n-*Total Induction*$_{\mathbf{j}}$) denote Total $\mathrm{Induction}_{\mathbf{j}}$ restricted to Σ_n (Π_n) $\mathbf{j}$-formulas.

We note that Total $\mathrm{Induction}_{\mathbf{j}}$ follows from $\mathrm{Induction}_{\mathbf{j}}$, as expected. However, because of the difficulties discussed in (2.1), we are unable to prove that Σ_n-Total $\mathrm{Induction}_{\mathbf{j}}$ follows, in general, from Σ_n-$\mathrm{Induction}_{\mathbf{j}}$. This limitation significantly reduces the usefulness of this variant of $\mathrm{Induction}_{\mathbf{j}}$. However, we can prove the implication for the case $n = 0$; we will make good use of this fact in the next section.

Proposition 4.3.

(1) *The theory* $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{Induction}_{\mathbf{j}}$ *proves each instance of* Total $\mathrm{Induction}_{\mathbf{j}}$.

(2) *The theory* ZFC + BTEE + Σ_0-Induction$_{\mathbf{j}}$ *proves each instance of* Σ_0-Total Induction$_{\mathbf{j}}$.

Proof. We prove (2). Let $\phi(x, \vec{y})$ be Σ_0 **j**-formula. Let $\psi(x, \vec{y})$ be given by

$$\psi(x, \vec{y}) \equiv \forall m \leq n\, \phi(m, \vec{y}).$$

Certainly ψ is Σ_0 (and this is where generalization to $\Sigma_k, k > 0$ fails). Work in ZFC + BTEE + Σ_0-Induction$_{\mathbf{j}}$. Let $\vec{a}$ be sets. We use Σ_0-Induction$_{\mathbf{j}}$ to prove $\forall n \in \omega\, \psi(n, \vec{a})$; this will complete the proof of the theorem. We assume

$$\forall n \in \omega \left[\phi(0, \vec{a}) \wedge \forall m < n\, \phi(m, \vec{a}) \implies \phi(n, \vec{a})\right]. \tag{4.3}$$

By (4.3), $\psi(0, \vec{a})$ holds. Assuming $\psi(n, \vec{a})$, we have

$$\forall m \leq n\, \phi(m, \vec{a}). \tag{4.4}$$

Again by (4.3), $\phi(m+1, \vec{a})$ must hold; it follows that $\psi(m+1, \vec{a})$ holds as well. This completes the induction step and the proof. ∎

We observed in [Co3] that, assuming Separation for **j**-formulas, the formula $\Phi(n, x, y)$ defined in (2.7) defines a class function. The same proof works assuming only ZFC+BTEE+Σ_1-Induction$_{\mathbf{j}}$. We outline the results here.

Proposition 4.4.

(1) *It is provable in* ZFC$_{\mathbf{j}}$ *that, for all* n, x, y, *there is at most one* f *for which* $\Theta(f, n, x, y)$ *(where* Θ *is as in (2.8)). That is*

$$\mathrm{ZFC}_{\mathbf{j}} \vdash \forall n \in \omega\, \forall x\, \forall y\, \forall f, g\, [\Theta(f, n, x, y) \wedge \Theta(g, n, x, y) \implies f = g].$$

(2) *It is provable in* ZFC + BTEE + Σ_1-Induction$_{\mathbf{j}}$ *that* $\Phi(n, x, y)$ *defines a class function. That is,*

$$\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_1\text{-Induction}_{\mathbf{j}} \vdash \forall n \in \omega\, \forall x\, \exists! y\, \Phi(n, x, y). \tag{4.5}$$

(3) *It is provable in* ZFC + BTEE + Σ_1-Induction$_{\mathbf{j}}$ *that* $\Psi(n, y)$ *defines a class function (where* $\Psi(n, y)$ *is as in (2.9)). That is,*

$$\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_1\text{-Induction}_{\mathbf{j}} \vdash \forall n \in \omega\, \exists! y\, \Psi(n, y). \tag{4.6}$$

Proof. We prove (1) and (2), and leave (3) to the reader. For (1), we begin by observing that the following holds in ZFC by a simple induction:

(4.7) Suppose $n \in \omega$, f, g are functions with domain $n+1$, $f(0) = g(0)$ and $f \neq g$. Then there is a least i with $1 \leq i \leq n$ for which $f(i) \neq g(i)$.

Thus, working in $\mathrm{ZFC}_{\mathbf{j}}$, if there are n, x, y, f, g for which $\Theta(f, n, x, y) \wedge \Theta(g, n, x, y)$ and $f \neq g$, then, since $f(0) = x = g(0)$, we obtain from (4.7) a least i for which $f(i) \neq g(i)$ (in other words, the "induction step" is already given to us by ZFC). This contradicts the definition of f and g since

$$f(i) = \mathbf{j}(f(i-1)) = \mathbf{j}(g(i-1)) = g(i).$$

For (2), we first establish the uniqueness part. It suffices to show that for all n, x there is at most one pair (f, y) for which $\Theta(f, n, x, y)$. So, assume there are two such pairs, (f_1, y_1) and (f_2, y_2). The argument in part (1) can be used again to show that $f_1 = f_2$. But uniqueness of f implies uniqueness of y since, by the definition of f, $f(n) = y$.

To complete the proof, we prove σ where

$$\sigma \equiv \forall n \in \omega\, \forall x\, \exists y\, \Phi(n, x, y).$$

We show that for each a,

$$\forall n \in \omega\, \gamma(n, a),$$

where

$$\gamma(n, x) \equiv \exists y\, \Phi(n, x, y).$$

We use the Σ_1 formula $\gamma(n, x)$ for the induction. For the induction step, let z satisfy $\Phi(n, a, z)$ with witness f having domain $n+1$. Setting $\hat{f} = f \cup \{(n+1, \mathbf{j}(f(n)))\}$, it is clear that $\hat{f}$ witnesses $\Phi(n+1, a, \mathbf{j}(z))$. Thus, by Σ_1-$\mathrm{Induction}_{\mathbf{j}}$, we have $\forall n\, \gamma(n, a)$. Since a was arbitrary, the result follows. ∎

We show in Corollary 5.4 that the theory $\mathrm{ZFC} + \mathrm{BTEE} + \Pi_1$-$\mathrm{Induction}_{\mathbf{j}}$ also suffices to obtain the conclusions of Proposition 4.4(2) and (3).

4.5 Remark. Part (3) of Proposition 4.4 tells us that, in the presence of Σ_1-$\mathrm{Induction}_{\mathbf{j}}$, Ψ defines the *class sequence* $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots \rangle$. However, in the absence of Σ_1-$\mathrm{Induction}_{\mathbf{j}}$, we have no guarantee that $\mathbf{j}^N(\kappa)$ is defined for each nonstandard integer N—recall Hatch's model—though for standard integers N we do have this assurance. Of course, such pathologies could arise only in non-wellfounded models. In particular, Ψ defines a class function within any transitive model of $\mathrm{ZFC} + \mathrm{BTEE}$. Similar observations apply to the formula Φ.

Proposition 4.4 suggests the correct version of the definition-by-induction theorem for sufficiently strong extensions of $\mathrm{ZFC}_{\mathbf{j}}$:

Theorem 4.6 (*Definition By Induction, One Variable*). *Suppose* $\mathbf{F} : V \to V$ *is a* $\mathbf{j}$*-class function defined by a* Σ_n $\mathbf{j}$*-formula* $\phi(x, y)$. *Then there is a unique* $\mathbf{j}$*-class function* $\mathbf{G} : \omega \to V$, *defined by a* Σ_{n+2} $\mathbf{j}$*-formula* $\psi(n, z)$ *such that for all* $n \in \omega$, $\mathbf{G} \restriction n$ *is a set and*

$$\mathbf{G}(n) = \mathbf{F}(\mathbf{G} \restriction n). \tag{4.8}$$

Proof. Let $\gamma(g, n, y)$ denote the following Π_{n+1} formula:

$$\gamma(g, n, y) \;\equiv\; \mathrm{dom}\; g = n + 1 \;\wedge\; \forall i \in \mathrm{dom}\; g\, \phi(g \restriction i, g(i)) \;\wedge\; g(n) = y.$$

We first show that

$$\forall n \in \omega\, \exists! g\, \exists! y\, \gamma(g, n, y). \tag{4.9}$$

For each n, the uniqueness of g and y can be shown exactly as in Proposition 4.4, using (4.7); no induction axioms for **j**-formulas are required for this part of the proof. To prove that g and y exist for each n, one proves the following by Σ_{n+2}-Induction$_{\mathbf{j}}$; the proof is similar to Proposition 4.4(2):

$$\forall n \in \omega\, \exists g\, \exists y\, \gamma(g, n, y).$$

This establishes (4.9). $\mathbf{G} : \omega \to V$ can now be defined as the union of the g's given by (4.9). The formula $\psi(n, y)$ that defines $\mathbf{G}$ is clearly Σ_{n+2}:

$$\psi(n, y) \;\equiv\; n \in \omega \implies \exists g\, \exists y\, \gamma(g, n, y). \tag{4.10}$$

For uniqueness of $\mathbf{G}$, suppose $\mathbf{G}'$ satisfies (4.8), defined by a **j**-formula $\psi'(n, y)$. Translating away the classes in (4.8) gives us

$$\forall n \in \omega\, \forall y\, \big(\psi'(n, y) \iff \exists g\, \gamma(g, n, y)\big).$$

It follows immediately, by uniqueness of such g, that

$$\forall n \in \omega\, \forall y\, \big(\psi(n, y) \iff \psi'(n, y)\big).\blacksquare$$

Remarks.

(1) We have not yet stated the theory in which the theorem is to be proven; technically, as in the ZFC case, it is a theorem schema—one theorem for each $\mathbf{F}$. As in [Ku1, p.25], the theorem says that given ϕ, we can explicitly define the formula ψ so that the class-free version of (4.8) is true; to establish this, the proof required Σ_{n+2}-Induction$_{\mathbf{j}}$ if ϕ is Σ_n. Thus, whenever $\mathbf{F}$ is Σ_n, the theorem for $\mathbf{F}$ is derivable from the theory ZFC + BTEE + Σ_{n+2}-Induction$_{\mathbf{j}}$.

(2) In the proof, we claimed that γ is merely a Π_{n+1} formula; this is because, as in (2.1), we cannot ignore the bounded quantification '$\forall i \in \mathrm{dom}\; g$' in computing complexity as we can in ZFC. Thus, the complexity of $\mathbf{G}$ jumps above that of $\mathbf{F}$ by 2.

(3) When $\mathbf{F}$ happens to be defined by a Σ_0 **j**-formula, notice that the bounded quantifier in this case does not increase complexity; thus, for such $\mathbf{F}$, $\mathbf{G}$ is defined by a Σ_1 formula.

(4) The critical sequence can be shown to be a class function in this scheme by defining $\mathbf{F}(x) = y$ iff $y = \emptyset$, unless x is a finite sequence s of ordinals; in that case, if z is the last term of s, then

set $y = \mathbf{j}(z)$. Of course, now that we have Theorem 4.6, we can define the critical sequence by the familiar clauses

$$h(0) = \kappa$$

$$h(n+1) = \mathbf{j}(h(n)).$$

In order to define $\Phi(n, x, y)$ using definition-by-induction, a two-variable version of Theorem 4.6 is necessary. We state the theorem and leave the proof to the reader.

Theorem 4.7 (*Definition By Induction, Two Variables*). *Suppose $\mathbf{F} : V \times V \to V$ is a $\mathbf{j}$-class function defined by a Σ_n $\mathbf{j}$-formula $\phi(u, x, y)$. Then there is a unique $\mathbf{j}$-class function $\mathbf{G} : \omega \times V \to V$, defined by a Σ_{n+2} $\mathbf{j}$-formula $\psi(n, w, z)$ such that for all $n \in \omega$ and all x,*

$$\mathbf{G}(n, x) = \mathbf{F}(n, \langle \mathbf{G}(0, x), \mathbf{G}(1, x), \ldots, \mathbf{G}(n-1, x) \rangle). \blacksquare \tag{4.11}$$

We conclude this section by considering some improvements of results in Section 2, upgrading "for each particular n" to "for all formal n" by means of Induction$_{\mathbf{j}}$. We begin with Theorem 2.12:

Proposition 4.8. ZFC + BTEE + Induction$_{\mathbf{j}}$ $\vdash$ *"κ is n-ineffable for every $n \in \omega$".*

Proof. The proof of Theorem 2.12 can be written in terms of formal $n \in \omega$ instead of standard n by using Induction$_{\mathbf{j}}$. $\blacksquare$

The next proposition shows that $\mathbf{j}^n$ is elementary for all formal n.

Proposition 4.9.

(1) *For each particular $m \geq 1$ and each Δ_m^{ZF} $\in$-formula $\phi(x_1, \ldots, x_k)$,*

$$\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_m\text{-Induction}_{\mathbf{j}} \vdash \forall n \geq 1\, \forall a_1, \ldots, a_k \text{ "}\mathbf{j}^n \textit{ preserves } \phi(a_1, \ldots, a_k)\text{"}.$$

(2) *For each $\in$-formula $\phi(x_1, \ldots, x_k)$,*

$$\mathrm{ZFC} + \mathrm{BTEE} + \text{Induction}_{\mathbf{j}} \vdash \forall n \geq 1\, \forall a_1, \ldots, a_k \text{ "}\mathbf{j}^n \textit{ preserves } \phi(a_1, \ldots, a_k)\text{"}.$$

Remarks.

(A) In Part (1), we have required $m \geq 1$ because for such m, Σ_m-Induction$_{\mathbf{j}}$ suffices to establish that $\Phi(n, x, y)$ is a class function — a fact that is needed when we apply various $\mathbf{j}^n$ to parameters $a_1, \ldots, a_k$. We establish a weaker version of this result for the case $m = 0$ in the next section.

(B) Note that if $\mathbf{j}^n$ "preserves ϕ" for every (Σ_r) formula ϕ, then $\mathbf{j}^n$ is (Σ_r-) elementary.

Proof. Part (2) follows from (1); we prove (1): Let $\phi_\forall(x_1, \ldots, x_k)$ be Π_m and $\phi_\exists(x_1, \ldots, x_k)$ be Σ_m such that

$$\mathrm{ZF} \vdash \forall a_1, \ldots, a_k \left[\phi(a_1, \ldots, a_k) \iff \phi_\forall(a_1, \ldots, a_k) \iff \phi_\exists(a_1, \ldots, a_k)\right].$$

Fix $a_1, \ldots, a_k$. We use the following Σ_m formula for induction:

$$\begin{aligned}\gamma(n, a_1, \ldots, a_k) \equiv &\phi_\forall(a_1, \ldots, a_k) \\ &\qquad \Longrightarrow \exists z_1, \ldots, z_k \left[z_1 = \mathbf{j}^n(a_1) \wedge \ldots \wedge z_k = \mathbf{j}^n(a_k) \wedge \phi_\exists(z_1, \ldots, z_k)\right].\end{aligned}$$

The case $n = 1$ is immediate. Assume $\gamma(n, a_1, \ldots, a_k)$ and $\phi_\forall(a_1, \ldots, a_k)$. By $\gamma(n, a_1, \ldots, a_k)$, we have $\phi_\exists(\mathbf{j}^n(a_1), \ldots, \mathbf{j}^n(a_k))$. By elementarity of $\mathbf{j}$, it follows that $\phi_\exists(\mathbf{j}^{n+1}(a_1), \ldots, \mathbf{j}^{n+1}(a_k))$, as required. By Σ_m-Induction$_\mathbf{j}$, we conclude that $\forall n \in \omega\, \gamma(n, a_1, \ldots, a_k)$. Since $a_1, \ldots, a_k$ were arbitrary, and since $\phi_\forall$ and $\phi_\exists$ are equivalent, the result follows. ∎

§5. The Theory $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Induction$_{\mathbf{j}}$

We restrict our focus in this section to $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Induction$_{\mathbf{j}}$ in order to lay the foundation for our results in Section 8. We begin by addressing the following question: What can be said about the critical sequence $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots \rangle$ using only Σ_0-Induction$_{\mathbf{j}}$? As we have seen, in the absence of Σ_1-Induction$_{\mathbf{j}}$, it is possible that $\mathbf{j}^n(x)$ does not exist for some n, x. We will show that Σ_0-Induction$_{\mathbf{j}}$ allows us to conclude that $\Phi(n, x, y)$ is as "close to" being a (total) class function as we need it to be for the results we wish to prove in Section nw. Using these initial observations, we show that Π_1-Induction$_{\mathbf{j}}$ suffices to establish that $\Phi(n, x, y)$ and $\Psi(n, \beta)$ are class functions.

In Section 3, we observed that in the canonical models $\mathrm{ZFC} + \mathrm{BTEE}$ obtained from an ω-Erdös cardinal, the critical sequence is cofinal in ON. We will see that *any* model of $\mathrm{ZFC} + \mathrm{WA}_0$ also has this property. However, this property does not hold in every model of $\mathrm{ZFC} + \mathrm{BTEE}$ — consider for example $\langle \mathbf{L}, \in, j \rangle$ obtained from $0^\#$. We show next that when this property fails, Ψ is a class function assuming only Σ_0-Induction$_{\mathbf{j}}$. We first give a name to this property in the form of an axiom:

Cofinal Axiom: $\forall \alpha\, \exists n \in \omega\, \exists \beta \left(\Psi(n, \beta) \wedge \alpha \leq \beta \right)$

Theorem 5.1. *The theory* $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Induction$_{\mathbf{j}}$ + ¬Cofinal Axiom *proves that* $\Psi(n, y)$ *defines a class function; that is,*

$$\forall n \in \omega\, \exists! \beta\, \Psi(n, \beta).$$

Proof. We work in the theory $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Induction$_{\mathbf{j}}$ + ¬Cofinal Axiom. The uniqueness part follows from Theorem 4.4(1). Let α be such that

$$\forall n \in \omega\, \forall \beta\, (\Psi(n, \beta) \Longrightarrow \alpha > \beta). \tag{5.1}$$

It follows that

$$\forall n \in \omega\, \forall \beta\, \forall f \left[\Theta(f, n, \kappa, \beta) \Longrightarrow f \in \alpha^{n+1} \right]. \tag{5.2}$$

Let $X = \alpha^{<\omega}$. We show by Σ_0-Induction$_{\mathbf{j}}$ that $\forall n \in \omega\, \gamma(n, X)$, where

$$\gamma(n, X) \equiv \exists f \in X\, \exists \beta < \alpha\, (\mathrm{dom}\ f = n + 1 \longrightarrow \Theta(f, n, \kappa, \beta)).$$

The case $n = 0$ is trivial. Assuming $\gamma(n, X)$, let $f_0 \in \alpha^{n+1}$ and β_0 be witnesses. Let $\beta = \mathbf{j}(\beta_0)$ and $f = f_0 \cup \{(n + 1, \beta)\}$. Then $\Theta(f, n + 1, \kappa, \beta)$ holds, and $f \in \alpha^{n+2}$, as required. ∎

We can obtain the same result without any assumption concerning the Cofinal Axiom if we modify Ψ slightly. Recall that we say $\mathbf{j}^n(x)$ *exists* or *is defined* just in case there is some y for which $\Phi(n, x, y)$.

Let $\mathbf{F}$ denote the $\mathbf{j}$-class function defined by $\Psi(n, y)$. Define $\mathbf{G}$ by

$$\mathbf{G}(n) = \begin{cases} \mathbf{F}(n) & \text{if } \mathbf{j}^n(\kappa) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5.2. *The theory* ZFC + BTEE + Σ_0-Induction$_{\mathbf{j}}$ *proves that* $\mathbf{G}$ *is a class function.*

Proof. The uniqueness part follows from a proof like the one used in Theorem 4.4(1). The existence part follows because $\mathbf{G}$ was tailor-made to have value 0 wherever $\mathbf{F}$ is undefined. ∎

In the presence of the Cofinal Axiom, $\mathbf{G}$ is a satisfactory substitute for $\mathbf{F}$ because any set x is contained in some $\mathbf{G}(n) = \mathbf{F}(n)$.

Hatch's model shows that, without Σ_1-Induction$_{\mathbf{j}}$, $\mathbf{j}^n(x)$ may not be defined for various n, x. The next proposition describes conditions under which $\mathbf{j}^n(x)$ does exist, assuming only Σ_0-Induction$_{\mathbf{j}}$. Certainly, for all standard n, $\mathbf{j}^n(x)$ always exists. The results below show that, whenever $\mathbf{j}^n(x)$ exists and k is standard, then $\mathbf{j}^{n+k}(x)$ exists, as do all $\mathbf{j}^n(y)$ for which the rank of y is at most ρ^{+m}, where $\text{rank}(x) = \rho$, for some standard m. Some of these arguments could be simplified under the additional assumption that $\mathbf{j}(\alpha) \geq \alpha$ for all ordinals α.

Proposition 5.3. *The theory* ZFC + BTEE + Σ_0-Induction$_{\mathbf{j}}$ *proves the following: For all formal* $n \in \omega$.

(1) *Suppose* $\mathbf{j}^n(x)$ *exists. Then for all* $m \leq n$, $\mathbf{j}^m(x)$ *exists, and for each particular* $k \in \omega$, $\mathbf{j}^{n+k}(x)$ *exists.*
(2) *If* $\mathbf{j}^n(V_\alpha)$ *exists, then* $\mathbf{j}^n(x)$ *exists for all* $x \in V_\alpha$.
(3) *Suppose* $\mathbf{j}^n(x)$ *exists. Then* $\mathbf{j}^n(\text{rank}(x))$ *exists and* $\mathbf{j}^n(\text{rank}(x)) = \text{rank}(\mathbf{j}^n(x))$.
(4) *If* $\mathbf{j}^n(V_\alpha)$ *exists, then* $\mathbf{j}^n\alpha$ *exists and* $\mathbf{j}^n(V_\alpha) = V_{\mathbf{j}^n(\alpha)}$.
(5) *If* α *is an ordinal and* $\mathbf{j}^n(\alpha)$ *exists, then* $\mathbf{j}^n(V_\alpha)$ *exists.*
(6) *If* $\mathbf{j}^n(V_\alpha)$ *exists,* $\mathbf{j}^n(V_{2^\alpha})$ *exists.*
(7) *If* $\mathbf{j}^n(x)$ *exists, there is an ordinal* α *such that* $x \in V_\alpha$ *and* $\mathbf{j}^n(V_\alpha)$ *exists.*
(8) *Assume the* Cofinal Axiom. *Then for each set* x, *there is* $n \in \omega$ *such that both* $\mathbf{j}^n(V_\kappa)$ *and* $\mathbf{j}^n(V_{\mathbf{j}(\kappa)})$ *exist, and* $x \in \mathbf{j}^n(V_\kappa)$. *Moreover, for such* n, $\mathbf{j}^n(V_\omega)$ *exists.*

Proof of (1). If f, y are such that $\Theta(f, n, x, y)$, then, for each $m < n$, $\Theta(f \upharpoonright m+1, m, x, f(m))$ is true (and so $\mathbf{j}^m(x)$ exists). For the second part, if $\mathbf{j}^n(x)$ exists and k is a particular natural number, clearly $\mathbf{j}^k(\mathbf{j}^n(x))$ must also exist (since $\mathbf{j}^k$ is defined everywhere). To see that $\mathbf{j}^k(\mathbf{j}^n(x)) = \mathbf{j}^{n+k}(x)$, we use Σ_0-Induction$_{\mathbf{j}}$. Let g, y and h, z be such that $\Theta(g, n, x, y)$ and $\Theta(h, k, y, z)$. Define h' on $[n, n+k)$ by $h'(m) = h(m-n)$. Define u on $n+k$ by $u = g \cup h'$. By Σ_0-Induction$_{\mathbf{j}}$ on $i \leq k$, it is easy to see that $\Theta(u, n+k, x, z)$ must be true (one shows that for $0 < i \leq k$, $\Theta(u, n+i, x, h'(i))$ must be true). This completes the proof. ∎

Proof of (2). Let $x \in V_\alpha$ and assume $\mathbf{j}^n(V_\alpha)$ exists.

