

# The Axiom of Infinity, Quantum Field Theory, and Large Cardinals



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# The Quest for an Axiomatic Foundation For Large Cardinals

- Gödel believed natural axioms would be found to supplement ZFC (or other foundational theories), from which existence of large cardinals could be derived
- Till such axioms are found, Gödel suggested heuristics for making existence of large cardinals a reasonable assumption:
  - *Generalization*
  - *Reflection*

# The Quest for an Axiomatic Foundation For Large Cardinals

- Modern approach: Evidence for large cardinals can be found in the consequences derivable from them.
  - Example: Every Borel set is determined. What about analytic sets? If there is a measurable cardinal, every analytic set is determined.
  - Provides evidence for existence of measurable cardinals.

# The Quest for an Axiomatic Foundation For Large Cardinals

- Modern approach: Evidence for large cardinals can be found in the consequences derivable from them.
  - Example: Every Borel set is determined. What about analytic sets? If there is a measurable cardinal, every analytic set is determined.
  - Provides evidence for existence of measurable cardinals.
- *Fresh Approach*: Take another look at the Axiom of Infinity and use the insight that motivated this axiom to suggest what sorts of "infinity" we expect to find in the universe.

# What Does the Axiom of Infinity Tell Us?

## Axiom of Infinity

$$\exists \mathbf{I} (\emptyset \in \mathbf{I} \wedge \forall x \in \mathbf{I} ((x \cup \{x\}) \in \mathbf{I}))$$

*There is an inductive set*

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### Characteristics:

- Formulation is natural and economical
- Says just enough to guarantee existence of  $\omega$

# What It Does *Not* Tell Us

- Gives no intuition about what "infinite sets" really are
- Therefore, our Axiom of Infinity does not *naturally suggest* large cardinal properties, though with extra work these were discovered, like:
  - $\omega$  is a regular strong limit
  - $\omega$  has the tree property
  - $\omega$  admits a nonprincipal  $\omega$ -complete ultrafilter

# An Intuition About "Infinite Sets" Using a QFT Approach

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# An Intuition About "Infinite Sets" Using a QFT Approach

- ❑ Physics asks: What are the "ultimate building blocks" of matter?
- ❑ Classical view: Everything is made of *particles*.
- ❑ Quantum field theory, the most successful theory of physics to date, offers a deeper view:  
*The underlying reality of "matter" – of particles – is that they arise as precipitations of unbounded quantum fields.*

# The QFT View Applied to Infinite Sets

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# The QFT View Applied to Infinite Sets

- ❑ The Axiom of Infinity tells us the collection  $\{0, 1, 2, \dots\}$  is a set. This fundamental "infinite" is seen as a collection of discrete numbers.
- ❑ Alternative view inspired by QFT: The "underlying reality" of the discrete values that form infinite sets is, in some way, the dynamics of an unbounded "field" and discrete values are "precipitations" of this field
- ❑ What would the Axiom of Infinity look like if this is used as the motivating intuition?

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- A set  $A$  is *Dedekind-infinite* if there is a bijection from  $A$  to one of its proper subsets.
- A map  $j: A \rightarrow A$  is a *Dedekind self-map* if  $j$  is injective but not surjective. An element  $a \in A$  not in the range of  $j$  is a *critical point of  $j$* .

# A New Axiom of Infinity

*There is a Dedekind Self-Map  $j: A \rightarrow A$*

It is well-known that the theory ZFC – Infinity proves the equivalence of the usual Axiom of Infinity with this New Axiom of Infinity



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***The QFT Analogy:*** We think of  $j$  as an analogy for "dynamics of an unbounded field". We anticipate that the discrete values of an infinite set will arise as "precipitations" of these dynamics.

# Precipitations of a Dedekind Self-Map

- Let  $a$  be a critical point of  $j: A \rightarrow A$ . Let  $B = j''A$ . Then  $B$  is Dedekind infinite and  $j|_B: B \rightarrow B$  is a Dedekind self-map with critical point  $j(a)$ .

