THE WHOLENESS AXIOM AND LAVER SEQUENCES

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Abstract. In this paper we introduce the Wholeness Axiom (WA), which asserts that there is a nontrivial elementary embedding from V to itself. We formalize the axiom in the language $\{\in, \mathbf{j}\}$, adding to the usual axioms of ZFC all instances of Separation, but no instance of Replacement, for $\mathbf{j}$-formulas, as well as axioms that ensure that $\mathbf{j}$ is a nontrivial elementary embedding from the universe to itself. We show that WA has consistency strength strictly between I_3 and the existence of a cardinal that is super-n-huge for every n. ZFC + WA is used as a background theory for studying generalizations of Laver sequences. We define the notion of Laver sequence for general classes $\mathcal{E}$ consisting of elementary embeddings of the form $i : V_\beta \to M$, where M is transitive, and use five globally defined large cardinal notions—strong, supercompact, extendible, super-almost-huge, superhuge—for examples and special cases of the main results. Assuming WA at the beginning, and eventually refining the hypothesis as far as possible, we prove the existence of a strong form of Laver sequence (called *special* Laver sequences) for a broad range of classes $\mathcal{E}$ that include the five large cardinal types mentioned. We show that if κ is globally superstrong, if $\mathcal{E}$ is Laver-closed at beth fixed points, and if there are superstrong embeddings i with critical point κ and arbitrarily large targets such that $\mathcal{E}$ is weakly compatible with i, then our standard constructions are $\mathcal{E}$-Laver at κ. (In particular, if κ is super-almost-huge (superhuge, super-2-huge), there is an extendible (super-almost-huge, superhuge) Laver sequence at κ.) In addition, in most cases our Laver sequences can be made special if $\mathcal{E}$ is upward λ-closed for sufficiently many λ.

§1. Introduction.

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In this paper, we discuss a new large cardinal axiom, which we call the *Wholeness Axiom (*WA*)*, and apply it to study several questions concerning Laver sequences.

The Wholeness Axiom asserts the following:

(1.1) "there is a nontrivial elementary embedding from V to itself"

The axiom was first introduced in [7] as a candidate for the "right" strengthening of the Axiom of Infinity, strong enough to provide a foundational theory that could accommodate virtually all known large cardinal axioms. We argue in [7] that some form of (1.1) is natural, based on criteria of elegance, simplicity, generalization, and other "first principles." As is well known, Reinhardt [26] asked whether such an embedding could exist, and shortly thereafter, K. Kunen impressively demonstrated in [21] that the existence of such an embedding is inconsistent with ZFC. Convinced of the naturalness of such an axiom, however, we formulated WA in an effort to provide the minimal weakening of (1.1) that avoids the inconsistency given in Kunen's Theorem, yet retains sufficient strength to provide an umbrella theory for large cardinals.

So far in the literature, efforts to obtain weakenings of (1.1) that are still strong but not inconsistent have focused primarily on the fact that Kunen's proof of the inconsistency of a $j : V \to V$ depends on the Axiom of Choice; a central open question that has remained is whether some sort of inconsistency proof could be found for the existence of such a j using only ZF. In unpublished work, Woodin has obtained the following results which show that a global version of such an axiom is extremely strong, even without AC:

1.1 Theorem (Woodin)**.** *Assume ZF. Assume there is an initial ordinal κ such that for all $\alpha \in ON$ there is an elementary embedding $j : V \to V$ with critical point κ such that $j(\kappa) > \alpha$. Then there are forcing extensions M, N such that*

(1) *in M, the Axiom of Choice holds and there are j, λ such that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an elementary embedding with critical point κ (and $\lambda > \kappa$); and*
(2) *there is an elementary embedding $j : N \to N$ with critical point κ and $< \lambda^+$-DC holds in N, where $\lambda = sup(\{j^n(\kappa) : n \in \omega\})$.*

Another approach has been to weaken the notion of "elementary embedding" to try to achieve consistency. Blass and, independently, Trnková, obtained the following result along these lines:

1.2 Theorem (Blass [5], Trnková [28])**.** *The following are equivalent:*

(1) *There is an exact functor from the category of sets to itself which is not naturally isomorphic to the identity functor.*
(2) *There exists a measurable cardinal.*

Our approach has been to weaken (1.1) by removing every "shred" of definability of $\mathbf{j}$ in V. In order to make this precise, it is helpful to study Kunen's Theorem more carefully. The assertion that there is no nontrivial elementary embedding from the universe to itself is, as is well-known, not formalizable in ZFC alone. In particular, in order to arrive at a contradiction in the proof, it is necessary to take the supremum of the sequence $\langle j^n(\kappa) : n \in \omega\rangle$, and this step involves the use of an instance of Replacement in which 'j' occurs. Whether such an instance of Replacement is true cannot be determined in ZFC and must be viewed as an assumption, as part of a theory that extends ZFC. (We note here, in accord with Hamkins [15], that the content of Kunen's Theorem is not merely the assertion that no nontrivial elementary embedding from the universe to itself is *definable*—this fact can be proved with considerably less effort. Rather, as we argue here, Kunen's proof entails an essential application of Replacement in an extended theory. See Hamkins' excellent discussion of this point in [15].)

We suggest here perhaps the simplest way to formalize Kunen's Theorem and describe how we can obtain WA in this formal context. We add a single unary function symbol $\mathbf{j}$ to the usual language $\{\in\}$ of ZFC and add to the axioms of ZFC other axioms asserting that $\mathbf{j}$ is an elementary embedding ($\phi(x_1, \ldots, x_n) \iff \phi(\mathbf{j}(x_1), \ldots, \mathbf{j}(x_n))$ for each formula ϕ) and that Separation and Replacement hold for all formulas, including those with occurrences of $\mathbf{j}$. Let us call this extended theory, in this extended language, $\mathrm{ZFC}^{\mathbf{j}}$. With this theoretical framework, Kunen's result can be re-stated as follows:

$$\mathrm{ZFC}^{\mathbf{j}} \vdash \text{“}\mathbf{j} \text{ must be the identity.”} \tag{1.2}$$

Perhaps more of the flavor of Kunen's result is captured by wording the result this way:

$$\text{“There is no model of } \mathrm{ZFC}^{\mathbf{j}} \text{ in which } \mathbf{j} \text{ is not the identity.”} \tag{1.3}$$

The notion that, for a subcollection X of a model M of ZF, the expanded model (M, X) satisfies all instances of Replacement for formulas of the extended language has been studied by Enayat [13] and others (see references in [13]). Under these conditions, Enayat calls such an X a *class* in M; we do not use this terminology here since we prefer to follow the more common convention in set theory that a class is just a definable subcollection of M. Nevertheless, isolating this concept is very useful here, and so we will say, for our purposes, that such an X is *weakly definable in* M. (As we show in Section 3, weak definability is appropriately named in that it is implied by definability.) With these notions, Kunen's Theorem can be re-stated as follows:

(1.4) “If M is a model of ZFC, there is no nontrivial elementary embedding from M to M that is weakly definable in M.”

Our approach to obtaining a version of (1.1), therefore, is, the following: In the context of the extended language $\{\in, \mathbf{j}\}$, add to the usual axioms of ZFC all instances of Separation but no instance of Replacement, for formulas in which $\mathbf{j}$ occurs. We also add to the axioms the assertion of nontriviality: $\exists x\,(\mathbf{j}(x) \neq x)$. And we refer collectively to the axioms that we have added to ZFC as the *Wholeness Axiom* or WA. Assuming WA, the step in Kunen's proof in which one obtains the supremum of $\langle \kappa, j(\kappa), j^2(\kappa), \ldots\rangle$ cannot be taken, so the theory ZFC + WA cannot be proven inconsistent (at least not with methods that are currently known). Using the notion of weak definability, we can re-state WA — in contrast to (1.4) — as follows:

(1.5) "There is a nontrivial elementary embedding from V to V."

Then, by Kunen's Theorem, any such embedding must not be weakly definable in V.

A familiar setting in which WA holds arises when there is a nontrivial elementary embedding $j : V_\lambda \to V_\lambda$; in that case, $\langle V_\lambda, \in, j\rangle \models$ ZFC + WA. Indeed, this is our "intended model" of WA. However, there are other much weaker examples of elementary embeddings from a model of set theory to itself that are not weakly definable: for instance, $j : L \to L$ or $j : K \to K$ (where L is Gödel's constructible universe and K is the core model). These do not provide models of WA, however, precisely because we have included the axiom schema of Separation for **j**-formulas; indeed, to exclude these types of embeddings, it suffices to require that for each set x, $j \restriction x$ is also a set. (Our definition of WA in earlier preprints of this paper used the latter as an axiom rather than Separation for **j**-formulas for just this reason.[3]) Moreover, WA has much stronger consequences than such embeddings: We show in Section 3 that if $j : V \to V$ is a witness to WA, then its critical point κ must be the κth cardinal that is super-n-huge for every $n \in \omega$; in that section we prove several other results that highlight the strength of the axiom.

In Section 4, we use the theory ZFC + WA as a framework for studying Laver sequences. By now, Laver sequences have appeared enough in the literature to merit investigation as objects of interest in their own right. Laver sequences were first introduced in [22], where Laver used them in his proof that it is consistent, relative to a supercompact κ, that supercompactness cannot be destroyed by κ-directed closed forcing. Gitik and Shelah [14] obtained a similar result for strong cardinals: Assuming a strong cardinal and using a Laver sequence for strong cardinals, they build a model in which strong-ness is indestructible under κ^+-weakly closed forcings that satisfy the Prikry condition. Barbanel [2] used a 2-huge cardinal and a local version of a Laver sequence for huge cardinals to build a model in which the following holds: Whenever there is a model M obtained by κ-directed closed forcing (using a partial order of cardinality λ less than the target of κ) in which hugeness of κ is destroyed, there is, in M a λ^+-directed closed forcing that restores the hugeness of κ. From a different direction, Hamkins [16] uses a version of Laver sequences for supercompact and strong cardinals to construct models in which κ-concerned forcing (forcing which preserves $\kappa^{<\kappa}$ and κ^+ but not $P(\kappa)$) always destroys even measurability. Finally, one of the earliest and most important applications of Laver sequences was the proof of the consistency of PFA from a supercompact (cf. [11]).

Questions about Laver sequences that seem natural to ask include: Is there a direct construction of a Laver sequence (Laver's original proof was by contradiction)? Which large cardinals admit their own brand of Laver sequence and under what large cardinal hypotheses can they be

[3] J. Hamkins [15] has formulated a hierarchy of Wholeness Axioms in which the weaker of these two versions of WA is located at the bottom and the stronger is located at the top, with ω many refinements in-between. He shows that they have strictly increasing consistency strengths.

built? Can Laver sequences be constructed with special properties (definable or undefinable in V_κ; fast-growing; absolute for various inner models)?

In tackling these questions, we found that the theory ZFC + WA provided a useful context and interesting answers. To begin, WA can be used to motivate a direct construction of a Laver sequence that admits an easy generalization to other globally defined large cardinals; in this paper, we consider specifically strong, extendible, super-almost-huge, superhuge cardinals, and of course, supercompact cardinals (the original context for the study of Laver sequences). Moreover, we generalize the construction further by considering classes $\mathcal{E}$ of set embeddings of the form $i : V_\beta \to M$ (where M is a transitive set) in place of specific large cardinals. Because of the strong consequences of WA, we are able to provide a single construction and, essentially, a single proof for the existence of Laver sequences corresponding to a broad collection of classes of embeddings. Moreover, the construction we use, and the proof that it works under WA, has the advantage that the resulting Laver function f can be forced to agree with an arbitrary function $t : \kappa \to V_\kappa$ on a normal measure 1 set. This flexibility allows us to construct quite a variety of Laver sequences. For instance, under WA our construction can be adapted to give a Laver function f such that the function $\alpha \mapsto |f(\alpha)|$ dominates each $h \in {}^\kappa\kappa$ definable in V_κ on a normal measure 1 set. Moreover, this function has strong properties from which we prove that it is consistent for κ to be the κth extendible cardinal.

In Sections 5 and 6, we carry out the program of Section 4 with the aim of weakening the large cardinal hypothesis as far as possible. The theme of Section 5 is to answer the question: To what extent can the theorems of Section 4 be proven using elementary embeddings of the form $j : V \to N$ instead of the embedding $j : V \to V$ given by WA? We obtain a strong form of a Laver sequence for each of the five large cardinals under fairly modest hypotheses. At the same time, we provide abstract conditions on a class $\mathcal{E}$ of embeddings under which such a class admits its own brand of Laver sequence. In Section 6, we show how to obtain essentially the same results, but with much less work, by modifying the construction slightly.

Our work in Sections 5 and 6 is also intended as a beginning step in a related line of research that seems timely. Given the considerable number of large cardinal axioms that have emerged, it would seem natural to have an abstract theory of large cardinals from which one could deduce a substantial portion of the current body of knowledge concerning large cardinals. (Perhaps a useful comparison is the historical development of abstract group theory: The abstract theory began to emerge on the basis of relatively few examples of groups.) The problem of generalizing Laver sequences is a reasonable starting point for this program because, without having at least some kind of general context, it isn't possible to define what one means by "Laver sequence" for large cardinals in general. Our "abstract objects" of study in this program (analogous to abstract groups) have been classes $\mathcal{E}$ of embeddings. What we found in attempting to prove the existence of general kinds of Laver sequences for such classes $\mathcal{E}$ was that certain properties of these classes appeared particularly relevant ("coherence", "Laver-closure", "compatibility", to name a few); these properties, as one would expect, are carefully formulated generalizations of properties that are familiar in the context of one or more specific large cardinals. We also found it useful, again, to have, as a starting point, the theory ZFC + WA as our background theory, allowing us to reason about properties in a simple context before attempting to optimize hypotheses. From this starting point, we hope to obtain a much richer abstract theory. One direction for further work is to obtain abstract conditions on classes $\mathcal{E}$ for which $\mathcal{E}$-Laver sequences are indestructible after certain preparatory forcing has been done (in analogy with the results on supercompact cardinals).

Section 7 is devoted to answering a number of technical questions that arise in Sections 4-6 but that are not really part of the main thread of ideas. And Section 2 is devoted to preliminaries. Many of the results there are known (though some have not appeared in published form); others represent slight improvements of known results. There are also a number of technical results in Section 2 that serve as lemmas for work in Sections 4-6, but belong in a section on preliminaries.

Finally, the Appendix consists of a few corrections that needed to be made to our work in [9]; that paper built on the results of the present paper but was published before the refereeing process of the present paper was complete. Thus, some errors propagated from earlier versions of this paper into [9]. The Appendix addresses these issues.

To conclude this introduction, I would like to gratefully acknowledge the logicians and set theorists who took the time to discuss some of these ideas with me, including M. Benedikt, T. Drucker, J. Hamkins, M. Jahn, A. Kanamori, K. Kunen, R. Laver, M. McKinzie, M. Magidor, S. Shelah, R. Solovay, C. Tuckey, and H. Woodin. Finally, I would like to thank the referees who have reviewed this paper through several incarnations: The first of these referees suggested a more elegant formalization of WA and a better direct construction of a Laver sequence, both of which I have used. The second referee spotted quite a number of serious errors and made countless helpful suggestions.

§2. Preliminaries.

This section is dedicated to giving the notation, definitions and background theorems needed for the rest of the paper. Many of the propositions stated here are small refinements of known (but in some cases unpublished) results; one or two are new and perhaps of independent interest. All are stated here for use in later sections.

We begin with some terminology and notational conventions. A *beth fixed point* is a cardinal α such that $\alpha = |V_\alpha|$. It is well known that there is a sentence σ such that "$\forall\alpha\, V_\alpha \models \sigma$ iff α is a beth fixed point" is provable in ZFC (see [27, p. 86]); we shall call such a sentence a *beth fixed point sentence*. Many of our results will involve indexed sequences of elementary embeddings (in working with almost-huge cardinals, for instance), and we will denote the codomains of these embeddings by subscripting M or N with ordinals. (Moreover, in this paper, an expression like M_λ will *never* have the meaning $(V_\lambda)^M$.)

Some Large Cardinal Axioms

If M and N are transitive classes (possibly proper), if $j : M \to N$ is elementary, and some ordinal is moved by j, we denote the least ordinal moved by j—the *critical point* of j—by $\mathrm{cp}(j)$.

We proceed to the definitions of the large cardinal concepts we will study in this paper; more information about these notions can be found in [12], [17], [18], [19], [20], [26], [27].

Our terms *super-almost-huge* and *globally superstrong* do not seem to have appeared in the literature; they are defined as the global versions of almost huge and superstrong, in the same spirit as superhuge is the natural global version of huge (see [4]). Our version of "strong" and "superstrong" follows [24].

2.1 Definition. Suppose κ is an infinite cardinal.

(1) For $\lambda > \kappa$, κ is λ- *supercompact* if there is an inner model M and an elementary embedding $j : V \to M$ such that M is closed under λ-sequences, $\mathrm{cp}(j) = \kappa$, and $j(\kappa) > \lambda$; the embedding j is called a *λ-supercompact embedding.* κ is *supercompact* if κ is λ-supercompact for every $\lambda > \kappa$.

(2) For $\lambda > \kappa$, κ is *λ-strong* if there is an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_\lambda \subset M$; the map j is called a *λ-strong embedding.* κ is *strong* if κ is λ-strong for every $\lambda > \kappa$.

(3) A cardinal κ is *superstrong* if there is an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subset M$. The map j is called a *superstrong embedding.* κ is *globally superstrong* if for each $\gamma > \kappa$, there is a superstrong embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and $j(\kappa) \geq \gamma$.

(4) For $\eta \geq 0$, κ is *η-extendible* if there are ζ and an elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ such that $\mathrm{cp}(j) = \kappa$ and $\eta < j(\kappa) < \zeta$. κ is *extendible* if κ is η-extendible for every η.

(5) κ is *almost huge* if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and, for each $\alpha < j(\kappa)$, M is closed under α-sequences; $j(\kappa)$ is called the *a.h. target* of j and j is called an *a.h. embedding.* κ is *super-almost- huge* if, for each γ there are $\lambda > \gamma$ and $j_\lambda : V \to M_\lambda$ such that $\mathrm{cp}(j_\lambda) = \kappa$, $j_\lambda(\kappa) = \lambda$ and for all $\alpha < \lambda, M_\lambda$ is closed under α-sequences.

(6) For each $n \in \omega$, κ is n-huge if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$ and M is closed under $j^n(\kappa)$-sequences; $j(\kappa)$ is called the *target* of j and j is called an *n-huge embedding.* κ is super-n-huge if, for every $\lambda > \kappa$, there is an n-huge embedding j such that $\mathrm{cp}(j) = \kappa$ and $j(\kappa) > \lambda$. (See [4] for an equivalent definition.)

(7) $I_3(\kappa)$ is the statement that for some $\lambda > \kappa$ there is an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point κ.

(8) $I_1(\kappa)$ is the statement that for some $\lambda > \kappa$ there is an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ.

(9) $I_0(\kappa)$ is the statement that there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point κ, where $\lambda = \sup\{j^n(\kappa) \mid n \in \omega\}$.

Supercompact Cardinals

Recall that κ is λ-supercompact if and only if there is a normal ultrafilter over $P_\kappa\lambda$ (see [18]). In particular, if $j : V \to M$ is a λ-supercompact embedding, the set

$$U = \{X \subset P_\kappa\lambda : j''\lambda \in j(X)\}$$

is a normal ultrafilter over $P_\kappa\lambda$; U will be called the *normal ultrafilter over $P_\kappa\lambda$ derived from j*. Conversely, given a normal ultrafilter U over $P_\kappa\lambda$, the elements of the ultrapower by U—as well as its transitive collapse—will be denoted $[g]_U$. The *canonical embedding determined by U* is the map $i_U : V \to V^{P_\kappa\lambda}/U \cong M_\lambda$ defined by $i_U(x) = [c_x]_U$ where c_x is the constant function with value x; i_U is a λ-supercompact embedding.

For each $\alpha \leq \lambda$, define functions r_α, t_α over $P_\kappa\lambda$ by

$$r_\alpha(x) = x \cap \alpha;$$
$$t_\alpha(x) = \mathrm{ot}(x \cap \alpha).$$

Then in M_λ, $[r_\alpha]_U = j''\alpha$ and $[t_\alpha]_U = \alpha$.

If $j : V \to M$ is a λ-supercompact embedding, U the derived normal ultrafilter over $P_\kappa\lambda$, and $M_\lambda \cong V^{P_\kappa\lambda}/U$, then there is an embedding $k : M_\lambda \to M$ defined by $k([g]_U) = j(g)(j''\lambda)$ with the properties that $j = k \circ j_U$ and $k \restriction \lambda^+ = \mathrm{id}_{\lambda^+}$. In the special case in which $j = i_W$ where W is a normal ultrafilter over $P_\kappa\mu, \mu \geq \lambda$, k can be defined by $k([g]_U) = [\tilde{g}]_W$ where $\tilde{g}(x) = g(x \cap \lambda)$. Moreover, it can be shown that $U = W \mid \lambda$ where we define $X \mid \lambda \subset P_\kappa\lambda$ and $W \mid \lambda$ by:

$$X \mid \lambda = \{x \cap \lambda : x \in X\};$$
$$W \mid \lambda = \{X \mid \lambda : X \in W\}.$$

Strong and Superstrong Cardinals

We obtain alternative definitions of strong and (globally) superstrong cardinals using extenders. To fix notation, we review the basics of the theory of extenders; the proofs of the preliminary propositions below appear in [24].