Claim. Suppose $m \leq n$ is such that $\mathbf{j}^m(x)$ exists. Then $\mathbf{j}^m(x) \in \mathbf{j}^m(V_\alpha)$.

Proof of Claim. Let f, y be such that $\Theta(f, m, x, y)$, and let g, z be such that $\Theta(g, m, V_\alpha, z)$. We use Σ_0-Bounded Induction$_{\mathbf{j}}$ to prove

$$\forall k \leq m\, f(k) \in g(k).$$

The case $k = 0$ is just the assertion $x \in V_\alpha$, and is therefore true. Assume the formula holds at $k < m$; then $f(k) \in g(k)$. Using the definitions of f and g and the induction hypothesis, we have $f(k+1) = \mathbf{j}(f(k)) \in \mathbf{j}(g(k)) = g(k+1)$, as required. This completes the proof of the claim. ■

Continuing the proof of (2), let δ be a limit ordinal such that $\mathbf{j}^n(V_\alpha) \in V_\delta$, and let $Y = V_\delta$. We use Σ_0-Bounded Induction$_{\mathbf{j}}$ to prove

$$\forall m \leq n\, \exists f, w \in Y\, \Theta(f, m, x, w). \tag{5.3}$$

By the Claim, proving (5.3) will complete the proof of (2), since the claim implies that any f, w that witness the existence of $\mathbf{j}^n(x)$ must lie in V_δ. The induction proof is straightforward; the Claim is used in the induction step to ensure that for any function $f \in Y$ witnessing the existence of $\mathbf{j}^m(x)$, the extension of f obtained by adding to it the ordered pair $(m+1, \mathbf{j}(f(m)))$ still lies in Y. ■

Proof of (3). Let f, y be such that $\Theta(f, n, x, y)$. Define g on $n+1$ by $g(m) = \text{rank}(f(m))$. To complete the proof of (3), it suffices to prove the following claim:

Claim. For all $m \leq n$, $\mathbf{j}^m(\text{rank}(x))$ exists, and $g(m) = \mathbf{j}^m(\text{rank}(x))$.

Proof of Claim. Let $\alpha = \text{rank}(x)$. Let $\delta > \text{rank}(\mathbf{j}^n(x))$ be a limit ordinal, and let $Y = V_\delta$. To establish the claim, we prove by Σ_0-Bounded Induction$_{\mathbf{j}}$ that

$$\forall m \leq n\, \exists h, \beta \in Y\, \big(\Theta(h, m, \alpha, \beta) \wedge g(m) = h(m)\big).$$

The case $m = 0$ asserts that $\text{rank}(x) = \alpha$, which is true. Let $m < n$ and let $h_0, \beta_0 \in Y$ be such that $\Theta(h_0, m, \alpha, \beta_0)$ and $g(m) = h_0(m)$. Let $\beta = \mathbf{j}(\beta_0)$. Then

$$\beta = \mathbf{j}(h_0(m)) = \mathbf{j}(g_0(m)) = \mathbf{j}(\text{rank}(f(m)) = \text{rank}(\mathbf{j}(f(m))) = \text{rank}(f(m+1)) = g(m+1).$$

This shows that $\beta \in Y$; it follows that the function $h = h_0 \cup \{(m+1), \beta)\}$ has the required properties. This completes the induction, the proof of the claim, and the proof of (3). ■

Proof of (4). By (3), since $\mathbf{j}^n(V_\alpha)$ exists, so does $\mathbf{j}^n(\alpha)$. Let f, y, g, z be such that $\Theta(f, n, V_\alpha, y)$ and $\Theta(g, n, \alpha, z)$. Define h on $n+1$ by

$$h(m) = V_{g(m)}.$$

We use Σ_0-Bounded Induction$_{\mathbf{j}}$ to show

$$\forall m \leq n\, f(m) = h(m).$$

The case $m = 0$ is immediate. Assuming $f(m) = h(m)$ for $m < n$, we have

$$f(m+1) = \mathbf{j}(f(m)) = \mathbf{j}(h(m)) = \mathbf{j}(V_{g(m)}) = V_{\mathbf{j}(g(m))} = V_{g(m+1)} = h(m+1), \tag{5.4}$$

as required. ∎

Proof of (5). Let g, β be such that $\Theta(g, n, \alpha, \beta)$. Define h on $n + 1$ by

$$h(m) = V_{g(m)}.$$

Let δ be a limit ordinal such that $h \in V_\delta$. Let $X = V_\alpha$ and $Y = V_\delta$. To complete the proof of (5), we use Σ_0-Bounded Induction$_{\mathbf{j}}$ to show

$$\forall m \leq n\, \exists f, y \in Y\, \big(\Theta(f, m, X, y) \,\wedge\, f(m) = h(m)\big). \tag{5.5}$$

The case $m = 0$ is trivial. Assume (5.5) holds for $m < n$, with witnesses $f_0, y_0 \in Y$. Let $y = \mathbf{j}(y_0)$. Certainly $\mathbf{j}(y_0) = \mathbf{j}(h(m))$, and the steps in (5.4) can be used here to verify that $\mathbf{j}(h(m)) = h(m+1)$. By the choice of δ, $y \in Y$, letting $f = f_0 \cup \{(m+1, y)\}$, it is clear that $f \in Y$ and $f(m+1) = h(m+1)$. This completes the induction and the proof of (5). ∎

Proof of (6). By (3), $\mathbf{j}^n(\alpha)$ exists. Using an argument like the one for (5) above, one proves that

$$\mathbf{j}^n(2^\alpha) \text{ exists and } \mathbf{j}^n(2^\alpha) = 2^{\mathbf{j}^n(\alpha)}.$$

By (5), the result follows. ∎

Proof of (7). Let $\beta = \text{rank}(x)$. By (3), $\mathbf{j}^n(\beta)$ exists. By (5), $\mathbf{j}^n(V_\beta)$ exists. By (6), $\mathbf{j}^n(V_{2^\beta})$ exists. Now $\alpha = 2^\beta$ satisfies the conclusion of part (7).∎

Proof of (8). Given x, let $\alpha = \text{rank}(x)$. By the Cofinal Axiom, there is $n \in \omega$ such that $\mathbf{j}^n(\kappa)$ exists and exceeds α. It follows from (5) that $\mathbf{j}^n(V_\kappa)$ exists. By (1), $\mathbf{j}^{n+1}(V_\kappa)$ also exists. Using the fact that $\mathbf{j}(V_\kappa) = V_{\mathbf{j}(\kappa)}$ and Lemma 2.18, it follows that $\mathbf{j}^{n+1}(V_\kappa) = \mathbf{j}^n(V_{\mathbf{j}(\kappa)})$ (since $\Phi(n, \mathbf{j}(V_\kappa), y)$ is equivalent to $\Phi(n+1, V_\kappa, y)$). This completes the proof of the main clause. The final clause now follows because of (2). ∎

We consider several corollaries to the theorem. The first is a slight modification of Hatch's proof that Σ_n-Induction$_{\mathbf{j}}$ implies Π_n-Induction$_{\mathbf{j}}$.

Corollary 5.4. *The theory* ZFC + BTEE + Π_1-Induction$_{\mathbf{j}}$ *proves that* $\Phi(n, x, y)$ *and* $\Psi(n, \beta)$ *are class functions.*

Proof. It suffices to prove the result for $\Phi(n, x, y)$. Since uniqueness follows from ZFC$_{\mathbf{j}}$, we need only prove that $\mathbf{j}^n(x)$ is defined for every set x and $n \in \omega$. We show that if this fails for some x, N

and Σ_0-Induction$_{\mathbf{j}}$ holds, then an instance of Π_1-Induction$_{\mathbf{j}}$ must fail. (We include Σ_0-Induction$_{\mathbf{j}}$ in the hypothesis so that we can use Proposition 5.3.)

Suppose x, N are such that $\mathbf{j}^N(x)$ does not exist. By Proposition 5.3(1), the class $\mathbf{C} = \{n : \mathbf{j}^n(x) \text{ exists}\}$ forms an initial segment of ω. Consider the following Π_1 formula:

$$\gamma(n, x, N) \equiv n \leq N \implies \neg\exists y\, \Phi(N - n, x, y).$$

The formula $\gamma(n, x, N)$ asserts that $\mathbf{j}^{N-n}(x)$ does *not* exist. We claim that Π_1-Induction$_{\mathbf{j}}$ fails for γ. Toward a contradiction, assume Π_1-Induction$_{\mathbf{j}}$ holds for γ. Certainly $\gamma(0, x, N)$ holds. Also, by Proposition 5.3(1) again, $\gamma(n, x, N)$ implies $\gamma(n + 1, x, N)$. By Π_1-Induction$_{\mathbf{j}}$, $\gamma(n, x, N)$ holds for all $n \in \omega$. Therefore, $\gamma(N, x, N)$ holds; but this says that $\mathbf{j}^0(x)$ does not exist, which is impossible. Thus, Π_1-Induction$_{\mathbf{j}}$ fails, and the result follows. ∎

Corollary 5.5. *Suppose $\phi(x_1, \ldots, x_k)$ is a Σ_0 $\in$-formula. Then*

$$\begin{aligned} \mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0\text{-Induction}_{\mathbf{j}} \vdash &\forall n \geq 1 \forall a_1, \ldots, a_k \big(\text{“}\mathbf{j}^n(a_1), \ldots, \mathbf{j}^n(a_k) \textit{ exist”} \implies \\ &\text{“}\mathbf{j}^n \textit{ preserves } \phi(a_1, \ldots, a_k)\text{”}\big). \end{aligned}$$

Proof. Given $a_1, \ldots, a_k$, there is some a_i of largest rank; use Proposition 5.3(7) to obtain a V_δ such that $a_i \in V_\delta$ and $\mathbf{j}^n(V_\delta)$ exists. The rest of the proof is the same as that for Proposition 4.9(1), except that we use the following formula for the Σ_0 induction:

$$\begin{aligned} \gamma(n, a_1, \ldots, a_k) \equiv &\phi(a_1, \ldots, a_k) \\ &\implies \exists z_1, \ldots, z_k \in X \left[z_1 = \mathbf{j}^n(a_1) \wedge \ldots \wedge z_k = \mathbf{j}^n(a_k) \wedge \phi(z_1, \ldots, z_k)\right], \end{aligned}$$

where $X = \mathbf{j}^n(V_\delta)$. (Note that the formulas $z_i = \mathbf{j}^n(a_i)$ can be expressed so that all quantifiers are bound by X; thus, γ is actually Σ_0.) ∎

Corollary 5.6. *Assume* ZFC + BTEE + Σ_0-Induction$_j$ + Cofinal Axiom. *Then the inaccessibles are unbounded in* ON.

Proof. Let α be an ordinal. By the Cofinal Axiom, for some $n \in \omega$, $\mathbf{j}^n(\kappa)$ exists and is greater than α. By Proposition 5.3(5), $\mathbf{j}^n(V_\kappa)$ exists, and by Proposition 5.3(6), $\mathbf{j}^n(V_{\kappa+\omega})$ exists. Let $Y = V_{\kappa+\omega}$. There is a formula that is Σ_0 in the parameters κ, Y which asserts that κ is inaccessible. By Corollary 5.6, $\mathbf{j}^n$ preserves this formula. Since $\mathbf{j}^n(Y) = V_{\mathbf{j}^n(\kappa)+\omega}$, (by Proposition 5.3(4)), it follows by absoluteness that $\mathbf{j}^n(\kappa)$ is inaccessible. Since α was arbitrary, the result follows. ∎

§6. The Least Ordinal Principle

In this section, we extend the induction axioms of the previous section into the transfinite by introducing the *Least Ordinal Principle*$_{\mathbf{j}}$. This axiom implies Induction$_{\mathbf{j}}$ and follows from Separation$_{\mathbf{j}}$. We will use the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to prove several lemmas that will be used in Section 7 where we study the theory $\mathrm{ZFC} + \mathrm{WA}_0$.

We begin with the definition of the Least Ordinal Principle$_{\mathbf{j}}$:

Least Ordinal Principle$_{\mathbf{j}}$: For any $\mathbf{j}$-formula $\phi(x, \vec{y})$ and sets $\vec{a}$,

$$\exists\alpha\left[\text{“}\alpha\text{ is an ordinal”} \wedge \phi(\alpha, \vec{a})\right] \Longrightarrow$$
$$\exists\alpha\left[\text{“}\alpha\text{ is an ordinal”} \wedge \phi(\alpha, \vec{a}) \wedge \forall\beta \in \alpha\left(\neg\phi(\beta, \vec{a})\right)\right].$$

The axiom says that, whenever there is an ordinal that satisfies the $\mathbf{j}$-formula ϕ, there is a least such ordinal. The Σ_n-*Least Ordinal Principle*$_{\mathbf{j}}$ (Π_n-*Least Ordinal Principle*$_{\mathbf{j}}$) is the Least Ordinal Principle$_{\mathbf{j}}$ restricted to Σ_n (Π_n) $\mathbf{j}$-formulas. (We continue to follow our convention of calling a formula Σ_n (Π_n) when it may only be $\Sigma_n^{\mathrm{ZFC}_{\mathbf{j}}}$ ($\Pi_n^{\mathrm{ZFC}_{\mathbf{j}}}$).)

We have the following easy proposition:

Proposition 6.1.

(1) *The* Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ *implies* Σ_0-Induction$_{\mathbf{j}}$.
(2) *For all* $n \in \omega$, *the* Σ_n-Least Ordinal Principle$_{\mathbf{j}}$ *implies* Π_n-Induction$_{\mathbf{j}}$.
(3) *The* Least Ordinal Principle$_{\mathbf{j}}$ *implies* Induction$_{\mathbf{j}}$.
(4) *For all* n, Σ_n-Separation$_{\mathbf{j}}$ *implies the* Σ_n-Least Ordinal Principle$_{\mathbf{j}}$.

Proof. Parts (1) and (3) follow from (2). To prove part (2), one argues indirectly in the usual way, using the Least Ordinal Principle$_{\mathbf{j}}$ to obtain the least natural number for which the induction assumptions hold but the given formula fails. For (4), given a Σ_n formula $\phi(x, \vec{y})$ and assuming $\exists\beta\, \phi(\beta, \vec{a})$ for some $\vec{a}$, use Σ_n-Separation$_{\mathbf{j}}$ to form the set $\{\gamma < \beta : \phi(\gamma, \vec{a})\}$. Now we can use ZFC to obtain the least member of this set, as required. ∎

We also observe that if, in the definition of the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$, we restrict ordinals to the finite ordinals, then this restricted version of the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ is equivalent to Σ_0-Induction$_{\mathbf{j}}$.

As with Induction$_{\mathbf{j}}$, the Least Ordinal Principle$_{\mathbf{j}}$ always holds in well-founded models:

Proposition 6.2. *Any well-founded model of* $\mathrm{ZFC}_{\mathbf{j}}$ *is also a model of the* Least Ordinal Principle$_{\mathbf{j}}$.

Proof. The proof is like that of Proposition 4.1. As in that proof, it suffices to prove the result for transitive models $\mathcal{M} = \langle M, \in, j\rangle$. Given a $\mathbf{j}$-formula $\phi(x, \vec{y})$ such that $\mathcal{M} \models \exists\alpha\left[\text{“}\alpha\text{ is an ordinal”} \wedge \phi(\alpha, \vec{a})\right]$, for some $a_1, \ldots, a_k \in M$, simply obtain the least ordinal in V for which ϕ^M holds. By transitivity of M, the result follows. ∎

We consider some convenient consequences of the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$.

Lemma 6.3. $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0$-Least Ordinal Principle$_{\mathbf{j}}$ $\vdash \forall\alpha\, \mathbf{j}(\alpha) \geq \alpha$.

Proof. Assume that for some α we have $\mathbf{j}(\alpha) < \alpha$. By the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$, we can find a least such α. But now by elementarity of $\mathbf{j}$ and the fact that $\mathbf{j}(\alpha) < \alpha$, we have $\mathbf{j}(\mathbf{j}(\alpha)) < \mathbf{j}(\alpha)$, which is a contradiction. ∎

A corollary to Lemma 6.3 and Proposition 6.2 is that if λ and $\mathcal{M} = \langle M, \in, j\rangle$ are such that $\mathcal{M}$ is a transitive model of $\mathrm{ZFC} + \mathrm{BTEE}$, λ is the supremum of Ψ in $\mathcal{M}$, and $j(\lambda) \neq \lambda$, then $j(\lambda) > \lambda$.

To establish the conclusion of Lemma 6.3, Σ_0-Induction$_{\mathbf{j}}$ does not suffice: Though one can prove from Σ_0-Induction$_{\mathbf{j}}$ that, if $\mathbf{j}(\alpha) < \alpha$, there is a $\mathbf{j}$-class $\{\alpha, \mathbf{j}(\alpha), \mathbf{j}^2(\alpha), \ldots\}$ such that $\alpha > \mathbf{j}(\alpha) > \mathbf{j}^2(\alpha)\ldots$, without an additional instance of Separation for $\mathbf{j}$-formulas, one cannot prove that this class is a set to get the expected contradiction. The next proposition shows that, relative to the existence of an ω-Erdös cardinal, "$\mathbf{j}(\alpha) < \alpha$" is consistent with Σ_0-Induction$_{\mathbf{j}}$. This result is a slight improvement of an observation made by the referee, who outlined a proof of the result to the author assuming the existence of $0^{\#}$.

Proposition 6.4. Con(ZFC + *"there is an ω-Erdös cardinal"*) *implies* Con(ZFC + BTEE+ Σ_0-Induction$_{\mathbf{j}}$ $+ \exists\alpha\, \mathbf{j}(\alpha) < \alpha$).

Proof. Let $\mathcal{M} = \langle M, E\rangle$ be a model of ZFC + "λ is an ω-Erdös cardinal" having a nonstandard integer q. In $\mathcal{M}$, there is a set $I \subset L_\lambda$ of indiscernibles for L_λ, having ordertype ω. Still in $\mathcal{M}$, define the Skolem hull $B = \mathfrak{H}^{L_\lambda}(I)$. Let $I_E = \{x \in M \mid \mathcal{M} \models x\, E\, I\}$. Since the integers in $\mathcal{M}$ are nonstandard, their ordertype in V is, as usual, that of $N + Z \cdot A$, where $(A, <)$ is some unbounded dense linearly ordered set of integers (see [Ke, Chapter 6]); therefore this is the ordertype of I_E. Write

$$I_E = \{s_n : n \in \omega^V\} \cup \{s_\xi : \xi \in Z \cdot A\}.$$

Define $i_E : I_E \to I_E$ so that it satisfies:

$$i_E(s_n) = s_{n+1};$$
$$i_E(s_q) < s_q; \text{ and}$$
$$i_E \text{ is order-preserving}$$

Now we may define $i : I \to I$ in $\mathcal{M}$ by

$$\big(\mathcal{M} \models i(x) = y\big) \quad \text{iff} \quad i_E(x) = y$$

Now, in $\mathcal{M}$, i lifts to an elementary embedding $i : B \to B$ in the usual way. Because, in $\mathcal{M}$, i moves some of the indiscernibles (one of them downward), and because, according to $\mathcal{M}$, $\langle B, E\rangle$ is well-founded, we have

$$\big(\langle B, E, i\rangle \models \mathrm{ZFC} + \mathrm{BTEE} + \exists\alpha\, \mathbf{j}(\alpha) < \alpha\big)^{\mathcal{M}}.$$

from which it follows that

$$\mathcal{B} = \langle B_E, \in, i_E\rangle \models \mathrm{ZFC} + \mathrm{BTEE} + \exists\alpha\,\mathbf{j}(\alpha) < \alpha.$$

We observe next that $\mathcal{B}$ satisfies Σ_0-Induction$_{\mathbf{j}}$ as well: Suppose $\phi(x,\vec{y})$ is a Σ_0 $\mathbf{j}$-formula, $\vec{a}$ are sets, and we have

$$\mathcal{B} \models \psi(0,\omega,\vec{a}),$$

where

$$\psi(0,\omega,\vec{a}) \equiv \phi(0,\vec{a}) \wedge \forall n \in \omega\,[\phi(n,\vec{a}) \Longrightarrow \phi(n+1,\vec{a})].$$

In $\mathcal{M}$, the model $\langle B, E, i\rangle$ also models $\psi(0,\omega,\vec{a})$. Because ψ is Σ_0 and because i is a set in M, $\psi(0,\omega,\vec{a})$ holds in $\mathcal{M}$ as well, and so by ordinary induction in $\mathcal{M}$, $\mathcal{M} \models \forall n \in \omega\,\phi(n,\vec{a})$. By absoluteness again, $\langle B, E, i\rangle$ also satisfies $\phi(n,\vec{a})$. It therefore follows that

$$\mathcal{B} \models \phi(n,\vec{a}),$$

as required. ■

A corollary to Lemma 6.3 is that, in the presence of Σ_0-Least Ordinal Principle$_{\mathbf{j}}$, we may restrict the schema of Elementarity to its Σ_1 instances (this was pointed out to the author by Joel Hamkins), as we now show. Let Σ_1-Elementarity denote the schema of Σ_1 instances of Elementarity:

Corollary 6.5. *For each $\in$-formula $\phi(x_1,\dots,x_k)$,*

$$\mathrm{ZFC} + \Sigma_1\text{-Elementarity} + \text{Critical Point} + \Sigma_0\text{-Least Ordinal Principle}_{\mathbf{j}} \vdash$$
$$\forall a_1,\dots,a_k\,\big(\phi(a_1,\dots,a_k) \Longleftrightarrow \phi(\mathbf{j}(a_1,\dots,a_k))\big)$$

.

Proof. Kanamori gives an easy induction argument in [Ka] showing that if a Σ_1 elementary embedding $j : V \to M$ satisfies the property

(6.1) for each set $x \in M$ there is a set $y \in V$ such that $x \subseteq j(y)$,

then j is fully elementary. For this proof only, we call embeddings satisfying (6.1) *cofinal* embeddings. In the present context, suppose $\mathbf{j}$ satisfies the hypotheses of the corollary; to prove the result, it suffices to show that $\mathbf{j}$ is a cofinal embedding. For each ordinal α, we have

a. $\mathbf{j}(V_\alpha) = V_{\mathbf{j}(\alpha)}$, by Σ_1 elementarity (recall that the V_α have a Π_1 definition), and
b. $V_\alpha \subseteq V_{\mathbf{j}(\alpha)}$ by Lemma 6.3.

Thus, for any x, we can obtain y such that $x \subseteq \mathbf{j}(y)$ by letting $y = V_\alpha$ where α is greater than $\mathrm{rank}(x)$. ■

The next lemma says that whenever f is a witness for $\Phi(n,x,y)$ (as defined at the end of Section 2) and x is an ordinal, then f is a nondecreasing sequence of ordinals.

Proposition 6.6. ZFC + BTEE + Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ $\vdash \forall f, n, x, y\, \big[\Theta(f, n, x, y) \wedge$ *"x is an ordinal"* $\Longrightarrow$ *"f is a nondecreasing sequence of ordinals"*$\big]$.

Proof. Using Lemma 6.3, this is an immediate corollary to Proposition 2.17. ■

We close this section with an application to $\mathbf{j}$-inherited models (see the definition given in Section 2).