# Precipitations of a Dedekind Self-Map

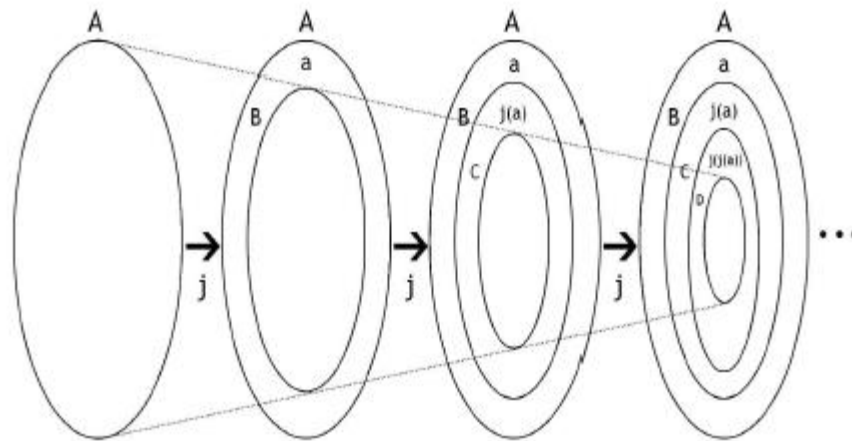
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- Let  $C = j''B$ . Then  $C$  is Dedekind infinite and  $j|_C: C \rightarrow C$  is a Dedekind self-map with critical point  $j(j(a))$ .

# Precipitations of $j: A \rightarrow A$ .

- Leads to formation of the *critical sequence* of  $j$ :

$$a, j(a), j(j(a)), \dots$$

– a *precursor* or *blueprint* for the natural numbers.



Transformational dynamics of a Dedekind self-map  $j: A \rightarrow A$

# Generating $\omega$ and $s: \omega \rightarrow \omega$

- $W = \{a, j(a), j(j(a)), \dots\}$  can be defined formally as the *smallest  $j$ -inductive subset of  $A$* .

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- Let  $(N, \epsilon)$  be the Mostowski collapse of  $(W, E)$  [ $\pi: (W, E) \cong (N, \epsilon)$ ]  
where  $x E y$  iff there are  $x = x_0, x_1, \dots, x_k = y$  with  $x_{r+1} = j(x_r)$  [ $E$  and the Mostowski collapsing map can be defined without relying on natural numbers]

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- The unique map  $s: N \rightarrow N$  making the diagram below commute is defined by  $s(x) = x \cup \{x\}$ . In particular,  $N = \omega$ .

$$\begin{array}{ccc}
 W & \xrightarrow{j \upharpoonright W} & W \\
 \downarrow \pi & & \downarrow \pi \\
 N & \xrightarrow{s} & N
 \end{array}$$

# Dynamics of $j: A \rightarrow A$ Produce a Blueprint of $\omega$

- Let  $f = j \upharpoonright W : W \rightarrow W$ ,  $W = \{a, j(a), j(j(a)), \dots\}$ .
- Let  $\mathcal{E} = \{i_1, i_2, \dots\}$  be functionals; for any self-map  $g$ , we define:  
$$i_n(g) = \pi \circ g^n.$$
- We say  $(f, a, \mathcal{E})$  is a *blueprint for  $\omega$*  because, for every  $n \in \omega$ , there is  $i$  in  $\mathcal{E}$  such that  
$$i(f)(a) = n$$



# Characteristics of a Dedekind Self-Map

Given a Dedekind self-map  $j: A \rightarrow A$  with critical point  $a$ .

- $j$  preserves essential characteristics of its domain
  - image of  $j$ , like its domain, is Dedekind infinite
  - restrictions of  $j$  to successive images are, like  $j$ , Dedekind self-maps
  
- A blueprint of the natural numbers is generated by interaction between  $j$  and its critical point.