2.2 Definition. Let Y be a transitive set and let κ be a cardinal. An *extender with critical point κ and support Y* is a system $E = \langle E_a : a \in {}^{<\omega}[Y]\rangle$ with the following properties:

(a) Each E_a is a κ-complete measure on ${}^aV_\kappa$, and at least one E_a is not κ^+- complete.

(b) The E_a are *compatible*; i.e., if $X \subseteq {}^aV_\kappa, E_a(X) = 1$, and $a \subseteq b$, then

$$E_b(\{\, f : f \restriction a \in X \,\}) = 1.$$

(c) $E_a\big(\{\, f \mid f : (a, \in) \cong (\mathit{range}(f), \in) \,\}\big) = 1.$

(d) If $F : {}^aV_\kappa \to V_\kappa$ and

$$E_a\big(\{\, f : F(f) \in \cup(\mathit{range}(f)) \,\}\big) = 1,$$

then there is a $y \in Y$ such that

$$E_{a\cup\{y\}}\big(\{\, f \mid F(f \restriction a) = f(y)\,\}\big) = 1.$$

(e) $\mathrm{Ult}(V, E)$ is well-founded.

2.3 Remark. In part (e) of the above definition, $\mathrm{Ult}(V, E)$ is defined as follows: For all $a, b \in {}^{<\omega}[Y]$, and all $F : {}^{a}V_\kappa \to V, G : {}^{b}V_\kappa \to V$, say $F \sim G$ iff there is $c \in {}^{<\omega}[Y], c \supseteq a \cup b$, such that

$$E_c\big(\{\, h : F(h \restriction a) = G(h \restriction b)\,\}\big) = 1.$$

Let $[F]$ denote the set of all members of minimal rank of the $\sim$-equivalence class of F. Similarly, for all $a, b \in {}^{<\omega}[Y]$, and all $F : {}^{a}V_\kappa \to V, G : {}^{b}V_\kappa \to V$, define the membership relation R by $[F]\, R\, [G]$ iff there is $c \in {}^{<\omega}[Y], c \supseteq a \cup b$, such that

$$E_c\big(\{\, h : F(h \restriction a) \in G(h \restriction b)\,\}\big) = 1.$$

Now $\mathrm{Ult}(V, E)$, the *ultrapower of V by E*, is the proper class model having domain $\big\{\, [F]_E : (\exists a \in {}^{<\omega}[Y])(F : {}^{a}V_\kappa \to V)\,\big\}$ and having membership relation R. We will write $[F]$ for $[F]_E$ when there is no possibility of confusion. As usual, Łoś' Theorem holds, giving us the *canonical elementary embedding* $i : V \to \mathrm{Ult}(V, E) : x \mapsto [c^a_x]$ for some (any) $a \in {}^{<\omega}[Y]$, where $c^a_x : {}^{a}V_\kappa \to V : f \mapsto x$ is the constant function with value x. In this paper, $\mathrm{Ult}(V, E)$ will always be well-founded; we identify each $[F] \in \mathrm{Ult}(V, E)$ with its image under the Mostowski collapsing isomorphism, making $\mathrm{Ult}(V, E)$ a transitive class model of ZFC containing all the ordinals.

2.4 Proposition. *Let $E = \langle E_a : a \in {}^{<\omega}[Y]\rangle$ and let $i_E : V \to Ult(V, E)$ be the canonical embedding.*

(1) $i_E \mid V_\kappa = \mathrm{id}_{V_\kappa}$ but $i_E(\kappa) > \kappa$.
(2) For each $y \in Y$ and $a \in {}^{<\omega}[Y]$ with $y \in a$, let $H^a_y(f) = f(y)$, for all $f \in {}^{a}V_\kappa$. Then in $\mathrm{Ult}(V, E)$, $y = [H^a_y]$.

2.5 Proposition. *Suppose $j : V \to N$ is an elementary embedding with N transitive and with critical point κ. Let Y be a transitive set with $\kappa \in Y \subseteq V_{j(\kappa)} \cap N$. Define $E = \langle E_a : a \in {}^{<\omega}[Y]\rangle$ as follows: For each a and all $X \subseteq {}^{a}V_\kappa$, let $E_a(X) = 1$ iff $j^{-1} \restriction j(a) \in j(X)$. Then*

(a) *E is an extender.*

(b) *Define $k : Ult(V, E) \to N$ by*

$$k([F]) = \big(j(F)\big)\big(j^{-1} \restriction j(a)\big)$$

whenever $F : {}^{a}V_\kappa \to V$. Then k is an elementary embedding, $k \restriction Y = \mathrm{id}_Y$, and $j = k \circ i_E$.

2.6 Definition. The extender E defined from j in the above proposition will be called the *extender derived from j with support Y*.

Thus, given a λ-strong embedding with critical point κ, we can obtain an extender E with critical point κ and support V_λ. The converse is also true, but, given E, the embedding i_E will not in general be the required embedding since it is possible for $i_E(\kappa) = \lambda$ (note, though, that λ cannot be a successor ordinal in this case); whenever this is the case, one can use $i_E \circ i_E$ instead; we provide more details in the comments following Definition 2.16. The parallel equivalence for superstrong cardinals is immediate. Summing up:

2.7 Proposition. *Suppose κ is an infinite cardinal.*

(1) *Suppose $\kappa < \lambda$. Then κ is λ-strong if and only if there is an extender E with crtical point κ and support V_λ.*

(2) *κ is superstrong if and only if there are λ, E such that $\lambda > \kappa$ and E is an extender with critical point κ and support V_λ, and $i_E(\kappa) = \lambda$.*■

Super-almost-huge Cardinals

There is a standard ultrafilter definition of almost-huge cardinals: Recall (see [27]) that, for ordinals $\kappa < \lambda$, a sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ is *coherent* if for each η, U_η is a normal ultrafilter over $P_\kappa\eta$ and if $\kappa \leq \eta < \zeta < \lambda$, $U_\eta = U_\zeta \mid \eta$. For each such η, let M_η denote the transitive collapse of the ultrapower by U_η and let $j_\eta : V \to M_\eta$ denote the canonical embedding. Let $k_{\eta\zeta} : M_\eta \to M_\zeta$ denote the embeddings (described earlier) such that $j_\zeta = k_{\eta\zeta} \circ j_\eta$ and $k_{\eta\zeta} \restriction \eta + 1 = \mathrm{id}_{\eta+1}$. Let M denote the direct limit of the directed system $\{M_\eta; k_{\eta\zeta} : \kappa \leq \eta < \zeta < \lambda\}$; we identify M with its transitive collapse. There are embeddings $k_\eta : M_\eta \to M$ and an embedding $j : V \to M$ so that for each η, $k_\eta \circ j_\eta = j$.

We will call the following condition on a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ the **SRK**-*Criterion*:

> **SRK**(κ, λ)**.** *λ is inaccessible and for all η, ρ for which $\kappa \leq \eta < \lambda$ and $\eta \leq \rho < j_\eta(\kappa)$ there is ζ such that $\eta \leq \zeta < \lambda$ and $k_{\eta\zeta}(\rho) = \zeta$.*

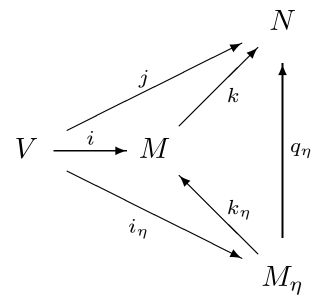
In [3], Barbanel isolates an equivalent condition in terms of ultrafilters; we modify his condition slightly and call it *Barbanel's Criterion* ($\mathcal{B}(\kappa, \lambda)$); in Section 6, we show that this condition is equivalent to the condition formulated by Barbanel in [3], and also to **SRK**(κ, λ).

> $\mathcal{B}(\kappa, \lambda)$**.** *$\lambda$ is inaccessible and for all η for which $\kappa \leq \eta < \lambda$ and all $h : P_\kappa\eta \to ON$, if $\{x \in P_\kappa\eta : \mathrm{ot}(x) \leq h(x) < \kappa\} \in U_\eta$ then there is ζ such that $\eta \leq \zeta < \lambda$ and $\{x \in P_\kappa\zeta :$ ot $x = h(x \cap \eta)\} \in U_\zeta$.*

If $\kappa < \lambda$ and there is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ as above that satisfies $\mathcal{B}(\kappa, \lambda)$, then κ is almost huge and the embedding $j : V \to M$ obtained from the direct limit construction is an a.h. embedding.

Conversely, starting from an a.h. embedding $j : V \to M$, one obtains a coherent sequence $\langle U_\eta : \kappa \leq \eta < j(\kappa)\rangle$ by putting $X \in U_\eta$ if and only if $j''\eta \in j(X)$. Moreover, the sequence satisfies $\mathcal{B}(\kappa, j(\kappa))$. We shall say that $\langle U_\eta : \kappa \leq \eta < j(\kappa)\rangle$ is the coherent sequence *derived from* j. Summarizing:

2.8 Proposition ([3], [27]). *Suppose κ is an infinite cardinal and $\lambda > \kappa$. Then κ is almost huge with a.h. target λ if and only if there is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ satisfying $\mathcal{B}(\kappa, \lambda)$.* ∎



In analogy with many of the smaller large cardinals, if we obtain $\langle U_\eta : \kappa \leq \eta < j(\kappa)\rangle$ from an a.h. embedding $j : V \to N$ and then obtain the embedding $i : V \to M$ from the direct limit of ultrapowers by the U_η, as described above, one can always find an embedding $k : M \to N$ such $j = k \circ i$ and $k \restriction V_{j(\kappa)} = \mathrm{id}_{V_{j(\kappa)}}$. To see this, assume M is the direct limit of the directed system $\{M_\eta; k_{\eta\zeta} : \kappa \leq \eta < \zeta < \lambda\}$, where $\lambda = j(\kappa)$; and that we have, for each η, embeddings $i_\eta : V \to M_\eta$ and $k_\eta : M_\eta \to M$ such that $i = k_\eta \circ i_\eta$. Using the standard result for supercompact embeddings, we can find, for each η, an embedding $q_\eta : M_\eta \to N$ satisfying $j = q_\eta \circ i_\eta$ and $q_\eta \restriction \eta^+ = \mathrm{id}_{\eta^+}$. Using the universal property of direct limits, there must be $k : M \to N$ such that $q_\eta = k \circ k_\eta$. We have

$$\begin{aligned} k \circ i &= k \circ k_\eta \circ i_\eta \\ &= q_\eta \circ i_\eta \\ &= j. \end{aligned}$$

It remains to show that k is the identity on V_λ. But notice that if $\kappa \leq \xi < \eta < \lambda$, then

$$\begin{aligned} \xi &= q_\eta(\xi) \\ &= k(k_\eta(\xi)) \\ &= k(\xi). \end{aligned}$$

By inaccessibility of λ, the result follows. We have shown:

2.9 Proposition. *Suppose $j : V \to N$ is an a.h. embedding with critical point κ and $\langle U_\eta : \kappa \leq \eta < j(\kappa)\rangle$ is the coherent sequence derived from j. Then if $i : V \to M$ is the a.h. embedding obtained from the direct limit of the ultrapowers by the U_η, $\kappa \leq \eta < j(\kappa)$, there is an embedding $k : M \to N$ such that $j = k \circ i$ and $k \restriction V_{j(\kappa)} = \mathrm{id}_{V_{j(\kappa)}}$.* ∎

In working with the direct limit construction, we will avoid as far as possible the particulars of the construction. Occasionally, however, these details will be necessary; we pause here to set up the notation we will be using and to prove a few technical lemmas.

Assuming we have the directed system $\langle M_\eta; k_{\eta\zeta} : \kappa \le \eta < \zeta < \lambda\rangle$ and the embeddings j_η, we carry out the construction of M and give the actual definitions of the k_η and j. M is obtained as a collection of equivalence classes $[\eta, x]$ of members (η, x) of the disjoint union of the M_η; two members (η, x), (ν, y) are equivalent iff there is $\zeta > \eta, \gamma$ such that $k_{\eta\zeta}(x) = k_{\nu\zeta}(y)$. The embeddings k_η for each η are defined by $k_\eta(\eta, x) = [\eta, x]$. The map $j : V \to M$ is then defined to be $k_\eta \circ j_\eta$ for any η (and j is well-defined). A "membership" relation E is then defined on M as follows:

$$[\eta, x]\, E\, [\nu, y] \iff \exists \zeta > \eta, \nu\, (k_{\eta\zeta}(x) \in k_{\nu\zeta}(y));$$

one verifies that E is well-defined. One shows that (M, E) is a well-founded model of ZFC, and that the k_η, and hence j, are elementary embeddings. As mentioned earlier, M is identified with its transitive collapse. It is shown in [27] that j is in fact an a.h. embedding with critical point κ and target λ.

2.10 Proposition. *Given a directed system $\langle M_\eta; k_{\eta\zeta} : \kappa \le \eta < \zeta < \lambda\rangle$ with direct limit M and embeddings $k_\eta : M_\eta \to M$ as above, for each η, $\kappa \le \eta < \lambda$, $k_\eta \restriction \eta + 1 = \mathrm{id}_{\eta+1}$.*

Proof. Given η, we prove by induction on $\alpha \le \eta$ that $k_\eta(\alpha) = \alpha$, using the fact that $k_{\eta\zeta} \restriction \eta + 1 = \mathrm{id}_{\eta+1}$ whenever $\eta < \zeta < \lambda$. By elementarity, $k_\eta(0) = 0$. Assume $k_\eta(\beta) = \beta$ for all $\beta < \alpha$. Since, by elementarity again, $k_\eta(\alpha) \ge \alpha$, it suffices to show that

$$\forall [\xi, x] \in M \left([\xi, x]\, E\, [\eta, \alpha] = k_\eta(\alpha) \Longrightarrow \exists \beta < \alpha\, ([\xi, x] = \beta)\right).$$

Thus, suppose $[\xi, x]\, E\, [\eta, \alpha]$. Then there is $\zeta > \eta, \alpha$ such that $k_{\xi\zeta}(x) \in k_{\eta\zeta}(\alpha) = \alpha$. Thus, there is $\beta < \alpha$ such that $k_{\xi\zeta}(x) = \beta = k_{\eta\zeta}(\beta)$. Pulling back gives us $[\xi, x] = [\eta, \beta]$. Thus,

$$[\xi, x] = [\eta, \beta] = k_\eta(\beta) = \beta,$$

as required. ∎

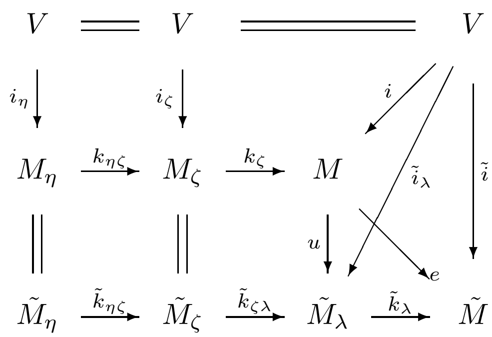
An application of this proposition that we will need in Section 5 is the following:

2.11 Theorem. *Suppose $j : V \to N$ is an a.h. embedding with target $\tilde\lambda$, and let $\langle U_\eta : \kappa \le \eta < \tilde\lambda\rangle$ be the coherent sequence derived from j. Suppose further that for some $\lambda, \kappa < \lambda < \tilde\lambda$, the restriction $\langle U_\eta : \kappa \le \eta < \lambda\rangle$ satisfies $\mathcal{B}(\kappa, \lambda)$. Let $i : V \to M$ denote the canonical a.h. embedding obtained from $\langle U_\eta : \kappa \le \eta < \lambda\rangle$ (as described above). Then there is $\tilde e : M \to N$ such that $\tilde e \restriction V_\lambda = \mathrm{id}_{V_\lambda}$ and $j = \tilde e \circ i$.*

Proof. We begin by fixing notation: Let $\langle \tilde M_\eta; \tilde k_{\eta\zeta} : \kappa \le \eta < \zeta < \tilde\lambda\rangle$ be the usual directed system of ultrapowers by $\langle U_\eta : \kappa \le \eta < \tilde\lambda\rangle$ with maps $\tilde i_\eta : V \to \tilde M_\eta$ such that $\tilde k_{\eta\zeta} \circ \tilde i_\eta = \tilde i_\zeta$, whenever $\eta < \zeta < \tilde\lambda$. Let $\tilde M$ denote (the transitive collapse of) the direct limit of this system with maps $\tilde k_\eta : \tilde M_\eta \to \tilde M$. Let $\tilde i : V \to \tilde M$ be defined by $\tilde i = \tilde k_\eta \circ \tilde i_\eta$, for any η.

For $\kappa \le \eta \le \zeta < \lambda$, let $M_\eta = \tilde M_\eta$, $i_\eta = \tilde i_\eta$, and $k_{\eta\zeta} = \tilde k_{\eta\zeta}$. Let M be the direct limit of the system $\langle M_\eta; k_{\eta\zeta} : \kappa \le \eta < \zeta < \lambda\rangle$, with maps $k_\eta : M_\eta \to M$. Let $i : V \to M$ be defined by

$i = k_\zeta \circ i_\zeta$ for any ζ; i is an a.h. embedding because the coherent sequence from which it was defined satisfies $\mathcal{B}(\kappa, \lambda)$.



By the universal property of direct limits, there is an elementary embedding $u : M \to \tilde{M}_\lambda$ such that $u \circ k_\zeta = \tilde{k}_{\zeta\lambda}$ whenever $\kappa \leq \zeta < \lambda$. Now $u \circ i = \tilde{i}_\lambda$, since

$$\begin{aligned} \tilde{i}_\lambda &= \tilde{k}_{\zeta\lambda} \circ \tilde{i}_\zeta \\ &= \tilde{k}_{\zeta\lambda} \circ i_\zeta \\ &= u \circ k_\zeta \circ i_\zeta \\ &= u \circ i. \end{aligned}$$

Let $e = \tilde{k}_\lambda \circ u$. Notice that

$$\tilde{i} = e \circ i \tag{2.1}$$

(since $\tilde{i} = \tilde{k}_\lambda \circ \tilde{i}_\lambda = \tilde{k}_\lambda \circ u \circ i = e \circ i$.)

We prove that $e \restriction \zeta + 1 = \mathrm{id}_{\zeta+1}$ whenever $\kappa \leq \zeta < \lambda$: By Proposition 2.10, $\tilde{k}_\zeta \restriction \zeta + 1 = \mathrm{id}_{\zeta+1}$. For $\kappa \leq \eta \leq \zeta$,

$$\eta = \tilde{k}_{\zeta\lambda}(\eta) = u(\tilde{k}_\zeta(\eta)) = u(\eta),$$

and so $u \restriction \zeta + 1 = \mathrm{id}_{\zeta+1}$. Finally, for $\kappa \leq \eta \leq \zeta$, $\eta = \tilde{k}_\lambda(u(\eta)) = e(\eta)$. Now, since λ is inaccessible, it follows that $e \restriction V_{\lambda_0} = \mathrm{id}_{V_{\lambda_0}}$, whenever $\kappa \leq \lambda_0 < \lambda$; thus,

$$e \restriction V_\lambda = \mathrm{id}_{V_\lambda}.$$

We have obtained an $e : M \to \tilde{M}$ that is the identity on V_λ and for which (2.1) holds. To complete the proof, we use Proposition 2.9 to obtain $\tilde{k} : \tilde{M} \to N$ such that $j = \tilde{k} \circ \tilde{i}$ and $\tilde{k} \restriction V_{\tilde{\lambda}} = \mathrm{id}_{V_{\tilde{\lambda}}}$. Then $\tilde{e} = \tilde{k} \circ e$ is the required embedding. ∎

Our final proposition in this sub-section will be useful for computing the size of $i(V_\gamma)$ for $\gamma > i(\kappa)$, where i is an a.h. embedding obtained from a coherent sequence of normal ultrafilters (see Proposition 2.28).

2.12 Proposition. *Suppose κ is almost huge with target λ. Suppose $\gamma > \lambda$ is a regular cardinal. Then $j(\gamma) = \sup\{j(\alpha) : \alpha < \gamma\}$.*

Proof. Note that for each η,

$$\begin{aligned} j(\gamma) &= k_\eta(j_\eta(\gamma)) \\ &= [\eta, j_\eta(\gamma)]. \end{aligned}$$

We show that for each x below $j(\gamma)$ there is, by regularity of γ, an $\alpha < \gamma$ such that $x < j(\alpha)$; that is, we show that whenever $[\xi, x]E[\eta, j_\eta(\gamma)]$, there is $\alpha < \gamma$ such that

$$[\xi, x]\, E\, [\eta, j_\eta(\alpha)].$$

Now $[\xi, x]\, E\, [\eta, j_\eta(\gamma)]$ implies there is $\zeta > \xi, \eta$ such that

$$k_{\xi\zeta}(x) \in k_{\eta\zeta}(j_\eta(\gamma)) = j_\zeta(\gamma).$$

Thus, as an element of M_ζ, $k_{\xi\zeta}(x)$ is represented by a function $f : P_\kappa\zeta \to \gamma$; because γ is regular and $> \lambda$, there is $\alpha < \gamma$ such that $f : P_\kappa\zeta \to \alpha$, and so $[f]_{U_\zeta} \in j_\zeta(\alpha)$. Thus, $k_{\xi\zeta}(x) \in k_{\eta\zeta}(j_\eta(\alpha))$; pulling back, $[\xi, x]\, E\, [\eta, j_\eta(\alpha)]$, as required. ■

Super n-huge Cardinals

There is also an ultrafilter characterization of n-huge cardinals:

2.13 Proposition. *Suppose κ is an uncountable cardinal. Then κ is n-huge if and only if there is a κ-complete normal ultrafilter U over some $P(\lambda)$ and cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \lambda$ so that for each $i < n$,*

$$\{x \in P(\lambda) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in U. ■$$

Given an n-huge embedding $j : V \to N$ with $\mathrm{cp}(j) = \kappa$, obtain the required U by putting $X \in U$ if and only if $X \subseteq P(j^n(\kappa))$ and $j''j^n(\kappa) \in j(X)$; U is called the *normal ultrafilter over $P(j^n(\kappa))$ derived from j*. Conversely, given U as in the proposition, form the ultrapower $V^{P(\lambda)}/U$; then the canonical embedding is n-huge.