Proposition 6.7. *Assume the universe $\langle V, \in, j\rangle$ satisfies* ZFC + Elementarity + Nontriviality + Σ_0-Least Ordinal Principle$_{\mathbf{j}}$. *Then there is no $\mathbf{j}$-inherited countable transitive model of* ZFC + Elementarity + Nontriviality.

Proof. Assume there is such a model $\langle M, \in, i\rangle$, where $i = j \restriction M : M \to M$. There is, therefore, some $x \in M$ that is moved by i and hence by j. Apply the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ on the formula γ defined by

$$\gamma(\alpha) \equiv \exists x \in M\, \big(\mathbf{j}(x) \neq x \wedge \alpha = \mathrm{rank}(x)\big). \tag{6.2}$$

to obtain a least α for which $\gamma(\alpha)$ holds. Let x be such that $\mathrm{rank}(x) = \alpha$. Since α is countable, $i(\alpha) = \mathbf{j}(\alpha) = \alpha$. Thus, by elementarity, $\mathrm{rank}(x) = \mathrm{rank}(i(x)) = \mathrm{rank}(\mathbf{j}(x))$. By Proposition 2.5, we have a contradiction.■

The Least Ordinal Principle$_{\mathbf{j}}$ allows us to carry out arguments by transfinite induction. However, in extensions of ZFC + BTEE + Cofinal Axiom, we cannot prove the corresponding definition-by-transfinite-recursion theorem—if we could, we would be able to define a class sequence like this:

$$\begin{aligned} x_0 &= \kappa \\ x_{\alpha+1} &= \mathbf{j}(x_\alpha) \\ x_\lambda &= \sup\{x_\alpha : \alpha < \lambda\} \quad (\lambda \text{ a limit}). \end{aligned}$$

Of course, the Cofinal Axiom prevents such a sequence from being well-defined. The problem is that the proof of the definition-by-recursion theorem makes essential use of Replacement at limit stages; however, as we shall prove in Sections 9 and 10, very little of Replacement for $\mathbf{j}$-formulas is consistent with the theory ZFC + BTEE + Cofinal Axiom. Thus, we must abide by the following guideline:

(6.3) *The method of definition by transfinite recursion is not allowed.*

Exceptions to this rule have to be established on a case-by-case basis.

§7. The Cofinal Axiom And Inconsistency

In this section, we will isolate axioms that will lead to the Kunen inconsistency. As we remarked in the Introduction, the two most familiar embeddings of a model of set theory to itself are given by an I_3 embedding $j : V_\lambda \to V_\lambda$ and an embedding $j : \mathbf{L} \to \mathbf{L}$. In the first case, the critical sequence is cofinal in the ordinals; in the second case, the critical sequence is bounded. These examples suggest a dichotomy, marked by the notion of the cofinality of the critical sequence. An $\omega+\omega$-Erdös cardinal suffices to build a transitive model of Cofinal Axiom as well as of $\neg$Cofinal Axiom. Therefore, in this section, we seek a minimal set of axioms necessary to produce an inconsistent extension of $\mathrm{ZFC}+\mathrm{BTEE}+$ Cofinal Axiom, and another minimal set of axioms that will yield an inconsistent extension of $\mathrm{ZFC}+\mathrm{BTEE}+\neg$Cofinal Axiom.

Two general themes that will start to become apparent in this section are:

(1) Inconsistency of a set of axioms about $\mathbf{j}$ (even "natural" axioms) is not always due to the fact that the large cardinal strength has become "too big".

(2) If we wish to consider statements of the form "there is an elementary embedding $M \to M$ having certain properties" as a hierarchy of statements having ever greater consistency strengths, like large cardinals, then the direction toward greater consistency strength lies in adding instances of $\mathrm{Separation}_{\mathbf{j}}$ but not instances of Replacement for $\mathbf{j}$-formulas.

We begin with the observation that, under mild hypotheses, models of Cofinal Axiom and of $\neg$Cofinal Axiom can be constructed:

Proposition 7.1. *Assume there is an $\omega+\omega$-Erdös cardinal. Then there are transitive models of both* $\mathrm{ZFC}+\mathrm{BTEE}+\mathrm{Induction}_j+$ Cofinal Axiom *and* $\mathrm{ZFC}+\mathrm{BTEE}+\mathrm{Induction}_j+\neg$Cofinal Axiom

Proof. For Cofinal Axiom, we simply observe that our standard construction of a transitive model of $\mathrm{ZFC}+\mathrm{BTEE}$ also satisfies Cofinal Axiom, and by transitivity, $\mathrm{Induction}_{\mathbf{j}}$ holds as well. (This construction required only an ω-Erdös cardinal.) To obtain a model of $\neg$Cofinal Axiom, recall that from an $\omega+\omega$-Erdös cardinal λ, we can obtain a set $I \subseteq \lambda$ of indiscernibles of type $\omega+\omega$ with the property that the ωth indiscernible is the supremum of the previous indiscernibles. We can take the transitive collapse of the Skolem hull of I in L_λ; the resulting model must be some L_α generated by a set $J \subseteq \alpha$ of indiscernibles isomorphic to I. Enumerate J by $J=\{\beta_\xi : \xi<\omega+\omega\}$. Define $f : J \to J$ so that $f(\beta_n)=\beta_{n+1}$ for each $n \in \omega$. Extend f to an elementary embedding $j : L_\alpha \to L_\alpha$ in the usual way. Clearly, $\langle L_\alpha, \in, j\rangle \models \mathrm{ZFC}+\mathrm{BTEE}+\mathrm{Induction}_j$ (see arguments of this kind in Section 3). Since the critical sequence of j is $\langle \beta_n : n \in \omega\rangle$ and is bounded in L_α, $\neg$Cofinal Axiom must also hold in the model, as required. ∎

A weak instance of Replacement for $\mathbf{j}$-formulas suffices to push the theory $\mathrm{ZFC}+\mathrm{BTEE}+$ Cofinal Axiom to inconsistency. We start by defining the axiom schema $\mathrm{Replacement}_j$ as follows:

Replacement$_j$: For each $\mathbf{j}$-formula $\psi(x, y, \vec{u})$,

$$\forall A\, \forall \vec{a}\, \big(\forall x \in A\, \exists ! y\, \psi(x, y, \vec{a}) \Longrightarrow \exists Y\, \forall z\, [z \in Y \Longleftrightarrow (\exists x \in A\, \psi(x, z, \vec{a}))]\big). \tag{7.1}$$

We let Σ_n-Replacement$_j$ (Π_n-Replacement$_j$) denote the restriction of the Replacement$_j$ schema to Σ_n (Π_n) $\mathbf{j}$-formulas ψ. We define the *Critical Instance* (CI) of Replacement$_j$ that leads to inconsistency as follows:

Critical Instance (CI):

$$\forall n \in \omega\, \exists ! y\, \Psi(n, y) \Longrightarrow \exists Y\, \forall z\, [z \in Y \Longleftrightarrow (\exists n \in \omega\, \Psi(n, z))],$$

where Ψ is defined as in (2.9).

Clearly, CI is a Σ_1 instance of Replacement$_{\mathbf{j}}$. In order for CI to be potent at all, each $\mathbf{j}^n(\kappa)$ must exist; otherwise CI is vacuously true. Existence of each $\mathbf{j}^n(\kappa)$ can be established with either Σ_1-Induction$_{\mathbf{j}}$ (Proposition 4.4) or Π_1-Induction$_{\mathbf{j}}$ (Corollary 5.4). Therefore, we have:

Proposition 7.2. The following theories are inconsistent:

(1) $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_1$-Induction$_{\mathbf{j}}$ + Cofinal Axiom + CI

(2) $\mathrm{ZFC} + \mathrm{BTEE} + \Pi_1$-Induction$_{\mathbf{j}}$ + Cofinal Axiom + CI.

Proof. We prove (1) and (2) simultaneously. Use either Σ_1-Induction$_{\mathbf{j}}$ or Π_1-Induction$_{\mathbf{j}}$ to establish that Ψ is a class function. Therefore, by CI, the critical sequence is a set, and therefore has a supremum, and this contradicts the Cofinal Axiom.■

Corollary 7.3. There is no transitive model of ZFC + BTEE + Cofinal Axiom + CI.■

Proposition 7.2 is, in an obvious sense, trivial: of course the critical sequence cannot be simultaneously bounded (because of CI) and unbounded (by Cofinal Axiom) in the ordinals. The significance of the proposition, though, is that the axioms CI and Cofinal Axiom arise in different ways — CI from Replacement$_{\mathbf{j}}$, Cofinal Axiom from Separation$_{\mathbf{j}}$. Recall from [Co3] that we denote BTEE + Separation$_{\mathbf{j}}$ by WA; as we showed there, ZFC + WA is not inconsistent (since $\langle V_\lambda, \in, j\rangle$ is a model whenever $j : V_\lambda \to V_\lambda$ is an I_3 embedding); however, adding this single instance of Replacement$_{\mathbf{j}}$ — CI — does render the theory inconsistent. The proposition shows that the inconsistency that we find in ZFC + WA + CI is already present in ZFC + BTEE + CI together with two consequences of Separation$_{\mathbf{j}}$: Σ_1-Induction$_{\mathbf{j}}$ and Cofinal Axiom. Indeed, the set $\{\Sigma_1\text{-Induction}_{\mathbf{j}}, \mathrm{CI}\}$ is in a sense a *minimal* set of axioms that can be added to ZFC + BTEE + Cofinal Axiom to obtain inconsistency because each of the theories ZFC + BTEE + Cofinal Axiom + $\neg\Sigma_1$-Induction$_{\mathbf{j}}$ + CI (Hatch's model), ZFC + BTEE + Cofinal Axiom + Induction$_{\mathbf{j}}$ (canonical indiscernible models from Section 3), and ZFC + BTEE + Induction$_{\mathbf{j}}$ + CI (the model $\mathcal{N}$ in Proposition 7.6) is consistent.

Notice that the inconsistency we obtain in Proposition 7.2 does not, in this case, require Kunen's argument (though, as we will see in the next section, his argument *is* used to prove Cofinal Axiom from Separation$_{\mathbf{j}}$). Here, the difficulty lies in the combination of the fact that the critical sequence is cofinal and, at the same time, it is required to satisfy an instance of Replacement$_{\mathbf{j}}$. A familiar corollary is the fact that the critical sequence of an I_3 embedding $j : V_\lambda \to V_\lambda$ must not be weakly definable in V_λ (see Section 1).

Inconsistency in this case does not arise because we have combined very strong axioms of infinity. As we observed above, the theory ZFC + BTEE + Induction$_{\mathbf{j}}$ + Cofinal Axiom is quite weak. Similarly, the consistency of a measurable cardinal suffices for the consistency of ZFC + BTEE + Induction$_{\mathbf{j}}$ + CI, as Example 7.6 below shows.

We turn now to extensions of the theory ZFC + BTEE + $\neg$Cofinal Axiom. The statement $\neg$Cofinal Axiom asserts that the critical sequence has an upper bound; in the discussion below (in this section only), the Greek letter λ will signify such an upper bound. (Thus, for example, the statement "$\mathbf{j} \restriction \lambda$ is a set" is short for "whenever $\mathbf{j}^n(\kappa)$ exists, $\mathbf{j}^n(\kappa) < \lambda$, and $\mathbf{j} \restriction \lambda$ is a set".) We consider now several statements related to $\neg$Cofinal Axiom and the relationships between them. The discussion will bring to light several interesting examples of models of ZFC + BTEE + $\neg$Cofinal Axiom. We will conclude with the promised inconsistency result for this theory.

We introduce the following terminology, so that we can talk about the "supremum" of the critical sequence even when it may not be totally defined, or not a set. An ordinal δ is said to be the *supremum* of Ψ if the following is true:

$$\forall n \in \omega \forall \beta \,[\Psi(n,\beta) \Longrightarrow \beta < \delta] \wedge$$
$$\forall \delta\prime \left[\left((\forall n \in \omega \forall \beta \,[\Psi(n,\beta) \Longrightarrow \beta < \delta'] \right) \Longrightarrow \delta \leq \delta' \right]$$

Consider the following statements:

(A) $\mathbf{j}^n(\kappa)$ exists for every $n \in \omega$.

(B) if δ is the supremum of Ψ, $\mathbf{j}(\delta) = \delta$

(C) $\neg$Cofinal Axiom

(D) Ψ has a supremum

(E) the (range of the) critical sequence is a set

(F) $\mathbf{j} \restriction \lambda$ is a set

(G) $\mathbf{j}''(\lambda)$ is a set

Proposition 7.4. *The theory* ZFC + BTEE *proves the following:*

(1) (C) + Σ_0-Induction$_{\mathbf{j}}$ $\Rightarrow$ (A).

(2) (D) $\Rightarrow$ (C).

(3) (C) + Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ $\Rightarrow$ (D).

(4) (E) $\Rightarrow$ (B) $\wedge$ (D).
(5) (F) $\Rightarrow$ (E).
(6) (F) $\Leftrightarrow$ (G).

Proof. (1) was proved in Proposition 5.1, and (2) is obvious. For (3), let λ be an upper bound for the critical sequence, given by $\neg$Cofinal Axiom, and let $A = \lambda^{<\omega}$, the set of finite sequences $n \to \lambda$, $n \in \omega$. Then the following holds:

$$\forall n \in \omega \forall f \in A \,(\mathrm{dom}\ f = n + 1 \longrightarrow \exists \beta < \lambda\, \Theta(f, n, \kappa, \beta)).$$

By Σ_0-Least Ordinal Principle$_{\mathbf{j}}$, there is a least such λ; clearly, this least λ is the supremum of Ψ. For (4), (E) $\Rightarrow$ (B) is obvious. For the other implication, assume the critical sequence $z : \omega \to \delta$ is a set with supremum δ. Then $\mathbf{j}(\delta) = \sup(\mathrm{ran}\ (\mathbf{j}(z)))$. For each n, $\mathbf{j}(z)(n) = \mathbf{j}(z(n))$. From Proposition 2.19, we may conclude that for each n, $\mathbf{j}(z(n)) = z(n + 1)$. It follows that $\sup(\mathrm{ran}\ (\mathbf{j}(z))) = \sup(\mathrm{ran}\ (z)) = \delta$, as required. For (5), let $g = \mathbf{j} \restriction \lambda$. Then using just ZFC, we may form the set $\{\kappa, g(\kappa), g^2(\kappa), \ldots, g^n(\kappa), \ldots\}$, as required. For (6), note that $\mathbf{j}''(\lambda)$ is the range of $\mathbf{j} \restriction \lambda$, and $\mathbf{j} \restriction \lambda$ is the increasing enumeration of $\mathbf{j}''\lambda$. ∎

None of the implications here is reversible (unless otherwise indicated, as in part (6)). In particular, the example given in Proposition 6.4 shows that Σ_0-Induction$_{\mathbf{j}}$ + (C) $\not\Rightarrow$ (D). Example 7.5 shows (D) $\not\Rightarrow$ (B). The model $\mathcal{M}$ of Example 7.6 below shows (B) $\wedge$ (D) $\not\Rightarrow$ (E), whereas the model $\mathcal{N}$ of Example 7.6 shows (E) $\not\Rightarrow$ (F).

Example 7.5. *A model of* ZFC+BTEE+Least Ordinal Principle$_{\mathbf{j}}$+ "Ψ *has supremum* λ"+$\mathbf{j}(\lambda) \neq \lambda$, *from an* $\omega + \omega$*-Erdös cardinal.* For the example, assume there is an $\omega + \omega$-Erdös cardinal δ. Let $I = \{\alpha_\xi : \xi < \omega + \omega\} \subseteq \delta$ be a set of indiscernibles for L_δ of ordertype $\omega + \omega$ satisfying the conclusion of Corollary 3.4(2) — in particular, $\alpha_\omega = \sup\{\alpha_n : n \in \omega\}$. Let $B = \mathfrak{H}^{L_\delta}(I)$. Let $f : I \to I$ be any order-preserving function such that $f(\alpha_\xi) = \alpha_{\xi+1}$ whenever $\xi \leq \omega$, and lift f to $i : B \to B$ in the usual way. As usual, the transitive collapse of B must be an L_β, and the collapsing map π induces an elementary embedding $j : L_\beta \to L_\beta$. Let $J = \pi''I = \{\beta_\xi : \xi < \omega + \omega\}$. Clearly, $\langle L_\beta, \in, j\rangle \models$ ZFC + BTEE + Induction$_j$. Let $\lambda = \sup\{\beta_n : n \in \omega\} = \beta_\omega$. By definition of f, $j(\lambda) > \lambda$. Since $\langle \beta_n : n \in \omega\rangle$ is the critical sequence of j, we have shown that $\langle L_\beta, \in, j\rangle$ has all the required properties. ∎

The next example was discovered by Joel Hamkins, who communicated it to the author. With his permission, we present his results here.

Example 7.6 (Hamkins). *Assuming a measurable cardinal, there are models* $\mathcal{M}$ *and* $\mathcal{N}$ *with*

(1) $\mathcal{M} \models$ ZFC + BTEE + Least Ordinal Principle$_{\mathbf{j}}$ + "Ψ has a supremum λ" + $\mathbf{j}(\lambda) = \lambda$ + "the critical sequence is *not* a set".

(2) $\mathcal{N} \models$ ZFC+BTEE+Least Ordinal Principle$_{\mathbf{j}}$+ "the critical sequence is a set" $+\forall\lambda\,($"λ bounds the critical sequence" $\longrightarrow \neg\exists z\,(z = \mathbf{j}\restriction\lambda))$.

Proof. We start with a measurable cardinal κ and a normal measure U on κ. $\mathcal{M} = \langle M, \in, j\rangle$ will be an embedding of an iterated ultrapower, and $\mathcal{N} = \langle M[S], \in, \hat{j}\rangle$ will be obtained from M by adding a Prikry-generic sequence S, and lifting j to $M[S]$. Since both models will be transitive

For the first model, let $M_0 = V$ and let

$$M_0 \xrightarrow{i_{01}} M_1 \xrightarrow{i_{12}} M_2 \to \ldots \to M_n \xrightarrow{i_{n,n+1}} M_{n+1} \to \ldots \to M_\omega$$

be the usual sequence of ultrapowers defined by U, where $M_n = \mathrm{Ult}^{(n)}(U)$ is the (transitive collapse of the) nth ultrapower, $i_{n,n+1} = i^U_{n,n+1}$ is the canonical embedding by $i_{0,n}(U)$, and

$$M_\omega = \mathrm{limdir}_{n\in\omega}\,\{M_n; i^U_{m,n} \mid 0 \le m \le n\}, \tag{7.2}$$

where $i^U_{m,n}$ is the composition $i_{n-1,n} \circ \ldots \circ i_{m,m+1}$. For each $n \le \omega$, let $i_n = i_{0,n}$, $\kappa^{(0)} = \kappa$, $\kappa^{(n)} = i_n(\kappa)$, and $U^{(n)} = i_n(U)$. The following facts are well-known (see [Je1]):

$$\langle \kappa^{(n)} : n \le \omega\rangle \text{ is increasing and continuous.} \tag{7.3}$$

Also, for all $X \in M_\omega$ for which $X \subseteq \kappa^{(\omega)}$,

$$X \in U^{(\omega)} \quad \text{iff} \quad \exists n \in \omega\,(X \supseteq \{\kappa^{(k)} : n \le k < \omega\}). \tag{7.4}$$

It is straightforward to verify that $i_2 = i_1 \cdot i_1$ (where $\cdot$ is application of embeddings), and in general $i_{n+1} = i_1 \cdot i_n$. It follows that, for each $n < \omega$,

$$\mathrm{cp}(i_{n+1}) = i_1^n(\kappa) = i_n(\kappa), \tag{7.5}$$

where i_1^n is the nth iterate of i_1 under composition.

The following is a commutative diagram of elementary embeddings:

$$\begin{tikzcd} V \arrow[r, "i_\omega"] \arrow[d, "i_1"'] & M_\omega \arrow[d, "i_1 \restriction M_\omega"] \\ M_1 \arrow[r, "i_1 \cdot i_\omega"] & i_1(M_\omega) \end{tikzcd}$$

We observe that, in the diagram, $i_1(M_\omega) = M_\omega$: If we apply i_1 to (7.2), we obtain

$$\begin{aligned} i_1(M_\omega) &= \mathrm{limdir}_{n\ge 1}\{\mathrm{Ult}^{(n)}(U); i^U_{m,n} \mid 1 \le m \le n\} \\ &= M_\omega. \end{aligned}$$

We let $j = i_1 \restriction M_\omega : M_\omega \to M_\omega$. Let $\mathcal{M} = \langle M_\omega, \in, j \rangle$. Since j is elementary with critical point κ, it follows that $\mathcal{M} \models \mathrm{ZFC} + \mathrm{BTEE}$. By (7.5), for each $n \in \omega$, $j^n(\kappa) = \kappa^{(n)}$. Thus, the critical sequence for j is $\langle \kappa^{(0)}, \kappa^{(1)}, \ldots, \kappa^{(n)}, \ldots \rangle$. Since, in V, $\sup\{\kappa^{(n)} : n \in \omega\} = \kappa^{(\omega)}$, it follows that $\mathcal{M} \models$ "Ψ has a supremum". Because $\{\kappa^{(n)} : n \in \omega\}$ is a set in V, the argument in Proposition 7.4(4) can be used to show $j(\kappa_\omega) = \kappa_\omega$. We have proven all the desired properties of $\mathcal{M}$, except for the fact that the critical sequence is not a set; we prove this in the context of describing the properties of the model $\mathcal{N}$.

For the model $\mathcal{N}$, note that, in M_ω, $\kappa^{(\omega)} = i_{0,\omega}(\kappa)$ is a measurable cardinal and $U^{(\omega)}$ is a normal measure on $\kappa^{(\omega)}$. Therefore, in M_ω, we let P denote Prikry forcing with respect to $\kappa^{(\omega)}$ and $U^{(\omega)}$:

$$P = \{(s, A) \mid s \in [\kappa^{(\omega)}]^{<\omega} \text{ and } A \in U^{(\omega)}\}$$

$$(t, B) \le (s, A) \text{ iff } s \text{ is an initial segment of } t,\ B \supseteq A, \text{ and } t - s \subset A$$

Recall (see [Je1, Theorem 21.14]) that for sets $S \subset \kappa^{(\omega)}$ (in V) of ordertype ω,

$$S \text{ is } P\text{-generic over } M_\omega \text{ iff for every } X \in U^{(\omega)},\ S - X \text{ is finite.} \tag{7.6}$$

Moreover, given P-generic G, the set $S = S_G = \bigcup\{s : (s, A) \in G\}$ is also P-generic; given a generic $S \subset \kappa^{(\omega)}$ of ordertype ω, the set $G = G_S = \{(s, A) \in P \mid s$ is an initial segment of S and $S - s \subset A\}$ is generic. In both cases, $M_\omega[G] = M_\omega[S]$. In particular, it is well-known (see [Je1, Theorem 21.15]) that if S is the critical sequence of j, $S = \{\kappa^{(n)} : n \in \omega\}$, then S is P-generic over M_ω. But this means that $S \notin M_\omega$, and so we have established the final property of the model $\mathcal{M}$.