# An Intuition for Moving Toward Large Cardinals

- If the QFT intuition is correct and the truth about infinite sets lies in the existence of an "underlying field" – a Dedekind self-map – then existence of large cardinals boils down to existence of the right kind of Dedekind self-map.

What is the "right kind" of Dedekind self-map?

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What is the "right kind" of Dedekind self-map?

- Defined as  $j: V \rightarrow V$
- $j: V \rightarrow V$  exhibits strong *preservation properties*
- $j: V \rightarrow V$  produces a blueprint for a key collection of sets, perhaps a blueprint for all of  $V$ .

# Dedekind Self-Maps of the Universe

- If  $V$  is a model of ZFC-Infinity, a Dedekind self-map  $j: V \rightarrow V$  need not imply existence of an infinite set (consider the successor function  $x \rightarrow x \cup \{x\}$ )
- Dedekind self-maps  $j: V \rightarrow V$  with reasonable preservation properties do, however, imply existence of infinite sets

# Some Preservation Properties for Self-Maps $V \rightarrow V$

Suppose  $j : V \rightarrow V$  is any function.

- (1)  $j$  is said to *preserve disjoint unions* if, whenever  $X, Y \in V$  are disjoint,  $j(X), j(Y)$  are also disjoint and  $j(X \cup Y) = j(X) \cup j(Y)$ .
- (2)  $j$  is said to *preserve coproducts* if, whenever  $X, Y$  are disjoint,  $|j(X \cup Y)| = |j(X) \cup j(Y)|$ .  $j$  *preserves products* if for all  $X, Y$ ,  $|j(X \times Y)| = |j(X) \times j(Y)|$ .
- (3)  $j$  *preserves singletons* if, for any  $X$ ,  $j(\{X\}) = \{j(X)\}$ .
- (4)  $j$  *preserves terminal objects* if  $j$  maps singleton sets to singleton sets since.

# Infinite Sets from Dedekind Self-Maps

**Theorem 1.** (ZFC – Infinity) Suppose  $j : V \rightarrow V$  is a class Dedekind self-map. Suppose also that  $j$  preserves disjoint unions, the empty set, and singletons. Then there is an infinite set.

- In Theorem 1, the infinite set that is obtained as the transitive closure of the critical point

# Infinite Sets from Dedekind Self-Maps

**Theorem 2.** (ZFC – Infinity) Suppose  $j : V \rightarrow V$  is a class Dedekind self-map with a strong critical point (some  $Z$  for which  $|Z| \neq |j(Z)|$ ). Suppose  $j$  preserves finite coproducts and terminal objects. Then there is an infinite set.

- In Theorem 2, the infinite set obtained is the strong critical point itself
- Assuming that  $\omega$  exists, a  $j: V \rightarrow V$  with the properties listed in Theorem 2 can be obtained.

# More Preservation Properties, Leading to Inaccessible Cardinals

Suppose  $j : V \rightarrow V$  is a function.

- (1)  $j$  *preserves images* if, whenever  $f : X \rightarrow Y$  is a function,  $j(f)$  is a function and  $j(\text{ran } f) = \text{ran } j(f)$ .
- (2) If  $j$  preserves ordinals, then  $j$  *preserves unboundedness* if, whenever  $\alpha \in \text{ON}$  and  $A \subseteq \alpha$  is unbounded in  $\alpha$ , then, if  $j(A) \subseteq j(\alpha)$ , then  $j(A)$  is unbounded in  $j(\alpha)$ .
- (3)  $j$  *preserves power sets* if, for all  $X$ ,  $j(\mathcal{P}(X)) = \mathcal{P}(j(X))$ .
- (4)  $j$  is a *functor* if  $j$  preserves identity maps and composition of functions.
- (5)  $j$  *preserves functions* if for all  $f, x$ ,  $j(f(x)) = j(f)(j(x))$ .
- (6) The *equalizer*  $E_{f,g}$  of two functions  $f, g$  with common domain and codomain is the set on which the functions agree. A functor  $j$  *preserves equalizers* if, for all such  $f, g$ ,  $j(E_{f,g}) = E_{j(f),j(g)}$ .
- (7) Coequalizers and definition of “preservation” of coequalizers are obtained by dualizing.