Suppose $j : V \to N$ is an n-huge embedding with critical point κ and let U be the normal ultrafilter over $P(j^n(\kappa))$ derived from j. Form the ultrapower and let M denote the transitive collapse and $i : V \to M$ the canonical embedding. One shows that for all $\alpha \le j^n(\kappa)$, α is represented in M by the function $x \mapsto \mathrm{ot}\,(x \cap \alpha)$. Then, as usual, there is an embedding $k : M \to N$ defined by

$$k[g]_U = j(g)[j''j^n(\kappa)],$$

satisfying

$$j = k \circ i; \quad k \restriction j^n(\kappa) = \mathrm{id}_{j^n(\kappa)}.$$

I_3, I_1, I_0, and Inconsistency

In [21], Kunen showed that the natural "limit" of large cardinal axioms described by elementary embeddings $V \to M$, with increasingly inclusive M, is inconsistent with ZFC:

2.14 Theorem [21], Kunen's Theorem**.** *There is no elementary embedding $V \to V$ other than the identity.*■

His proof also shows the following:

2.15 Corollary (Kunen)**.** *There is no elementary embedding of the form $j : V_{\lambda+2} \to V_{\lambda+2}$ other than the identity.* ■

At this time, the strongest hypothesis not known to be inconsistent (and that has received significant attention) is Woodin's I_0 (as in Definition 2.1(9) above). It is well known that

$$I_0 \Longrightarrow I_1 \Longrightarrow I_3$$

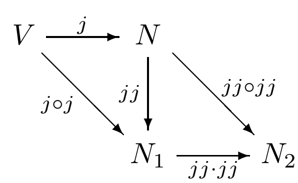
and that the implications are strong (cf. [18]).

Properties of Large Cardinals

Elementary embeddings can be combined not only by ordinary composition, but by direct application as well; we list some facts that we will need concerning these two ways of combining embeddings:

2.16 Definition. $(j \cdot j)$. Suppose $j : V \to N$ is an elementary embedding with critical point κ. We define $j \cdot j$ by

$$j \cdot j = \bigcup_{\alpha \in \mathrm{ON}} j(j \upharpoonright V_\alpha).$$



As in the diagram,

$$\begin{aligned} N_1 &= (j \cdot j)(N) \\ N_2 &= [(j \cdot j) \cdot (j \cdot j)](N_1) \\ &= (j \cdot j)\big[(j \cdot j)(N)\big] \end{aligned}$$

As a simple application of this notion, we can complete the proof of Proposition 2.7(1): Given an extender E with critical point κ and support V_λ, we wish to obtain a λ-strong embedding i. As we observed earlier, $i_E(\kappa) \geq \lambda$. In case $i_E(\kappa) > \lambda$, we set $i = i_E$. If $i_E(\kappa) = \lambda$, we set $i = (i_E \cdot i_E) \circ i_E$. Noting that $\mathrm{cp}(i_E \cdot i_E) = i_E(\kappa) = \lambda$, the result follows.

If U is the normal ultrafilter over κ derived from j, then write

$$\begin{aligned} U^j &= j(U) \\ &= \{X \subset j(\kappa) : j(\kappa) \in (j \cdot j)(X)\} \\ U^{jj} &= (j \cdot j)[j(U)] \\ &= \{X \subset j^2(\kappa) : j^2(\kappa) \in [(j \cdot j) \cdot (j \cdot j)](X)\}. \end{aligned}$$

U^j will be called the normal ultrafilter over $j(\kappa)$ derived from $j \cdot j$ in N. Likewise, U^{jj} will be called the normal ultrafilter over $j^2(\kappa)$ derived from $(j \cdot j) \cdot (j \cdot j)$ in N_1.

2.17 Proposition.

(1) $(j \cdot j) \circ j = j \circ j$.
(2) $[(j \cdot j) \cdot (j \cdot j)] \circ (j \cdot j) = (j \cdot j) \circ (j \cdot j)$.
(3) *In N, U^j is a normal ultrafilter over $j(\kappa)$.*
(4) *In N_1, U^{jj} is a normal ultrafilter over $j^2(\kappa)$.*
(5) *If $V_{j(\kappa)+1} \subset N$, then*

$$V \models \text{“}U^j \text{ is a normal ultrafilter over } j(\kappa)\text{”};$$
$$N \models \text{“}U^{jj} \text{ is a normal ultrafilter over } j^2(\kappa)\text{”}.$$

Note that hypothesis of (5) holds if j is a huge embedding with critical point κ. Next, we prove some facts concerning the relative strengths of the large cardinals that will concern us in later sections. In [18], Kanamori proves the following theorem:

2.18 Theorem. *Suppose κ is an infinite cardinal.*

(1) *If κ is strong or supercompact, then $V_\kappa \prec_2 V$.*
(2) *If κ is inaccessible and there are arbitrarily large $\lambda > \kappa$ for which λ is inaccessible and $V_\kappa \prec V_\lambda$, then $V_\kappa \prec_3 V$.* ∎

2.19 Theorem. *If κ is strong (supercompact) and $V_\kappa \models$ “α is strong (supercompact)” , then α is strong (supercompact).*

Proof. Kanamori [18] shows that the statement “α is γ-supercompact”is Δ_2^{ZF} by noting that it can be expressed both in the form $\exists x\ [x = V_{\gamma+5} \wedge \phi(\alpha, \gamma, x)]$ and $\forall x\ [x = V_{\gamma+5} \rightarrow \phi(\alpha, \gamma, x)]$, where ϕ is a Σ_0 formula asserting the existence of a normal ultrafilter over $P_\alpha\gamma$, with quantifiers bound to $x = V_{\gamma+5}$. A similar observation (also mentioned in [18]) shows that “α is γ-strong” is Δ_2^{ZF}. From these observations, it follows that “α is not strong (not supercompact)” is Σ_2^{ZF} and therefore, by the previous theorem, relativizes down to V_κ. The result follows. ∎

We now obtain similar results for some of the cardinals that are larger than supercompact. These results will follow from some general observations about large cardinal properties that can be formulated as “globalized” local properties. Let us recall that a property $P(x)$ is *local* if it has the form $\exists\delta\,(V_\delta \models \psi(x))$. In [27] (see remarks following Definition 2.6 of that paper), it is proved that local properties are precisely those that can be expressed by a Σ_2 formula. Let us call a property $R(x)$ a *globalized local property* if it is of the form

$$\forall y\,\exists\beta > rank(y)\ (V_\beta \models \psi(x,y))$$

for some formula ψ. We show that a property is globalized local iff it can be expressed as a Π_3 formula. It is clear that such properties are Π_3. Conversely, if $S(x)$ is a Π_3 property given by $\forall y\,\exists z\,Q(x,y,z)$ where Q is Π_1, it is easy to see that

$$S(x) \iff \forall y\,\exists\beta > rank(y)\ \big[V_\beta \models \sigma \wedge \exists z\,Q(x,y,z)\big],$$

where σ is a beth fixed point sentence (recall the beginning of Section 2). Such large cardinal properties as globally superstrong, super-almost-huge, extendible, and superhuge are globalized local. The next proposition allows us to do for these large cardinals what was done for strong and supercompact in Theorems 2.18 and 2.19.

2.20 Proposition. *Suppose $R(x)$ is a globalized local property. Suppose κ is inaccessible and there are arbitrarily large $\lambda > \kappa$ such that λ is inaccessible and $V_\kappa \prec V_\lambda$. Then*

$$[V_\kappa \models R(\alpha)] \Longrightarrow R(\alpha).$$

Proof. Since $\neg R(\alpha)$ is given by a Σ_3 formula, the result follows from Theorem 2.19(2). ■

2.21 Corollary. *Suppose $\alpha < \kappa$ are infinite cardinals such that*

(i) *κ is globally superstrong (extendible, super-almost-huge, superhuge);*

(ii) *$V_\kappa \models$ “α is globally superstrong (extendible, super-almost-huge, superhuge)”.*

Then α is globally superstrong (extendible, super-almost-huge, superhuge).

Proof. Each of these large cardinal properties satisfies the hypothesis of Proposition 2.20. ■

2.22 Theorem. *Let κ be an infinite cardinal.*

(1) *Suppose κ is 2-huge and $j : V \to N$ is a 2-huge embedding with $cp(j) = \kappa$. Then $V_{j(\kappa)} \models$ “κ is superhuge”. Moreover, if κ is super 2-huge then κ is superhuge and if j is a 2-huge embedding as above, $N \models$ “κ is superhuge”.*

(2) *Suppose κ is huge and $j : V \to N$ is a huge embedding with $cp(j) = \kappa$. Then $V_{j(\kappa)} \models$ “κ is super-almost- huge”. Moreover, if κ is superhuge, then κ is super-almost-huge, and if j is a huge embedding as above, $N \models$ “κ is super-almost-huge”.*

(3) *Suppose κ is almost huge and $j : V \to N$ is an almost huge embedding with $cp(j) = \kappa$. Then $V_{j(\kappa)} \models$ “κ is extendible ”. Moreover, if κ is super-almost-huge, then κ is extendible, and if j is an almost huge embedding as above, $N \models$ “κ is extendible”.*

(4) *Suppose κ is supercompact and there is an elementary embedding $j : V_\eta \to V_\zeta$ with $\mathrm{cp}(j) = \kappa$ and $\kappa < \eta < j(\kappa) < \zeta$. Then $V_{j(\kappa)} \models$ "κ is supercompact ". Moreover, if κ is extendible, then κ is supercompact, and for every ξ, there are $\eta > \xi$ and an elementary embedding $j : V_\eta \to V_\zeta$ as above so that $V_\zeta \models$ "κ is supercompact."*

(5) *Suppose κ is superstrong and $j : V \to N$ is a superstrong embedding with $\mathrm{cp}(j) = \kappa$. Then $V_{j(\kappa)} \models$ "κ is strong". Moreover, if κ is globally superstrong, then κ is strong, and if j is a superstrong embedding as above, $N \models$ "κ is strong".*

Proof of (1). The first part is proved in Theorem 5c of [4]. For the second part, note that for any 2-huge embedding $j : V \to N$, if D is the normal ultrafilter over κ derived from j then D contains the set $S = \{\alpha < \kappa : V_\kappa \models \text{"}\alpha \text{ is superhuge"}\}$. Since κ is superhuge, by Corollary 2.21, each $\alpha \in S$ is superhuge too. The result follows. ∎

Proof of (2). Let U^j be as in Definition 2.16 and let $N_1 = (j \cdot j)(N)$. Let $S = \{\beta < j(\kappa) : V_{j(\kappa)} \models$ "β is an a.h.target for κ"$\}$. It suffices to show that $S \in U^j$. Note that

$$S \in U^j \iff V^N_{j^2(\kappa)} = V^{N_1}_{j^2(\kappa)} \models \text{"}j(\kappa) \text{ is an a.h. target for } \kappa.\text{"} \tag{2.2}$$

Because $j \restriction V_{j(\kappa)} \in N$, a coherent sequence $\langle U_\eta : \kappa \le \eta < \lambda\rangle$ satisfying Barbanel's Criterion $\mathcal{B}(\kappa, \lambda)$ can be defined in $V^N_{j^2(\kappa)}$. Note that $\mathcal{B}(\kappa, \lambda)$ is absolute for $V^N_{j^2(\kappa)}$. This establishes (2.2) and completes the proof of the first part.

For the second part, argue as in part (1) using Corollary 2.21. ∎

Proof of (3). Let $j : V \to N$ be an almost-huge embedding, and let U^j be the normal ultrafilter over $j(\kappa)$ derived from $j \cdot j$ in N (as in Definition 2.16). Note that U^j is a $j(\kappa)$-complete filter in V, and hence is uniform. Let $\psi(\alpha, \beta, \gamma)$ be the formula

$$\begin{aligned} \exists\sigma < \beta\forall\eta\exists\xi < \gamma\exists i\big[\sigma < \eta < \beta \Longrightarrow \ & \mathrm{cp}(i) = \alpha \wedge i(\alpha) > \eta \wedge \\ & \text{"}i : V_\eta \to V_\xi \text{ is an elementary embedding"}\big]. \end{aligned}$$

Let $S = \{\beta < j(\kappa) : \psi(\kappa, \beta, j(\kappa))\}$. Note that $\psi(\kappa, \beta, j(\kappa))$ is absolute for N, so $S \in N$. To finish the proof of the first part, it suffices to prove that S is unbounded in $j(\kappa)$; since U^j is a uniform filter in V, it suffices to prove $S \in U^j$:

$$\begin{aligned} S \in U^j &\iff j(\kappa) \in (j \cdot j)(S) \\ &\iff N_1 \models \psi(\kappa, j(\kappa), j^2(\kappa)) \\ &\iff N \models \psi(\kappa, j(\kappa), j^2(\kappa)). \end{aligned}$$

By almost-hugeness, for all $\eta, \kappa < \eta < j(\kappa)$, we have $j \restriction V_\eta \in N$. Thus, we can verify "$N \models \psi(\kappa, j(\kappa), j^2(\kappa))$" by setting $\sigma = \kappa$ and for each $\eta < j(\kappa)$, $\xi = j(\eta)$ and $i = j \restriction V_\eta$.

For the second part, note that a super-almost-huge cardinal is supercompact. Also, if $\lambda > \kappa$ is supercompact and κ is extendible, then $V_\lambda \models$ "κ is extendible"; this can be proved by using the fact that "κ is not extendible" is Σ^{ZF}_3 (see [18, Ex. 23.9(a)]) and applying Theorem 2.18. Thus, we can argue as in [18, 24.13] as follows: Given any ordinal η, let $j : V \to N$ be an almost huge embedding with $\text{cp}(j) = \kappa$ and $j(\kappa) > \eta$. Then $j \restriction V_{\kappa+\eta} \in N$ and so $N \models$ "κ is η-extendible". As $N \models$ "$j(\kappa)$ is supercompact", by the above remark, "κ is η-extendible" holds in $V^N_{j(\kappa)} = V_{j(\kappa)}$. Thus κ is really η- extendible; since η was arbitrary, it follows that κ is extendible.

Finally, to show that for any super-almost-huge embedding $j : V \to N$, $N \models$ "κ is extendible", argue as in part (1), using Corollary 2.21. ∎

Proof of (4). The first part is proven in [18, Proposition 23.8]. The fact that every extendible is supercompact is proven in [18, Proposition 23.6]. To complete the proof, note that since κ is extendible, there is a proper class of inaccessibles; for each inaccessible $\eta > \kappa$ let $j : V_\eta \to V_\zeta$ be an elementary embedding such that cp(j)= κ and $\eta < j(\kappa) < \zeta$. Let $D = \{X \subset \kappa : \kappa \in j(X)\}$ and observe that if $S = \{\alpha < \kappa : V_\kappa \models$ "α is supercompact"$\}$ then $S \in D$. Since η is inaccessible,

$$V_\eta \models \text{“}\kappa \text{ is supercompact”} \wedge \forall \alpha \in S\, (V_\kappa \models \text{“}\alpha \text{ is supercompact”}).$$

Now use Theorem 2.19 inside V_η to conclude that, for each $\alpha \in S$, $V_\eta \models$ "α is supercompact". It follows from elementarity that

$$V_\zeta \models \text{“}\kappa \text{ is supercompact”}.\blacksquare$$

Proof of (5). Given a superstrong embedding $j : V \to N$ and $\lambda < j(\kappa)$, let E be the extender with support $V_{\lambda+1}$ derived from j; $E \in V_{j(\kappa)}$ is a witness to λ-strongness in $V_{j(\kappa)}$ (see the remarks preceding Proposition 2.7).

If κ is globally superstrong, it is clearly strong as well. Argue as in (1), using Theorem 2.19, to conclude that in N, κ is strong. ∎

<u>*Laver Sequences*</u>

2.23 Definition A *Laver sequence* is a function $g : \kappa \to V_\kappa$ such that for each x and each $\lambda \geq \max(\kappa, |\text{TC}(x)|)$, there is a normal ultrafilter U over $P_\kappa\lambda$ such that $x = i_U(g)(\kappa)$.

Laver introduced the concept of a Laver sequence in [22]; he proves such sequences exist whenever κ is supercompact. (In the literature, the condition "$\lambda \geq \max(\kappa, |\text{TC}(x)|)$" in Definition 2.23 is sometimes abbreviated to "$\lambda \geq |\text{TC}(x)|$"; we note here, however, that this abbreviation is technically inaccurate since there is no nontrivial normal ultrafilter over $P_\kappa\alpha$ when $\alpha < \kappa$.) In [14], Shelah and Gitik note without proof that an argument similar to Laver's can be carried out to establish the existence of an analogue to Laver sequences for strong cardinals, replacing normal ultrafilters over index sets of the form $P_\kappa\lambda$ by (κ, λ) ultrafilters. By now, the notion of *extenders* has become more popular (and somewhat more useful) than (κ, λ) ultrafilters (cf. [1], [24]); we give a definition of Laver sequences for strong cardinals using extenders and in Theorem 2.30(2) provide a proof that strong cardinals always admit this type of Laver sequence.

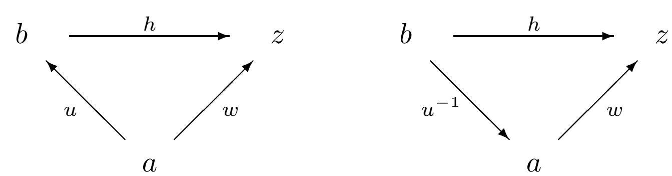
2.24 Definition *Strong Laver Sequences* A *strong Laver sequence* is a function $g : \kappa \to V_\kappa$ such that for each x and each $\lambda > \max(\kappa, rank(x))$, there is an extender E with critical point κ and support V_λ such that $x = i_E(g)(\kappa)$.

Before proving the existence of strong Laver sequences, we extract a couple of useful lemmas implicit in Laver's original proof that will be useful for the proof and for generalizations later.

We shall say that, for any transitive class M (possibly proper) and any $a \in M$, *M is a-closed* if, for each function f, $f \in M$ whenever there is $y \in M$ such that $f : a \to y$.

2.25 Lemma. *Suppose M is transitive, closed under composition of functions and under inverses of bijections. Suppose $a, b \in M$ and $|a| = |b|$. Then M is a-closed iff M is b-closed.*

Proof. By symmetry, it suffices to prove just one direction. Let $u : a \to b$ be a bijection and assume M is a-closed. Because of M's closure properties, both u and u^{-1} must be elements of M. Given $h : b \to z$ with $z \in M$, we show $h \in M$. Let $w = h \circ u : a \to z$. By a-closure again, $w \in M$. Since M is closed under composition of functions, $h = w \circ u^{-1} \in M$. ■

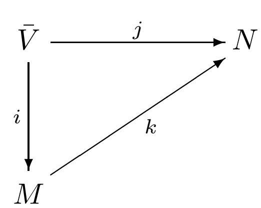


A typical application of Lemma 2.25 arises when, for some λ, $a = V_\lambda$ and $b = |V_\lambda|$. Note that the assertion

$$^{\lambda}M \subset M \tag{2.3}$$

generally asserts more than "M is λ-closed", so when (2.3) is needed, we will be explicit about it.

2.26 Lemma. *Suppose $\bar{V}, M, N$ are transitive classes (possibly proper), where $\bar{V}$ is either V itself or some V_β. Let $\kappa < \lambda$ be ordinals in $\bar{V}$ and suppose $g : \kappa \to V_\kappa \in \bar{V}$. Suppose the following is a commutative diagram of elementary embeddings*



so that $\mathrm{cp}(j) = \kappa = \mathrm{cp}(i)$. Let $x = j(g)(\kappa)$. Suppose further that one of I, II holds, where

(I) (a) $k \restriction (\lambda + 1) = \mathrm{id}_{\lambda+1}$;
 (b) $|\mathrm{TC}(x)| \leq \lambda$; and
 (c) M is λ-closed.

(II) (a) $k \restriction V_\lambda \cap M = \mathrm{id}_{V_\lambda \cap M}$;

(b) $rank(x) < \lambda$.

Then,

$$x \in M,\ k(x) = x, \text{ and } x = i(g)(\kappa).$$

Remarks.

(1) Note that in (I) the computation of $|\mathrm{TC}(x)|$ is in V and in (II), $rank(x)$ is computed in V.