Let

$$S' = i_1(S) = \{\kappa^{(n)} : n \ge 1\}.$$

By (7.6), S' is P-generic over M_ω. Let $G' = G_{S'}$. Using (7.4), one shows that $j(U^{(\omega)}) = U^{(\omega)}$. Therefore, $A \in U^{(\omega)}$ iff $j(A) \in U^{(\omega)}$, so

$$\text{if } S - s \subset A, \text{ then } S' - j(s) \subset j(A) \tag{7.7}$$

It follows that

$$p = (s, A) \in G \Longrightarrow j(p) = (j(s), j(A)) \in G'.$$

Thus, defining $\hat{j} : M_\omega[G] \to M_\omega[G']$ by

$$\hat{j}(\sigma_G) = (j(\sigma))_{G'}$$

yields a well-defined elementary embedding. But since $M[G] = M[S] = M[S'] = M[G']$, we have that $\hat{j} : M[G] \to M[G]$. It follows that

$$\langle M[S], \in, \hat{j} \rangle \models \mathrm{ZFC} + \mathrm{BTEE} + \text{"the critical sequence is a set".}$$

Let $\mathcal{N} = \langle M[S], \in, \hat{j}\rangle$. Since Induction$_{\mathbf{j}}$ holds in the model and the critical sequence is a set, $\mathcal{N}$ satisfies CI in a nontrivial way.

Finally, we observe that, in $\mathcal{N}$, if λ bounds the critical sequence, then $j \restriction \lambda$ is *not* a set. This follows by Proposition 7.7; indeed, adding the axiom "$\mathbf{j} \restriction \lambda$ is a set" to Th($\mathcal{N}$) would render the theory inconsistent. ∎

We conclude by showing that any extension of ZFC + BTEE + ¬Cofinal Axiom that includes the assertion that $\mathbf{j} \restriction \lambda$ is a set, where λ bounds the critical sequence, is inconsistent. The set $\{\exists z\,(z = \mathbf{j} \restriction \lambda)\}$ is *minimal* among sets of axioms that render ZFC + BTEE + ¬Cofinal Axiom inconsistent in a couple of ways. First, replacing $\exists z\,(z = \mathbf{j} \restriction \lambda)$ with any of the sentences (A) – (E) above yields a consistent theory, as the model $\mathcal{N}$ from the previous example shows. Secondly, as we show in Proposition 9.11, for each particular n, the theory ZFC+BTEE+¬Cofinal Axiom+$(\exists z\,(z = \mathbf{j} \restriction \mathbf{j}^n(\kappa)))$ is consistent (relative to an $n+2$-huge cardinal).

Proposition 7.7. *The following theory is inconsistent:*

$$\text{ZFC} + \text{BTEE} + \neg\text{Cofinal Axiom} + \exists z\,(z = \mathbf{j} \restriction \lambda),$$

where λ is any bound for the critical sequence.

Proof. Since $\mathbf{j} \restriction \lambda$ is a set, we know by Proposition 7.4 that the critical sequence is a set, it has a supremum δ, $\mathbf{j}(\delta) = \delta$, and there is a set $H = \mathbf{j}''\delta$ (since $\mathbf{j} \restriction \delta$ must also be a set). Note that δ is a strong limit cardinal of cofinality ω. As in Kunen's proof, let $F : {}^{\omega}\delta \to \delta \in V_{\delta+2}$ be an ω-Jonsson function (that is, F has the property that for all $A \in [\delta]^{\delta}$, $F''\left({}^{\omega}A\right) = \delta$). Since $\mathbf{j}(F)$ is also such a function, we have $\mathbf{j}(F)''({}^{\omega}H) = \delta$, leading to the contradiction that, for some $s : \omega \to H$,

$$\begin{aligned}
\kappa &= \mathbf{j}(F)(s) \\
&= \mathbf{j}(F)(\mathbf{j}(t)) \text{ for some } t : \omega \to \delta \\
&= \mathbf{j}(F(t)). \blacksquare
\end{aligned}$$

As we will show in Section 9, any axiom of the form "$\exists z\,(z = \mathbf{j} \restriction \alpha)$," where $\alpha \geq \kappa^+$ has significant large cardinal strength — at least that of a strong cardinal. In the presence of ¬Cofinal Axiom, such an axiom leads to inconsistency, as we have just seen, when α is large enough. But when added to extensions of ZFC + BTEE + Cofinal Axiom, no such inconsistency arises, though the consistency strength of the theory grows tremendously. In the next section, we study the theory ZFC + BTEE + Cofinal Axiom + $\forall\alpha\,\exists z\,(z = \mathbf{j} \restriction \alpha)$ and show that such a theory is strong enough to prove the existence of all known large cardinals having consistency strength below an I_3 embedding.

§8. Separation Axioms And Amenability

In the last section, we showed that in the presence of ¬Cofinal Axiom, a statement of the form $\exists z\,(z = \mathbf{j} \restriction \lambda)$ leads to inconsistency, and we mentioned that an axiom of this kind (where $\lambda \geq \kappa^+$), always has significant large cardinal consequences. This section is dedicated to investigating the theory ZFC + BTEE + $\forall x \exists z\,(z = \mathbf{j} \restriction x)$; one expects much stronger large cardinal consequences from such a theory. The new axiom is called Amenability$_{\mathbf{j}}$:

Amenability$_{\mathbf{j}}$: For every set x, there is a set z such that $z = \mathbf{j} \restriction x$.

As we will show, Amenability$_{\mathbf{j}}$ is a consequence of ZFC + WA (recall the definition of WA and WA$_n$ from Section 1; more details are given below). We study this apparent weakening of WA here as part of one of the paper's themes, to see to what extent axioms of the type "there exists an elementary embedding from M to M having certain properties" can be viewed as a hierarchy of assertions that are parallel to the hierarchy of large cardinal axioms. So far, in this paper, none of the models or theories we have considered have had consistency strength beyond a measurable cardinal. We will see in this section that the theory ZFC + BTEE + Amenability$_{\mathbf{j}}$ has consistency strength beyond a super-n-huge cardinal. In the subsequent section, we will explore axioms that produce consistency strengths somewhere between these two.

Another reason for studying this axiom in some detail is to verify a conjecture shared by the author and Hamkins, which arose during the writing of [Ha1]: That paper began as an attempt to improve the hypotheses of a consistency result obtained in [Co2]; in the latter it was shown that, assuming an I_1 embedding, $V = \mathrm{HOD}$ is consistent with ZFC + WA. Seeking to weaken the I_1 hypothesis, Hamkins eventually established the relative consistency result for Amenability$_{\mathbf{j}}$ rather than WA: If ZFC + WA$_0$ is consistent, so is ZFC + WA$_0$ + $V = \mathrm{HOD}$. The conjecture in this case was that this weaker theory ZFC + WA$_0$ is almost as strong as ZFC + WA. In this section, we verify the conjecture by showing that all large cardinal consequences that are known for ZFC + WA are also consequences of ZFC + WA$_0$.

We begin by setting up notation and giving the necessary definitions. In general, we recall from Section 1 that for each $n \in \omega$, we denote the Σ_n-Separation axioms Σ_n-*Separation*$_j$, and we denote full separation by *Separation*$_j$. Recall that WA = BTEE + Separation$_j$ and WA$_n$ = BTEE + Σ_n-Separation$_j$.

We recall from Section 2 that an *instance* of Separation$_{\mathbf{j}}$ is a formula

$$\forall A\, \forall \vec{a}\, \exists z\, \forall u\, [u \in z \longleftrightarrow u \in A \wedge \phi(u, A, \vec{a})],$$

where ϕ is a $\mathbf{j}$-formula. When ϕ is Σ_n (Π_n), we call this instance an instance of Σ_n-Separation$_{\mathbf{j}}$ (Π_n-Separation$_{\mathbf{j}}$). Given a (Σ_n, Π_n) $\mathbf{j}$-formula ϕ and sets $A, \vec{a}$, we may also refer to the formula

$$\exists z\, \forall u\, [u \in z \longleftrightarrow u \in A \wedge \phi(u, A, \vec{a})]$$

as a(n) (Σ_n, Π_n) instance of Separation$_{\mathbf{j}}$. We continue to follow our convention of calling a formula Σ_n (Π_n) when it may only be $\Sigma_n^{\mathrm{ZFC}_{\mathbf{j}}}$ ($\Pi_n^{\mathrm{ZFC}_{\mathbf{j}}}$).

We now show the connection between Separation$_{\mathbf{j}}$ and Amenability$_{\mathbf{j}}$. This observation is mentioned in [Ha1]. Enayat points out that a result of this kind is known in a much broader context. We need the following easy lemma:

Lemma 8.1. *For each particular (metatheoretic) natural number $n \geq 1$,*

$$\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{Amenability}_{\mathbf{j}} \vdash \forall x \, \exists z \, (z = \mathbf{j}^n \restriction x).$$

Proof. Proceed by induction on $n \geq 1$ in the metatheory. The case $n = 1$ follows from Amenability$_{\mathbf{j}}$. Assume the proposition holds for $n \geq 1$. By our definition of $\mathbf{j}$-terms (see the beginning of Section 2), $\mathbf{j}^{n+1} = \mathbf{j} \circ \mathbf{j}^n$. Let x be a set. Since $\mathbf{j}^n \restriction x$ is a set, it has a range y, and certainly $\mathbf{j} \restriction y$ is a set by Amenability$_{\mathbf{j}}$. Let

$$f = (\mathbf{j} \restriction y) \circ (\mathbf{j}^n \restriction x).$$

The fact that f is a set follows from ZFC$_{\mathbf{j}}$. Clearly, $f(u) = \mathbf{j}^{n+1}(u)$ for all $u \in x$, and the result follows. ∎

Theorem 8.2 [Ha1]**.** *The theory* ZFC + BTEE *proves*

$$\mathrm{Amenability}_{\mathbf{j}} \Longrightarrow \Sigma_0\text{-Separation}_{j}.$$

Proof. Suppose $\phi(x, \vec{u})$ is a Σ_0 formula. The idea is this: Given a set A, replace occurrences of $\mathbf{j}$ in ϕ with the restrictions $\mathbf{j} \restriction V_\delta$, where δ is large enough; by Amenability$_{\mathbf{j}}$, each such restriction is a set; by ordinary Separation, the subclass of A defined by ϕ must be a set. Here are the details:

Let $\rho(z, w)$ be the $\mathbf{j}$-formula asserting that $z = \mathbf{j} \restriction V_w$. Let $\gamma(z, w)$ say that "z is a function with domain V_w". Let $\theta(x, \vec{u}, z)$ be the $\in$-formula obtained from ϕ by replacing each occurrence of $\mathbf{j}$ with[4] the variable z. Let

$$\sigma(x, \vec{u}, w) \equiv \exists z \left(\rho(z, w) \,\wedge\, \gamma(z, w) \,\wedge\, \theta(x, \vec{u}, z)\right).$$

Note the σ says that z plays the role of $\mathbf{j} \restriction V_w$ in ϕ. Let $\vec{a}$ be a finite sequence of parameters. Let m denote the number of occurrences of $\mathbf{j}$ in ϕ. Amenability$_{\mathbf{j}}$ implies that

$$\forall \beta > \mathrm{rank}(\vec{a}) \, \forall x \in V_\beta \, \forall \delta \left[\delta = \mathbf{j}^{m+1}(\beta) \Longrightarrow \left[\phi(x, \vec{a}) \Longleftrightarrow \sigma(x, \vec{a}, \delta)\right]\right]. \tag{8.1}$$

---

[4] More precisely, each occurrence of a $\mathbf{j}$-term $\mathbf{j}^m(v_0)$ is replaced with an appropriate variation of $\exists v_m \, \eta_m(v_0, v_m, z)$ where $\eta_m(v_0, v_m, z) \equiv \exists v_1 \ldots v_{m-1} \left[(v_0, v_1) \in z \,\wedge\, (v_1, v_2) \in z \,\wedge\, \ldots \,\wedge\, (v_{m-1}, v_m) \in z\right]$. Thus, for example, an atomic formula such as $x \in \mathbf{j}^m(v_0)$ would be replaced by $\exists v_m \, (\eta_m(v_0, v_m, z) \,\wedge\, x \in v_m)$.

(The proof of this equivalence is straightforward: One first proves it when ϕ is an atomic formula composed of $\mathbf{j}$-terms; one then proceeds by induction on the complexity of ϕ, establishing the result for quantifier-free formulas and then all bounded formulas. In every case, one shows that δ is large enough so that the witness to σ can play the role of $\mathbf{j}$.)

Notice that ZFC proves that for any $A, \vec{a}, f, \delta$,

$$\exists X\, \forall x\, [x \in X \iff x \in A \,\wedge\, \gamma(f, \delta) \,\wedge\, \theta(x, \vec{a}, f)]. \tag{8.2}$$

We now work in ZFC+BTEE+Amenability$_{\mathbf{j}}$. Given $A, \vec{a}$, let $\beta > \mathrm{rank}(\{A, \vec{a}\})$ and let $\delta = \mathbf{j}^{m+1}(\beta)$. There is a set $f = \mathbf{j} \restriction V_\delta$. It therefore follows from (8.2) that

$$\exists X\, \forall x\, [x \in X \iff x \in A \,\wedge\, \gamma(f, \delta) \,\wedge\, \theta(x, \vec{a}, f) \,\wedge\, \rho(f, \delta)],$$

and so

$$\exists X\, \forall x\, [x \in X \iff x \in A \,\wedge\, \sigma(x, \vec{a}, \gamma)]. \tag{8.3}$$

Combining,(8.1) and (8.3),

$$\exists X\, \forall x\, (x \in X \iff x \in A \,\wedge\, \phi(x, \vec{a})).$$

In particular, $\{x \in A : \phi(x, \vec{a})\}$ is a set. ∎

By the theorem, we may now establish consequences of the theory ZFC+BTEE+Amenability$_{\mathbf{j}}$ by working instead in the theory $\mathrm{ZFC} + \mathrm{WA}_0$. We begin by showing that WA_0 suffices to prove the Cofinal Axiom. We start with an important lemma:

Proposition 8.3. $\mathrm{ZFC} + \mathrm{WA}_0 \vdash \forall \alpha\, \exists z_1, z_2\, (z_1 = \mathbf{j}''\alpha \,\wedge\, z_2 = \mathbf{j} \restriction \alpha)$.

Proof. $\mathbf{j}''\alpha$ is a Σ_0 definable subset of $\mathbf{j}(\alpha)$:

$$\mathbf{j}''\alpha = \{\gamma \in \mathbf{j}(\alpha) : \exists \beta \in \alpha\, (\gamma = \mathbf{j}(\beta)\}.$$

Also, $\mathbf{j} \restriction \alpha$ is a Σ_0 definable subset of $\alpha \times \mathbf{j}''\alpha$: Let $z_1 = \mathbf{j}''\alpha$. Then

$$\mathbf{j} \restriction \alpha = \{(\gamma, \beta) \in \alpha \times z_1 : \mathbf{j}(\gamma) = \beta\}. ∎$$

Proposition 8.4. $\mathrm{ZFC} + \mathrm{WA}_0 \vdash$ Cofinal Axiom.

Proof. If this fails, there is a model of $\mathrm{ZFC} + \mathrm{WA}_0 + \neg$Cofinal Axiom; in particular, for some λ that bounds the critical sequence, the sentence $\exists z\, (z = \mathbf{j} \restriction \lambda)$ holds in the model. But Proposition 7.7 shows that this is impossible. ∎

Corollary 8.5. *Any well-founded model of* ZFC + WA$_0$ *satisfies the following:*

(1) $\langle \kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots \rangle$ *is cofinal in* ON.

(2) $V = \bigcup_{n \in \omega} V_{j^n(\kappa)}$.

Proof. We can represent $\Psi(n, \beta)$ as the class sequence $\langle \kappa, j(\kappa), \ldots \rangle$ because Induction$_{\mathbf{j}}$ holds in such models. Now both parts follow from Proposition 8.4. ∎

Note that in Corollary 8.5, $\mathbf{j}^n(\kappa)$ exists for every n because the model is well-founded.

We now start working toward a proof that

$$V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec V_{\mathbf{j}^2(\kappa)} \prec \ldots \prec V$$

assuming WA$_0$. Since we do not have Σ_1-Induction$_{\mathbf{j}}$, we may not assume that Ψ or Φ are class functions, and have to take account of the possibility that $\mathbf{j}^n(x)$ may not be defined for certain n and x. We recall our terminology from Section 2: we say that $\mathbf{j}^n(x)$ *exists* or *is defined* if $\exists y\, \Phi(n, x, y)$.

We observe here that if $\mathcal{M} = \langle M, E, j \rangle$ is a model of ZFC + BTEE + Amenability$_{\mathbf{j}}$ with a nonstandard membership relation, Amenability$_{\mathbf{j}}$ does *not* say that $j \restriction x \in M$ for each $x \in M$; this is because the restriction operator is not absolute in this case. Of course, what the axiom does guarantee is that, for each $x \in M$,

$$\exists i \in M \left[i = (j \restriction x)^{\mathcal{M}} \right]. \tag{8.4}$$

Lemma 8.6.

(1) ZFC+WA$_0 \vdash \forall n \geq 1 \, \forall x \left(\text{“}\mathbf{j}^n(x) \text{ exists”} \Longrightarrow \exists z \, (z = \mathbf{j}^n \restriction x) \right)$. *In particular,* ZFC+BTEE $\vdash \Sigma_0$-Separation$_{\mathbf{j}} \Longleftrightarrow$ Amenability$_{\mathbf{j}}$.

(2) ZFC + WA$_0$ $\vdash$ $\forall n \geq 1 \, \forall M \left(\text{“}\mathbf{j}^n(M) \text{ exists”} \Longrightarrow \text{“}\mathbf{j}^n \restriction M : M \to \mathbf{j}^n(M) \text{ is an elementary embedding”} \right)$.

(3) ZFC + WA$_0 \vdash \forall n \geq 1 \, \mathbf{j}^n \restriction V_\kappa = \mathrm{id}_{V_\kappa}$.

Proof of (1). One direction was proved in Theorem 8.2. For the other direction, first notice that $\mathbf{j} \restriction x$ is Σ_0-definable from $x \times \mathbf{j}(x)$ and is therefore a set:

$$\mathbf{j} \restriction x = \{(r, s) \in x \times \mathbf{j}(x) : r \in x \wedge \mathbf{j}(r) = s\}.$$

Assume the result fails for some $n \in \omega$ and some x; by Proposition 5.3(4), we may assume $x = V_\alpha$ for some α. By Proposition 5.3(3) $\mathbf{j}^n(\alpha)$ exists. Let $\delta > \mathbf{j}^n(\alpha)$ and let $X = V_\delta$. We have the following:

$$\begin{aligned} \forall z \in X \, \exists y \in X \, \neg \Big[y \in z \Longleftrightarrow \big(&\text{“}y \text{ is an ordered pair”} \wedge \\ &(y)_0 \in X \wedge \exists f \in X \, \Theta(f, n, (y)_0, (y)_1) \big) \Big]. \end{aligned} \tag{8.5}$$

Since (8.5) is Σ_0, we can use the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to obtain the least n for which the formula holds. Notice that for this least n, the fact that there is no $z \in V_\delta$ for which $z = \mathbf{j}^n \upharpoonright V_\alpha$ implies that there is no such z at all since any such restriction would have to lie in V_δ. By Proposition 5.3(1),(2), $\mathbf{j}^m(V_\alpha)$ exists for all $m < n$, and $\mathbf{j}^m(u)$ exists for all $m \leq n$ and $u \in V_\alpha$.

Because Amenability$_{\mathbf{j}}$ holds, we have that $n > 1$. Again by Amenability$_{\mathbf{j}}$, $\mathbf{j} \upharpoonright V_{\mathbf{j}^{n-1}(\alpha)}$ is a set. Because of the leastness of n, $\mathbf{j}^{n-1} \upharpoonright V_\alpha$ is also a set. We therefore have the following equation, which demonstrates that $\mathbf{j}^n \upharpoonright V_\alpha$ is a set as well:

$$\mathbf{j}^n \upharpoonright V_\alpha = \mathbf{j} \upharpoonright V_{\mathbf{j}^{n-1}(\alpha)} \circ \mathbf{j}^{n-1} \upharpoonright V_\alpha.$$

By our orginal assumption, this is impossible. Therefore, the theorem is proven. ∎

Proof of (2) We first obtain the result for the case $n = 1$. Work in ZFC + WA$_0$. Let M be a set; by (1), $i = \mathbf{j} \upharpoonright M$ is also a set. We show $i : \langle M, \in \rangle \to \langle \mathbf{j}(M), \in \rangle$ is, formally, an elementary embedding. Suppose p is a formal $\in$-formula in V_ω. Let $b : \mathrm{rank}(p) \to M$. Then $\mathbf{j}(p) = p$ and $\mathbf{j}(b) : \mathrm{rank}(p) \to \mathbf{j}(M)$. By elementarity of $\mathbf{j}$, we have

$$\mathrm{Sat}(p, M, b) \iff \mathrm{Sat}(p, \mathbf{j}(M), \mathbf{j}(b)),$$

as required.

For general n, we will apply the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to a formula that asserts that $\mathbf{j}^n$ is not elementary, and arrive at a contradiction. We begin with a formula that makes this assertion, but that is not Σ_0. Then, by binding all quantified variables to a large enough set, we will devise an equivalent formula that is Σ_0, and then apply Σ_0-Least Ordinal Principle$_{\mathbf{j}}$.

Let $\mathrm{Fmla}_\in \subset V_\omega$ denote the set of all formal formulas (formulas coded as sets in one of the standard ways; see [Dr]). Consider the following formula:

$$\begin{aligned}\hat{\gamma}(n, M) \equiv \exists p \in \mathrm{Fmla}_\in \, \exists b \in {}^{\mathrm{rank}(p)}M \Big(&\big[\mathrm{Sat}(p, M, b) \wedge \neg\mathrm{Sat}(p, \mathbf{j}^n(M), \mathbf{j}^n(b))\big] \vee \\ &\big[\neg\mathrm{Sat}(p, M, b) \wedge \mathrm{Sat}(p, \mathbf{j}^n(M), \mathbf{j}^n(b)\big]\Big).\end{aligned}$$

The formula $\hat{\gamma}(n, M)$ says that $\mathbf{j}^n \upharpoonright M$ is not, formally, an elementary embedding. Note that $\mathbf{j}^n(V_\omega)$ is defined (and hence $\mathbf{j}^n \upharpoonright V_\omega$ can be applied to p). We observe also that $\mathbf{j}^n(b)$ is defined for any $b : \mathrm{rank}(p) \to M$. This follows because, by Proposition 5.3(4),(7), we can find a limit ordinal δ such that $M \in V_\delta$ and $\mathbf{j}^n(V_\delta)$ exists; but any such b must lie in V_δ. Also, in order for the formula $\mathrm{Sat}(p, \mathbf{j}^n(M), \mathbf{j}^n(b))$ to make sense, $\mathbf{j}^n(b)$ must be a function $\mathrm{rank}(p) \to \mathbf{j}^n(M)$. By Proposition 5.5, Σ_0-Induction$_{\mathbf{j}}$ (whence WA$_0$) suffices to establish that $\mathbf{j}^n$ is Σ_0-elementary (relative to parameters at which it is defined). Therefore $\mathbf{j}^n(p) = p$ and $\mathbf{j}^n(m) = m$ for all $m \in \omega$. It follows that $\mathrm{rank}(p) = \mathrm{rank}(\mathbf{j}^n(p))$. Thus, by Σ_0-elementarity again, $\mathbf{j}^n(b) : \mathrm{rank}(p) \to \mathbf{j}^n(M)$.