# Arriving at Inaccessible Cardinals

**Theorem 3.** Suppose  $j : V \rightarrow V$  is a Dedekind self-map that preserves  $\in$  and preserves ordinals. In this case, there must exist a  $\kappa$  that is the *least ordinal moved*.

- (A) Suppose  $j$  preserves countable disjoint unions, the empty set, and singletons. Then  $\kappa > \omega$ .
- (B) Assume (A) and suppose  $j$  also preserves cardinals, unboundedness, functions, and images. Then  $\kappa$  is an uncountable regular cardinal.
- (C) Assume (B) and suppose  $j$  also preserves power sets. Then  $\kappa$  is an inaccessible cardinal.

□ In Theorem 3 (C), the inaccessible obtained is the least critical point of  $j$ .

# Arriving at Measurable Cardinals

**Theorem 4.** There is a measurable cardinal if and only if there is a Dedekind self-map  $j : V \rightarrow V$  with a strong critical point, which has the following preservation properties:

- (1)  $j$  is a functor
- (2)  $j$  preserves equalizers and finite products
- (3)  $j$  preserves coequalizers and finite coproducts

- Given a  $j: V \rightarrow V$  with the properties of the Theorem, the least ordinal moved by  $j$  is at least as large as the least measurable (which must exist).
- The least among all strong critical points of Dedekind self-maps  $j: V \rightarrow V$  satisfying the properties of the theorem is the least measurable cardinal.

# Preserving *All* Properties

- As long as  $j$  is *definable* in  $V$ , we cannot require  $j$  to preserve all first order properties (by Kunen's Theorem). We study this notion by adding an extra function symbol to the language

The theory BTEE (*Basic Theory of Elementary Embeddings*) is the collection of the following axioms in the language  $\{\in, \mathbf{j}\}$ :

- (a) *Elementarity Schema*. Each of the following  $\mathbf{j}$ -sentences is an axiom, where  $\phi(x_1, x_2, \dots, x_m)$  is an  $\in$ -formula,

$$\forall x_1, x_2, \dots, x_m (\phi(x_1, x_2, \dots, x_m) \iff \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \dots, \mathbf{j}(x_m))).$$

- (b) *Least Ordinal Moved*.

$$\exists \kappa [\kappa \text{ is the least ordinal moved by } \mathbf{j}]$$

**Fact:** The consistency strength of ZFC + BTEE is less than 0-sharp. The critical point of a BTEE embedding is ineffable and totally indescribable.

# The Theory ZFC + BTEE + MUA

**Measurable Ultrafilter Axiom (MUA).** The class  $\{X \subseteq \kappa \mid \kappa \in \mathbf{j}(X)\}$  is a set.

## **Theorem 5.**

- (1) If  $j : V \rightarrow V$  is an MUA-embedding obtained in the theory ZFC + BTEE + MUA, the least ordinal moved by  $j$  is a measurable cardinal of high Mitchell order.
- (2) If  $\kappa$  is  $2^\kappa$ -supercompact, there is a transitive model of ZFC + BTEE + MUA.

# Blueprint in ZFC+BTEE + MUA

An MUA-embedding  $j : V \rightarrow V$  with critical point  $\kappa$  produces a blueprint  $(\ell, \kappa, \mathcal{E})$  for the rank  $V_{\kappa+1}$ . That is, for every  $x \in V_{\kappa+1}$  there is  $i \in \mathcal{E}$  with

$$i(\ell)(\kappa) = x.$$

Note:  $\ell$  is obtained from a partial Laver sequence for  $j$  and elements of  $\mathcal{E}$  are ultra-power maps  $V \rightarrow V^\kappa/U \cong N$  restricted to  $V_{\kappa+1}$ .