(2) Condition (I) actually implies $k \restriction \lambda^+ = \mathrm{id}_{\lambda^+}$. This follows from condition (Ic) (which shows that $\lambda^+ = (\lambda^+)^M$) and the easily proved fact that the critical point of k must be a cardinal in M. On the other hand, this statement is not generally true assuming condition (II) instead, since it is possible in that case for $\mathrm{cp}(k) = \lambda$. (An example of this situation occurs when κ is almost huge with two targets $\lambda, \tilde{\lambda}$, as described in the hypothesis of Theorem 2.11. In the notation of that theorem, Lemma 2.26(II) holds for $\tilde{e}, \lambda$ and $\mathrm{cp}(\tilde{e}) = \lambda$. The hypotheses of Theorem 2.11 are shown in Theorem 5.20 to follow from the existence of a huge cardinal.)

Proof. Begin by setting $y = i(g)(\kappa) \in M$ and noting that since $k(\kappa) = \kappa$, $k(y) = x$. Also observe that $\lambda \in M$ since $\lambda \leq i(\lambda) \in M$; likewise $\lambda \in N$.

Assuming (I), we observe first that the terms $\mathrm{TC}(\{y\})$, $\mathrm{TC}(y)$ are defined in (and absolute for) M; this follows because

$$y = i(g)(\kappa) \in i(V_\kappa) = V^M_{i(\kappa)} = H^M_{i(\kappa)}$$

where, for each cardinal ν, H_ν is the set of all w for which $|\mathrm{TC}(w)| < \nu$.

We show by $\in$-induction (in V) that for all $z \in \mathrm{TC}(\{y\})$, $k(z) = z$. Assume $k(u) = u$ for all $u \in z$ and let $f : \lambda \to z$ be a surjection. Since $z \in M$, by (Ic), $f \in M$. Then $k(\mathrm{dom}\ f) =$ dom f and $k(f)(\beta) = k[f(\beta)] = f(\beta)$ by induction hypothesis. Thus, $k(f) = f$, and so

$$k(z) = k(range(f)) = range(k(f)) = range(f) = z.$$

Thus, $k(y) = y$. We have therefore shown that $x = k(y) = y = i(g)(\kappa) \in M$.

To obtain the result assuming (II), note that $N \models rank(x) < \lambda$ (since $x = j(g)(\kappa) \in N$ and $x \in V_\lambda$ by (II)). Now by elementarity of k, $M \models rank(y) < \lambda$. We have, therefore, using (IIa), $x = k(y) = y = i(g)(\kappa) \in M$, as required. ■

Notice that the definition of Laver sequences for supercompact cardinals involves the notion of transitive closure whereas the parallel definition for strong cardinals uses ranks. In Section 4 (Definition 4.14) when we give a generalized definition for Laver sequences, we will use ranks instead of transitive closures; arguments involving this definition will therefore often invoke Lemma 2.26(II). On the other hand, we will have occasion to use Lemma 2.26(I) in arguments concerning the standard Laver sequences for supercompact cardinals.

The following observation will be useful when we need to compare the result of performing an ultrapower construction in V with that of doing the same construction in some inner model (or transitive set) N.

2.27 Lemma. *Suppose N is either an inner model of* ZFC *or a transitive set satisfying* ZFC $-$ *Replacement* $+\forall x\,\exists\alpha\,(x\in V_\alpha)$.

(1) *Suppose U is a normal ultrafilter over $P_\kappa\lambda$, $U\in N$, and ${}^{P_\kappa\lambda}N\subset N$. Then whenever $h:P_\kappa\lambda\to N$, h represents the same element in $V^{P_\kappa\lambda}/U$ as in $\left(V^{P_\kappa\lambda}/U\right)^N$.*

(2) *Suppose E is an extender with critical point κ and support Y, E and Y are in N, and ${}^\kappa N\subset N$. Then for each $a\in{}^{<\omega}[Y]$ and each $F:{}^aV_\kappa\to N$, F represents the same element in $Ult(V,E)$ as it does in $\left(Ult(V,E)\right)^N$.*

(3) *Suppose U is a normal ultrafilter over $P(\lambda)$ so that $\{X\in P(\lambda):\mathrm{ot}(X)=\kappa\}\in U$, $U\in N$, and ${}^{P(\lambda)}N\subset N$. Then whenever $h:P(\lambda)\to N$, h represents the same element in $V^{P(\lambda)}/U$ as in $\left(V^{P(\lambda)}/U\right)^N$.*

Proof. The proofs of each case are similar; we prove (2) and leave the proofs of (1) and (3) to the reader. To prove (2), first observe that the elements of $Ult(V,E)$ represented by a function $F:{}^aV_\kappa\to N$ form a transitive class, so we may proceed by $\in$-induction over this class. Given $F:{}^aV_\kappa\to N$, notice that for each $G:{}^bV_\kappa\to N$,

$$Ult(V,E)\models[G]_E\in[F]_E \text{ iff } \left(Ult(V,E)\models[G]_E\in[F]_E\right)^N \tag{2.4}$$

since the statement $Ult(V,E)\models[G]_E\in[F]_E$ is equivalent to

$$\forall c\in{}^{<\omega}[Y]\left[(c\supseteq a\cup b)\Longrightarrow\Big(E_c\big(\{\,h:G(h\restriction b)\in F(h\restriction a)\,\}\big)=1\Big)\right],$$

which is absolute for N and V. By the induction hypothesis, the conclusion of (2) holds for all $G:{}^bV_\kappa\to N$ for which $[G]_E\in[F]_E$. It now follows from (2.4) that $[F]_E$ and $[F]_E^N$ have exactly the same elements. ∎

The next proposition tells us that when we form various ultrapowers with suitably large V_γ's instead of with V, the resulting universes look the same below V_γ, and we can get a definable upper bound on the codomain of the canonical embeddings.

2.28 Proposition. *Suppose $\kappa<\lambda<\beta\le\gamma<\rho$, where γ is a regular cardinal and ρ is a beth fixed point. Suppose i and $\tilde{i}$ are elementary embeddings with critical point κ. Also, suppose one of the following conditions holds:*

(A) *U is a normal ultrafilter over $P_\kappa\lambda$ and $\gamma>2^{\lambda^{<\kappa}}$; $\tilde{i}:V\to V^{P_\kappa\lambda}/U\cong\tilde{M}$ and $i:V_\gamma\to V_\gamma^{P_\kappa\lambda}/U\cong M$ are the canonical embeddings.*

(B) *E is an extender with critical point κ and support V_λ, and $\tilde{i}:V\to Ult(V,E)\cong\tilde{M}$ and $i:V_\gamma\to Ult(V_\gamma,E)\cong M$ are the canonical embeddings.*

(C) *$\langle U_\eta:\kappa\le\eta<\lambda\rangle$ is a coherent sequence of normal ultrafilters satisfying $\mathcal{B}(\kappa,\lambda)$, $\tilde{M}$ is the direct limit of the system $\langle\tilde{M}_\eta;\tilde{k}_{\eta\zeta}:\kappa\le\eta<\zeta<\lambda\rangle$, where $\tilde{M}_\eta\cong V^{P_\kappa\eta}/U_\eta$, and M is the direct limit of the system $\langle M_\eta;k_{\eta\zeta}:\kappa\le\eta<\zeta<\lambda\rangle$, where $M_\eta\cong V_\gamma^{P_\kappa\eta}/U_\eta$, and $\tilde{i}:V\to\tilde{M}$, $i:V_\gamma\to M$ are the respective canonical embeddings.*

(D) $\gamma > 2^{2^\lambda}$, *U is a normal ultrafilter over* $P(\lambda)$ *containing the set* $\{X \in P(\lambda) : \mathrm{ot}(X) = \kappa\}$, *and* $\tilde{i} : V \to V^{P(\lambda)}/U \cong \tilde{M}$, $i : V_\gamma \to V_\gamma^{P(\lambda)}/U$ *are the canonical embeddings.*

Then

(1) $\tilde{i} \restriction V_\beta = i \restriction V_\beta$;

(2) $\tilde{i}(V_\beta) = i(V_\beta)$ *if* $\beta < \gamma$, *and* $\tilde{i}(V_\gamma) = M$;

(3) *both* $\tilde{i}(V_\beta)$ *and* $\tilde{i} \restriction V_\beta$ *are members of* V_ρ.

Proof. First we consider the conditions (A), (B), and (D). We first observe that in each of these cases,

$$\tilde{i}(\gamma) = \sup\{\tilde{i}(\alpha) : \alpha < \gamma\}; \tag{2.5}$$

to prove this, let $[f] \in \tilde{i}(\gamma)$, where $f : I \to \gamma$ (and I is either $P_\kappa\lambda$, $P(\lambda)$, or ${}^aV_\kappa$ for some $a \in [V_\lambda]^{<\omega}$). By regularity of γ and its size relative to I, there is $\alpha < \gamma$ such that $f : I \to \alpha$, whence $[f] < \tilde{i}(\alpha)$, as required.

Note that since γ is regular and large enough, V_γ satisfies the hypotheses for the class N in Lemma 2.27(1)-(3). The conclusion of Lemma 2.27 gives us (1) and the first part of (2) of the present theorem immediately, for $\beta < \gamma$. To complete the proof of (2), use (2.5) to observe that

$$\begin{aligned}\tilde{i}(V_\gamma) &= V^{\tilde{M}}_{\tilde{i}(\gamma)} \\ &= \bigcup_{\alpha < \tilde{i}(\gamma)} V^{\tilde{M}}_\alpha \\ &= \bigcup_{\alpha < \tilde{i}(\gamma)} V^M_\alpha \\ &= M.\end{aligned}$$

Part (3) requires a computation. Assuming condition (A),

$$\begin{aligned}\max\bigl(rank(i \restriction V_\beta), rank(i(V_\beta))\bigr) &< i(\beta) + \omega \\ &< (\beta^{\lambda^{<\kappa}})^+ + \omega \\ &< \rho.\end{aligned}$$

For (B),

$$\begin{aligned}\max\bigl(rank(i \restriction V_\beta), rank(i(V_\beta))\bigr) &< i(\beta) + \omega \\ &< \Bigl(\bigl|\sum_{|V_\lambda|} \beta^\kappa\bigr|\Bigr)^+ + \omega \\ &< \rho.\end{aligned}$$

For (D),

$$\begin{aligned}\max\bigl(rank(i \restriction V_\beta), rank(i(V_\beta))\bigr) &< i(\beta) + \omega \\ &< (\beta^{2^\lambda})^+ + \omega \\ &< \rho.\end{aligned}$$

Finally, we consider the condition (C). We will make use of the details of the direct limit construction, as described earlier in this section. We review the relevant notation for the direct limit $\tilde{M}$; the parallel notation will be understood for M. $\tilde{M}$ is obtained as a collection of equivalence classes $[\eta, x]$; there are embeddings $\tilde{k}_\eta$ for each η defined by $\tilde{k}_\eta(x) = [\eta, x]$. The embedding $\tilde{i} : V \to \tilde{M}$ is defined by $\tilde{k}_\eta \circ \tilde{i}_\eta$ for any η.

By (A), for each η, $\tilde{i}_\eta \restriction V_\gamma = i_\eta \restriction V_\gamma$ and $\tilde{i}_\eta(V_\gamma) = M_\eta$; from this, it follows that, for $\kappa \le \eta < \zeta < \lambda$,

$$\tilde{k}_{\eta\zeta} \restriction M_\eta = k_{\eta\zeta} \restriction M_\eta. \tag{2.6}$$

We wish to show that for all $x \in M_\eta$, $\tilde{k}_\eta(x) = k_\eta(x)$, i.e., $[\eta, x]_{\tilde{M}} = [\eta, x]_M$. It suffices to show that for all ξ and all $y \in \tilde{M}_\xi$, $(\xi, y) \sim_{\tilde{M}} (\eta, x)$ if and only if $y \in M_\xi$ and $(\xi, y) \sim_M (\eta, x)$. If $y \in \tilde{M}_\xi$ and $(\xi, y) \sim_{\tilde{M}} (\eta, x)$, then there is $\zeta > \xi, \eta$ such that $\tilde{k}_{\xi\zeta}(y) = \tilde{k}_{\eta\zeta}(x) = k_{\eta\zeta}(x) \in M_\zeta$. But $\tilde{k}_{\xi\zeta}(y) \in M_\zeta$ implies $y \in M_\xi$ by (2.6). Thus $(\xi, y) \sim_M (\eta, x)$. The proof of the converse is similar but easier. Thus $\tilde{k}_\eta \restriction M_\eta = k_\eta$.

Now let $x \in V_\gamma$. Then

$$\tilde{i}(x) = \tilde{k}_\eta \circ \tilde{i}_\eta(x) = k_\eta \circ i_\eta(x) = i(x),$$

and so $\tilde{i} \restriction V_\gamma = i$.

To complete the proof of (2), we can argue as in the other cases, using Proposition 2.12, that $\tilde{i}(V_\gamma) = \bigcup_{\alpha < \tilde{i}(\gamma)} i(V_\alpha) = M$.

To prove (3) for this case, we perform the following computation:

$$\begin{aligned} \max\bigl(rank(i \restriction V_\beta), rank(i(V_\beta))\bigr) &< i(\beta) + \omega \\ &< \sum_{\kappa \le \eta < \lambda} (\beta^{\eta^{<\kappa}})^+ + \omega \\ &< \rho, \end{aligned}$$

and we are done. ∎

The next proposition applies Lemma 2.27 and will be used in Section 5.

2.29 Proposition. *Suppose κ is a strong cardinal, $\alpha < \kappa$, and $g : \alpha \to V_\alpha$ is a function.*

(1) *If g is not a (supercompact) Laver sequence, then there are $x \in V_\kappa$ and $\lambda < \kappa$ with $\lambda \ge \max(\alpha, |\mathrm{TC}(x)|)$, such that for all normal ultrafilters U over $P_\alpha\lambda$, $i_U(g)(\alpha) \neq x$.*

(2) *If g is not a strong Laver sequence, then there are $x \in V_\kappa$ and $\lambda < \kappa$ with $\lambda \ge \max(\alpha, rank(x))$, such that for all extenders E with critical point α and support V_λ, $i_E(g)(\alpha) \neq x$.*

Proof. The method of proof in each case is similar; both cases make use of Lemma 2.27. We prove (2) and leave the proof of (1) to the reader. Since g is not strong Laver at α, there is a set x and a $\lambda > \max(\alpha, rank(x))$ such that for each extender E with critical point α and support

V_λ, $x \neq i_E(g)(\alpha)$. Let $j : V \to N$ be an elementary embedding with critical point κ, $j(\kappa) > \lambda^+$ and $V_{\lambda^+} \subset N$. Now V_λ, together with every extender with critical point α and support V_λ, belong to N. By Lemma 2.27, for all such extenders E, $i_E(g)(\alpha) = i_E^N(g)(\alpha)$; thus the following holds in N:

$$\exists x\, \exists \lambda < j(\kappa)\, [\max(\alpha, rank(x)) < \lambda \text{ and for all extenders } E \text{ with critical point } \alpha \text{ and support } V_\lambda,\ i_E(g)(\alpha) \neq x].$$

By elementarity of j, the following is true:

$$\exists x\, \exists \lambda < \kappa\, [\max(\alpha, rank(x)) < \lambda \text{ and for all extenders } E \text{ with critical point } \alpha \text{ and support } V_\lambda,\ i_E(g)(\alpha) \neq x],$$

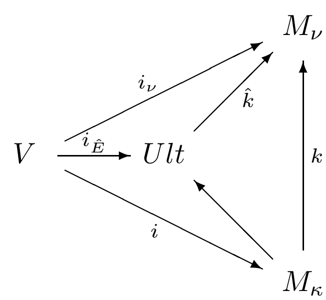
as required.■

The proof of Lemma 2.29 shows that we can pick λ to be as large as we like, below κ.

2.30 Theorem (Laver [22]). *Suppose κ is an infinite cardinal.*

(1) *If κ is supercompact, there is a Laver sequence at κ.*

(2) *If κ is strong, there is a strong Laver sequence at κ.*



Proof. Part (1) was proved in [22]. For (2), we proceed as in [22] using extenders instead of normal ultrafilters as follows. Assume the theorem is false. For each $f : \kappa \to V_\kappa$, let λ_f be the least ordinal such that for some set x with $\max(\kappa, rank(x)) < \lambda_f$, $i_E(f)(\kappa) \neq x$ for any extender E with critical point κ and support V_{λ_f}. Let ν be a regular cardinal greater than $|V_{\lambda_f}|$ for all λ_f.

Let $i_\nu : V \to M_\nu$ be an elementary embedding with critical point κ, $i_\nu(\kappa) > \nu$ and $V_\nu \subseteq M_\nu$. Let $\phi(g, \delta)$ denote the following formula:

> "There exists a cardinal α with $g : \alpha \to V_\alpha$ and δ is the least ordinal for which there is a set x with $\max(\alpha, rank(x)) < \delta$ such that for all extenders E with critical point κ and support V_δ, $i_E(g)(\kappa) \neq x$."

Claim. *For all $f \in {}^\kappa V_\kappa$, $M_\nu \models \phi(f, \lambda_f)$.*

Proof of Claim. We have chosen ν large enough to guarantee the absoluteness in M_ν of all relevant notions. It suffices to show that, for any extender E with critical point κ and support V_{λ_f},

$$i_E^{M_\nu}(f)(\kappa) = i_E(f)(\kappa). \tag{2.7}$$

Since $f : \kappa \to V_\kappa \in M_\nu$, it follows from Lemma 2.27 that

$$i_E^{M_\nu}(f) \;=\; [c_f^a]_E^{M_\nu} \;=\; [c_f^a]_E \;=\; i_E(f).\blacksquare$$

Now let $U_\kappa = \{B \subseteq \kappa \mid \kappa \in i_\nu(B)\}$, and let $i : V \to M_\kappa \cong V^\kappa/U_\kappa$ be the canonical elementary embedding. As usual (cf. [17]), defining $k : M_\kappa \to M_\nu$ by $k([f]) = i_\nu(f)(\kappa)$ makes k an elementary embedding such that $k \circ i = i_\nu$.

Now

$$M_\nu \models \big(\forall f : \kappa \to V_\kappa\big)\,\big(\exists \lambda < i_\nu(\kappa)\big)\,\phi(f, \lambda).$$

Since $i_\nu = k \circ i, k(\kappa) = \kappa$, and $k(V_\kappa) = V_\kappa$,

$$M_\kappa \models \big(\forall f : \kappa \to V_\kappa\big)\,\big(\exists \lambda < i(\kappa)\big)\,\phi(f, \lambda).$$

By Łoś' Theorem, there is $D \in U_\kappa$ such that for all $\alpha \in D$,

$$\big(\forall f : \alpha \to V_\alpha\big)\,\big(\exists \lambda < \kappa\big)\,\phi(f, \lambda).$$

Define $g : \kappa \to V_\kappa$ recursively as follows: Assuming $g \restriction \alpha$ has been defined, put $g(\alpha) = 0$, unless $\alpha \in D$ and $g \restriction \alpha : \alpha \to V_\alpha$, in which case let $g(\alpha)$ be a witness for $\phi(g \restriction \alpha, \lambda_{g\restriction\alpha})$.

Applying i_ν to the formula

$$\forall \alpha \in D\,\big(\text{“}g(\alpha) \text{ witnesses } \phi(g \restriction \alpha, \lambda_{g\restriction\alpha})\text{”}\big),$$

and noting that $\kappa \in i_\nu(D)$ and $i_\nu(g) \restriction \kappa = g$, we obtain

$$M_\nu \models \text{“}x \text{ witnesses } \phi(g, \lambda_g),\text{”}$$

where $x = i_\nu(g)(\kappa)$. Using (2.7) as in the proof of the claim, it follows that

$$i_\nu(g)(\kappa) \neq i_E(g)(\kappa) \tag{2.8}$$

for any extender E with critical point κ and support V_{λ_g}. We obtain a contradiction by showing that the definition of g gives rise to an extender $\hat{E}$ that violates (2.8).

Let $\hat{E}$ denote the extender derived from i_ν with support V_{λ_g}, let $i_{\hat{E}} : V \to \mathrm{Ult}(V, \hat{E})$ be the canonical embedding and let $\hat{k} : \mathrm{Ult}(V, \hat{E}) \to M_\nu$ be defined as in Proposition 2.5. Since $\hat{k} \restriction V_{\lambda_g} = \mathrm{id}_{V_{\lambda_g}}$ and $rank(x) < \lambda_g$, we can use Lemma 2.26(II) to conclude that $x = i_{\hat{E}}(g)(\kappa)$; this is a contradiction. $\blacksquare$

We now give a kind of ultrafilter characterization of Laver sequences and use it to show that the statement “f is a Laver sequence at κ” is Π_3^{ZFC}. This syntactic classification will be useful for proving reflection properties of Laver sequences in Section 5.

We begin with an observation. Suppose $g : \kappa \to V_\kappa, \lambda > \kappa, h : P_\kappa\lambda \to V, U$ is a normal ultrafilter over $P_\kappa\lambda$, and, in the (transitive collapse of the) ultrapower M_λ by U, we let $x = [h]_U$.

Let $i_U : V \to M_\lambda$ be the canonical embedding. Recall that the function $t_\kappa : P_\kappa\lambda \to V : P \mapsto \text{ot}(P \cap \kappa)$ represents κ in M_λ (see the subsection *Supercompact Cardinals*). We have the following equivalences:

$$(*)_{sc} \qquad \begin{aligned} i_U(g)(\kappa) = x &\iff M_\lambda \models [c_g]([t_\kappa]) = [h] \\ &\iff \{P \in P_\kappa\lambda : g(\text{ot}(P \cap \kappa) = h(P)\} \in U \\ &\iff \{P \in P_\kappa\lambda : g(P \cap \kappa) = h(P)\} \in U. \end{aligned}$$

(The last equivalence uses the fact that $\{P : \text{ot}(P \cap \kappa) = P \cap \kappa\} \in U$, which follows from the fact that $i_U''\kappa = \kappa$.)