Assume now that there is an $n \in \omega$ for which $\hat{\gamma}(n, M)$ is true. By (1), $\mathbf{j}^n \upharpoonright M$ is a set. By Corollary 5.6, we can find $\delta > \operatorname{rank}(\{\mathbf{j}^n \upharpoonright M, \mathbf{j}^n({}^{\omega}M)\})$ such that δ is inaccessible. Let $X = V_\delta$. Since Sat is Δ_1^{ZF}, we can obtain Δ_0 formulas $\phi(x, u, v, w)$ and $\psi(y, u, v, w)$ such that

$$\text{(8.6)} \qquad \mathrm{ZF} \vdash \forall u, v, w\big(\exists x\, \phi(x, u, v, w) \iff \mathrm{Sat}(u, v, w)\big),$$

and

$$\text{(8.7)} \qquad \mathrm{ZF} \vdash \forall u, v, w\big(\forall y\, \psi(y, u, v, w) \iff \mathrm{Sat}(u, v, w)\big).$$

Let $h : \mathrm{Fmla}_\in \to \omega$ be defined by $h(x) = \operatorname{rank}(x)$. Let $A = \mathbf{j}^n(M)$. With these constants, we can bound all necessarily variables by X and transform $\gamma(n, M)$ into the following equivalent Σ_0 formula $\gamma(n, M, X, h, A)$:

$$\begin{aligned}\exists p, b, c, f \in X \Big(p \in \mathrm{Fmla}_\in \,\wedge\, \text{“}b \text{ is a function”} \,\wedge\, \operatorname{dom}(b) = h(p) \,\wedge\, \Theta(f, n, b, c) \,\wedge \\ \big(\big[\exists x \in X\, \phi(x, p, M, b) \,\wedge\, \exists y \in X \neg\psi(y, p, A, c)\big] \vee \\ \big[\exists y \in X \neg\psi(y, p, M, b) \,\wedge\, \exists x \in X\, \phi(x, p, A, c)\big]\big)\Big).\end{aligned}$$

The Σ_0 $\mathbf{j}$-formula $\Theta(f, n, x, y)$, which asserts that $\mathbf{j}^n(x) = y$ with witness f, is defined in Section 2. Note that $c = \mathbf{j}^n(b), A = \mathbf{j}^n(M)$. Since Δ_1^{ZF} formulas are absolute for transitive models of ZFC and $\langle X, \in\rangle$ is such a model, it follows from (8.6) and (8.7) that, for the given choices of n, X, h, A, $\hat{\gamma}(n, M)$ is equivalent (in $\mathrm{ZFC}_{\mathbf{j}}$) to the Σ_0 formula $\gamma(n, M, X, h, A)$.

By the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$, we can find a least $k \le n$ for which $\gamma(k, M, X, h, A)$ is true. For this choice of k, $\hat{\gamma}(k, M)$ holds. Note that $k > 1$ since we have already established the $n = 1$ case. Let $p \in \mathrm{Fmla}_\in$ and $b \in {}^{\operatorname{rank}(p)}M$ be witnesses for $\hat{\gamma}(k, M)$. By the leastness of k, we have

$$\text{(8.8)} \qquad \mathrm{Sat}(p, M, b) \iff \mathrm{Sat}(p, \mathbf{j}^{k-1}(M), \mathbf{j}^{k-1}(b)).$$

By elementarity of $\mathbf{j}$ we have, as before, $\mathbf{j}(p) = p$ and $\mathbf{j}^k(b) : \operatorname{rank}(p) \to \mathbf{j}^k(M)$; applying $\mathbf{j}$ to the formula $\mathrm{Sat}(p, K, L)$, with $K = \mathbf{j}^{k-1}(M)$ and $L = \mathbf{j}^{k-1}$ yields:

$$\text{(8.9)} \qquad \mathrm{Sat}(p, \mathbf{j}^{k-1}(M), \mathbf{j}^{k-1}(b)) \iff \mathrm{Sat}(p, \mathbf{j}^k(M), \mathbf{j}^k(b)).$$

Combining (8.8) and (8.9) yields

$$\mathrm{Sat}(p, M, b) \iff \mathrm{Sat}(p, \mathbf{j}^k(M), \mathbf{j}^k(b)),$$

and this contradicts $\hat{\gamma}(k, M)$. This completes the proof. ■

Proof of (3). We first prove the result for $n = 1$. Let $M = V_\kappa$. Consider the following Σ_0 formula:

$$\text{(8.10)} \qquad \gamma(\alpha) \equiv \exists x \in M\, \big(\mathbf{j}(x) \neq x \,\wedge\, (\alpha = \operatorname{rank}(x))^M\big).$$

We have relativized the formula "$\alpha = \text{rank}(x)$" to M in order to ensure that γ is Σ_0. Notice that if there is an $x \in M$ for which $\mathbf{j}(x) \neq x$, its rank must lie in M. Therefore, if $\gamma'(\alpha)$ is the formula obtained by replacing the subformula "$(\alpha = \text{rank}(x))^M$" with the formula "$\alpha = \text{rank}(x)$", then for all α, $\gamma'(\alpha) \Rightarrow \gamma(\alpha)$.

Applying the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to γ, we obtain the least α for which $\gamma(\alpha)$ holds. Let x be such that $\text{rank}(x) = \alpha$. By elementarity, $\text{rank}(x) = \text{rank}(\mathbf{j}(x))$. By Proposition 2.5, we have a contradiction.

For general n, an easy Σ_0-Induction$_{\mathbf{j}}$ on the following Σ_0 formula

$$\rho(n) : \forall x \in M\, (\mathbf{j}^n(x) = x)$$

(recalling $M = V_\kappa$) establishes the result. It is not immediately obvious that "$\mathbf{j}^n(x) = x$" is equivalent to a Σ_0 formula. Certainly, this formula is equivalent to $\exists f\, \Theta(f, n, x, x)$. But in the present context, the existential quantifier can be bound by V_κ, making it Σ_0. (The referee points out that this step may also be proved by observing that $\Theta(f_x, n, x, x)$ holds for each $x \in V_\kappa$, where $f_x(i) = x$ for $0 \leq i \leq n$.)

The induction now shows that for all (formal) n, $\forall x \in V_\kappa\, (\mathbf{j}^n(x) = x)$. It follows that $\mathbf{j}^n \restriction V_\kappa$ is a set, namely, the function id_{V_κ}. ■

We remark that part (1) of the lemma gives us that "$z = \mathbf{j}^n \restriction x$" is Σ_0, by the usual proof. Thus,

$$\text{the formula "}z = \mathbf{j}^n \restriction x\text{" is } \Sigma_0^{\text{ZFC+WA}_0}. \tag{8.11}$$

In part (3), we did not require the hypothesis "if $\mathbf{j}^n(\kappa)$ exists" in order to obtain the result. However, in order to conclude that $\mathbf{j}^n \restriction V_\kappa$ is elementary, part (2) is needed, and then an assumption of this kind is necessary; by Proposition 5.3(5), the hypothesis that $\mathbf{j}^n(\kappa)$ exists suffices.

A handy corollary to Lemma 8.6(1) is the following:

Corollary 8.7. *The theory* ZFC + WA$_0$ *proves the following:*

(1) *For any set A, $\mathbf{j}''A \subset A \implies \mathbf{j} \restriction A = \text{id}_A$.*
(2) $\forall \alpha \geq \kappa\, \mathbf{j}(\alpha) > \alpha$.
(3) *For any set A,*

$$|A| = |\mathbf{j}(A)| \iff A = \mathbf{j}(A) \iff A \in V_\kappa.$$

Proof of (1). Let $a \in A$ be such that $\mathbf{j}(a) \neq a$. Then $a \notin V_\kappa$. By the Cofinal Axiom, there is $n \in \omega$ such that $\mathbf{j}^n(\kappa)$ exists and $A \in V_{\mathbf{j}^n(\kappa)}$. Since $\text{rank}(a) \geq \kappa$, one shows by Σ_0-Induction$_{\mathbf{j}}$ that there is a least $m \leq n$ such that $\mathbf{j}^m(a) \notin A$ and $m > 0$. Then $a' = \mathbf{j}^{m-1}(a) \in A$ (here, we let $\mathbf{j}^0(a)$ denote a), but $\mathbf{j}(a') \notin A$. Thus $\mathbf{j}''A \not\subset A$. ■

Proof of (2). Assume the conclusion fails. We can use the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to obtain the least $\alpha > \kappa$ for which $\mathbf{j}(\alpha) = \alpha$. By the Cofinal Axiom, there is an n such that $\alpha < \mathbf{j}^n(\kappa)$. Translating this inequality into a Σ_0 statement using Θ, and using the fact that the $\mathbf{j}^m(\kappa)$ are increasing for $m \leq n$ (by Proposition 6.6), we can use the Σ_0-Least Ordinal Principle$_{\mathbf{j}}$ to obtain the largest $m \leq n$ for which $\mathbf{j}^m(\kappa) \leq \alpha$; let $\beta = \mathbf{j}^m(\kappa)$. Then $\beta \leq \alpha$ but $\mathbf{j}(\beta) > \mathbf{j}(\alpha)$, contradicting the fact that $\mathbf{j}$ is nondecreasing. ∎

Proof of (3). Since the implications

$$A \in V_\kappa \implies A = \mathbf{j}(A) \implies |A| = |\mathbf{j}(A)|$$

are obvious, it suffices to prove that if $A \notin V_\kappa$, then $|A| < |\mathbf{j}(A)|$. Given $A \notin V_\kappa$, let λ be such that $|A| = \lambda > \kappa$. By (2), we have $|\mathbf{j}(A)| = \mathbf{j}\big(|A|\big) = \mathbf{j}(\lambda) > \lambda = |A|$, as required. ∎

Theorem 8.8.

(1) $\mathrm{ZFC} + \mathrm{WA}_0 \vdash \forall n \in \omega \Big((\text{“}\mathbf{j}^n(\kappa) \textit{ exists”}) \implies \big[V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec \ldots \prec V_{\mathbf{j}^n(\kappa)}\big]\Big)$.

(2) $\mathrm{ZFC} + \mathrm{WA}_0 \vdash V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec V_{\mathbf{j}^2(\kappa)} \prec \ldots \prec V$.

Remarks.

(A) The ellipsis in part (2) has an unusual interpretation in the present context — we understand the statement in (2) to mean that the chain of $V_{\mathbf{j}^n(\kappa)}$'s extends as far as $\mathbf{j}^n(\kappa)$ exists, and that each member in this chain below V is an elementary submodel of V. In the proof, we give a precise statement. The notation is justified by the fact that for every set x, there is an n such that $\mathbf{j}^n(\kappa)$ exists and $x \in V_{\mathbf{j}^n(\kappa)}$, by the Cofinal Axiom. In other contexts in which Σ_1-Induction$_{\mathbf{j}}$ holds (and hence, in which Φ is a class function), we will understand (2) to have its usual meaning (namely, that n ranges over all of ω). In particular, the proofs of (1) and (2) in $\mathrm{ZFC} + \mathrm{WA}_1$ are identical to those given in [Co3, Proposition 3.12].

(B) Part (2) is actually a schema. For each $\in$-formula $\phi(\vec{x})$ we show that $\phi[\vec{a}]$ iff $\mathrm{Sat}(\ulcorner \phi \urcorner, V_{\mathbf{j}^n(\kappa)}, b)$ for all sufficiently large n (for which $\mathbf{j}^n(\kappa)$ exists) and suitable b. We cannot improve this to a statement about all *formal* formulas since there is no formal definition of truth in V.

Proof of (1). Fix $n \geq 1$. Suppose p, M, N, r, ρ are such that $\rho = \max\{\mathrm{rank}(M), \mathrm{rank}(N)\}$, $\mathbf{j}^n(\rho)$ exists, $r = \mathrm{rank}(p)$, and

$$\forall b \in {}^rM \,\big[\mathrm{Sat}(p, M, b) \iff \mathrm{Sat}(p, N, b)\big].$$

The displayed formula says that the $\in$-formula coded by p is absolute for M, N.

By Proposition 5.3(6),(7), we can find a cardinal δ such that $\rho < \delta$ and $\mathbf{j}^n(V_\delta)$ exists; let $X = V_\delta$. By Lemma 8.6(2), $i = \mathbf{j}^n \upharpoonright X$ is elementary. Applying i, we have

$$\forall b \in {}^r\big(\mathbf{j}^n(M)\big) \,\big[\mathrm{Sat}(p, \mathbf{j}^n(M), b) \iff \mathrm{Sat}(p, \mathbf{j}^n(N), b)\big]. \tag{8.12}$$

(Notice that $\mathbf{j}^n(X)$ is large enough to ensure that (8.12) is absolute for $\mathbf{j}^n(X)$; see for example [De, Lemma 1.9.10]. Notice also that we must restrict $\mathbf{j}^n$ to a set to ensure its elementarity.) The displayed formula says that the $\in$-formula coded by p is absolute for $\mathbf{j}^n(M), \mathbf{j}^n(N)$. Since p was arbitrary, we have shown formally that $M \prec N$ implies $\mathbf{j}^n(M) \prec \mathbf{j}^n(N)$.

By hypothesis, $\mathbf{j}^n(V_\kappa)$ exists, and as in Proposition 5.3(8), $\mathbf{j}^n(V_{\mathbf{j}(\kappa)})$ exists as well. By Lemma 8.6(2), $\mathbf{j}^n \upharpoonright V_\kappa : V_\kappa \to \mathbf{j}^n(V_\kappa) = V_{\mathbf{j}^n(\kappa)}$ is elementary. By Lemma 8.6(3), $\mathbf{j}^n \upharpoonright V_\kappa = \mathrm{id}_{V_\kappa}$. It follows that $V_\kappa \prec V_{\mathbf{j}(\kappa)}$. Now setting $M = V_\kappa$ and $N = V_{\mathbf{j}(\kappa)}$ in the previous paragraph, we conclude $V_{\mathbf{j}^n(\kappa)} \prec V_{\mathbf{j}^{n+1}(\kappa)}$, as required. ∎

Proof of (2). In this case, we argue as in [Co3, Proposition 3.12] by induction on the complexity of an $\in$-formula ϕ; in the present context, we must take care to extend the elementary chain only as far as $\mathbf{j}^n(\kappa)$ exists. To this end, we define

$$\mathrm{CondSat}(n,p,A,b) \equiv \big(\text{“}\mathbf{j}^n(A) \text{ exists”}\big) \Longrightarrow \mathrm{Sat}(p, \mathbf{j}^n(A), b).$$

$$\mathrm{CondSatLimInf}(k,p,A,p) \equiv \mathrm{Sat}(p, \mathbf{j}^k(A), b) \,\wedge\, \forall n \geq k \;\mathrm{CondSat}(n,p,A,b).$$

We show that

$$\begin{aligned} \mathrm{ZFC} + \mathrm{WA}_0 \vdash \forall r, b \Big[&(r = \mathrm{rank}(\ulcorner \phi \urcorner) \,\wedge\, \text{“}b \text{ is a function with domain } r\text{”} \\ &\Longrightarrow \big[\phi(b(1), \dots, b(m)) \iff \exists k \in \omega \, \mathrm{CondSatLimInf}(k, \ulcorner \phi \urcorner, V_\kappa, b)\big]\Big]. \end{aligned}$$

We prove the atomic and existential quantifier cases.

For the forward direction in the case of atomic formulas $\phi(x_1, x_2)$ and assignment $\langle b(1), b(2) \rangle$, we can find, by the Cofinal Axiom, a $k \in \omega$ for which $b(1), b(2) \in V_{\mathbf{j}^k(\kappa)}$, whence $\mathrm{Sat}(\ulcorner \phi \urcorner, V_{\mathbf{j}^k(\kappa)}, b)$; it follows easily that $\forall n \geq k \,\mathrm{CondSat}(n, \ulcorner \phi \urcorner, V_\kappa, b)$. The converse is immediate.

For the existential quantifier case, assume

$$\phi(x_1, \dots, x_m) \equiv \exists y \, \psi(x_1, \dots, x_m, y)$$

and let $r = \mathrm{rank}(\ulcorner \phi \urcorner)$ and let b be a function defined on r. For one direction, if $k \in \omega$ is such that $\mathrm{Sat}(\ulcorner \phi \urcorner, V_{\mathbf{j}^k(\kappa)}, b)$, and for all $n \geq k$, $\mathrm{CondSat}(n, \ulcorner \phi \urcorner, V_\kappa, b)$, let $c \in V_{\mathbf{j}^k(\kappa)}$ and b' be such that

$$b' \upharpoonright (r \setminus \{m+1\}) = b \upharpoonright (r \setminus \{m+1\}) \quad \text{and} \quad b'(m+1) = c, \tag{8.13}$$

and $\mathrm{Sat}(\ulcorner \psi \urcorner, V_{\mathbf{j}^k(\kappa)}, b')$. By part (1), it follows that for all $n \geq k$, $\mathrm{CondSat}(n, \ulcorner \psi \urcorner, V_\kappa, b')$. By the induction hypothesis, $\psi(b'(1), \dots, b'(m), b'(m+1))$ holds, and hence so does $\phi(b(1), \dots, b(m))$.

For the other direction, assume $\phi(b(1), \dots, b(m))$ holds and let c, b' be such that b' is as in (8.13) and $\psi(b'(1), \dots, b'(m), b'(m+1))$. Using the induction hypothesis, one can find $k \in \omega$ such that $\mathrm{Sat}(\ulcorner \psi \urcorner, V_{\mathbf{j}^k(\kappa)}, b')$ and for each $n \geq k$, $\mathrm{CondSat}(n, \ulcorner \psi \urcorner, V_\kappa, b')$. The result follows. ∎

As a first application of Theorem 8.8, we improve upon Theorem 2.14(2) and Proposition 5.6:

Corollary 8.9. *Suppose $A(x)$ is a large cardinal property expressible in the language $\{\in\}$. Suppose* $\mathrm{ZFC} + \mathrm{WA}_0 \vdash A(\kappa)$. *Then*

$$\mathrm{ZFC} + \mathrm{WA}_0 \vdash \forall\alpha\,\exists\lambda > \alpha\; A(\lambda).$$

Proof. By Theorem 8.8(2) and Theorem 2.14(1),

$$V_{\mathbf{j}(\kappa)} \models A(\kappa) \wedge \text{“}\{\alpha < \kappa : A(\alpha)\} \text{ is unbounded in } \kappa\text{”}. \tag{8.14}$$

For each n for which $\mathbf{j}^n(\kappa)$ exists, $i = \mathbf{j}^n \restriction V_{\mathbf{j}^2(\kappa)}$ exists and is an elementary embedding. Applying i to (8.14), we have

$$V_{\mathbf{j}^{n+1}(\kappa)} \models A(\mathbf{j}^n(\kappa)) \wedge \text{“}\{\alpha < \mathbf{j}^n(\kappa) : A(\alpha)\} \text{ is unbounded in } \mathbf{j}^n(\kappa)\text{”}.$$

Since the class of ordinals β such that $\beta = \mathbf{j}^n(\kappa)$ for some $n \in \omega$ are cofinal in ON, the proof is complete. ∎

We turn to the proof that WA_0 has essentially the same large cardinal consequences as those that are known to follow from WA itself. We first recall the notions of n-huge and super-n-huge cardinals: For each $n \in \omega$, κ is n-huge if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and M is closed under $j^n(\kappa)$-sequences; $j(\kappa)$ is called the *target* of j and j is called an *n-huge embedding*. For any cardinal ν, κ is n-huge ν times if there is a one-one function $f : \nu \to \mathrm{ON}$ and there are elementary embeddings j_α, $\alpha < \nu$ such that for each α, the target of j_α is $f(\alpha)$. Finally, κ is super-n-huge if, for every cardinal $\lambda > \kappa$, κ is n-huge λ times. It is well-known (see [Co3] or [Ka]) that the existence of a huge cardinal implies the consistency of many strong, supercompact, and extendible cardinals; that a superhuge cardinal is also strong, supercompact, extendible, and, of course, huge; that consistency of $n+1$-huge implies consistency of super-n-huge; and that the property of being super-n-huge for every n is the strongest among these variants of hugeness.

Let $\kappa_0 = \kappa$ and for all $n \geq 1$ for which $\mathbf{j}^n(\kappa)$ exists, we let $\kappa_n = \mathbf{j}^n(\kappa)$. We need two other notions. For any $\mathbf{j}$-class $\mathbf{C}$, we define $\mathbf{j} \cdot \mathbf{C}$ by

$$\mathbf{j} \cdot \mathbf{C} = \bigcup_{\alpha \in \mathrm{ON}} \mathbf{j}(\mathbf{C} \cap V_\alpha).$$

In ZFC + WA, such definitions make sense since $\mathbf{C} \cap V_\alpha$ is a set. We call $\cdot$ *application.* It is easy to see that

$$\mathbf{j} \cdot \mathbf{j} = \bigcup_{\alpha \in \mathrm{ON}} \mathbf{j}(\mathbf{j} \restriction V_\alpha).$$

The definition of $\mathbf{j} \cdot \mathbf{j}$ makes essential use of $\mathrm{Separation}_{\mathbf{j}}$; for this reason, it is not definable in weaker theories such as ZFC + BTEE + $\mathrm{Induction}_{\mathbf{j}}$. Using straightforward variants of the definition of application, it is also not possible to define iterates of application in $\mathrm{ZFC} + \mathrm{WA}_0$ — for instance, we could try to define $\mathbf{j} \cdot \mathbf{j} = \{(x, y) : \exists X\, [x \in \mathbf{j}(X) \wedge \mathbf{j}(\mathbf{j} \upharpoonright X)(x) = y]\}$. But since $\mathbf{j} \cdot \mathbf{j}$ has a Σ_1 definition, $(\mathbf{j} \cdot \mathbf{j}) \upharpoonright Y$ is not, in general a set in $\mathrm{ZFC} + \mathrm{WA}_0$ for arbitrary sets Y since it requires a Σ_1 instance of Separation$_j$. To handle the problem, one might try the definition $\mathbf{j} \cdot \mathbf{j} = \{(x, y) : x \in \mathbf{j}(x) \wedge \mathbf{j}(\mathbf{j} \upharpoonright \mathbf{j}(x))(x) = y]\}$. With this approach, $\mathbf{j} \cdot \mathbf{j}$ becomes Σ_0-definable, but it is not defined everywhere. (Under this definition, $\mathbf{j} \cdot \mathbf{j}$ *is* defined on every ordinal α and rank V_α for which $\alpha \geq \kappa$, but is not defined on members of V_κ, on any finite set, nor on any set A for which $A \notin \operatorname{ran}(\mathbf{j})$ and $|A| < \kappa$.) Therefore, $\mathbf{j} \cdot (\mathbf{j} \cdot \mathbf{j})$ is not definable in any obvious way in $\mathrm{ZFC} + \mathrm{WA}_0$.

The following results are well-known and easily proven in $\mathrm{ZFC} + \mathrm{WA}$; we verify that only Σ_0-Separation$_\mathbf{j}$ is required:

Proposition 8.10. *The following can be formalized and proven within* $\mathrm{ZFC} + \mathrm{WA}_0$*:*

(1) $\operatorname{cp}(\mathbf{j} \cdot \mathbf{j}) = \mathbf{j}(\kappa)$.
(2) *for all* $n \geq 1$, *if* κ_n *exists,* $\mathbf{j} \cdot \mathbf{j}(\kappa_n) = \kappa_{n+1}$.
(3) $(\mathbf{j} \cdot \mathbf{j}) \circ \mathbf{j} = \mathbf{j} \circ \mathbf{j}$.