# The Theory ZFC + WA

**Wholeness Axiom (WA).** WA provides  $j$  with “as much definability as possible”:

$$\text{WA} = \text{BTEE} + \text{Separation}_j.$$

**Theorem 6.**

- (1) Suppose  $j : V \rightarrow V$  is a WA-embedding with critical point  $\kappa$ . Then  $\kappa$  is the  $\kappa$ th cardinal that is super- $n$ -huge for every  $n \in \omega$ .
- (2) *Blueprint for  $V$ .* A WA-embedding produces a blueprint for all of  $V$ . In particular, there is  $\ell : V_\kappa \rightarrow V_\kappa$  and  $\mathcal{E}$  such that  $(\ell, \kappa, \mathcal{E})$  is a blueprint for  $V$ .
- (3) *Blueprint for  $\omega$ .* Let  $f = j \upharpoonright \{\kappa, j(\kappa), j(j(\kappa)), \dots\}$ . Then  $f$  yields another blueprint for  $\omega$ , but now every element of the critical sequence is a WA-cardinal.

*Note.*  $\ell$  is obtained from an extendible Laver sequence and  $\mathcal{E}$  is the class of extendible embeddings, restricted to  $V_{\kappa+1}$ .

# Conclusion

- ❑ Based on the QFT perspective that particles are in reality precipitations of unbounded fields, we offer a New Axiom of Infinity based on the intuition that the "underlying reality" of infinite sets should be thought of as the "dynamics of an unbounded field" – realized as a Dedekind self-map
- ❑ A Dedekind self-map  $j: A \rightarrow A$  with critical point  $a$  generates an infinite set through interaction between  $j$  and  $a$ , exhibits strong preservation properties, and generates a blueprint for the set of natural numbers

# Conclusion

- We then sought a Dedekind self-map  $j: V \rightarrow V$ , with strong preservation properties, whose precipitations, obtained from  $j$  interacting with its critical point, result in large cardinals and also in a blueprint for a "significant" class of sets
- We found, ultimately that a WA-embedding  $j: V \rightarrow V$ 
  - Preserves *all* first order properties of its domain
  - Has a canonical critical point that has all known large cardinal properties below an  $I_3$  cardinal
  - Generates a blueprint of the universe.



# Appendix: The QFT View of Particles

“Quantum foundations are still unsettled, with mixed effects on science and society. By now it should be possible to obtain consensus on at least one issue: Are the fundamental constituents fields or particles? As this paper shows, experiment and theory imply unbounded fields, not bounded particles, are fundamental. . . Particles are epiphenomena arising from fields.”

Hobson, American Journal of Physics, 2013

# Appendix: Formalizing the Idea of a "Blueprint"

## Plan

- Examine carefully how a Dedekind self-map "produces" the natural numbers, from the theory ZFC-Infinity, and, more importantly, how it "generates" the usual successor function  $S: \omega \rightarrow \omega$
- Extract essential features to obtain a *definition* of blueprint
- Make Part B of the Conjecture (concerning blueprints) more precise.

# Obtaining $s: \omega \rightarrow \omega$ from $j: A \rightarrow A$

Given a Dedekind self-map  $j: A \rightarrow A$  with critical point  $a$ .

1. Define  $W \subseteq A$  (intuitively,  $W = \{a, j(a), j(j(a)), \dots\}$ )
  - A set  $B \subseteq A$  is  $j$ -inductive if
    - (1)  $a \in B$
    - (2) whenever  $x \in B$ ,  $j(x)$  is also in  $B$ .
  - Let  $\mathcal{I} = \{B \subseteq A \mid B \text{ is } j\text{-inductive}\}$ . Let  $W = \bigcap \mathcal{I}$ .  
Note:  $W$  is  $j$ -inductive.