2.31 Lemma. *Suppose κ is supercompact. Then for each x and each $\lambda \geq \max(\kappa, |\text{TC}(x)|)$, there exist a normal ultrafilter U over $P_\kappa\lambda$ and a function $h : P_\kappa\lambda \to V_\kappa$ such that h represents x in the ultrapower by U.*

Proof. The point is to show that it is always possible to find a representing h for x having range in V_κ. By supercompactness, let $g : \kappa \to V_\kappa$ be a Laver function and let U be a normal ultrafilter over $P_\kappa\lambda$ such that $x = i_U(g)(\kappa)$. Let $\hat{h}$ represent x in the ultrapower. Then by $(*)_{sc}$, there is a U-measure 1 set of $P \in P_\kappa\lambda$ such that $\hat{h}(P) \in V_\kappa$. Now define $h : P_\kappa\lambda \to V_\kappa$ so that it agrees with $\hat{h}$ on a U-measure 1 set. ∎

For the next theorem, we need the following formulas $Q_{sc}(g, \kappa, \lambda, x)$ and $Q'_{sc}(g, \kappa, \lambda, x)$:

$Q_{sc}(g, \kappa, \lambda, x)$: $\lambda \geq \max(\kappa, |\text{TC}(x)|)$ implies that there exist a normal ultrafilter U over $P_\kappa\lambda$ and a function $h : P_\kappa\lambda \to V_\kappa$ such that $\{P : g(P \cap \kappa) = h(P)\} \in U$ and, if M denotes the transitive collapse of the ultrapower $V^{P_\kappa\lambda}/U$, then in M, $x = [h]_U$.

$Q'_{sc}(g, \kappa, \lambda, x)$: $\lambda \geq \max(\kappa, |\text{TC}(x)|)$ implies that there exist a normal ultrafilter U over $P_\kappa\lambda$ and a function $h : P_\kappa\lambda \to V_\kappa$ such that $\{P : g(P \cap \kappa) = h(P)\} \in U$ and, if N denotes the transitive collapse of the ultrapower $V_\gamma^{P_\kappa\lambda}/U$, where $\gamma > 2^{\lambda^{<\kappa}}$ is a regular cardinal, then $x = [h]_U$.

2.32 Theorem. *Suppose κ is an infinite cardinal and $g : \kappa \to V_\kappa$ is a function. Then the following are equivalent:*

(1) *g is Laver at κ;*

(2) $\forall x \forall \lambda\, Q_{sc}(g, \kappa, \lambda, x)$;

(3) $\forall x \forall \lambda\, Q'_{sc}(g, \kappa, \lambda, x)$

Proof. (1) ⇔ (2) follows from $(*)_{sc}$. To prove (2) ⇔ (3), use Lemma 2.27(1), noting that V_γ and $h : P_\kappa\lambda \to V_\kappa$ satisfy the hypotheses of that proposition. ∎

2.33 Corollary. *The statement "g is a Laver function at κ" is* Π_3^{ZFC}.

Proof. In light of the last theorem, it is easy to see that, in ZFC, $Q'_{sc}(g,\kappa,\lambda,x)$ is equivalent to the following:

$$\exists Y\,\exists Z\,\exists\gamma\,\exists\beta\,\big[Y = V_\gamma \wedge Z = P(\lambda)\ \wedge\ \text{"}\beta \text{ and } \gamma \text{ are regular cardinals"}\ \wedge\ \lambda < |Z| < \beta < \gamma\ \wedge \\ \exists U, X, h, t, \pi, M \in Y\ \sigma[U, X, h, t, \pi, M, \kappa, \lambda, \beta, x]\big],$$

where σ says that U is a normal ultrafilter over $P_\kappa\lambda$, X is the ultrapower of $V_\beta^{P_\kappa\lambda}$ by U, $\pi : X \to M$ is the collapsing isomorphism, $t = [h]_U$, $\pi(t) = x$, and $\{P : g(P\cap\kappa) = h(P)\} \in U$. Note that γ has been chosen large enough so that all variables in σ can be bound to $Y = V_\gamma$. Because the formulas "$Y = V_\gamma$", "$Z = P(\lambda)$" and "β,γ are regular cardinals" are Π_1^{ZFC}, it is easy to see that the displayed formula is indeed Σ_2^{ZFC}; prefixing the formula with the universal quantifiers '$\forall x\forall\lambda$' turns it into a Π_3^{ZFC} formula, as required. ∎

Note that we have not claimed the formula in the proof is Π_3^{ZF} since the Axiom of Choice is used in obtaining an h from each triple (x,λ,U).

2.34 Remark. The results just presented for supercompact cardinals carry over to strong cardinals, and, to a lesser extent, to super-almost-huge and superhuge cardinals. These results can be used to show that Laver sequences corresponding to each of these large cardinals (to be defined in Section 4) are also Π_3^{ZFC}. This syntactic result will be obtained in a somewhat slicker, less laborious way in Section 5, so the details below can be safely skipped by the reader who wants to move quickly to the main results.

(1) *Strong cardinals.* First, we have the following analogue to $(*)_{sc}$: Suppose $g:\kappa\to V_\kappa, \lambda > \kappa, G : {}^aV_\kappa \to V, E$ is an extender with critical point κ and support $V_{\lambda+1}$, and, in the ultrapower $Ult(V,E)$ we let $x = [G]_E$. Let i_E be the canonical embedding. Recall that for each $b \in {}^{<\omega}[V_{\lambda+1}]$, with $\kappa\in b$, κ is represented by $H^b_\kappa : h \mapsto h(\kappa)$ in $Ult(V,E)$. We have, for all $a,b,c,d \in {}^{<\omega}[V_{\lambda+1}]$, with $\kappa \in b$ and $a\cup b\cup c \subseteq d$:

$$(*)_{str} \qquad i_E(g)(\kappa) = x \iff Ult(V,E) \models [c^a_g]([H^b_\kappa]) = [G] \\ \iff E_d\Big(\{h : g(h(\kappa)) = G(h \restriction c)\}\Big) = 1.$$

The analogue to Lemma 2.31 holds. The analogue to Q_{sc} is the following:

$Q_{str}(g,\kappa,\lambda,x)$: $\lambda > \max(\kappa, rank(x))$ implies that there exist an extender E with critical point κ and support $V_{\lambda+1}$ and a function $G : {}^aV_\kappa \to V_\kappa$ such that, for all $c,d \in {}^{<\omega}V_\kappa$ with $\kappa\in d$ and $c\subseteq d$, $E_d\Big(\{h : g(h(\kappa)) = G(h\restriction c)\}\Big) = 1$, and $x = [G]_E$.

As in the supercompact case, we let $Q'_{str}(g,\kappa,\lambda,x)$ be the same as $Q_{str}(g,\kappa,\lambda,x)$ except that we consider the ultrapower $Ult(V_\gamma, E)$ where $\gamma > \lambda$ is a regular cardinal. With these definitions, the analogue to Theorem 2.32 can be proven for strong Laver sequences using Lemma 2.27(2); it follows that "g is a strong Laver function at κ" is Π_3^{ZFC}.

(2) *Super-almost-huge cardinals.* Although we have not attempted to give a definition of a Laver sequence for these cardinals, we can still carry out most of the details of the work above for later use. (The interested reader can verify, after reading Definition 4.14, that the corresponding notion of a Laver sequence turns out to be Π_3^{ZFC}; this is proven in Section 5 using other methods.) We again begin by formulating an analogue to $(*)_{sc}$: Suppose $g : \kappa \to V_\kappa, \lambda > \kappa$ is inaccessible, $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ is a coherent sequence satisfying $\mathcal{B}(\kappa, \lambda)$, and x is a set and η is an ordinal with $\max(\kappa, |V_{rank(x)}|) < \eta < \lambda$. Let $\langle M_\eta; k_{\eta\zeta} : \kappa \leq \eta < \zeta < \lambda\rangle$ be the usual sequence of ultrapowers over the U_η with canonical embeddings i_η; let M be the direct limit of the M_η; let $k_\eta : M_\eta \to M$ be the usual embeddings for which $k_\eta = k_\zeta \circ k_{\eta\zeta}$ for $\eta < \zeta$; and let $i : V \to M$ be defined by $i = k_\eta \circ i_\eta$ for all η. Let $h : P_\kappa\eta \to V_\kappa$ represent x in M_η. Then

$$(*)_{sah} \qquad \begin{aligned} i(g)(\kappa) = x &\Longleftrightarrow (k_\eta \circ i_\eta)(g)(\kappa) = x \\ &\Longleftrightarrow k_\eta(i_\eta(g))\, k_\eta(\kappa) = k_\eta(x) \\ &\Longleftrightarrow i_\eta(g)(\kappa) = x \\ &\Longleftrightarrow \{P \in P_\kappa\eta : g(P \cap \kappa) = h(P)\} \in U_\eta. \end{aligned}$$

The analogue to Q_{sc} is the following:

$Q_{sah}(g, \kappa, \lambda, x)$: $\lambda > \max(\kappa, rank(x))$ and λ is inaccessible and there is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ satisfying $\mathcal{B}(\kappa, \lambda)$ and an η such that $\max(\kappa, |V_{rank(x)}|) < \eta < \lambda$ and a function $h : P_\kappa\eta \to V_\kappa$ such that that $\{P : g(P \cap \kappa) = h(P)\} \in U_\eta$ and, if M_η denotes the transitive collapse of the ultrapower $V^{P_\kappa\eta}/U_\eta$, then in M_η, $x = [h]_{U_\eta}$.

The formula $Q'_{sah}(g, \kappa, \lambda, x)$ is obtained from $Q_{sah}(g, \kappa, \lambda, x)$ by replacing M_η by N_η where N_η is the transitive collapse of the ultrapower $V_\gamma^{P_\kappa\eta}/U_\eta$ and γ is a regular cardinal $> \lambda$. With this definition, one proves as in Theorem 2.32, that for all g, κ, λ, x, $Q_{sah}(g, \kappa, \lambda, x) \Longleftrightarrow Q'_{sah}(g, \kappa, \lambda, x)$. We note that the analogy with Theorem 2.32 stops here: a "super-almost-huge Laver sequence" g at κ *cannot* be characterized by either of the formulas $\forall x \forall \lambda\, Q_{sah}$, $\forall x \forall \lambda\, Q'_{sah}$. This point will become apparent in Section 4 when we give a more general definition of Laver sequences.

(3) *Superhuge cardinals.* As with super-almost-huge cardinals, the definition of superhuge Laver sequences will remain undisclosed for the moment, but we can obtain some useful information as in (2) above. The displayed equivalences $(*)_{sh}$ are obtained by mimicking $(*)_{sc}$ verbatim, but replacing the index set by $P(\lambda)$ and the normal ultrafilter over $P_\kappa\lambda$ by a normal U over $P(\lambda)$ that contains the set $\{P \in P(\lambda) : \mathrm{ot}\ P = \kappa\}$. Also:

$Q_{sh}(g, \kappa, \lambda, x)$: $\lambda > \max(\kappa, rank(x))$ and λ inaccessible and there exist a normal ultrafilter U over $P(\lambda)$ such that $\{P \in P(\lambda) : \text{ot } P = \kappa\} \in U$ and a function $h : P(\lambda) \to V_\kappa$ such that $\{P : g(P \cap \kappa) = h(P)\} \in U$ and, if M denotes the transitive collapse of the ultrapower $V^{P(\lambda)}/U$, then in M, $x = [h]_U$.

Now obtain $Q'_{sh}(g, \kappa, \lambda, x)$, taking ultrapowers over some V_γ where γ is regular and $> 2^{2^\lambda}$, and observe that for all g, κ, λ, x,

$$Q_{sh}(g, \kappa, \lambda, x) \Longleftrightarrow Q'_{sh}(g, \kappa, \lambda, x).$$

As in the super-almost-huge case, the analogy with Theorem 2.32 stops here: a "superhuge Laver sequence" g at κ *cannot* be characterized by either of the formulas $\forall x \forall \lambda\, Q_{sh}$, $\forall x \forall \lambda\, Q'_{sh}$. A syntactic classification of such Laver sequences will be obtained in Theorem 5.15.

§3. The Wholeness Axiom.

We formalize WA by adding a single unary function symbol **j** to the usual language $\{\in\}$ of ZFC, and by adding to the axioms of ZFC other axioms which assert that **j** is a nontrivial elementary embedding and that Separation (but not Replacement) holds even for formulas with occurrences of **j**. We will call a formula in the extended language an $\in$-*formula* if it has no occurrence of **j**, and a **j**-*formula* if it has such an occurrence; if we wish to leave open either possibility, we will call it a $\{\in, \mathbf{j}\}$-formula.

3.1 Definition. (ZFC + WA) The axiom system ZFC + WA consists of the usual axioms of ZFC together with the following:

- **(1)**$_\phi$ (*Separation Schema for* **j**-*formulas*). Each instance of the usual Separation schema involving ϕ is an axiom (where ϕ is a **j**-formula).
- **(2)**$_\phi$ (*Elementarity Schema for* $\in$-*formulas*). Each of the following **j**-sentences is an axiom, where $\phi(x_1, x_2, \ldots, x_m)$ is an $\in$-formula,

$$\forall x_1, x_2, \ldots, x_m \left(\phi(x_1, x_2, \ldots, x_m) \Longleftrightarrow \phi(\mathbf{j}(x_1), \mathbf{j}(x_2), \ldots, \mathbf{j}(x_m))\right);$$

- **(3)** (*Nontriviality*). $\exists x\, (\mathbf{j}(x) \neq x)$.

Clearly, the theory ZFC + WA is a recursive extension of ZFC. When we informally talk about "adding WA to ZFC," or when we say in a proof "assume WA", we mean that we are working in the language $\{\in, \mathbf{j}\}$ and in the theory ZFC + WA as described above. When we interpret **j** in a model, the resulting elementary embedding will be denoted j and will often be called the WA-*embedding*. In this section, and this section only, we will be careful to use 'j' only for the WA-embedding, and to observe the distinction between **j** and j.

In the sequel, we will often have occasion to speak of elementary embeddings and elementary substructures, sometimes obtained by restricting the WA-embedding j—or through some other construction involving j—and sometimes unrelated to j; in every case, the "elementarity" will be with respect to $\{\in\}$-formulas only.

Let us assume now that we have a fixed Gödelization of the syntax of ZFC + WA and have built up a set $\mathcal{L}$ of hereditarily finite sets consisting of analogues to the basic symbols and formulas of the language of set theory (without constants); in addition, we assume we have a class $\mathcal{L}_V \supseteq \mathcal{L}$ that also includes a "constant" $\chi(a)$ for each set a, where χ is a simply definable class function. Assume that for each of these languages, we have defined relations $vble, fmla, \in\text{-}fmla, \in\text{-}sent$ (and other such sets) that correspond to the sets of variables, formulas, $\in$-formulas, and $\in$-sentences, and that are absolute for models of ZFC. Since $\mathcal{L}$ has no constants, we may assume that these relations, as defined for $\mathcal{L}$, are subsets of V_ω. Note that every $\{\in, \mathbf{j}\}$-formula $\Phi(x_1, \dots, x_n)$ has an $\mathcal{L}$- counterpart in V_ω which we denote (just in this section of the paper) $\phi(x_1, \dots, x_n)$. Assume further that we have defined a "satisfaction" predicate $Sat(x, y)$ for $\mathcal{L}_V$ so that for each $\in$-formula $\Phi(x_1, x_2, \dots, x_m)$ having $\mathcal{L}$-counterpart $\phi(x_1, x_2, \dots, x_m)$

$$\begin{aligned}\text{ZFC} \vdash \forall M\, \forall a_1 \in M\, \forall a_2 \in M\, \dots \forall a_m \in M\, \big(\Phi^M[a_1, a_2, \dots, a_m] \\ \iff Sat(M, \phi(\chi(a_1), \chi(a_2), \dots, \chi(a_m)).\end{aligned}$$

For details concerning this approach to formalizing syntax, see [10]. We will make use of the formalisms described above to prove some of the basic facts about WA, and relax into a less fussy standard in Sections 4-8.

We begin our sequence of proofs of basic facts about WA with the observation that for each n and each set x, $j^n(x)$ is also a set. We express the relation $j^n(x) = y$ as a 3-place relation added by definitional extension: Let $\Phi(n, x, y)$ be the following formula:

$$\begin{aligned}n \in \omega \Longrightarrow \exists! f\, [\text{“}f \text{ is a function”} \wedge \text{dom } f = n+1 \wedge f(0) = x \wedge \\ \forall i\, \big(0 < i \le n \Longrightarrow f(i) = \mathbf{j}(f(i-1))\big) \wedge f(n) = y].\end{aligned}$$

The next proposition shows that Φ actually defines a binary function.

3.2 Proposition. ZFC + WA $\vdash \forall n \in \omega\, \forall x\, \exists! y\, \Phi(n, x, y)$.

Proof. Proceed by induction on n within ZFC + WA. The case $n = 0$ is trivial. For the induction step, let x be arbitrary, and let y be unique such that $\Phi(n, x, y)$ with unique witness f having domain $n+1$. Define $\hat{f}$ on $n+2$ so that it extends f and $\hat{f}(n+1) = \mathbf{j}(f(n))$. Now $\hat{f}$ is the unique function witnessing $\Phi(n+1, x, \mathbf{j}(y))$. Moreover, $\mathbf{j}(y)$ witnesses $\exists y\, \Phi(n+1, x, y)$, and $\mathbf{j}(y)$ is the only such witness. This establishes the induction step and completes the proof. ∎

We can now show that the WA-embedding has a critical point:

3.3 Proposition. *The following are provable in* ZFC + WA*:*

(1) $\forall \alpha\, [\alpha \in ON \Longrightarrow \mathbf{j}(\alpha) \ge \alpha]$*;*
(2) $\exists \alpha\, [\alpha \in ON \wedge \mathbf{j}(\alpha) > \alpha]$.

Proof. For (1), this follows from elementarity, as usual. For (2), assume that no ordinal is moved; use the Separation Schema for **j**-formulas to obtain x of least rank such that $\mathbf{j}(x) \neq x$. Note that $rank(x) = rank(\mathbf{j}(x))$. Then

$$y \in x \iff \mathbf{j}(y) \in \mathbf{j}(x) \iff y \in \mathbf{j}(x),$$

giving the contradiction.■

We will denote the critical point of **j** by cp(**j**), and, in this section only, we will reserve the letter 'κ' for this critical point. The term κ is to be thought of as a constant, added to ZFC + WA by a one-step extension by definitions.

Next, we show that for each set x and each n, $\mathbf{j}^n \restriction x$ is also a set; this coincides with one of our intuitive requirements in formulating WA (see §1):

3.4 Proposition. ZFC + WA $\vdash \forall n \forall X \exists F$ [*"F is a function"* $\wedge$ $dom\, F = X \wedge \forall x \in X\, (\mathbf{j}^n(x) = F(x))$].

Proof. Given n, X, let $Y = \mathbf{j}^n(X)$. Then use the Separation Schema for **j**-formulas to obtain $F = \{(x, y) \in X \times Y : y = \mathbf{j}^n(x)\}$. ■

Informally, the F of the preceding proposition is the restriction $\mathbf{j}^n \restriction X$. The proposition can be used to perform another one-step extension by definitions obtained by adding the three-place relation $F = \mathbf{j}^n \restriction X$.

The next proposition shows that restrictions of the WA-embedding are themselves elementary embeddings.

3.5 Proposition. ZFC + WA $\vdash \forall M$ "$\mathbf{j} \restriction M : M \to \mathbf{j}(M)$ *is an elementary embedding*".

Proof. Informally, applying j yields, for any $\in$-formula $\Phi(x, y)$ and any a, b, $M \models \Phi[a, b] \iff j(M) \models \Phi[j(a), j(b)]$. As usual, this can be formalized to yield a proof from ZFC + WA of the sentence

$$\forall \phi \forall x \forall y \Big[[\phi \in (\in\text{-}fmla) \wedge \{x, y\} \subset vble] \Longrightarrow \\ \big[Sat\big(M, \phi(\chi(x), \chi(y))\big) \iff Sat\big(\mathbf{j}(M), \phi(\chi(\mathbf{j}x), \chi(\mathbf{j}y))\big)\big]\Big].$$■

We can now show that the sequence $\langle \kappa, \mathbf{j}(\kappa), \dots, \mathbf{j}^n(\kappa), \dots \rangle$ is cofinal in ON (note that this sequence is a legitimate class for ZFC + WA, defined as $\{(x, y) : x \in \omega \wedge y = \mathbf{j}^x(\kappa)\}$).

3.6 Proposition. ZFC + WA $\vdash \forall \alpha \exists n \in \omega\, [\mathbf{j}^n(\kappa) \geq \alpha]$.

Proof. If this fails (in some model), then $\lambda = \sup\{j^n(\kappa) : n \in \omega\}$ exists, and since j^n respects rank, it follows from elementarity that $j \restriction V_{\lambda+2} : V_{\lambda+2} \to V_{\lambda+2}$. By Proposition 3.5, $j \restriction V_{\lambda+2}$ is a nontrivial elementary embedding, contradicting Kunen's Theorem (see Theorem 2.14).■

Proposition 3.6 implies that several predicates of interest are not weakly definable in any model of ZFC; we first review the definition of weak definability in a model of set theory:

3.7 Defintion. Suppose M is a model of ZF and $X \subset M$. Then X is *weakly definable in M* if the expanded model $\langle M, \in, X\rangle$ satisfies all instances of Replacement for formulas of the expanded language.