Proof of (1). Let $\alpha > \kappa$. Let $i = \mathbf{j} \upharpoonright V_\alpha$. Since $\operatorname{cp}(i) = \kappa$, we have, by elementarity, that

$$\forall \beta < \mathbf{j}(\kappa)\, (\mathbf{j}(i)(\beta) = \beta),$$

and

$$\mathbf{j}(i)(\mathbf{j}(\kappa)) > \mathbf{j}(\kappa).$$

Thus, $\operatorname{cp}(\mathbf{j} \cdot \mathbf{j}) = \mathbf{j}(\kappa)$. ∎

Proof of (2). Let $\alpha > \kappa_{n-1}$. Let $i = \mathbf{j} \upharpoonright V_\alpha$. Applying $\mathbf{j}$ to the formula

$$i(\kappa_{n-1}) = \kappa_n$$

yields

$$\mathbf{j} \cdot \mathbf{j}(\kappa_n) = \mathbf{j}(i)(\mathbf{j}(\kappa_{n-1})) = \mathbf{j}(\kappa_n) = \kappa_{n+1}.$$

($\mathbf{j}$ was applied in the middle step. Notice that Induction$_\mathbf{j}$ is not required for the argument.)∎

Proof of (3). Let x be a set and α an ordinal such that $x \in V_\alpha$. Let $i = \mathbf{j} \upharpoonright V_\alpha$ and let $y = \mathbf{j}(x)$. Then, applying $\mathbf{j}$ to the formula

$$i(x) = y$$

yields

$$(\mathbf{j} \cdot \mathbf{j})(\mathbf{j}(x)) = \mathbf{j}(i)(\mathbf{j}(x)) = \mathbf{j}(y) = (\mathbf{j} \circ \mathbf{j})(x),$$

as required. ($\mathbf{j}$ was applied in the middle step.) ∎

In later sections, we will need to consider the self-applicative iterates

$$\mathbf{j}\cdot(\mathbf{j}\cdot\mathbf{j}), \mathbf{j}\cdot(\mathbf{j}\cdot(\mathbf{j}\cdot\mathbf{j})), \ldots$$

defined in ZFC + WA, and so we give the relevant definitions and preliminary lemmas here; these will not be used in the rest of this section.

We work in the theory ZFC + WA; in particular, full Induction$_{\mathbf{j}}$ holds. Using the definition-by-induction theorem (Theorem 4.7), we may define the two-variable $\mathbf{j}$-class sequence $\langle \mathbf{j}_{(n)} : n \in \omega\rangle$; we begin by defining auxiliary class functions **F, G** and **H**:

$$\begin{aligned}
\mathbf{F}(x) &= V_{\mathrm{rank}(x)+1};\\
\mathbf{G}(0,x) &= \mathbf{j}\restriction \mathbf{F}(x)\\
\mathbf{G}(n+1,x) &= \mathbf{j}(\mathbf{G}(n,x));\\
\mathbf{H}(n,x) &= \mathbf{G}(n,x)(x);
\end{aligned}$$

Now we define $\mathbf{j}_{(n)}$ by

$$\mathbf{j}_{(n)}(x) = \mathbf{H}(n,x).$$

Note in particular that $\mathbf{j}_{(0)} = \mathbf{j}$, $\mathbf{j}_{(1)} = \mathbf{j}\cdot\mathbf{j}$, and $\mathbf{j}_{(2)} = \mathbf{j}\cdot(\mathbf{j}\cdot\mathbf{j})$.

Using the ideas in the proof of Proposition 8.10, one uses Induction$_{\mathbf{j}}$ to prove the following:

Proposition 8.11. *The theory* ZFC + WA *proves the following:*

(1) $\forall n \in \omega\, \mathrm{cp}(\mathbf{j}_{(n)}) = \kappa_n$.

(2) $\forall n \in \omega\, \forall r \geq n\, \mathbf{j}_{(n)}(\kappa_r) = \kappa_{r+1}$.∎

We also obtain the following:

Proposition 8.12. *Suppose* $\bar{\mathcal{M}} = \langle M, E, j\rangle \models$ ZFC + WA *and suppose* $\bar{\mathcal{M}} \models n \in \omega$. *Let* $k : M \to M$ *be defined by*

$$k(x) = y \iff \bar{\mathcal{M}} \models j_{\{n\}}(x) = y.$$

Then $\langle M, E, k\rangle \models$ ZFC + WA.

Proof. The fact that k is a nontrivial elementary embedding is obvious. The fact that the new model satisfies Separation$_{\mathbf{j}}$ follows from the fact that $j_{\{n\}}$ is defined from j in $\bar{\mathcal{M}}$. ∎

See [La] for many results concerning application (in the context of embeddings $j : V_\lambda \to V_\lambda$).

We also define:

$$U^{\mathbf{j}} = \{X \in P(\mathbf{j}(\kappa)) : \mathbf{j}(\kappa) \in \mathbf{j}\cdot\mathbf{j}(X)\}.$$

Notice that $U^{\mathbf{j}}$ is a Σ_0-definable subset of $P(\mathbf{j}(\kappa))$. One verifies easily that $U^{\mathbf{j}} = \mathbf{j}(U)$, where $U = \{X \in P(\kappa) : \kappa \in \mathbf{j}(X)\}$.

Finally, since Φ and Ψ are not guaranteed to be total class functions in the context of ZFC + WA_0, we define the following notion: Let $\mathbf{A} = \{n \in \omega : \kappa_n \text{ exists}\}$. $\mathbf{A}$ may be a proper class. By the Cofinal Axiom, every set is contained in a V_{κ_n} for some $n \in \mathbf{A}$. Working in the context of ZFC + WA_0, we shall say that, for any property $P(n)$, P is true *for all n that matter* if for all $n \in \mathbf{A}$, $P(n)$ holds.

Proposition 8.13. ZFC+$\mathrm{WA}_0 \vdash$ *κ is the κth cardinal that is super-n-huge for all n that matter.*

Proof. We first show that the expected proof of n-hugeness goes through under the given hypotheses. Let $n \in \mathbf{A}$. By Proposition 8.3, $\mathbf{j}''\kappa_n$ is a set. We obtain the usual n-huge ultrafilter U as follows:

$$U = \{X \in P(P(\kappa_n)) : \exists z \in P(\kappa_n)\, (z = \mathbf{j}''\kappa_n \ \wedge \ z \in \mathbf{j}(X))\}.$$

Since $\mathbf{j}''\kappa_n$ is Σ_0, the defining formula for U is clearly also Σ_0, and hence U is a set by Σ_0-Separation$_j$. It is necessary to verify that U is κ-complete, fine, closed under diagonal intersections, and also contains all collections of the form $C_i = \{x \in P(\kappa_n) : \mathrm{ot}(x \cap \kappa_{i+1}) = \kappa_i\}$ for each $i < n$. The usual proofs work as long as the usual collections are actually sets. The verification of the first three of these involves only straightforward applications of elementarity of $\mathbf{j}$. That each of the C_i is a set follows immediately from $\mathrm{ZFC}_{\mathbf{j}}$.

Next, for each $n \in \mathbf{A}$ we show that for all $m \in \mathbf{A}$, κ is n-huge with κ_m targets. Since the κ_m for $m \in \mathbf{A}$ are cofinal in ON, this suffices to establish super-n-hugeness. Define

$$S_1 = \big\{ \alpha < \mathbf{j}(\kappa) : \alpha \text{ is a target of some } n\text{-huge embedding having critical point } \kappa \big\}.$$

Clearly, S_1 is a set. Also, $S_1 \in U^{\mathbf{j}}$ since $\mathbf{j}(\kappa)$ is a target of an n-huge embedding having critical point κ, as we just showed. Hence, S_1 is stationary.

We wish to define by recursion on the natural numbers in $\mathbf{A}$

$$S_{m+1} = (\mathbf{j} \cdot \mathbf{j})(S_m). \tag{8.15}$$

and observe by elementarity that each S_m is stationary in κ_m for each $m \in \mathbf{A}$, thereby completing the proof of super-n-hugeness. The standard formula for defining the S_m by recursion is Σ_1 (since it asserts the existence of functions that describe the build-up of the S_m and since $\mathbf{A}$ is Σ_1); to obtain the construction using only Σ_0 notions, we proceed indirectly, as in Theorem 8.4: Assume that for some $m \in \mathbf{A}$, κ is not n-huge with κ_m targets. In other words,

$$\begin{aligned}\forall \alpha \geq \kappa_m\, \forall X \subseteq \alpha \ \mid \{\gamma \in X : & \gamma \text{ is a target of an } n\text{-huge} \\ & \text{embedding with critical point } \kappa\} \mid < \kappa_m.\end{aligned} \tag{8.16}$$

Let δ be a cardinal such that $\delta = |V_\delta|$ and $\kappa_m < \delta$. Let $z = \mathbf{j} \restriction V_\delta$. We use Σ_0-Induction$_{\mathbf{j}}$ to prove the following:

$$\forall k \in \omega\, \gamma(k, m, n, z, S_1, V_\delta),$$

where

$$\begin{aligned}\gamma(k,m,n,v,w,Y) \equiv k \le m \Longrightarrow \\ &\Big[\exists X \in Y\, \exists f \in Y^{[1,k]}\big(\text{"}f, v \text{ are functions"} \wedge \operatorname{dom} v = Y \wedge \\ &f(1) = w \wedge \forall i\big(2 \le i \le k \Longrightarrow f(i) = \mathbf{j}(v)(f(i-1))\big) \wedge f(k) = X \wedge \\ &\exists \beta \in Y\, \exists g \in Y^{k+1}(\Theta(g,k,\kappa,\beta) \wedge \text{"}X \text{ is stationary in } \beta\text{"}) \wedge \\ &\forall \nu \in X\, \exists U \in Y\, \mathbf{target}(\kappa,n,\nu,U)\big)\Big],\end{aligned}$$

where $\mathbf{target}(\kappa,n,\nu,U)$ asserts that U is an n-huge ultrafilter on $P(\nu)$ with critical point κ.

We observe the following:

(a) The definition of f starts at 1 rather than at 0.

(b) In the formula $\gamma(k,m,n,z,S_1,V_\delta)$, $\mathbf{j}(z)(f(i-1))$ must agree with $(\mathbf{j}\cdot\mathbf{j})(f(i-1))$ for each i because the domain of z is large enough.

(c) As in previous arguments (such as Theorem 4.4(1)), uniqueness of f and X are guaranteed by $\mathrm{ZFC}_{\mathbf{j}}$.

(d) The final clauses in the definition of $\gamma(k,m,n,z,S_1,V_\delta)$ say that $f(k) = X$ is a stationary subset of κ_k and that X consists of targets of n-huge embeddings with critical point κ, formalized in terms of ultrafilters. Note that δ was chosen large enough to contain all such ultrafilters and to ensure that all of their relevant properties, including stationarity of X, are expressible as Σ_0 properties relative to V_δ.

For the induction, the case $k = 1$ is easy. Assuming $\gamma(k,m,n,z,S_1,V_\delta)$ and $1 \le k < m$, we obtain witnesses f_k, X_k where $f_k : [1,k] \to V_\delta$ and X_k is a stationary subset of κ_k. Define $X = \mathbf{j}(z)(X_k)$ and $f = f_k \cup \{(k+1, X)\}$. Certainly, $f : [1,k+1] \to V_\delta$. Also $X = (\mathbf{j}\cdot\mathbf{j})(X_k) \subseteq \kappa_{k+1}$, and by elementarity, X is stationary. Since $\mathbf{j}\cdot\mathbf{j}$ fixes κ, we still have that $\forall \nu \in X\, \exists U \in Y\, \mathbf{target}(\kappa,n,\nu,U)$. This completes the induction.

It now follows that we have witnesses $f : [1,m] \to V_\delta$ and $X \subseteq \kappa_m$ for γ. The fact that X is a stationary subset of κ_m consisting of targets of n-huge embeddings having critical point κ contradicts (8.16). We may therefore conclude that κ is n-huge κ_m times. Since n, m were arbitrary and the κ_m are cofinal, it follows that κ is super-n-huge for all n that matter.

Finally, to prove that κ is the κth such cardinal, we apply Theorem 2.14(1). ∎

Because $\mathrm{ZFC} + \mathrm{WA}_0$ appears to have the same large cardinal consequences as $\mathrm{ZFC} + \mathrm{WA}$, it is natural to ask whether the theories are the same, or if not, equiconsistent. Hamkins [Ha1] shows that $\mathrm{WA}_0 \not\equiv \mathrm{WA}_1$ by obtaining a forcing extension of a model of $\mathrm{ZFC} + \mathrm{WA}_0$ in which the latter holds, but in which a Σ_1-definable subclass of ω fails to be a set. Another example, which underscores the need to keep track of which $\mathbf{j}^n(\kappa)$ are defined in our arguments under $\mathrm{ZFC} + \mathrm{WA}_0$, is a modification of Hatch's model (Section 4) (also observed independently by Hamkins): Start with a nonstandard model $\mathcal{M} = \langle M, E, j\rangle$ of $\mathrm{ZFC} + \mathrm{WA}_0$, and define $\mathcal{N} = \langle N, E, i\rangle$ by letting $N = \{x \in M : \exists n \in \omega\, \mathcal{M} \models \mathrm{rank}(x) < \mathbf{j}^n(\kappa)\}$, and $i = j \restriction N$, as in Hatch's model. As before, $\mathcal{N}$ is a model of $\mathrm{ZFC} + \mathrm{BTEE}$, and as in [H], Σ_1-Induction$_j$ fails for the formula Ψ: in particular, there are nonstandard n for which $i^n(\kappa)$ fails to be defined. Finally, it is easy to see that,

$$\forall x \in N\; i \restriction x \in N. \tag{8.17}$$

Therefore, WA_0 holds in $\mathcal{N}$. This shows that $\mathrm{ZFC} + \mathrm{WA}_0$ fails to prove one of the consequences of $\mathrm{ZFC} + \mathrm{WA}_1$; in particular, it shows that Σ_1-Induction$_j$ is not provable from $\mathrm{ZFC} + \mathrm{WA}_0$.

§9. Intermediate Axioms

The results of the last section show that there is a significant gap between the consistency strengths of the theories ZFC + BTEE and ZFC + WA_0. We would like to close this gap by carefully selecting individual instances of Σ_0-Separation$_j$ to add to ZFC + BTEE, or possibly other axioms, with the hope of gently increasing the strength of the theory and in this way obtain a ladder of extensions of ZFC + BTEE having the full spectrum of consistency strengths. However, as Theorem 8.2 shows, even apparently weak consequences of Σ_0-Separation$_j$ can turn out to be very strong.

As a first step, we give a simple characterization of $0^{\#}$. We also consider several natural candidates for such axioms. The most fruitful approach that we have discovered for obtaining intermediate strength axioms is by restricting Amenability$_{\mathbf{j}}$ to *local* versions of Amenability — in other words, axioms of the form $\exists z\,(z = \mathbf{j}\restriction x)$, for various sets x. Axioms of this kind produce extensions of ZFC + BTEE having consistency strengths with lower bounds ranging from a strong cardinal to a huge cardinal, and beyond. So far, however, we do not know how to provide tight upper bounds for many of these theories, except for those at the upper end of the spectrum. On the other hand, we can provide better bounds for the theory obtained by adding an axiom that asserts the existence of the ultrafilter on κ derived from $\mathbf{j}$: We show that the theory ZFC + BTEE, augmented by this new axiom, has consistency strength somewhere between a measurable cardinal of high Mitchell order and a cardinal κ that is 2^κ-supercompact.

Proposition 9.1. *The following are equivalent:*

(1) $0^{\#}$ *exists.*

(2) *There is a model* $\langle M, \in, j\rangle$ *which satisfies the theory* ZFC + BTEE *such that* M *is a transitive class containing all the ordinals.*

(3) *There is a transitive set* M *and an elementary embedding* $i : M \to M$ *with critical point* κ *such that* ON^M *is an uncountable cardinal.*

Proof. Assuming $0^{\#}$ exists, we can obtain nontrivial elementary embeddings $\mathbf{L} \to \mathbf{L}$ and $L_\lambda \to L_\lambda$ for any uncountable cardinal λ (see Theorem 3.2); this establishes (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). The converse in each case is obtained by restricting the given embedding $M \to M$ to $\mathbf{L}^M \to \mathbf{L}^M$. ∎

In our present framework, it becomes apparent that the proof of the equivalence between the existence of $0^{\#}$ and the existence of a nontrivial elementary embedding $\mathbf{L} \to \mathbf{L}$ requires an additional assumption that is not usually mentioned in the literature: in order for the existence of a $j : \mathbf{L} \to \mathbf{L}$ to imply $0^{\#}$, j must be "sufficiently" definable in V (though not in $\mathbf{L}$). Certainly, requiring that the model $\langle \mathbf{L}, \in, j\rangle$ be sharp-like is sufficient, but even the assumption that j satisfies all Separation axioms is enough. Without such an assumption, the equivalence can fail:

Example 9.2 . *A model in which there is an elementary embedding* $\mathbf{L} \to \mathbf{L}$ *and* $0^{\#}$ *does not exist*. Assume that the universe $\langle V, \in, j\rangle$ is a model of $\mathrm{ZFC} + \mathrm{BTEE} +$ "$0^{\#}$ does not exist" (such a model is easy to obtain: if there is an ω-Erdös cardinal, there is one in $\mathbf{L}$, and so we can carry out the argument in Proposition 3.5 to obtain the required model). We obtain an inherited model $\langle \mathbf{L}, \in, j \restriction \mathbf{L}\rangle$ by restriction of j. Now we have $j : \mathbf{L} \to \mathbf{L}$ and $0^{\#}$ does not exist.

Of course, there is no such model which is *sharp-like*. We turn to a brief discussion of four candidates for intermediate axioms; the formulation of each of these axioms is natural in the present context. The first of these provides a natural restriction of $\text{Amenability}_{\mathbf{j}}$:

Ordinal $\textit{Amenability}_{\mathbf{j}}$: $\forall\alpha\left[\text{“}\alpha \text{ is an ordinal”} \Longrightarrow \exists z\,(z = \mathbf{j} \restriction \alpha)\right]$.

We also consider restrictions of $\text{Amenability}_{\mathbf{j}}$ to sets. Since κ is the only legitimate constant in the language, such local versions need to be formulated in terms of κ. We shall write $\text{LOA}_{\mathbf{j}}$ as an abbreviation for Local Ordinal Amenability.

$\textit{LOA}_{\mathbf{j}}(\kappa^+)$: $\exists z\,(z = \mathbf{j} \restriction \kappa^+)$.

$P(\kappa)$-$\textit{Amenability}_{\mathbf{j}}$: $\exists z\,(z = \mathbf{j} \restriction P(\kappa))$.

We also consider a syntactically natural axiom: It says that one of the $\mathbf{j}$-classes determined by the atomic $\mathbf{j}$-formula "$\kappa \in \mathbf{j}(X)$" is a set:

Measurable Ultrafilter Axiom: The class $\{X \subseteq \kappa : \kappa \in \mathbf{j}(X)\}$ is a set.

At the end of this section, we will discuss a sequence of additional axioms that generalize Measurable Ultrafilter Axiom. We now discuss the relative consistency strengths of the axioms described above, as far as these are known. First, we show that Ordinal $\text{Amenability}_{\mathbf{j}}$ is in fact equivalent to $\text{Amenability}_{\mathbf{j}}$; this observation is due to the referee:

Proposition 9.3. *The following are equivalent in* $\mathrm{ZFC} + \mathrm{BTEE}$*:*

(1) $\text{Amenability}_{\mathbf{j}}$

(2) Ordinal $\text{Amenability}_{\mathbf{j}}$.

Proof. We prove (2) $\Rightarrow$ (1). Assume Ordinal $\text{Amenability}_{\mathbf{j}}$. Given a set A, let $\pi : A \to \gamma$ be a bijection, where γ is a cardinal. Clearly, $\mathbf{j}(\pi) : \mathbf{j}(A) \to \mathbf{j}(\gamma)$. By Ordinal $\text{Amenability}_{\mathbf{j}}$, there is a set function $f = \mathbf{j} \restriction \gamma$. Let $p = \mathbf{j}(\pi)$. Define g on A by

$$g = \{(x, y) \mid x \in A \text{ and } f(\pi(x)) = p(y)\}.$$

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[d, "g=\mathbf{j}\restriction A"'] & \gamma \arrow[d, "f=\mathbf{j}\restriction\gamma"] \\
\mathbf{j}(A) \arrow[r, "p=\mathbf{j}(\pi)"] & \mathbf{j}(\gamma)
\end{tikzcd}$$

Clearly, $g = \mathbf{j} \restriction A$.■

The proof shows slightly more: For any infinite cardinal λ, the proof gives us that

$$\exists z\,(z = \mathbf{j} \restriction \lambda) \iff \forall A\,(|A| = \lambda \longrightarrow \exists z\,(z = \mathbf{j} \restriction A)). \tag{9.1}$$

Also, by the proposition, we have:

$$\text{Amenability}_{\mathbf{j}} \Longrightarrow \text{Ordinal Amenability}_{\mathbf{j}} \Longrightarrow P(\kappa)\text{-Amenability}_{\mathbf{j}} \Longrightarrow \text{LOA}(\kappa^+).$$

We also have that $P(\kappa)$-Amenability$_{\mathbf{j}}$ $\Longrightarrow$ Measurable Ultrafilter Axiom (use ordinary Separation to define $U = \{X \in P(\kappa) : \kappa \in (\mathbf{j} \restriction P(\kappa))(X)\}$).

Next, we show that the existence of $0^\#$ is derivable from LOA(κ^+), but then observe that both this axiom and $P(\kappa)$-Amenability$_{\mathbf{j}}$ are actually much stronger than this lower bound suggests.

Proposition 9.4. ZFC + BTEE + LOA$_{\mathbf{j}}(\kappa^+) \vdash$ "$0^\#$ *exists*".

Proof. Using LOA$_{\mathbf{j}}(\kappa^+)$, obtain the restriction $i = \mathbf{j} \restriction L_\alpha : L_\alpha \to L_{\mathbf{j}(\alpha)}$, where $\alpha = \kappa^+$. Certainly $\mathbf{j}(\kappa^+)$ is a cardinal. By Theorem 3.2, the result follows. ■

M. Zeman pointed out to the author that, using standard techniques from inner model theory, many natural candidate axioms that are restrictions of Amenability$_{\mathbf{j}}$ can be shown to have consistency strength in the vicinity of the strongest large cardinals for which there is a good core model theory. With his permission, I have given below an outline of results of this kind for the axioms LOA$_{\mathbf{j}}(\kappa^+)$ and $P(\kappa)$-Amenability$_{\mathbf{j}}$. These proofs assume some background in inner model theory which we do not provide here; see [Ze].

Proposition 9.5 [Zeman]**.** ZFC + BTEE + LOA$_{\mathbf{j}}(\kappa^+) \vdash$ *"there is an inner model of a strong cardinal."*

Proof (*Outline*) Assume there is no inner model of a strong cardinal and build the corresponding core model $\mathbf{K}$. Clearly, $\mathbf{j} \restriction \mathbf{K} : \mathbf{K} \to \mathbf{K}$. It is easy to show that from LOA$_{\mathbf{j}}(\kappa^+)$, one gets the existence of the set $z = \mathbf{j} \restriction \mathcal{J}^E_{\kappa^+}$, where E is the extender sequence for $\mathbf{K}$. Let $\nu = \sup(\mathbf{j}''(\kappa^+))$. Because κ^+ has uncountable cofinality, one defines from z in a canonical way a class embedding $\pi : \mathbf{K} \to N$; moreover, it follows that $N = \mathbf{L}[E']$ for some extender sequence E' for which $\mathcal{J}^{E'}_\nu = \mathcal{J}^E_\nu$. The Iteraration Map Theorem (which holds in the context of building inner models of a single strong cardinal) implies that π must be an iteration map. This yields a contradiction because the extender sequences for $\mathbf{K}$ and N must agree on $[\kappa, \mathrm{o}(\kappa))$. (They clearly do not agree here since, for example there must be a β in this interval for which $E_\beta \neq \emptyset$ for $\mathbf{K}$, but any such E_β must be empty for N because $\mathbf{j}(\kappa) > \beta$.) The result follows. ■

Proposition 9.6 [Zeman]**.** ZFC + BTEE + $P(\kappa)$-Amenability$_{\mathbf{j}}$ $\vdash$ *"there is an inner model of ω Woodin cardinals."*

Remark. This proposition represents a sample of what is possible; since it is known that the construction of $\mathbf{K}^c$ does not break down under the assumption that there is no inner model of ω Woodin cardinals, we have used this particular large cardinal assertion here. In fact, any such large cardinal assertion whose negation admits a successful construction of $\mathbf{K}^c$ could be used here.