# Obtaining $s: \omega \rightarrow \omega$ from $j: A \rightarrow A$

2. Define an order relation  $E$  on  $W$  as follows (intuitively,  $x E y$  iff there are  $x=x_0, x_1, \dots, x_k = y$  with  $x_{r+1} = j(x_r)$ ). For all  $x, y \in W$ ,  $x E y$  iff there is  $F \subseteq W$  satisfying:

(i)  $x, y, \in F$

(ii) for some  $v \in F, y = j(v)$

(iii) there is no  $u \in F$  for which  $x = j(u)$

(iv) if  $u \in F$  and  $u \neq y$ , then  $j(u) \in F$

(v) if  $v \in F$  and  $v \neq x$ , there is  $u \in F$  such that  $v = j(u)$

The set  $F$  is said to *join*  $x$  to  $y$ , and is called a *joining set*. Intuitively,  $F$  is *finite*, but we do not state this in the definition, nor try to prove it

# Obtaining $s: \omega \rightarrow \omega$ from $j: A \rightarrow A$

3. **Proposition.**  $E$  well-orders  $W$ .
4. **Definition.** Let  $\pi: W \rightarrow N$  be the Mostowski collapsing isomorphism, applied to the relation  $E$ :

$$\pi(x) = \{\pi(y) \mid y E x\}$$

(The Mostowski collapsing function is defined in terms of  $j$  and  $\alpha$ , using  $j$ -induction for the necessary recursion).

# Obtaining $s: \omega \rightarrow \omega$ from $j: A \rightarrow A$

## 5. Proposition:

- (a)  $\pi(a) = \emptyset$
- (b) The unique map  $s: N \rightarrow N$  making the diagram below commute is defined by  $s(x) = x \cup \{x\}$ . In particular,  $N = \omega$ .

$$\begin{array}{ccc} W & \xrightarrow{j \upharpoonright W} & W \\ \downarrow \pi & & \downarrow \pi \\ N & \xrightarrow{s} & N \end{array}$$

- (c) Let  $k = j \upharpoonright W$ . Then, for each  $x \in \omega$ , there is  $n \in \omega$  such that  $(\pi \circ k^n)(a) = x$ .

# Motivation for Blueprint Definition

- **Definition.** For  $n \in \omega$  and  $g \in W^W$  we let

$$i_n(g) = \pi \circ g^n \in \omega^W$$

and let  $\mathcal{E} = \{i_1, i_2, \dots\}$ .

- **Theorem.** For every  $x \in \omega$ , there is  $i \in \mathcal{E}$  such that

$$i(j \mid W)(a) = x.$$

- **Note.** The maps in  $\mathcal{E}$  have several nice preservation properties. A map  $i$  will be called *weakly elementary* if it has properties such as: [full list of properties not shown]
  - whenever  $f$  is injective (surjective),  $i(f)$  is injective (surjective)
  - whenever  $f$  preserves disjoint unions (or intersections),  $i(f)$  preserves disjoint unions (or intersections)

# Blueprints

**Definition** (Blueprints). Suppose  $j : A \rightarrow A$  is a Dedekind self-map with critical point  $a$ , and suppose  $X$  is a set. A  $j$ -*blueprint* (or simply a *blueprint*) for  $X$  is a triple  $(f, a, \mathcal{E})$  having the following properties:

- (1) For some set  $B$ ,  $f$  is either a Dedekind self-map  $B \rightarrow B$  with critical point  $y$  or a co-Dedekind self-map with co-critical point  $y$  (the value  $y$  may or may not be equal to  $a$ ).
- (2)  $\mathcal{E}$  is a set of weakly elementary functionals, compatible with  $j$ , such that, for each  $i \in \mathcal{E}$ :
  - (a) there exist  $C_i, D_i$  so that  $i : B^B \rightarrow D_i^{C_i}$
  - (b)  $C_i \supseteq B$
  - (c) If  $C_i \neq D_i$ , then there is a bijection  $\pi_i : D_i \rightarrow C_i$  that is definable from  $j, a$ .
  - (d)  $a \in \text{dom } i(f)$ .
- (3) (*Encoding*) The self-map  $f$  is *defined from*  $\mathcal{E}, j, a$ .
- (4) (*Decoding*)  $f$  generates  $X$  in the following sense:

For every  $x \in X$ , there is  $i \in \mathcal{E}$  such that  $i(f)(a) = x$ .