We show that the notion of weak definability has been appropriately named:

3.8 Theorem. *Suppose M is a model of set theory and $X \subset M$. Suppose $W \subset M$ is weakly definable in M and X is definable in $\langle M, \in, W\rangle$. Then X is weakly definable in M. In particular, if $X \subset M$ is definable in M then X is weakly definable in M.*

Outline of Proof. We restrict our attention to standard models $\langle M, \in\rangle$. Let $W \subset M$ be as in the hypothesis, and suppose $X \subset M$ is definable in $\langle M, \in, W\rangle$ by $\Phi(x, y)$ with parameter $a \in M$. Let $\psi(x, y)$ be an X-formula and suppose $A \in M$ is such that

$$\langle M, \in, X\rangle \models \forall x \in A\, \exists! y\, \psi(x, y).$$

Obtain a W-formula $\bar{\psi}$ equivalent in $\langle M, \in, X, W\rangle$ to ψ by replacing each occurrence of '$z \in X$' with '$\Phi(z, a)$'. Then

$$\langle M, \in, W\rangle \models \forall x \in A\, \exists! y\, \bar{\psi}(x, y).$$

Since all instances of Replacement involving W-formulas hold in $\langle M, \in, W\rangle$, we can use Replacement to obtain $Y \in M$ such that

$$\langle M, \in, W\rangle \models \forall x \in A\, \exists y \in Y\, \bar{\psi}(x, y).$$

Replacing $\bar{\psi}$ with ψ yields:

$$\langle M, \in, X\rangle \models \forall x \in A\, \exists y \in Y\, \psi(x, y),$$

as required. ■

3.9 Proposition. *Suppose $\langle M, \in, j\rangle$ is a model of* ZFC + WA. *Then the sequence $\langle \kappa, j(\kappa), j^2(\kappa), \dots\rangle$ is not weakly definable in $\langle M, \in\rangle$.*

Proof. Suppose F is weakly definable in M, where $F = \langle \kappa, j(\kappa), j^2(\kappa), \dots\rangle$. Then, by weak definability we can use Replacement to obtain:

$$\langle M, \in, F\rangle \models \exists z\, [z = F''\omega \,\wedge\, \exists \lambda\, (\lambda = \sup z)].$$

Letting z, λ be witnesses, we can prove

$$\langle M, \in\rangle \models \text{“}j \restriction V_{\lambda+2} : V_{\lambda+2} \to V_{\lambda+2} \text{ is a nontrivial embedding”},$$

and we have a contradiction. ∎

From Proposition 3.9, we can show other familiar collections are not weakly definable:

3.10 Metatheorem. *If* $\langle M, \in, j\rangle$ *is a model of* ZFC + WA, *none of the following subcollections of* M *is weakly definable in* $\langle M, \in\rangle$*:*

(1) j;
(2) $j \upharpoonright ON : ON \to ON$;
(3) $j''ON$;
(4) $j''M$.

Proof. All four parts are proved using Theorem 3.8. For (1), notice that if j is weakly definable in $\langle M, \in\rangle$, then, since the sequence $\langle \kappa, j(\kappa), j^2(\kappa), \dots\rangle$ is definable from j (i.e. in $\langle M, \in, j\rangle$), Theorem 3.8 shows that $\langle \kappa, j(\kappa), j^2(\kappa), \dots\rangle$ is weakly definable in $\langle M, \in\rangle$, contradicting Proposition 3.9. Part (2) follows using exactly the same reasoning. For (3), note that $j \upharpoonright ON$ is definable from $j''ON$: it is in fact the increasing enumeration of $j''ON$. Finally, for (4), note that $j''ON$ is definable from $j''V$. ∎

We now show that

$$V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec \dots \prec V_{\mathbf{j}^n(\kappa)} \prec \dots \prec V$$

forms an elementary chain. We do this in two steps; this allows us to prove that the claim can be formalized and proven within ZFC + WA.

3.11 Proposition. ZFC + WA $\vdash \forall n \in \omega\, [V_\kappa \prec V_{\mathbf{j}(\kappa)} \prec \dots \prec V_{\mathbf{j}^n(\kappa)}]$.

Proof. Proceed by induction on n within ZFC + WA. Start with the fact that $\mathbf{j} \upharpoonright V_\kappa$ is an elementary inclusion map, and hence $V_\kappa \prec V_{\mathbf{j}(\kappa)}$. For the induction step, apply $\mathbf{j}$. ∎

We should perhaps mention that in the argument above, we have not made the mistake of thinking that elementary substructures preserve $\mathbf{j}$-formulas. For instance, we were able to conclude that $\mathbf{j} \upharpoonright V_\kappa : V_\kappa \prec V_{\mathbf{j}(\kappa)}$ by using Proposition 3.5 and the fact that

$$\text{ZFC} \vdash \forall i\, \forall M\, [\text{“}i \text{ is elementary”} \wedge M \in \text{dom } i \wedge i \upharpoonright M = \text{id}_M \Longrightarrow i \upharpoonright M : M \prec i(M)].$$

We will use arguments of this kind throughout the sequel without special mention. The next proposition shows that for all n, $V_{\mathbf{j}^n(\kappa)} \prec V$. Technically, the result is a schema and is proven by induction in the metatheory.

3.12 Proposition. *(Metatheorem) For each* $\in$*-formula* $\Phi(x_1, x_2, \dots, x_m)$ *and corresponding* $\mathcal{L}$*-formula* $\phi(x_1, x_2, \dots, x_m)$

$$\text{ZFC} + \text{WA} \vdash \forall a_1 \forall a_2 \dots \forall a_m \Big[\Phi(a_1, a_2, \dots, a_m) \Longleftrightarrow \exists k \in \omega\, \forall n \geq k\, Sat\big(V_{\mathbf{j}^n(\kappa)}, \phi(\chi(a_1), \chi(a_2), \dots, \chi(a_m))\big)\Big].$$

Proof. Proceed by induction on the complexity of Φ in the metatheory. As in the usual proof (cf. [7]), the only interesting case is when $\Phi(x_1, x_2, \ldots, x_m) \equiv \exists y\, \Psi(y, x_1, x_2, \ldots, x_m)$. If $k \in \omega$ is such that for all $n \geq k$, $Sat\big(V_{\mathbf{j}^n(\kappa)}, \phi(\chi(a_1), \chi(a_2), \ldots, \chi(a_m))\big)$, let b be such that for all $n \geq k$, $Sat(V_{\mathbf{j}^n(\kappa)}, \psi(\chi(b),\ \chi(a_1), \chi(a_2), \ldots, \chi(a_m)))$, where ψ is the $\mathcal{L}$-formula corresponding to Ψ (by Proposition 3.11, such a b can be found). By the induction hypothesis, it follows that $\Psi(b, a_1, a_2, \ldots, a_m)$, whence $\Phi(a_1, a_2, \ldots, a_m)$.

Conversely, assume $\Phi(a_1, a_2, \ldots, a_m)$ and let b be such that $\Psi(b, a_1, a_2, \ldots, a_m)$. Using the induction hypothesis, one can find $k \in \omega$ such that for each $n \geq k$, $b \in V_{\mathbf{j}^n(\kappa)}$ and $Sat\big(V_{\mathbf{j}^n(\kappa)}, \psi(\chi(b), \chi(a_1), \chi(a_2), \ldots, \chi(a_m))\big)$, and the result follows. ∎

Proposition 3.12 gives us a formula **Tr** which is provably (in ZFC + WA) a truth definition for ZFC; namely, for each $\phi \in V_\omega$, define **Tr** by requiring that, if $\phi \in \mathcal{L}$ and $\phi \in (\in\text{-}sent)$, then

$$\mathbf{Tr}(\phi) \Longleftrightarrow Sat(V_\kappa, \phi).$$

Note that since κ is definable from $\mathbf{j}$, **Tr** is definable without parameters. On the other hand, note that κ is not definable by $\in$-formulas ϕ with parameters in V_κ: If there were $a \in V_\kappa$ and a formula $\phi(x, a)$ such that for all β,

$$\beta \text{ is the critical point of } \mathbf{j} \Longleftrightarrow (V, \in) \models \phi(\beta, a),$$

then applying $\mathbf{j}$ to $(V, \in) \models \phi(\kappa, a)$ yields $(V, \in) \models \phi(\mathbf{j}(\kappa), a)$, which is impossible. (Our observation here of course fails if we relax the requirement that $a \in V_\kappa$; $a = \kappa$ is a counterexample.)

We now turn to some results on WA that establish bounds on its consistency strength.

3.13 Theorem.

(1) $\mathrm{ZFC} + I_3 \vdash Con(\mathrm{ZFC} + \mathrm{WA})$;
(2) $\mathrm{ZFC} + I_3 \not\vdash Con(\mathrm{ZFC} + \mathrm{WA}) \Longrightarrow Con(\mathrm{ZFC} + I_3)$, *unless* $\mathrm{ZFC} + I_3$ *is inconsistent.*
(3) $\mathrm{ZFC} + \mathrm{WA} \vdash Con(\mathrm{ZFC} + \mathrm{WA}) \Longrightarrow Con(\mathrm{ZFC} + \mathrm{WA} + \neg I_3)$.

Proof. To prove (1), assume $I_3(\kappa)$, and let $i : V_\lambda \to V_\lambda$ be elementary with critical point κ. It is easy to see that $\langle V_\lambda, \in, i\rangle \models \mathrm{ZFC} + \mathrm{WA}$.

For (2), putting (1) together with $\mathrm{ZFC} + I_3 \vdash Con(\mathrm{ZFC} + \mathrm{WA}) \Longrightarrow Con(\mathrm{ZFC} + I_3)$ would give us $\mathrm{ZFC} + I_3 \vdash Con(\mathrm{ZFC} + I_3)$, which, by Gödel's Incompleteness Theorem, implies $\mathrm{ZFC} + I_3$ is inconsistent.

For part (3), we use a trick mentioned in [20, 6.9]. Let us first observe that, assuming $I_3(\kappa)$, there is a λ that is definable from κ for which there is an elementary embedding $i : V_\lambda \to V_\lambda$; we let λ_κ denote the least such. Now, working in ZFC + WA, define the class $M = \{x : \forall\kappa(I_3(\kappa) \Longrightarrow x \in V_{\lambda_\kappa})\}$. Now if $\forall\kappa\, \neg I_3(\kappa)$, then $M = V$; but if $\exists\kappa\, I_3(\kappa)$, then $M = V_{\lambda_\kappa}$ where κ is least for which $I_3(\kappa)$. In either case, $M \models \mathrm{ZFC} + \mathrm{WA} + \neg I_3$. ∎

Although consistency-wise, WA is weaker than all the axioms I_3-I_0, if κ is the critical point of the WA-embedding, then κ is always larger than the least critical point of an I_m-embedding (for $0 \leq m \leq 3$), whenever both kinds of embeddings exist:

3.14 Theorem. *Assume* WA *and let κ be the critical point of the* WA*-embedding* $\mathbf{j}$. *Suppose further that $A(x)$ is a large cardinal property (having just one free variable) and $\exists\lambda\, A(\lambda)$. Then*

(1) there is $\lambda < \kappa$ such that $A(\lambda)$; and
(2) $A(\kappa) \Longrightarrow |\{\alpha < \kappa : A(\alpha)\}| = \kappa$.

Proof. For (1), since $V_\kappa \prec V$, it follows that there is $\lambda < \kappa$ such that $A(\lambda)$. For (2), use the embedding $\mathbf{j}$ to define a normal ultrafilter D over κ; then if $A(\kappa)$ holds, then $\{\alpha < \kappa : A(\alpha)\} \in D$.■

To prove that WA implies super-n-huge for every n, we invoke a lemma that shows how to apply the WA-embedding $\mathbf{j}$ to itself in various ways to obtain other elementary embeddings having arbitrarily large $\mathbf{j}^k(\kappa)$ as critical points with image $\mathbf{j}^{k+n}(\kappa)$ for arbitrarily large n; the lemma will prove useful in other contexts later.

3.15 Lemma. *Assume* WA *and let* $\mathbf{j}$ *be the* WA*-embedding. For each $k \geq 0, m > k$, and $n \geq 1$, there is an elementary embedding $i = i_{k,m,n} : V_\beta \to V_{\mathbf{j}^n(\beta)}$ such that*
(1) *$cp(i) = \mathbf{j}^k(\kappa)$;*
(2) *$\beta = \mathbf{j}^m(\kappa)$;*
(3) *$i(\mathbf{j}^\ell(\kappa)) = \mathbf{j}^{\ell+n}(\kappa)$, for all $\ell \geq k$ for which $\mathbf{j}^\ell(\kappa) \in V_\beta$.*

Proof. Define $i_{k,m,n}$ inductively with respect to k as follows:

$$i_{0,m,n} = \mathbf{j}^n \upharpoonright V_{\mathbf{j}^m(\kappa)}, \qquad \text{for } m > 0 \text{ and } n \geq 1;$$
$$i_{k+1,m,n} = \mathbf{j}\big(i_{k,m-1,n} \upharpoonright V_{\mathbf{j}^{m-1}(\kappa)}\big), \qquad \text{for } m > k+1 \text{ and } n \geq 1.$$

That $i_{k,m,n}$ satisfies $(1)-(3)$ is proved by a simple induction.■

Three examples of embeddings as defined in Lemma 3.15 that we will use later are:

$$i_{0,1,1} = \mathbf{j} \upharpoonright V_{\mathbf{j}(\kappa)};$$
$$i_{1,2,1} = \mathbf{j}\big(\mathbf{j} \upharpoonright V_{\mathbf{j}(\kappa)}\big);$$
$$i_{2,3,1} = i_{1,2,1}(i_{1,2,1}).$$

Note that $i_{1,2,1}$ is the analogue of $\mathbf{j} \cdot \mathbf{j}$ for definable elementary embeddings (as in Definition 2.16), and that $i_{2,3,1}$ is the analogue of $(\mathbf{j} \cdot \mathbf{j}) \cdot (\mathbf{j} \cdot \mathbf{j})$. It follows readily that if we let U denote the normal ultrafilter over κ derived from the WA-embedding $\mathbf{j}$, then, as in Definition 2.16,

$$U^{\mathbf{j}} = \{X \subset \mathbf{j}(\kappa) : \mathbf{j}(\kappa) \in i_{1,2,1}(X)\};$$
$$U^{\mathbf{jj}} = \{X \subset \mathbf{j}^2(\kappa) : \mathbf{j}^2(\kappa) \in i_{2,3,1}(X)\}.$$

As in Definition 2.16, $U^{\mathbf{j}}$ is a normal ultrafilter over $\mathbf{j}(\kappa)$ and $U^{\mathbf{jj}}$ is a normal ultrafilter over $\mathbf{j}^2(\kappa)$.

3.16 Theorem. *Assume* WA *and let* κ *be the critical point of the* WA*-embedding* $\mathbf{j}$. *Then* κ *is the* κ*th cardinal which is super-*n*-huge for every* n.

Proof. Let $\kappa_0 = \kappa$ and for each $n \geq 1$, let $\kappa_n = \mathbf{j}^n(\kappa)$. We first verify that κ is n-huge for every n. But this is easy since, for each n, the normal ultrafilter over $P(\mathbf{j}^n(\kappa))$ derived from $\mathbf{j}$ witnesses that κ is n-huge.

Now, to prove super-n-hugeness for every n, it suffices to show that for all $m, n \in \omega$, κ is n-huge with κ_m targets. We will first show that there is a stationary subset S_1 of $\mathbf{j}(\kappa)$ each of whose elements is the target of an n-huge embedding with critical point κ; then we apply a suitable elementary embedding repeatedly to S_1 to show that similar stationary sets exist below each κ_m.

Let $U^{\mathbf{j}}$ be as in the remarks following Lemma 3.15. Let

$$S_1 = \left\{ \alpha < \mathbf{j}(\kappa) : \alpha \text{ is a target of some } n\text{-huge embedding having critical point } \kappa \right\}.$$

Then $S_1 \in U^{\mathbf{j}}$ since $\mathbf{j}(\kappa)$ is a target of an n-huge embedding having critical point κ, as we just showed. Hence, S_1 is stationary.

For each $m > 0$, let $i_m = i_{1,m,1}$ as in Lemma 3.15. Now for each $m > 0$, inductively define

$$S_{m+1} = i_{m+1}(S_m).$$

By elementarity, S_m is a stationary subset of κ_m each of whose elements is a target of an n-huge embedding with critical point κ.

Finally, to see that κ is the κth cardinal that is super-n-huge for every n, apply Proposition 3.14(2).■

The fact that $\mathbf{j}$ is not definable implies $\mathbf{j}(\kappa)$ must be quite large:

3.17 Proposition. *Suppose* $\langle M, \in, j \rangle$ *is a model of* ZFC + WA *and suppose that* $\mathbf{G}$: $ON^M \rightarrow ON^M$.

(1) *If* $\mathbf{G}$ *is definable in* M *with parameters in* V_κ^M *(and defined by an* $\in$*-formula), then* $j(\kappa) > \mathbf{G}^M(\kappa)$.
(2) *If* $\mathbf{G}$ *is weakly definable in* M, *then there is* $n \in \omega$ *such that for all* $m \in \omega$, $j^n(\kappa) > \left(\mathbf{G}^m(\kappa)\right)^M$.

Proof. For (1), arguing in M, since $V_{j(\kappa)} \prec V$, $\mathbf{G}(\kappa) = \mathbf{G}^{V_{j(\kappa)}}(\kappa) < j(\kappa)$. For (2), because all instances of Replacement involving occurrences of $\mathbf{G}$ (but not $\mathbf{j}$) hold in $\langle M, \in, j, \mathbf{G} \rangle$, the sequence $\langle \mathbf{G}^n(\kappa) : n \in \omega \rangle$ has a supremum λ in that model; let n be such that $j^n(\kappa) > \lambda$. ■

In Part (1) of Proposition 3.17, "definable" cannot be replaced by "weakly definable" since, for each model $\langle M, \in, j \rangle$ of ZFC + WA and each $n \in \omega$, there is a $\mathbf{G}_n : ON^M \rightarrow ON^M$, weakly definable in M, such that $j^n(\kappa) \leq \mathbf{G}_n(\kappa)$ (let $\mathbf{G}_n$ be the constant function with value $j^n(\kappa)$).

We conclude this section with a result that is related to the undefinability of $\mathbf{j}$: the WA-class $\{\kappa, \mathbf{j}(\kappa), \mathbf{j}^2(\kappa), \ldots\}$ is a class of indiscernibles for V. We again adopt the notation $\kappa_0 = \kappa$ and for each $n \geq 1$, $\kappa_n = \mathbf{j}^n(\kappa)$.

3.18 Theorem. *Assume* WA. *Suppose $m_1 < m_2 < \ldots < m_s$ and $n_1 < n_2 < \ldots < n_s$, and $\phi(x_1, \ldots, x_s)$ is an $\in$-formula with all free variables displayed. Then*

$$\phi(\kappa_{m_1}, \ldots, \kappa_{m_s}) \Longleftrightarrow \phi(\kappa_{n_1}, \ldots, \kappa_{n_s}).$$

To prove the theorem, we need the following lemma:

3.19 Lemma. *Assume* WA. *Given $n_1 < n_2 < \ldots < n_s$ and $r > \max(\{n_{m+1} - n_m : 1 \leq m < s\})$, there are $u < w$ and $i : V_{\kappa_u} \to V_{\kappa_w}$ such that i is an elementary embedding with critical point $> \kappa_{n_1}$ and for $1 < m \leq s$,*

$$i(\kappa_{n_m}) = \kappa_{n_1+(m-1)r}.$$

Proof. Let $\{n_m : 1 \leq m \leq s\}, r$ be as in the hypothesis. For each m with $2 \leq m \leq s$, let $t_m = n_{m-1} - n_m + r$. Note that for each such m, $t_m > 0$. We define cardinals $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_s$ and, for $1 \leq m \leq s$, elementary embeddings $i_m : V_{\lambda_m} \to V_{\lambda_{m+1}}$ so that the embedding i required by the lemma is $i_s \circ i_{s-1} \circ \ldots i_2 \circ i_1$, and $\kappa_u = \lambda_1, \kappa_w = \lambda_{s+1}$.

Let $\lambda_1 = \kappa_{n_1+sr} = \lambda_2$; λ_1 is big enough so that for all m, $\kappa_{n_m} \in V_{\lambda_1}$. Let $i_1 = \mathrm{id} : V_{\lambda_1} \to V_{\lambda_2}$. Pick $i_2 : V_{\lambda_2} \to V_{\lambda_3}$ so that $\mathrm{cp}(i_2) = \kappa_{n_2}$ and for all $\ell \geq n_2$, $\kappa_\ell \in V_{\lambda_2}$ implies $i_2(\kappa_\ell) = \kappa_{\ell+t_2}$. In the notation of Lemma 3.15, $i_2 = i_{n_2, n_1+sr, t_2}$.