Proof (*Outline*) Assume there is no inner model of ω Woodin cardinals. Build $\mathbf{K}^c$ by the usual inductive construction of N_α, M_α, and $E^{N_\alpha}_{\omega\beta_\alpha}$, with $M_\alpha = \mathrm{core}(N_\alpha)$. The clause in this construction that concerns us is the case in which α is a limit and we have defined E^{N_α}, and there is an extender F for which the following two conditions hold:

(a) $\langle \mathcal{J}^{E^{N_\alpha}}_{\beta_\alpha}, F\rangle$ is a premouse; and
(b) F is background certified.

Recall that in this case, the induction specifies that

$$N_\alpha = \langle \mathcal{J}^{E^{N_\alpha}}_{\beta_\alpha}, F\rangle. \tag{9.2}$$

Having constructed $\mathbf{K}^c$, we see again that $\mathbf{j} \restriction \mathbf{K}^c : \mathbf{K}^c \to \mathbf{K}^c$. Let $\lambda = \mathbf{j}(\kappa)$ and $\nu = \sup(\mathbf{j}''(\kappa^+))$; certainly λ is a cardinal. Let E denote the extender sequence for $\mathbf{K}^c$. Let F be the extender derived from $\mathbf{j} \restriction \mathbf{K}^c$. Note that F is definable from the set $\mathbf{j} \restriction P(\kappa)$ since, in fact, $F = \mathbf{j} \restriction P^W(\kappa)$ where $W = \mathbf{K}^c$.

Now observe there is α such that $E^{N_\alpha} = E \restriction \nu$ (where E^{N_α} was defined as above in the inductive definition). We verify that N_α is defined at this stage as in (9.2). Because F is derived from an embedding, condition (a) holds. Also, F is background certified (in a strong sense) because $F = F' \cap \mathcal{J}^E_\nu$ where F' is the extender derived from $\mathbf{j}$. Again, note that F' is a set because we have assumed $\mathbf{j} \restriction P(\kappa)$ exists. It follows that

$$N_\alpha = \langle \mathcal{J}^{E^{N_\alpha}}_{\beta_\alpha}, F\rangle = M_\alpha,$$

since $\omega\rho^n_{N_\alpha} = \omega\rho^\omega_{N_\alpha} = \lambda$. Now, we have

$$\rho_{\alpha\,\infty} = \lambda \text{ and } \tau_{\alpha,\infty} \geq \beta_\alpha.$$

Thus, M_α is an initial segment of $M_{\bar\alpha}$ whenever $\bar\alpha > \alpha$. But this is impossible, and the result follows. ■

At the end of this section, we build a transitive model of ZFC + BTEE + $P(\kappa)$-Amenability$_{\mathbf{j}}$ assuming a 2-huge cardinal. We next consider the Measurable Ultrafilter Axiom, and obtain upper and lower bounds. The bounds in this case are much sharper.

Proposition 9.7. *Let $T = \text{ZFC} + \text{BTEE} + \text{Measurable Ultrafilter Axiom}$.*

(1) $T \vdash$ *"κ is a measurable cardinal".*

(2) $T \vdash$ *"the measurables below κ form a normal measure 1 set".*

(3) *For each particular natural number $n \geq 1$,*

$$T \vdash \text{"the measurables below } \mathbf{j}^n(\kappa) \text{ form a normal measure 1 set"}.$$

Proof. (1) is clear, and (2) follows as usual because $\kappa \in \mathbf{j}(\{\alpha < \kappa : \alpha \text{ is measurable}\})$. (3) follows from (2) by elementarity of $\mathbf{j}^n$. ■

In fact, the theory $T = \text{ZFC} + \text{BTEE} + \text{Measurable Ultrafilter Axiom}$ has consistency strength at least a measurable of high Mitchell order. To show this, recall the Mitchell order on normal measures over a measurable cardinal κ is defined by

$$U_1 < U_2 \text{ iff } U_1 \in \text{Ult}(V, U_2),$$

where, as usual, $\text{Ult}(V, U_2)$ is identified with its transitive collapse. The order relation is a well-founded pre-order. For any normal measure on κ, $\text{o}(U)$ is the rank of U in $<$; also, $\text{o}(\kappa)$ — the Mitchell order of κ — is the height of $<$. See [Je1] for more discussion of basic results.

We define the *degree* $\deg(\kappa)$ of a cardinal κ inductively by the following clauses:

(a) $\deg(\kappa) \geq 0$ iff κ is measurable

(b) if $0 < \gamma < \kappa$, then $\deg(\kappa) \geq \gamma + 1$ iff for some normal measure U on κ,

$$\{\alpha < \kappa : \alpha \text{ is a cardinal and } \deg(\alpha) \geq \gamma\} \in U.$$

(c) if $0 < \gamma \leq \kappa$ is a limit, then $\deg(\kappa) \geq \gamma$ iff for some normal measure U on κ, $S_\delta \in U$ whenever $\delta < \gamma$ and $S_\delta = \{\alpha < \kappa : \alpha \text{ is a cardinal and } \deg(\alpha) \geq \delta\}$.

Lemma 9.8. *Suppose κ is an infinite cardinal and $0 \leq \gamma \leq \kappa$. Then $\deg(\kappa) \geq \gamma$ if and only if there is a normal measure U on κ for which $o(U) \geq \gamma$.*

Proof. We proceed by induction on γ to prove the following slightly stronger statement: For all normal measures U on κ and all $\gamma \leq \kappa$, U witnesses that $\deg(\kappa) \geq \gamma$ if and only if $\text{o}(U) \geq \gamma$. The case $\gamma = 0$ is obvious. If $\gamma > 0$ is a limit, then, for all normal measures U on κ,

$$\begin{aligned} U \text{ witnesses } \deg(\kappa) \geq \gamma \quad &\Longleftrightarrow \quad S_\delta \in U \text{ whenever } 0 \leq \delta < \gamma \\ &\Longleftrightarrow \quad \text{o}(U) \geq \delta \text{ whenever } 0 \leq \delta < \gamma \\ &\Longleftrightarrow \quad \text{o}(U) \geq \gamma. \end{aligned}$$

For the successor step, assume $\text{o}(U) \geq \gamma + 1$. Let $U' < U$, where $\text{o}(U') \geq \gamma$. By the induction hypothesis, U' witnesses that $\deg(\kappa) \geq \gamma$. Let $M = \text{Ult}(V, U)$. Since $U' \in M$ and $P(\kappa) \subset M$,

$M \models \deg(\kappa) \geq \gamma$. It follows that U witnesses $\deg(\kappa) \geq \gamma + 1$. Conversely, if $\deg(\kappa) \geq \gamma + 1$ with witness U and $M = \mathrm{Ult}(V, U)$, it follows that $M \models \deg(\kappa) \geq \gamma$, and in M there is a witness U'. Since $P(\kappa) \subset M$, we have in V that U' is a normal measure in V, $U' < U$, and U' witnesses $\deg(\kappa) \geq \gamma$. By the induction hypothesis, $\mathrm{o}(U') \geq \gamma$. Therefore, $\mathrm{o}(U) \geq \gamma + 1$.■

Proposition 9.9. ZFC + BTEE + Measurable Ultrafilter Axiom $\vdash o(\kappa) > \kappa$.

Proof. Let $\mathbf{U}$ be the ultrafilter on κ derived from $\mathbf{j}$. We prove by induction on γ that $\mathbf{U}$ is a witness for $\deg(\kappa) \geq \gamma$ for all $\gamma \leq \kappa$. The statement is obvious for $\gamma = 0$.

For the successor step, if $\gamma < \kappa$, we show $\deg(\kappa) \geq \gamma + 1$; this follows, as we show, because it is equivalent to the statement $\deg(\kappa) \geq \gamma$:

$$\begin{aligned} \deg(\kappa) \geq \gamma \quad &\iff \quad \kappa \in \{\alpha < \mathbf{j}(\kappa) : \deg(\alpha) \geq \gamma\} \\ &\iff \quad \{\alpha < \kappa : \deg(\alpha) \geq \gamma\} \in \mathbf{U} \\ &\iff \quad \deg(\kappa) \geq \gamma + 1. \end{aligned}$$

For the limit step, if $\gamma \leq \kappa$ is a limit, then, by induction hypothesis, $\mathbf{U}$ witnesses $\deg(\kappa) \geq \delta + 1$ for all $\delta < \gamma$, whence $\{\alpha < \kappa : \deg(\alpha) \geq \delta\} \in \mathbf{U}$; the result follows.■

On the other hand, we can build a model of ZFC+BTEE+Measurable Ultrafilter Axiom from a 2^κ-supercompact cardinal κ, using the method of Proposition 7.6, as follows.

Proposition 9.10. *If κ is 2^κ-supercompact, there is a transitive model of* ZFC + BTEE+ Measurable Ultrafilter Axiom.

Proof. Let κ be 2^κ-supercompact and let U be a normal measure on $P_\kappa(2^\kappa)$. We may obtain a sequence of iterated ultrapowers based on U, just as in Proposition 7.6:

$$M_0 \xrightarrow{i_{01}} M_1 \xrightarrow{i_{12}} M_2 \to \ldots \to M_n \xrightarrow{i_{n,n+1}} M_{n+1} \to \ldots \to M_\omega.$$

Using the same arguments, one obtains the following commutative diagram of elementary embeddings

$$\begin{tikzcd} V \arrow[r, "i_\omega"] \arrow[d, "i_1"'] & M_\omega \arrow[d, "i_1 \restriction M_\omega"] \\ M_1 \arrow[r, "i_1 \cdot i_\omega"] & i_1(M_\omega) \end{tikzcd}$$

and one shows, as before, that $j = i_1 \restriction M_\omega : M_\omega \to M_\omega$, and that the critical sequence for j is $\langle \kappa^{(0)}, \kappa^{(1)}, \ldots \rangle$, with supremum $\kappa^{(\omega)}$. As before, $\langle M_\omega, \in, j \rangle \models \mathrm{ZFC} + \mathrm{BTEE}$. We observe however that, in the present setting, we have

$$\big(P(P(\kappa))\big)^{M_\omega} = \big(P(P(\kappa))\big)^V, \tag{9.3}$$

where $\kappa = \kappa^{(0)}$. To see this, first note that since M_1 is closed under 2^κ-sequences, $(P(2^\kappa))^{M_1} = (P(2^\kappa))^V$. Also, because $i_{01}(\kappa) > 2^\kappa$, the standard argument shows that $i_{1\omega}(X) = X$ whenever $X \subseteq 2^\kappa$, and likewise, $i_{1\omega}(Y) = Y$ for all $Y \subseteq P(\kappa)$. Equation (9.3) follows, and therefore, $U = \{X \subset \kappa : \kappa \in j(X)\} \in M_\omega$. Therefore,

$$\langle M_\omega, \in, j\rangle \models \text{ZFC} + \text{BTEE} + \text{Measurable Ultrafilter Axiom.}\blacksquare$$

The techniques for establishing upper and lower bounds on Measurable Ultrafilter Axiom can be generalized in many ways; we illustrate with one example, which provides a sequence of additional axioms whose consistency strengths lie in the vicinity of n-huge cardinals. For each particular n, we define:

Huge Amenability$_n$: $\exists z\,(z = \mathbf{j}\restriction P(P(\mathbf{j}^n(\kappa))))$.

For the rest of this section, let $f = \mathbf{j}\restriction P(P(\mathbf{j}^n(\kappa)))$ and let $g = \mathbf{j}\restriction \mathbf{j}^n(\kappa)$. The existence of f allows us to define the n-huge ultrafilter W derived from $\mathbf{j}$:

$$W = \{X \in P(P(\mathbf{j}^n(\kappa))) \mid \text{range}(g) \in f(X)\}.$$

Moreover, if $\mathbf{U}$ denotes the normal measure on κ derived from $\mathbf{j}$, it is easy to see that

$$\{\alpha < \kappa \mid \alpha \text{ is } n\text{-huge}\} \in \mathbf{U}.$$

Therefore Huge Amenability Axiom$_n$ is bounded below by the existence of an n-huge cardinal with many n-huge cardinals below it (and, reasoning as we did for Measurable Ultrafilter Axiom, many n-huge cardinals above, as well).

For an upper bound, we can perform an iterated ultrapower construction, starting with an $n+2$-huge ultrafilter U. Following the development in the proof of Proposition 9.10, we obtain the sequence

$$M_0 \xrightarrow{i_{01}} M_1 \xrightarrow{i_{12}} M_2 \to \ldots \to M_n \xrightarrow{i_{n,n+1}} M_{n+1} \to \ldots \to M_\omega.$$

Letting $j = i_1 \restriction M_\omega : M_\omega \to M_\omega$, one shows as before that j has critical sequence $\langle \kappa^{(0)}, \kappa^{(1)}, \ldots\rangle$, with supremum $\kappa^{(\omega)}$, and that $\langle M_\omega, \in, j\rangle$ is a model of ZFC + BTEE. In the present context, we have that M_1 is closed under $j^{n+2}(\kappa)$-sequences, and so

$$\left(V_{j^{n+2}(\kappa)}\right)^{M_1} = V_{j^{n+2}(\kappa)}.$$

Since the critical point of $i_{1\omega}$ is $j^{n+2}(\kappa)$, it follows that for each $X \in V_{j^{n+2}(\kappa)}, i_{1\omega}(X) = X$. It follows, therefore, that

$$\left(V_{j^{n+2}(\kappa)}\right)^{M_\omega} = V_{j^{n+2}(\kappa)}.$$

We therefore have

$$j \restriction P(P(j^n(\kappa))) \in V_{j^{n+2}(\kappa)} \subset M_\omega,$$

and so

$$\langle M_\omega, \in, j\rangle \models \mathrm{ZFC} + \mathrm{BTEE} + \text{Huge Amenability Axiom}_n.$$

We summarize these observations in the following proposition:

Proposition 9.11. *Upper and lower bounds for* ZFC + BTEE + Huge Amenability Axiom$_n$ *are given by the following, for each particular n:*

(1) ZFC + BTEE + Huge Amenability Axiom$_n$ $\vdash$ "*κ is n-huge and admits a normal measure that contains the set of n-huge cardinals below κ.*".

(2) *Assuming an $n+2$-huge cardinal κ, there is a transitive M and an elementary embedding $j : M \to M$ such that* $\langle M, \in, j\rangle \models$ ZFC + BTEE + Huge Amenability Axiom$_n$. ∎

The "$n+2$-huge" upper bound can certainly be improved. Indeed, if one is willing to accept a more clumsy pair of axioms as a means to formulate the intuition that the n-huge ultrafilter derived from $\mathbf{j}$ exists (and so, replace Huge Amenability Axiom$_n$ with this alternative pair of axioms), one can get by with "$n+1$-huge": The first axiom asserts the existence of $g = \mathbf{j} \restriction \mathbf{j}^n(\kappa)$, and the second asserts the existence of the ultrafilter $W = \{X \in P(P(\mathbf{j}^n(\kappa)) \mid \mathrm{range}(g) \in \mathbf{j}(X)\}$. These two together still imply that κ is n-huge with a normal 1 measure set of n-huge cardinals below. But now, if we construct $\langle M_\omega, \in, j\rangle$ as above, starting from an $n+1$-huge, since both $j \restriction j^n(\kappa)$ and $\{X \in P(P(\mathbf{j}^n(\kappa)) \mid \mathrm{range}(g) \in \mathbf{j}(X)\}$ are elements of $V_{j^{n+1}(\kappa)}$, $\langle M_\omega, \in, j\rangle$ is a model of these two alternative axioms.

Finally, let us observe that our reasoning above shows easily that the model $\langle M_\omega, \in, j\rangle$ obtained from a 2-huge cardinal satisfies both LOA(κ^+) and $P(\kappa)$-Amenability, though one would expect that this bound is far from optimal.

§10. Replacement Axioms And Inconsistency.

In Section 7 we showed that, under mild large cardinal hypotheses, transitive models of both ZFC + BTEE + Cofinal Axiom and ZFC + BTEE + ¬Cofinal Axiom can be built. In Sections 8 and 9, we studied extensions of the first of these theories, and observed that these are the extensions that have consistency strengths that parallel those of the usual large cardinal axioms. In this section, we attempt to extend ZFC + BTEE + ¬Cofinal Axiom as far as possible by adding as much Replacement for **j**-formulas as possible. The strongest version of Replacement for **j**-formulas already implies Cofinal Axiom (even Separation$_{\mathbf{j}}$), and so in this case we cannot obtain a consistent theory if we include all the possible instances. However, we show that all possible instances of Collection for **j**-formulas *can* consistently be added. We also describe several inconsistent theories that combine low complexity Separation$_{\mathbf{j}}$ instances with low complexity instances of each of the following variants of Replacement: Replacement$_{\mathbf{j}}$, Collection$_{\mathbf{j}}$, Strong Replacement$_{\mathbf{j}}$, defined below.

Replacement$_j$: For each **j**-formula $\psi(x, y, \vec{u})$,

$$\forall A\, \forall \vec{a}\, \big(\forall x \in A\, \exists! y\, \psi(x, y, \vec{a}) \Longrightarrow \exists Y\, \forall z\, [z \in Y \Longleftrightarrow (\exists x \in A\, \psi(x, z, \vec{a}))]\big). \tag{10.1}$$

Strong Replacement$_j$: For each **j**-formula $\psi(x, y, \vec{u})$,

$$\forall A\, \forall \vec{a}\, \big(\forall x \in A\, \exists^* y\, \psi(x, y, \vec{a}) \Longrightarrow \exists Y\, \forall z\, [z \in Y \Longleftrightarrow (\exists x \in A\, \psi(x, z, \vec{a}))]\big). \tag{10.2}$$

where '$\exists^*$' is an abbreviation for "there exists at most one."

Collection$_j$: For each **j**-formula $\psi(x, y, \vec{u})$,

$$\forall A\, \forall \vec{a}\, \big(\forall x \in A\, \exists y\, \psi(x, y, \vec{a}) \Longrightarrow \exists Y\, \forall x \in A\, \exists y \in Y\, \psi(x, y, \vec{a})]\big). \tag{10.3}$$

As in Section 7, we let Σ_n-Replacement$_{\mathbf{j}}$ (Π_n-Replacement$_{\mathbf{j}}$) denote the restriction of the Replacement$_{\mathbf{j}}$ schema to Σ_n (Π_n) **j**-formulas ψ. (We continue to follow our convention of calling a formula Σ_n (Π_n) when it may only be $\Sigma_n^{\mathrm{ZFC}_{\mathbf{j}}}$ ($\Pi_n^{\mathrm{ZFC}_{\mathbf{j}}}$).) Similarly, we define $\Sigma_n(\Pi_n)$-Strong Replacement$_{\mathbf{j}}$ and $\Sigma_n(\Pi_n)$-Collection$_{\mathbf{j}}$.

We observe that Σ_n-Strong Replacement$_{\mathbf{j}}$ $\Longrightarrow$ Σ_n-Separation$_{\mathbf{j}}$: Given a Σ_n-formula $\phi(x, \vec{u})$, let $\psi(x, y, \vec{u}) \equiv \phi(x, \vec{u}) \wedge x = y$. Let A be a set and $\vec{a}$ a finite sequence. Then the set Y given by Σ_n-Strong Replacement$_{\mathbf{j}}$ for $\psi, A, \vec{a}$ is precisely $\{x \in A : \phi(x, \vec{a})\}$.

Implications between these versions of Replacement for **j**-formulas are the same as for the ZFC versions.

Proposition 10.1. *The theory* ZFC$_{\mathbf{j}}$ *proves the following implications:*

(1) Strong Replacement$_{\mathbf{j}}$ $\Longrightarrow$ Replacement$_{\mathbf{j}}$ $\wedge$ Collection$_{\mathbf{j}}$.

(2) Replacement$_{\mathbf{j}}$ + Σ_0-Separation$_{\mathbf{j}}$ $\Longrightarrow$ Strong Replacement$_{\mathbf{j}}$.

(3) $\text{Collection}_{\mathbf{j}} + \Sigma_0\text{-Separation}_{\mathbf{j}} \Longrightarrow \text{Strong Replacement}_{\mathbf{j}}$.

Proof. The proof of Strong $\text{Replacement}_{\mathbf{j}} \Longrightarrow \text{Replacement}_{\mathbf{j}}$ in (1) is easy, and the proof of Strong $\text{Replacement}_{\mathbf{j}} \Longrightarrow \text{Collection}_{\mathbf{j}}$ is the same as the ZFC version; see [Je2, pp. 72-73]. The proof of (3) can also be found in [Je2, p. 73]. We prove (2): Given a **j**-formula $\phi(x, y, \vec{u})$, sets A,$\vec{a}$, and the fact that $\forall x \in A\, \exists^* y\, \phi(x, y, \vec{a})$, define a (total) class function $\mathbf{F}$ by

$$\mathbf{F}(x) = \begin{cases} (1, y) & \text{if } \phi(x, y, \vec{a}) \\ (0, 0) & \text{otherwise} \end{cases}$$

By hypothesis, $\mathbf{F}$ is well defined, and so, by $\text{Replacement}_{\mathbf{j}}$, there is a set $Y_0 = \mathbf{F}''A$. Let $S = Y_0 \setminus \{(0, 0)\}$, let $\alpha = \text{rank}(S) + 1$, and let $W = V_\alpha$. Let $Y = \{y \in W \mid \exists w \in S\, (y = (w)_1)\}$. By $\Sigma_0\text{-Separation}_{\mathbf{j}}$, S and Y are sets. Now $Y = \{y \mid \exists x \in A\, \phi(x, y, \vec{a})\}$, as required. ∎

As claimed in Section 7, the axiom ¬Cofinal Axiom is a consequence of (a version of) Replacement:

Proposition 10.2. $\text{ZFC} + \text{BTEE} + \text{Collection}_{\mathbf{j}} \vdash \neg\text{Cofinal Axiom}$.

Proof. Since Ψ may not be a (total) class function, we define the class function $\mathbf{F}$ by

$$\mathbf{F}(x) = \begin{cases} y & \text{if } \Psi(x, y) \\ \kappa & \text{if } \neg\exists y\, \Psi(x, y) \end{cases}$$

Since $\mathbf{F}$ is total, we can apply $\text{Collection}_{\mathbf{j}}$ and obtain Y such that for all $n \in \omega$, there is $y \in Y$ with $y = \mathbf{F}(n)$. If $\alpha = \sup(Y)$, then the critical sequence is bounded by α. Hence, ¬Cofinal Axiom holds. ∎

Note that the class function $\mathbf{F}$ in the last proposition is definable by a Σ_1 formula:

$$y = \mathbf{F}(x) \quad \Longleftrightarrow \quad \exists f\, \Theta(f, x, \kappa, y) \,\vee\, y = \kappa.$$

Therefore ¬Cofinal Axiom is in fact derivable from $\Sigma_1\text{-Collection}_{\mathbf{j}}$.

We turn now to the program of consistently extending the theory ZFC + BTEE+ ¬Cofinal Axiom by adding instances of (versions of) Replacement for **j**-formulas. The next proposition generalizes an observation due to Hamkins:

Proposition 10.3. *Suppose $\langle M, \in\rangle$ is an inner model of* ZFC, *$M \neq V$, and $\langle M, \in, j\rangle$ is a sharp-like model of* $\text{ZFC}_{\mathbf{j}}$. *Then*

$$\langle M, \in, j\rangle \models \text{ZFC} + \text{Collection}_{\mathbf{j}}.$$

Remark. The condition "$M \neq V$" in the hypothesis of the proposition is redundant since it follows from the fact that $\mathcal{M}$ is sharp-like.

Proof. Let $\mathcal{M} = \langle M, \in, j\rangle$. Suppose $\phi(x, y, \vec{u})$ is a formula and $A \in M$. Suppose there are $\vec{a}$ such that

$$\mathcal{M} \models \forall x \in A\, \exists y\, \phi(x, y, \vec{a}).$$

In V, this means that for each $x \in A$, there exists y such that $\phi^M(x, y, \vec{a}) \wedge y \in M$. Since j and M are definable in V, we may apply ordinary Collection to ϕ^M to obtain W satisfying

$$\forall x \in A\, \exists y \in W\, \phi^M(x, y, \vec{a}) \wedge y \in M.$$

We may assume (using Separation in V if necessary) that $W \subset M$. Let $\delta > \mathrm{rank}(W)$. It follows that

$$\mathcal{M} \models \forall x \in A\, \exists Y\, \phi(x, y, \vec{a}),$$

where the witness Y is V_δ^M. ∎

Both the models $\mathcal{M}$ and $\mathcal{N}$ of Proposition 7.6 are examples of Proposition 10.4, obtained under the assumption of a measurable cardinal. These examples show that ZFC + BTEE + Collection$_{\mathbf{j}}$ does not decide whether the critical sequence is a set, even though ¬Cofinal Axiom is derivable. The next corollary was observed in [Co3 Metatheorem 2.5], but proved by different means.

Corollary 10.5. *There is no sharp-like model $\langle M, \in, j\rangle$ of* ZFC + WA *or* ZFC + WA$_0$ *for which $\langle M, \in\rangle$ is an inner model.*

Proof. If there were such a model $\mathcal{M}$, by Proposition 10.4 and Proposition 10.2, $\mathcal{M} \models$ ¬Cofinal Axiom; since ZFC + WA$_0 \vdash$ Cofinal Axiom, this is impossible. ∎

Note that the corollary does not forbid the inner model V itself from admitting a j for which $\langle V, \in, j\rangle \models$ ZFC + WA.

The proof of Proposition 10.6 also gives us the following:

Proposition 10.7. *Suppose $\langle M, \in\rangle$ is a transitive set model of* ZFC, ON^M *is a regular cardinal, and $j : M \to M$ is a function (not necessarily elementary). Then*

$$\langle M, \in, j\rangle \models \text{ZFC} + \text{Collection}_{\mathbf{j}}. ∎$$

Corollary 10.8. *There is no transitive set model $\langle M, \in, j\rangle$ of* ZFC + WA *or* ZFC + WA$_0$ *for which* ON^M *is a regular cardinal.* ∎

In contrast to Collection$_{\mathbf{j}}$, we cannot hope to add all axioms of Strong Replacement$_{\mathbf{j}}$ to ZFC + BTEE and obtain a consistent theory:

Proposition 10.9. *The theory* ZFC + Elementarity + Nontriviality + Strong Replacement$_{\mathbf{j}}$ *is inconsistent.*

Proof. Since Strong Replacement$_{\mathbf{j}}$ implies Separation$_{\mathbf{j}}$, the theory proves Cofinal Axiom. Since Strong Replacement$_{\mathbf{j}}$ also implies Collection$_{\mathbf{j}}$, the theory also proves ¬Cofinal Axiom. ∎

Recall that, if $\langle M, E\rangle$ is a model of ZFC, a set $A \subseteq M$ is said to be *weakly definable in M* if the extended structure $\langle M, E, A\rangle$ for the extended language in which there is an additional unary relation $\mathbf{U}$, satisfies Strong Replacement for $\mathbf{U}$-formulas.

Corollary 10.10. *If $\langle M, E\rangle \models \mathrm{ZFC}$, there is no weakly definable nontrivial elementary embedding $M \to M$.*

Proof. If there were such an embedding j, the structure $\langle M, E, j\rangle$ would satisfy the inconsistent theory $\mathrm{ZFC} + \mathrm{Elementarity} + \mathrm{Nontriviality} + \text{Strong Replacement}_{\mathbf{j}}$. ∎

We consider next several refinements of Proposition 10.9. These will lead to some partial results concerning the question, How much $\text{Replacement}_{\mathbf{j}}$ can be added to either of the theories $\mathrm{ZFC} + \mathrm{WA}_0$, $\mathrm{ZFC} + \mathrm{WA}$ without introducing inconsistency?

Lemma 10.11. $\mathrm{ZFC} + \mathrm{BTEE} + \Sigma_0\text{-Collection}_{\mathbf{j}} \vdash \Sigma_1\text{-Collection}_{\mathbf{j}}$.

Proof. Given a Σ_1 $\mathbf{j}$-formula $\psi(x, y, \vec{u})$, let $\theta(x, y, \vec{u}, z)$ be a Σ_0 $\mathbf{j}$-formula such that

$$\psi(x, y, \vec{u}) \equiv \exists z\, \theta(x, y, \vec{u}, z).$$

Working in $\mathrm{ZFC}_{\mathbf{j}}$, let A be a set and $\vec{a}$ be a finite sequence of parameters. Assume $\forall x \in A\, \exists y\, \psi(x, y, \vec{a})$. Then

$$\forall x \in A\, \exists y\, \exists z\, \theta(x, y, \vec{u}, z). \tag{10.4}$$

Clearly, (10.4) is equivalent to

$$\forall x \in A\, \exists w\, \theta'(x, (w)_0, \vec{u}, (w)_1, w),$$

where $(w)_0$ and $(w)_1$ are the zeroth and first coordinates of the ordered pair w, respectively, and $\theta'(x, y, \vec{u}, z, w)$ is the formula $\theta(x, y, \vec{u}, z) \wedge$ "w is an ordered pair". (Note that $\theta'(x, (w)_0, \vec{u}, (w)_1, w)$ is equivalent to a Σ_0 formula.) By $\Sigma_0\text{-Collection}_{\mathbf{j}}$, we can find a set Y such that

$$\forall x \in A\, \exists w \in Y\, \theta'(x, (w)_0, \vec{u}, (w)_1, w).$$

Without loss of generality, we may assume $Y = V_\gamma$ for some limit γ. Thus,

$$\forall x \in A\, \exists y \in Y\, \exists z \in Y\, \theta(x, y, \vec{a}, z),$$

whence,

$$\forall x \in A\, \exists y \in Y\, \psi(x, y, \vec{a}, z),$$

as required. ∎

Lemma 10.12. ZFC + Elementarity + Nontriviality + Π_1-Strong Replacement$_{\mathbf{j}}$ $\vdash$ Σ_0-Collection$_{\mathbf{j}}$.

Remark. As we show in Theorem 10.13, the theory ZFC + Elementarity + Nontriviality + Π_1-Strong Replacement$_{\mathbf{j}}$ is in fact inconsistent; however, to prove this, we need the preliminary step provided by this lemma.

Proof. Let $\phi(x, y, \vec{u})$ be a Σ_0 formula and let $A, \vec{a}$ be sets. Assume that

$$\forall x \in A \, \exists y \, \phi(x, y, \vec{a}).$$

The proof proceeds like the standard proof of Collection from Strong Replacement: for each $x \in A$, one forms the set $X_x = \{y : \phi(x, y, \vec{a})$ and y is of least possible rank$\}$. Letting $Y' = \{X_x : x \in A\}$, the required set Y is $\bigcup Y'$. What is needed here is to show that Π_1-Replacement$_j$ is sufficient to carry out the argument.

Consider the following Π_1 **j**-formulas:

$$\psi_1(x, \beta, v, \gamma, \vec{u}) \equiv \Big[\forall \delta < \beta \, \forall w \, (\mathrm{rank}(w) = \delta \Longrightarrow \neg\phi(x, w, \vec{u}))\Big] \wedge \Big[v = V_\gamma \, \wedge \, \gamma = \beta + 1 \, \wedge \, \exists w \in v \, \big(\phi(x, w, \vec{u})\big)\Big],$$

and

$$\psi_2(x, X, Z, \vec{u}) \equiv \forall w \in X \, [(\exists \beta \in Z \, \mathrm{rank}(w) = \beta) \, \wedge \, \phi(x, w, \vec{u})] \wedge \forall w \, \big[\big((\exists \beta \in Z \, \mathrm{rank}(w) = \beta) \, \wedge \, \phi(x, w, \vec{u})\big) \Longrightarrow w \in X\big].$$

The formula $\psi_1(x, \beta, v, \gamma, \vec{u})$ says that β is the least ordinal for which there is a w such that $\phi(x, w, \vec{u})$ holds and $\mathrm{rank}(w) = \beta$. The fact that ψ_1 is $\Pi_1^{\mathrm{ZFC}_{\mathbf{j}}}$ follows from the fact that "$z = \mathrm{rank}(x)$" is Δ_1^{ZF} and $v = V_\gamma$ is Π_1^{ZF}. It is easy to see that

$$\forall x \in A \, \exists! \beta \, \psi_1(x, \beta, v, \gamma, \vec{a}).$$

By Π_1-Replacement$_j$, there is a set Z such that

$$Z = \{\beta_x : x \in A\}, \tag{10.5}$$

where β_x is the unique β associated with a given $x \in A$.

Next, the formula $\psi_2(x, X, Z, \vec{u})$ asserts that X is the set of all w for which $\phi(x, w, \vec{u})$ holds and for which $\mathrm{rank}(w) \in Z$. Verification of the fact that $\psi_2(x, X, Z, \vec{u})$ is $\Pi_1^{\mathrm{ZFC}_{\mathbf{j}}}$ is straightforward; notice that the subformula "$\exists \beta \in Z \, \mathrm{rank}(w) = \beta$" is equivalent to a Π_1 formula because the bounded quantifier can be moved inside the scope of the unbounded quantifier in the Π_1 formulation of "$\mathrm{rank}(w) = \beta$." (This trick always works for $\in$-formulas, but not generally for $\mathcal{L}$-formulas, as pointed out in (2.1).)

Now, using the Z defined in (10.5) as a parameter in ψ_2, it is clear that

$$\forall x \in A\, \exists! X\, \psi_2(x, X, Z, \vec{a}).$$

By Π_1-Replacement$_j$, we can form the set

$$Y' = \{X_x : x \in A\},$$

where X_x is the unique set X associated with $x \in A$. But now we have

$$\forall x \in A\, \exists y \in Y\, \phi(x, y, \vec{a}),$$

where $Y = \bigcup Y'$, as required. ∎

Theorem 10.13. *Each of the following theories is inconsistent:*

(1) ZFC + Elementarity + Nontriviality + Σ_1-Strong Replacement$_{\mathbf{j}}$,
(2) ZFC + WA_0 + Σ_0-Collection$_{\mathbf{j}}$,
(3) ZFC + Elementarity + Nontriviality + Π_1-Strong Replacement$_{\mathbf{j}}$,
(4) ZFC + WA_0 + Σ_1-Induction$_{\mathbf{j}}$ + Σ_1-Replacement$_{\mathbf{j}}$
(5) ZFC + WA_0 + Π_1-Induction$_{\mathbf{j}}$ + Σ_1-Replacement$_{\mathbf{j}}$

Proof of (1). Σ_1-Strong Replacement$_{\mathbf{j}}$ implies Σ_0-Separation$_{\mathbf{j}}$, Σ_1-Induction$_{\mathbf{j}}$, and Σ_1-Replacement$_{\mathbf{j}}$. The first of these implies Cofinal Axiom; the third implies CI; and CI together with Σ_1-Induction$_{\mathbf{j}}$ implies ¬Cofinal Axiom. ∎

Proof of (2). Σ_0-Collection$_{\mathbf{j}}$ implies Σ_1-Collection$_{\mathbf{j}}$, which in turn implies ¬Cofinal Axiom. Since WA_0 implies Cofinal Axiom, the result follows.∎

Proof of (3). By Lemma 10.12, Π_1-Strong Replacement$_{\mathbf{j}}$ implies Σ_0-Collection$_{\mathbf{j}}$. On the other hand, Π_1-Strong Replacement$_{\mathbf{j}}$ implies Π_1-Separation$_{\mathbf{j}}$, which is equivalent to Σ_1-Separation$_{\mathbf{j}}$. Now the result follows from (2). ∎

Proof of (4) and (5). Use Σ_1-Induction$_{\mathbf{j}}$ or Π_1-Induction$_{\mathbf{j}}$ to ensure that each $\mathbf{j}^n(\kappa)$ exists, so that the hypothesis of CI holds. By Σ_1-Replacement$_{\mathbf{j}}$, CI holds, and it follows that the critical sequence is a set, whence we have ¬Cofinal Axiom. But WA_0 implies Cofinal Axiom. ∎

Theorem 10.13 leaves open two natural questions:

Question A. Is Replacement$_{\mathbf{j}}$ consistent with ZFC + WA_0 (or even with ZFC + BTEE)?

Question B. Is Σ_0-Replacement$_{\mathbf{j}}$ consistent with ZFC + WA?

For Question A, Hamkins has observed the following:

Proposition 10.14. *Relative to* ZFC + BTEE + Collection$_{\mathbf{j}}$, *it is consistent for* Replacement$_{\mathbf{j}}$ *to hold for all sets of size* $\leq \kappa$.

Proof. Let us recall the model $\mathcal{N}$ from Example 7.6: $\mathcal{N} = \langle M_\omega[S], \in, \hat{j}\rangle$ where M_ω is the direct limit of the ultrapower models $\langle M_n; i_{mn} : 0 \leq m \leq n < \omega\rangle$, starting from a normal measure in V; and $j = j_1 \restriction M_\omega$; $S = \{\kappa^{(n)} : n \in \omega\}$; and $\hat{j}$ is the usual lifting to the forcing extension. It is known (see [Je1, Theorem 21.15]) that

$$N = \bigcap_n M_n.$$

We have observed that $\mathcal{N} \models$ Collection$_{\mathbf{j}}$. We show that $\mathcal{N}$ satisfies Replacement$_{\mathbf{j}}$ for sets of size $\leq \kappa$. Suppose $\phi(x, y, \vec{u})$ is a $\mathbf{j}$-formula, A is a set that, in N, has cardinality $\leq \kappa = \kappa^{(0)}$, $\vec{a}$ are sets, and $\mathcal{N} \models \forall x \in A\, \exists! y\, (\phi(x, y, \vec{a}))$. Since each M_n is κ-closed, $N = \bigcap_n M_n$ is also κ-closed, and so $|A| \leq \kappa$ (in V). Since $\hat{j}$ is definable in V, we have in V

$$\forall x \in A\, \exists! y\, (y \in N \wedge \phi^{\mathcal{N}}(x, y, \vec{a})).$$

We can use ordinary Replacement in V to obtain Y such that

$$Y = \{y \in N \mid \exists x \in A\, \phi^{\mathcal{N}}(x, y, \vec{a})\}.$$

Now $Y \subset N$ and has cardinality $\leq \kappa$. Again since N is κ-closed, $Y \in N$. ∎

In light of the proposition, a reasonable conjecture is that, for each cardinal λ, there is an inner model $\mathcal{N}_\lambda = \langle N_\lambda, \in, j_\lambda\rangle$ satisfying ZFC + BTEE + Collection$_{\mathbf{j}}$ as well as "Replacement$_{\mathbf{j}}$ for all sets of size $\leq \lambda$". The strategy for showing this would be to perform the iterated ultrapower construction starting either with a λ-supercompact ultrafilter or a λ-strong extender. Most of the analogues to the theorems in the measurable case hold true in these other settings, except that it is not known whether $M_\omega[S] = \bigcap_n M_n$ — and this latter fact is needed (apparently) to show that $M_\omega[S]$ is λ-closed.

Our answer to Question B, however, will show that Replacement$_{\mathbf{j}}$ *for all sets* — in fact, Σ_0-Replacement$_{\mathbf{j}}$ for all sets — cannot hold in a transitive model of ZFC + WA$_0$.

Proposition 10.15. *The theory* ZFC + Σ_1-Induction$_{\mathbf{j}}$ + WA$_0$ + Σ_0-Replacement$_{\mathbf{j}}$ *is inconsistent. In particular,*

(1) *the theory* ZFC + WA + Σ_0-Replacement$_{\mathbf{j}}$ *is inconsistent*

(2) *there is no transitive model of* ZFC + WA$_0$ *in which* Σ_0-Replacement$_{\mathbf{j}}$ *holds.*

Proof. We prove the main part of the Proposition; parts (1) and (2) then follow immediately. Consider the following Σ_0 formula

$$\begin{aligned}\Psi_0(n, q) \quad \equiv \quad & n \in \omega \Longrightarrow \text{“}q \text{ is a an ordered pair} \wedge (q)_0 \text{ is a function with domain } n+1\text{”} \wedge \\ & (q)_0(0) = \kappa \wedge \forall i \left(0 < i \leq n \Longrightarrow (q)_0(i) = \mathbf{j}((q)_0(i-1))\right) \wedge (q)_0(n) = (q)_1.\end{aligned}$$

Using Σ_1-Induction$_{\mathbf{j}}$, one proves, as in Proposition 4.4, that

$$\forall n \in \omega \, \exists! q \, \Psi_0(n, q), \tag{10.6}$$

however, here, (10.6) is the hypothesis of a Σ_0 instance of Replacement$_{\mathbf{j}}$. Now the rank of the set Y that is given by Σ_0-Replacement$_{\mathbf{j}}$ bounds the critical sequence, and this contradicts Cofinal Axiom and hence WA_0. ■

§11. Open Questions.

The most interesting questions left open by our work here have to do with obtaining natural intermediate-strength extensions of $\mathrm{ZFC} + \mathrm{BTEE}$ to fill out the hierarchy of theories that we have begun to build. Having such a fine-grained ladder of theories could provide a useful alternative to the usual large cardinal axioms for measuring the strength of other theories in mathematics.

The first questions along these lines are concerned with pipointing a number of exact consistency strengths:

Question 1. What is the exact consistency strength of the theory ZFC + BTEE+ Measurable Ultrafilter Axiom?

Question 2. What are the exact consistency strengths of each of the theories $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{LOA}(\kappa^+)$ and $\mathrm{ZFC} + \mathrm{BTEE} + P(\kappa)\text{-Amenability}_{\mathbf{j}}$?

Question 3. What is the exact consistency strength of the theory ZFC + BTEE+ Huge Amenability Axiom$_n$ for each particular n?

Also, is there a natural way to fill out the hierarchy further? We have the following question:

Question 4. For each classical large cardinal axiom A, find a "natural" $\mathbf{j}$-axiom B such that $\mathrm{Con}(ZFC + A)$ is approximately equivalent to $\mathrm{Con}(\mathrm{ZFC} + \mathrm{BTEE} + B)$.

We showed in Proposition 10.14 that it is consistent with $\mathrm{ZFC} + \mathrm{BTEE} + \mathrm{Collection}_{\mathbf{j}}$ for $\mathrm{Replacement}_{\mathbf{j}}$ for all sets of size $\leq \kappa$ to hold. A natural question that we raised earlier is the following:

Question 5. Can the construction of $\mathcal{N}$ in Proposition 7.6 be modified to use supercompact ultrafilters (as in Proposition 9.10) so that we may conclude the following: For each cardinal $\lambda > \kappa$, there is an inner model N_λ and an elementary embedding $i_\lambda : N_\lambda \to N_\lambda$ such that

$$\langle N_\lambda, \in, i_\lambda \rangle \models \mathrm{ZFC} + \mathrm{BTEE} + \mathrm{Collection}_{\mathbf{j}} + \text{“Replacement}_{\mathbf{j}} \text{ for all sets of size } \leq \lambda\text{”?}$$

A technical question that is the key to answering Question 5, and obtaining other interesting consistency results is:

Question 6. In the construction of Proposition 9.10, is it possible to prove that

$$\bigcap_n M_n = M_\omega[S]?$$

Similarly, can this be proven when, instead of supercompact ultrafilters, we use huge ultrafilters? extenders for a strong or superstrong cardinal?

Our results on remarkability raise the following question (see Section 3):

Question 7. Does consistency of ZFC + BTEE imply consistency of a remarkable cardinal, or of the statement "$\mathbf{L}(R)$ is absolute (or absolute with ordinal parameters) under proper forcings"?

In [Co3], we showed that, assuming ZFC + WA, the critical sequence $\langle\kappa_0, \kappa_1, \kappa_2, \ldots\rangle$ is a $\mathbf{j}$-class of indiscernibles in V. It is natural to ask whether the same result holds for any of the subtheories of ZFC + WA studied here — in particular, for the theories ZFC + BTEE and ZFC + WA_0. The main observation in the proof of this result in ZFC + WA was the following (stated in paraphrased form):

Lemma 11.1. *Suppose $n_1 < n_2 < \ldots < n_s$ and $r > \max(\{n_{m+1} - n_m : 1 \le m < s\})$. A $\mathbf{j}$-class function $\mathbf{i}$, defined from $\mathbf{j}\cdot\mathbf{j}$, can be specified having the following properties:*

(1) *$\mathbf{i} : V \to V$ is an elementary embedding;*

(2) *$\mathrm{cp}(\mathbf{i}) > \kappa_{n_1}$;*

(3) *for $1 < m \le s$, $\mathbf{i}(\kappa_{n_m}) = \kappa_{n_1+(m-1)r}$.*

The lemma says that, given sequences $\kappa_{m_1} < \kappa_{m_2} < \ldots < \kappa_{m_s}$ and $\kappa_{n_1} < \kappa_{n_2} < \ldots < \kappa_{n_s}$, one can push these cardinals up high enough with the appropriate choice of $\mathbf{i}$ so that their transformed values agree; indiscernibility follows easily from this observation.

The proof does not work for weaker theories like ZFC + BTEE because the definition of $\mathbf{i}$ depends upon $\mathbf{j}\cdot\mathbf{j}$, and the latter is not definable in ZFC + BTEE alone since it requires the existence of sets of the form $\mathbf{j}\restriction X$ for arbitrary sets X (and this requires $\mathrm{Amenability}_{\mathbf{j}}$). The proof as it stands does not work in ZFC + WA_0 either because $\mathbf{j}\cdot(\mathbf{j}\cdot\mathbf{j})$ is not definable in that theory (even $\mathbf{j}\cdot\mathbf{j}\restriction X$ may not be defined). In Section 7, we repeatedly applied a trick to avoid such problems — proceed with an indirect argument and thereby obtain an upper bound of the form V_δ in which all the higher complexity arguments can be carried out in a Σ_0 way. When one attempts to apply this trick here, it is difficult to obtain the required upper bound. A good choice would be V_δ, where (using the notation of Lemma 11.1) $\delta > \kappa_{n_1+(s-1)r}$. However, assuming only WA_0, we have no guarantee that $\kappa_{n_1+(s-1)r}$ exists. Thus, new techniques will be needed to answer the following:

Question 8. Can the critical sequence be shown to be a $\mathbf{j}$-class of indiscernibles for either of the theories ZFC + BTEE or ZFC + WA_0?

Notice that if the answer to Question 9 is "no", at least for the theory ZFC + WA_0, then we would have on our hands an interesting property of $\mathbf{j}$ that holds in ZFC + WA but not in ZFC + WA_0; this result would be of some interest since it is still unknown whether there is a large cardinal property that follows from ZFC + WA but not from ZFC + WA_0.

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