In general, use Lemma 3.15 to choose $i_m : V_{\lambda_m} \to V_{\lambda_{m+1}}$ so that

$$\mathrm{cp}(i_m) = (i_{m-1} \circ i_{m-2} \circ \ldots \circ i_2 \circ i_1)(\kappa_{n_m});$$
$$i_m(\kappa_\ell) = \kappa_{\ell+t_m}, \text{ for } \kappa_\ell \geq \mathrm{cp}(i_m) \text{ and } \kappa_\ell \in V_{\lambda_m}.$$

Since we are using Lemma 3.15, notice that $\lambda_1, \lambda_2, \ldots, \lambda_{s+1}$ are completely determined by our choice of λ_1.

Claim 1. *For each $m, 2 \leq m \leq s$, for each k, $1 \leq k \leq m$,*

$$(i_k \circ i_{k-1} \circ \ldots \circ i_2 \circ i_1)(\kappa_{n_m}) = \kappa_{n_1+n_m-n_k+(k-1)r}.$$

Proof of Claim 1. This is an easy induction on k.■

Claim 2. *For each $m, 2 \leq m \leq s$,*

$$cp(i_m) = \kappa_{n_1+n_m-n_{m-1}+(m-2)r};$$
$$i_m(cp(i_m)) = \kappa_{n_1+(m-1)r}.$$

Proof of Claim 2. Recall that

$$\mathrm{cp}(i_m) = (i_{m-1} \circ i_{m-2} \circ \ldots \circ i_2 \circ i_1)(\kappa_{n_m});$$
$$i_m(\mathrm{cp}(i_m)) = (i_m \circ i_{m-1} \circ \ldots \circ i_2 \circ i_1)(\kappa_{n_m}).$$

By Claim 1, the result follows.■

To complete the proof of the lemma, let $i = i_s \circ i_{s-1} \circ \ldots i_2 \circ i_1 : V_{\kappa_u} \to V_{\kappa_w}$. Then, by Claim 2, for each m, $2 \le m \le s$,

$$\begin{aligned} i(\kappa_{n_m}) &= (i_s \circ i_{s-1} \circ \ldots i_2 \circ i_1)(\kappa_{n_m}) \\ &= \big(i_s \circ \ldots \circ i_{m+1}\big)\big[(i_m \circ \ldots \circ i_2 \circ i_1)(\kappa_{n_m})\big] \\ &= \big(i_s \circ \ldots \circ i_{m+1}\big)\big[i_m(\mathrm{cp}(i_m))\big] \\ &= \big(i_s \circ \ldots \circ i_{m+1}\big)\big(\kappa_{n_1+(m-1)r}\big) \\ &= \kappa_{n_1+(m-1)r}, \end{aligned}$$

as required.■

Proof of Theorem 3.18. Without loss of generality, assume $m_1 \le n_1$. If $m_1 < n_1$, then by applying $\mathbf{j}^{n_1-m_1}$ we have that

$$\phi(\kappa_{m_1}, \kappa_{m_2}, \ldots, \kappa_{m_s}) \Longleftrightarrow \phi(\kappa_{n_1}, \kappa_{m_2+n_1-m_1}, \ldots, \kappa_{m_s+n_1-m_1}),$$

and so we may assume that $n_1 = m_1$.

Let $r > \max(\{z : \exists k\,[1 \le k < s \wedge (z = n_{k+1} - n_k \vee z = m_{k+1} - m_k)]\})$. Then by the lemma, there are elementary embeddings $i : V_{\kappa_u} \to V_{\kappa_w}$ $\hat{i} : V_{\kappa_{\hat{u}}} \to V_{\kappa_{\hat{w}}}$ so that for each k, $1 \le k \le s$,

$$\begin{aligned} i(\kappa_{n_k}) &= \kappa_{n_1+(k-1)r} \\ \hat{i}(\kappa_{m_k}) &= \kappa_{n_1+(k-1)r}. \end{aligned}$$

Thus,

$$\begin{aligned} \phi(\kappa_{n_1}, \kappa_{n_2}, \ldots, \kappa_{n_s}) &\Longleftrightarrow \phi^{V_{\kappa_u}}(\kappa_{n_1}, \kappa_{n_2}, \ldots, \kappa_{n_s}) \\ &\Longleftrightarrow \phi^{V_{\kappa_w}}(\kappa_{n_1}, \kappa_{n_1+r}, \kappa_{n_1+2r}, \ldots, \kappa_{n_1+(s-1)r}) \\ &\Longleftrightarrow \phi(\kappa_{n_1}, \kappa_{n_1+r}, \kappa_{n_1+2r}, \ldots, \kappa_{n_1+(s-1)r}) \\ &\Longleftrightarrow \phi^{V_{\kappa_{\hat{w}}}}(\kappa_{n_1}, \kappa_{n_1+r}, \kappa_{n_1+2r}, \ldots, \kappa_{n_1+(s-1)r}) \\ &\Longleftrightarrow \phi^{V_{\kappa_{\hat{u}}}}(\kappa_{m_1}, \kappa_{m_2}, \ldots, \kappa_{m_s}) \\ &\Longleftrightarrow \phi(\kappa_{m_1}, \kappa_{m_2}, \ldots, \kappa_{m_s}), \end{aligned}$$

as required.■

We remark that a result like this does not hold for the familiar definable elementary embeddings $j : V \to M$ that we study here: First, suppose such a j is definable (in $\langle V, \in\rangle$) by a formula $\psi(x, y)$ (without extra parameters). If $\phi(z)$ is the one-parameter formula that asserts "z is the least ordinal such that $\exists y\, \psi(z, y) \wedge z \ne y$," then clearly $\phi(\kappa)$ is not equivalent to $\phi(j^n(\kappa))$ for any $n > 0$. Moreover, familiar embeddings $j : V \to M$ (such as measurable, λ-strong, λ-supercompact, almost huge, and huge), though not generally definable without parameters, can be so defined in special cases, assuming $V = \mathrm{HOD}$. (And it is known that $V = \mathrm{HOD}$ is consistent with these types of large cardinals, modulo extra hypotheses; see [8].)

§4. **Laver Sequences.**

As we mentioned in the Introduction, Laver sequences have been used in many contexts to achieve strong diamond-like reflection in a variety of arguments; in this section, we begin a study of Laver sequences themselves as interesting mathematical objects in their own right. As we shall see, the Wholeness Axiom, defined in Section 3, provides a natural context for a direct construction of Laver sequences, and for generalizing the concept to other kinds of large cardinals. We begin with a definition of Laver sequences and some natural questions about them.

4.1 Definition. Suppose κ is an infinite cardinal. A *Laver sequence at* κ is a function $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \geq \max(\kappa, |\mathrm{TC}(x)|)$ there is a normal ultrafilter U over $P_\kappa\lambda$ such that $x = i_U(f)(\kappa)$.

Question #1. *Is there a direct construction of a Laver sequence?* The only published construction of a Laver sequence—apart from minor modifications that have been devised—is Laver's original indirect proof that a Laver sequence must always exist at a supercompact cardinal [22]. The construction is not easy to modify for constructing Laver sequences with various properties. A direct construction would be useful.

Question #2. *Is there a Laver sequence* $f : \kappa \to V_\kappa$ *that is definable (undefinable) in* V_κ*?* The question of definability of Laver sequences is of interest because, as we show in Corollary 4.6, assuming WA, $V = HOD$ iff there is a Laver sequence definable in V_κ. Results like Theorems 3.14 and 3.18 and Proposition 3.17 suggest that if the construction of a Laver sequence depends "enough" on the WA-embedding, it will have to be undefinable.

Question #3. *Is there a Laver sequence* $f : \kappa \to V_\kappa$ *such that the function* $\alpha \mapsto |f(\alpha)|$ *dominates on a large set every function* $\kappa \to \kappa$ *that is definable in* V_κ*?* Note that a positive answer would give an example of an undefinable Laver sequence. The question is also motivated by the following observation: Let us call a class function $\mathbf{G} : ON \to ON$ *relatively bounded at* κ if the parameters of $\mathbf{G}$ are in V_κ and there is a $\lambda > \kappa$ such that for all transitive set models M of ZFC containing the parameters of $\mathbf{G}$, $\mathbf{G}^M(\kappa) < \lambda$. (As an example, using absoluteness of Δ_1^{ZF} formulas and an easy counting argument, one shows that every class function $ON \to ON$ defined by a Δ_1^{ZF} formula and having parameters in V_κ is relatively bounded at κ.) It can be shown that if $V_\kappa \prec V$ and $f : \kappa \to V_\kappa$ is Laver, $\alpha \mapsto |f(\alpha)|$ dominates, on a normal measure 1 set, every function $\mathbf{G} \restriction \kappa$ for which $\mathbf{G} : ON \to ON$ is relatively bounded at κ. Thus, Question #3 asks for a particular construction of an f that allows us to remove "relatively bounded" from the hypothesis of this proposition. (To prove the proposition, assume $V_\kappa \prec V$, let $f : \kappa \to V_\kappa$ be a Laver sequence, and let $\mathbf{G} : ON \to ON$ be relatively bounded at κ, with parameter $z \in V_\kappa$ and bound a cardinal λ. Let $\Phi(x, y, z)$ define $\mathbf{G}$. Let $g = \mathbf{G} \restriction \kappa$; note that $g : \kappa \to \kappa$. Let $i : V \to M$ be a λ^+-supercompact embedding with $i(f)(\kappa) = \lambda$. It's routine to verify that $i(g)(\kappa) = \mathbf{G}^N(\kappa) < \lambda = i(f)(\kappa) = |i(f)(\kappa)|$, where $N = V^M_{i(\kappa)}$. Let D be the normal ultrafilter on κ derived from i. Clearly, $\{\alpha < \kappa : g(\alpha) < |f(\alpha)|\} \in D$.)

Question #4. *Can Laver sequences be defined for large cardinals other than supercompact and strong?* In Section 2 we gave a proof of the existence of Laver sequences for strong cardinals; this was observed by Gitik and Shelah in [14] in obtaining indestructibility results for strong cardinals. In his analogous results on huge cardinals (giving conditions under which huge cardinals are resurrectable), Barbanel showed in [2] that, assuming κ is 2-huge with embedding j, there is an $f : \kappa \to V_\kappa$ such that for all $x \in V_{j(\kappa)}$ with $|x| \geq \kappa$, there is a huge embedding i such that $i(f)(\kappa) = x$. In a similar but more general vein, it would be interesting to see to what extent all large cardinals have Laver sequences, and to what extent the consequences of the existence of (supercompact) Laver sequences continue to hold for other large cardinals.

Question #5. *Are there strong hypotheses and interesting inner models N for which the statement "f is Laver at κ" is absolute?* The question arises in the following situation: Suppose $j : V \to N$ is an elementary embedding and D is the normal ultrafilter over κ derived from j. Suppose $g : \kappa \to V_\kappa$ is a function and $\{\alpha < \kappa : g \restriction \alpha \text{ is Laver at } \alpha\} \in D$. If j happened to be the WA-embedding, we could conclude that g is a Laver sequence at κ; for what N can we draw the same conclusion (that g is a real Laver sequence at κ)?

We begin with Question #1; the next proposition provides a hint about how to obtain a direct construction of a Laver sequence using WA:

4.2 Proposition. *Assume* WA. *Let j be the* WA*-embedding with critical point κ, and suppose $f : \kappa \to V_\kappa$ is a Laver sequence at κ. Then the set*

$$\left\{ \alpha < \kappa : f \restriction \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha \right\}$$

is stationary.

Proof. Let $D = \{A \subseteq \kappa : \kappa \in j(A)\}$. Since $j(f) \restriction \kappa = f$, we have

$$\left\{ \alpha < \kappa : f \restriction \alpha : \alpha \to V_\alpha \text{ is a Laver sequence at } \alpha \right\} \in D,$$

as required.∎

Thus we might expect to be able to directly build a Laver sequence f at the WA-critical point κ by arranging to have $f \restriction \alpha$ be Laver at α on a measure 1 set, and at other α, let $f(\alpha)$ witness the failure of this fact. We will use WA to carry out this idea, but first prove that the construction produces a Laver sequence under the much weaker assumption that κ is supercompact (and this provides an answer to Question #1); we will also discuss the reasons for considering a proof of the same result that requires a stronger hypothesis.

4.3 Canonical Construction CC(t) Given an arbitrary sequence $t : \kappa \to V_\kappa$, define $f : \kappa \to V_\kappa$ by recursion as follows:

$$f(\alpha) = \begin{cases} t_\alpha & \text{if } f \restriction \alpha \text{ is a Laver sequence at } \alpha \\ x \in V_\kappa & \text{if } f \restriction \alpha \text{ is not Laver and } \alpha \text{ is a cardinal,} \\ & \text{where } \exists \lambda < \kappa\, \phi(f \restriction \alpha, x, \lambda) \\ \emptyset, & \text{if } \alpha \text{ is not a cardinal,} \end{cases}$$

where $\phi(g, x, \lambda)$ denotes the following formula:

> "there exists a cardinal α such that $\max(\alpha, |\mathrm{TC}(x)|) \leq \lambda$ and $g : \alpha \to V_\alpha$, and for all normal ultrafilters U over $P_\alpha\lambda$, $i_U(g)(\alpha) \neq x$."

We prove that f is Laver, assuming that κ is supercompact. First notice that by Proposition 2.29, f is well-defined: whenever $f \upharpoonright \alpha$ is not Laver, there is an $x \in V_\kappa$ that witnesses this fact. Now suppose f is not a Laver sequence. Let x be a set such that for some λ, $\phi(f, x, \lambda)$ holds. Let μ be a strong limit $> 2^{\lambda^{<\kappa}}$, let U be a normal ultrafilter over $P_\kappa\mu$, and let $i_U : V \to M$ be the canonical embedding. M contains all normal ultrafilters over $P_\kappa\lambda$; thus $M \models \phi(f, x, \lambda)$. Let $D = \{X \subseteq \kappa : \kappa \in i_U(X)\}$. Notice that

$$\{\alpha < \kappa : \alpha \text{ is a cardinal and } f \upharpoonright \alpha \text{ is Laver at } \alpha\} \notin D; \tag{4.1}$$

otherwise, then $M \models$ "f is Laver at κ", from which it would follow that $M \models \neg\phi(f, x, \lambda)$. Thus,

$$\{\alpha < \kappa : \alpha \text{ is a cardinal and } \exists\lambda < \kappa\, \phi(f \upharpoonright \alpha, f(\alpha), \lambda)\} \in D,$$

and so $M \models \exists\lambda < i_U(\kappa)\, \phi(f, i_U(f)(\kappa), \lambda)$. Now it is easy to verify (using Lemma 2.26(I)) that $M \models i_{U|\lambda}(f)(\kappa) = i_U(f)(\kappa)$, in violation of ϕ.

The construction $\mathbf{CC}(t)$ can be modified slightly for strong cardinals, and a proof similar to the above (see Theorem 2.30(2)) can be given to show that assuming κ is strong, the modified construction yields a strong Laver sequence.

The proof just given is essentially the same as Laver's original proof. It has the interesting feature that the set $S = \{\alpha < \kappa : \alpha \text{ is a cardinal and } f \upharpoonright \alpha \text{ is Laver at } \alpha\}$ may or may not be large, depending on the strength of the large cardinal hypothesis involved: if κ is the least supercompact, S must be small (empty, in fact), but if κ is superhuge, S has normal measure 1 (see Theorems 5.13 and 5.35). This range of possiblities for S allows the proof to work even for the weakest possible hypothesis ("κ is supercompact"). On the other hand, it does not allow us the freedom to predetermine the values of f on a large set using the parameter t, and, as we shall see, it precludes the possibility of constructing certain types of Laver sequences that have strong properties. Also, the proof does not generalize to other globally defined large cardinals.

By contrast, our WA proof of this result forces S to be large, making use of our motivating result, Proposition 4.2. This fact guarantees that we can obtain Laver sequences with a wide variety of properties. It also provides a template for constructing functions like f and showing that they are Laver relative to other types of globally defined large cardinals. We will be able to use the strategy of proving, for each kind of globally defined large cardinal, the existence of a correponding type of Laver sequence under the assumption of WA, and then weakening the WA hypothesis as far as possible in order to obtain optimal results.

The next proposition is our "WA proof" that f is Laver at κ. In Section 5, we will turn to the task of weakening the large cardinal hypothesis that is used.

4.4 Theorem. *Assume* WA *and let* $j : V \to V$ *be the* WA-*embedding with critical point* κ. *Let* D *be the normal ultrafilter over* κ *derived from* j.

(1) *The function f given by the construction $CC(t)$ is a Laver sequence at κ.*
(2) *The sequence t of sets may be defined so that f is Laver at κ and the function $\alpha \mapsto |f(\alpha)|$ dominates, on a set in D, every function $\kappa \to \kappa$ that is definable in V_κ (with parameters).*
(3) *There exist λ, U such that $\lambda > \kappa$ and U is a normal ultrafilter over $P_\kappa\lambda$, D is the normal ultrafilter over κ derived from i_U, and $|\mathrm{TC}(i_U(f)(\kappa))| \leq \lambda$.*

Proof of (1) First note that, in the construction of f, whenever α is a cardinal for which $f \upharpoonright \alpha$ is not Laver at α, there is indeed a pair $(x, \lambda) \in V_\kappa$ such that $\phi(f \upharpoonright \alpha, x, \lambda)$, because (by WA) $V_\kappa \prec V$.

Now, suppose $j : V \to V$ is the WA-embedding with critical point κ. Let D be the normal ultrafilter over κ derived from j. We observe first that if the set $\{\alpha : f \upharpoonright \alpha \text{ is Laver at } \alpha\} \in D$, then, as $\kappa \in j(\{\alpha : f \upharpoonright \alpha \text{ is Laver at } \alpha\})$, f is Laver at κ. So, to complete the proof, it suffices to show that this set is indeed in D.

Assume that

$$\{\alpha : f \upharpoonright \alpha \text{ is Laver at } \alpha\} \notin D.$$

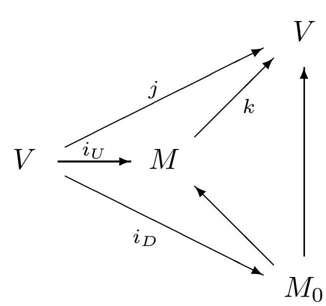
Then the set $\{\alpha < \kappa : \exists \lambda < \kappa\ \phi(f \upharpoonright \alpha, f(\alpha), \lambda)\} \in D$, whence $\phi(f, j(f)\kappa, \lambda)$ for some $\lambda < j(\kappa)$. Let $x = j(f)(\kappa)$. Let U be the normal ultrafilter over $P_\kappa\lambda$ derived from j. Now by Lemma 2.26(I), $i_U(f)(\kappa) = x$, and this is a contradiction.∎

Proof of (2) To ensure that $\alpha \mapsto |f(\alpha)|$ dominates, on a D-measure 1 set, every function $\kappa \to \kappa$ that is definable in V_κ (with parameters), we define the sequence t of sets in $\mathrm{CC}(t)$ as follows: Let $\langle h_\xi : \xi < \kappa \rangle$ enumerate the members of ${}^\kappa\kappa$ which are definable in V_κ. Then, whenever $f \upharpoonright \alpha$ is a Laver sequence at α, we let

$$t_\alpha = \sup\{h_\xi(\alpha) : \xi < \alpha\} + 1. \tag{4.2}$$

Since the set of α for which $f \upharpoonright \alpha$ is Laver at α has D-measure 1, it follows that f dominates each h_ξ on a D-measure 1 set. ∎

Proof of (3)



Let $\lambda = |\mathrm{TC}(j(f)(\kappa))|$. Notice that by (2), $\lambda > \kappa$. Define $U = \{X \subseteq P_\kappa\lambda : j''\lambda \in j(X)\}$ and let $i_U : V \to M$ be the canonical embedding. By Lemma 2.26(I), $i_U(f)(\kappa) = j(f)(\kappa)$. To see that

D is derived from i_U, let $k : M \to V$ be the usual embedding such that $k \circ i_U = j$. Then for all $X \subseteq \kappa$, because $k(\kappa) = \kappa$,

$$\begin{aligned} \kappa \in j(X) &\iff k(\kappa) \in k(i_U(X)) \\ &\iff \kappa \in i_U(X); \end{aligned}$$

the result follows. ∎

Note that we could have used the function $\alpha \mapsto rank(f(\alpha))$ in place of $\alpha \mapsto |f(\alpha)|$ in (2) and the proof would go through essentially unchanged.

We will call a Laver sequence satisfying the properties described in (2) and (3) above a *special Laver sequence.* More precisely:

4.5 Definition. *(Special Laver Sequences)* A Laver sequence $f : \kappa \to V_\kappa$ is *special* if there exist λ, U such that $\lambda > \kappa$, U is a normal ultrafilter over $P_\kappa\lambda$, $|\mathrm{TC}(i_U(f)(\kappa))| \leq \lambda$, and if D is the normal ultrafilter over κ derived from i_U, then for each $g : \kappa \to \kappa$ definable in V_κ, the function $\alpha \mapsto |f(\alpha)|$ dominates g on a set in D.

We observe for future use that an alternative definition of special Laver sequence is possible using rank instead of transitive closure. Let us say that a Laver sequence $g : \kappa \to V_\kappa$ is *special** if there are λ, U such that $\lambda > \kappa$, λ is a beth fixed point, U is a normal ultrafilter over $P_\kappa\lambda$, $rank(i_U(g)(\kappa)) < \lambda$, and if D is the normal ultrafilter over κ derived from i_U, then for each $h : \kappa \to \kappa$ definable in V_κ, the function $\alpha \mapsto rank(g(\alpha))$ dominates h on a set in D.

Our requirement that λ be a beth fixed point makes the definition of special* apparently somewhat stronger than a strict analogue of special. We have introduced this extra condition because, in the generalized context we shall consider in the next section, it lets us avoid certain bookkeeping issues. Yet, in that context, the added condition is innocuous in the sense that, if a λ with the other properties required for special* can be found at all, such a λ can be found that is a beth fixed point.

A proof similar to Theorem 4.4(3) can be given to show that WA implies that there are special* Laver sequences as well as special ones. Theorem 4.9 shows that special Laver sequences (as well as special* Laver sequences) are consistency-wise stronger than ordinary Laver sequences.

Theorem 4.4(2) gives an answer to Question #3; it also provides an example of an undefinable Laver sequence (Question #2). The following corollary shows that the existence of a definable Laver sequence is linked to the existence of a definable well-ordering of the universe.

4.6 Corollary. *Assume* WA. *Then the following are equivalent:*

(1) *There is a Laver sequence at κ that is definable in V_κ (without parameters).*

(2) $V = HOD$.

(In the Corollary, we mean the usual version of HOD obtained by $\in$-formulas.)

Proof. In the proof, the term 'definable' will always mean 'definable without parameters'. To begin, let us recall that HOD can be characterized as the largest transitive model of ZF in which there is a definable well-ordering of the universe (see [17]). To prove (1) $\Rightarrow$ (2), suppose $f : \kappa \to V_\kappa$ is a Laver sequence definable in V_κ. Define $g : V_\kappa \to \kappa$ by

$$g(x) = \text{ least } \alpha \text{ such that } f(\alpha) = x.$$

Let $h : \kappa \to g''V_\kappa$ be the increasing enumeration of $g''V_\kappa$. Define $k : \kappa \to V_\kappa$ by $k = g^{-1} \circ h$. Now k is a well-ordering of V_κ and since f is definable in V_κ, so is k. But now because $V_\kappa \prec V$, the defining formula for k defines a well-ordering of V as well.

For (2) $\Rightarrow$ (1), assume $V = HOD$ and that $\lhd$ is a definable well-ordering of V; since $V_\kappa \prec V$, the induced well-ordering $\lhd \restriction V_\kappa \times V_\kappa$ is definable in V_κ with the same formula. Use the construction CC(t) to define a Laver sequence $f : \kappa \to V_\kappa$, with $t_\alpha = \emptyset$ whenever $f \restriction \alpha$ is Laver at α, and with the additional refinement that, in case $f \restriction \alpha$ is not Laver at α, $f(\alpha)$ is chosen to be the $\lhd$-*least* set satisfying the second condition of the construction. Clearly, f is the required Laver sequence.∎

A natural question, which was brought to the author's attention originally by a referee (in a somewhat different context), is:

4.7 Open Question. Is $V = HOD$ consistent[4] with WA?

On the other hand, a model $M[G]$ of ZFC + WA + $V \neq HOD$ can be obtained from a model M of ZFC + WA by adding a Cohen real; the WA-embedding is preserved (since j fixes the Cohen order), and, in $M[G]$, $HOD \subset M$; see [17, p. 260].

Although part (2) of Theorem 4.4 provides an example of an undefinable Laver sequence, the proof requires κ to be much stronger than supercompact (since the first condition listed in the definition of f given in **CC(t)** must hold almost everywhere). It is easy to see, using a counting argument, that there must be undefinable Laver sequences whenever a Laver sequence exists at all: Let us say, for Laver sequences g, h, that

$$g \sim_L h \iff \forall \lambda, i \left[(\lambda \geq \kappa \wedge \text{“}i \text{ is } \lambda\text{-supercompact with critical point } \kappa\text{”}) \right. \\ \left. \Longrightarrow i(g)(\kappa) = i(h)(\kappa) \right]. \tag{4.3}$$

Then, given a Laver sequence $f : \kappa \to V_\kappa$ and any unbounded nonstationary subset A of κ, note that altering the values of f on A produces another Laver sequence that is $\sim_L$-equivalent to f. Since there are 2^κ ways to alter f on A and only κ definable (with parameters) functions $\kappa \to V_\kappa$, one such alteration must yield an undefinable Laver sequence. The next proposition exploits this idea to exhibit particular generic examples of undefinable Laver sequences:

4.8 Proposition. *If there is a Laver sequence at κ, there is also a Laver sequence at κ that is not definable in V_κ.*

Proof. Let $f : \kappa \to V_\kappa$ be a Laver sequence and let A denote the set of ordinals below κ that are not cardinals. Let

$$\mathcal{P} = \{p : A \to V_\kappa : p \text{ is a partial function and } |p| < \kappa\},$$

[4]Recent results [8] show that that the answer is yes, assuming the consistency of an I_1 embedding.

ordered by extension. Since $\mathcal{P}$ is $<\kappa$-closed, and since there are only κ dense subsets of $\mathcal{P}$ that are definable in V_κ, we can inductively build a filter G that meets them all. Let $g = \bigcup G : A \to V_\kappa$. We can use the definable dense sets to show that g is a total function on A which is not definable in V_κ (with parameters). Now define $h : \kappa \to V_\kappa$ by

$$h(\alpha) = \begin{cases} g(\alpha), & \text{if } \alpha \in A; \\ f(\alpha), & \text{otherwise.} \end{cases}$$

Clearly, $h \sim_L f$ and h is not definable in V_κ. ■

It follows from Proposition 4.8 (and our earlier observations) that each $\sim_L$-equivalence class has cardinality 2^κ. The canonical construction can also be used to show that there are 2^κ distinct $\sim_L$-equivalence classes (assuming WA): For each $X \in [\kappa]^\kappa$, let $t^X_\alpha = X \cap \alpha$ and let f_X be the function defined as in $\mathbf{CC}(t^X)$. Assuming WA, f_X is a Laver sequence. Let $j : V \to V$ be the WA-embedding and let D be the normal ultrafilter over κ derived from j. Clearly, if $X \neq Y$, then f_X and f_Y disagree on a set in D. Thus, for any λ with $\kappa < \lambda < j(\kappa)$, if U is the normal ultrafilter over $P_\kappa\lambda$ derived from j, $i_U(f_X)(\kappa) \neq i_U(f_Y)(\kappa)$ whenever $X \neq Y$, as required.

We can use these ideas to obtain the same result assuming only that a Laver sequence on κ exists: Thus, let f be any Laver sequence over κ, and define the functions $t^X, X \in [\kappa]^\kappa$, as in the last paragraph. We use f and the t^X to build 2^κ $\sim_L$-nonequivalent Laver sequences g_X directly. Let λ, U_0 be such that $\lambda \geq \kappa$, U_0 is a normal ultrafilter over $P_\kappa\lambda$, and $i_{U_0}(f)(\kappa) = 0$. Let D_0 be the normal ultrafilter over κ derived from i_{U_0}. Let $Y_0 = \{\alpha < \kappa : f(\alpha) = 0\}$. Clearly $Y_0 \in D_0$. Given $X \in [\kappa]^\kappa$, let U_X be normal over $P_\kappa\lambda$ so that $i_{U_X}(f)(\kappa) = X$, and let D_X be the normal ultrafilter over κ derived from i_{U_X}. Let $Y_X = \{\alpha < \kappa : f(\alpha) = t^X_\alpha\}$. Again, $Y_X \in D_X$. Note that $Y_0 \cap Y_X = \{\alpha : \alpha \leq \min X\}$. Now define g_X by

$$g_X(\alpha) = \begin{cases} 0 & \text{if } \alpha \in Y_X \setminus Y_0 \\ t^X_\alpha & \text{if } \alpha \in Y_0 \\ f(\alpha) & \text{otherwise.} \end{cases}$$

Certainly $i_{U_0}(g_X)(\kappa) = X$, and so if $X \neq X'$, $i_{U_0}(g_X) \neq i_{U_0}(g_{X'})$. Thus, we will be done when we have shown that g_X is Laver for each X. Let z be a set and $\lambda \geq \max(\kappa, |\mathrm{TC}(z)|)$. If $z \in \{0, X\}$, z can be captured by g_X by picking normal ultrafilters that work for f and switching them, as described in the previous paragraph. So, assume $z \notin \{0, X\}$. Let U_z be normal over $P_\kappa\lambda$ so that $i_{U_z}(f)(\kappa) = z$. Let D_z be the normal ultrafilter over κ derived from i_{U_z}. Notice that neither Y_0 nor Y_X is in D_z (otherwise, $i_{U_z}(f)(\kappa)$ would be either 0 or X). Hence, f and g_X agree on a set in D_z; it follows that $z = i_{U_z}(g)(\kappa)$, as required.

As promised earlier, we now show that the existence of special Laver sequences is consistency-wise much stronger than supercompactness:

4.9 Theorem. *If there is a special Laver sequence at κ, then there is a model of set theory in which there is a proper class of extendibles; moreover, it is consistent for κ to be the κth extendible.*

Proof. The argument makes use of the reasoning in [27, 6.3, 6.4, 8.5]. Let $f : \kappa \to V_\kappa$ be a special Laver sequence and let D be a normal measure on κ such that the function $\alpha \mapsto |f(\alpha)|$

dominates, on a D-measure 1 set, each function $\kappa \to \kappa$ that is definable in V_κ. Let λ, U be such that U is a supercompact ultrafilter over $P_\kappa\lambda$ so that D pulls back to U and $|\mathrm{TC}(i_U(f)(\kappa))| \leq \lambda$. Let $i_U : V \to M$ be the canonical supercompact embedding derived from U.

We will begin by building a sequence $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ of structures with the following properties:

(a) For all $\alpha < \kappa$, $\mathcal{M}_\alpha \in V_\kappa$.

(b) The sequence $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ is definable in V_κ.

(c) If $\alpha < \beta < \kappa$ and $i : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is an elementary embedding with critical point μ, then $V_\kappa \models$ "μ is extendible."

Once we have defined $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$, we will obtain a D-measure 1 set Δ with the property that if $\alpha, \beta \in \Delta, \alpha < \beta$, there is an elementary embedding $\mathcal{M}_\alpha \to \mathcal{M}_\beta$ with critical point α; by (c), the first part of the theorem will follow.

To obtain $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$, begin by defining a function $F : \kappa \to \kappa$ by

$$F(\alpha) = \begin{cases} \alpha, & \text{if } V_\kappa \models \text{“}\alpha \text{ is extendible”;} \\ \alpha + \beta, & \text{otherwise, where } \beta \text{ is least for which } V_\kappa \models \text{“}\alpha \text{ is not } \beta\text{-extendible.”} \end{cases}$$

Define $C = \{\alpha < \kappa : F''\alpha \subseteq \alpha\}$. For each $\alpha < \kappa$, let γ_α denote the least limit in C for which $\gamma_\alpha > \alpha$. Let $\mathcal{M}_\alpha = (V_{\gamma_\alpha}, \in, \{\alpha\}, C \cap \gamma_\alpha)$ for each $\alpha < \kappa$. To establish (c), we argue as follows:

Assume $j : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is elementary with critical point μ but $V_\kappa \not\models$ "μ is extendible." Then $F(\mu) = \rho = \mu + \delta$ for some $\delta > 0$. We obtain a contradiction by showing $V_\kappa \models$ "μ is δ-extendible," via the embedding $j \restriction V_\rho$. Note that because $\gamma_\alpha \in C$, we have

$$\mu < \rho < \gamma_\alpha.$$

Thus (in V_κ), $V_\rho \in \mathcal{M}_\alpha$, $\mathrm{cp}(j \restriction V_\rho) = \mu$, $j(\mu) < j(\rho)$, and $j \restriction V_\rho : V_\rho \to V_{j(\rho)}$. We will be done once we have shown that $j(\mu) > \rho$. To do this, we first show that $\mu \in C$: Suppose not. Let $\nu = \sup(C \cap \mu)$ and let ν_C be the least element of C greater than ν. Then $\nu < \mu < \nu_C < \gamma_\alpha$. Since $\mathrm{cp}(j) = \mu$, $j(\nu) = \nu$, and by the definition of ν_C from ν and C, $j(\nu_C) = \nu_C$. However, the least fixed point of j above its critical point is $\lambda = \sup\{j^n(\mu) : n \in \omega\}$, whence $\lambda \leq \nu_C$. But now, γ_α is a limit ordinal greater than λ, and this contradicts Kunen's results (Theorem 2.15). Thus, $\mu \in C$. Finally, since $C \cap \gamma_\alpha$ is a predicate for $\mathcal{M}_\alpha$, $j(\mu) \in C$; since

$$\mu \in j(\mu),\ F''j(\mu) \subseteq j(\mu), \text{ and } F(\mu) = \rho,$$

it follows that $\rho < j(\mu)$, as required. This completes the proof of (c).

For each $\alpha < \kappa$, let $X_\alpha = \{\xi :$ there is an elementary embedding $\mathcal{M}_\alpha \to \mathcal{M}_\xi$ with critical point $\alpha\}$ and let $T = \{\alpha < \kappa : X_\alpha \in D\}$. We will show that $T \in D$; we will then conclude that the diagonal intersection Δ of the X_α relative to T is also in D, yielding a proper class of V_κ-extendibles relative to the model V_κ.

To see that $T \in D$, first set

$$\begin{aligned} i_U(\langle \mathcal{M}_\xi : \xi < \kappa \rangle) &= \langle \mathcal{M}'_\xi : \xi < i_U(\kappa) \rangle; \\ i_U(\langle \mathcal{M}'_\xi : \xi < i_U(\kappa) \rangle) &= \langle \mathcal{M}''_\xi : \xi < i_U^2(\kappa) \rangle; \\ i_U(\langle X_\alpha : \alpha < \kappa \rangle) &= \langle X'_\alpha : \alpha < i_U(\kappa) \rangle. \end{aligned}$$

Note that $\xi < \kappa$ implies $\mathcal{M}_\xi = \mathcal{M}'_\xi$. Also, if $\alpha < \kappa$, $X_\alpha \in D$ iff $\kappa \in i_U(X_\alpha)$ iff $\mathcal{M}_\alpha$ is elementarily embeddable into $\mathcal{M}'_\kappa$ with critical point α. So, for $\alpha < i_U(\kappa)$, $X'_\alpha \in i_U(D)$ iff in M, $\mathcal{M}'_\alpha$ is elementarily embeddable into $\mathcal{M}''_{i_U(\kappa)}$ with critical point α. Thus, $T \in D$ iff $\kappa \in i_U(T)$ iff $X'_\kappa \in i_U(D)$ iff in M, $\mathcal{M}'_\kappa$ is elementarily embeddable into $\mathcal{M}''_{i_U(\kappa)}$ with critical point κ. Since $i_U \restriction \mathcal{M}'_\kappa : \mathcal{M}'_\kappa \to \mathcal{M}''_{i_U(\kappa)} = i_U(\mathcal{M}'_\kappa)$ is elementary, it suffices to show that $i_U \restriction \mathcal{M}'_\kappa \in M$. Define $h : \kappa \to \kappa$ by $h(\xi) = |\mathcal{M}_\xi|$. Since $\langle \mathcal{M}_\xi : \xi < \kappa \rangle$ is definable in V_κ, so is h; thus the function $\xi \mapsto |f(\xi)|$ dominates h on a D-measure 1 set, whence

$$i_U(h)(\kappa) < |i_U(f)(\kappa)| \leq \lambda.$$

Since $|\mathcal{M}'_\kappa| = i_U(h)(\kappa)$ and M is closed under λ-sequences, it follows that $i_U \restriction \mathcal{M}'_\kappa \in M$, as required.

For the "moreover" clause, first note that since $\Delta \in D$, it follows that

$$V^M_{i_U(\kappa)} \models \text{“}\kappa \text{ is extendible.”}$$

But this implies that for each $\alpha \in \Delta$,

$$V^M_{i_U(\kappa)} \models \text{“}\alpha \text{ is extendible,”}$$

for, if α were a counterexample, this fact would relativize down to V_κ (see Corollary 2.21). ■

We remark here that, with only minor modifications, the proof just given also shows that the conclusion of Theorem 4.9 follows from the existence of special* sequences as well: As in the proof, start with λ, U, D where, in this case, λ is a beth fixed point, and $rank(i_U(f)(\kappa)) < \lambda$. Build $\langle \mathcal{M}_\alpha : \alpha < \kappa \rangle$ as before; the proofs of (a) - (c) are the same. As before, to show that $T \in D$, it suffices to show that $i_U \restriction \mathcal{M}'_\kappa \in M$. Define $h : \kappa \to \kappa$ by $h(\xi) = rank(\mathcal{M}_\xi)$. We obtain

$$i_U(h)(\kappa) < rank(i_U(f)(\kappa)) < \lambda.$$

Since $rank(\mathcal{M}'_\kappa) = i_U(h)(\kappa)$ and λ is a beth fixed point, $|\mathcal{M}'_\kappa| < \lambda$. By λ-closure, $i_U \restriction \mathcal{M}'_\kappa \in M$. The rest of the proof is identical. This observation will prove useful when we generalize the notion of special Laver sequences to other large cardinals.

To conclude this section, we consider Question #4; we will make a few remarks concerning Question #5 in Section 5 (see Proposition 5.30, Corollary 5.31, and remarks following). Our canonical construction CC(t) and our proof of Theorem 4.4 will provide a template for our treatment of Laver sequences relative to other large cardinals. As the definitions, constructions, and proofs involved have many common features, we generalize to the setting of classes of elementary embeddings. As we shall see, if a class $\mathcal{E}$ of embeddings exhibits sufficient "compatibility" with the WA-embedding j, then the sequence f obtained using the canonical construction CC(t) relative to $\mathcal{E}$ will turn out to be an *$\mathcal{E}$-Laver sequence at κ*. Moreover, we will show that most of the familiar globally defined large can be characterized in terms of classes of embeddings that exhibit the required properties. One conclusion will be that, if κ is the WA-critical point, then κ admits superhuge, super-almost-huge, extendible, supercompact, and strong Laver sequences, and that, moreover, these Laver sequences can be constructed to exhibit the properties described in Theorem 4.4 above.

Our classes $\mathcal{E}$ will consist of elementary embeddings of the form $i : V_\beta \to M$, where M is transitive. We will impose certain conditions on these classes in order to study the notion of Laver sequences in an abstract setting. To motivate these conditions, we begin with a couple of observations about Laver sequences.

First, let us call a function $g : \kappa \to V_\kappa$ a *generalized Laver sequence* if for each x there is an elementary embedding $i : V \to M$ such that $x = i(g)(\kappa)$. Clearly, every Laver sequence and every strong Laver sequence is generalized Laver; indeed, generalized Laver-ness is the weakest form of Laver's original notion that retains the property of "capturing every set." As the next proposition shows, κ will not admit even a generalized Laver sequence unless κ is a strong cardinal:

4.10 Proposition. *Suppose κ is an infinite cardinal. Then the following are equivalent:*

(1) *κ is a strong cardinal;*

(2) *κ admits a strong Laver sequence;*

(3) *there is a function $g : \kappa \to V_\kappa$ such that for each x and each $\lambda > \max(\kappa, rank(x))$ there is an elementary embedding $j : V \to M$ such that $j(\kappa) > \lambda$ and $x = j(g)(\kappa)$;*

(4) *κ admits a generalized Laver sequence.*

Proof. (1) $\Rightarrow$ (2) was proven in Theorem 2.30(2). For (2) $\Rightarrow$ (3), let g be a strong Laver sequence, let x be a set, and let $\lambda > \max(\kappa, rank(x))$. Use (2) to obtain an extender E with critical κ and support $V_{\lambda+1}$ for which $i_E(g)(\kappa) = x$; clearly $i_E(\kappa) > \lambda$.

(3) $\Rightarrow$ (4) is immediate. For (4) $\Rightarrow$ (1), let $g : \kappa \to V_\kappa$ be a generalized Laver sequence and let $\lambda > \kappa$. Let $j : V \to M$ be an elementary embedding such that $j(g)(\kappa) = V_\lambda \in M$. By elementarity, $V_\lambda \in V^M_{j(\kappa)}$, whence $\lambda < j(\kappa)$.■

The proposition makes it natural to restrict our attention to classes $\mathcal{E}$ of embeddings for which $\{i(\kappa) : i \in \mathcal{E}\}$ is a proper class and for which, for any x, there is $i : V_\beta \to M \in \mathcal{E}$ with $x \in M$.

4.11 Proposition. *For each infinite cardinal κ and each function $g : \kappa \to V_\kappa$, the following are equivalent:*

(1) *g is Laver at κ;*

(2) *for every set x and every $\lambda > \max(\kappa, rank(x))$ there are ζ, U such that $\zeta > \lambda$ and U is a normal ultrafilter over $P_\kappa\zeta$ such that $x = i_U(g)(\kappa)$.*

Proof. (1) $\Rightarrow$ (2) is immediate; to prove (2) $\Rightarrow$ (1), let x be a set and let $\lambda \geq \max(\kappa, |\mathrm{TC}(x)|)$. Let ζ, U be such that $\zeta > \max(\lambda, rank(x))$ and U is a normal ultrafilter over $P_\kappa\zeta$, with $i_U(g)(\kappa) = x$. Now use Lemma 2.26(I) to show that $i_{U|\lambda}(g)(\kappa) = x$. ■

Propositions 4.10 and 4.11 make it clear that the requirement on a function f—that for every x and *arbitrarily large* λ, there is an appropriately defined embedding i such that $i(f)(\kappa) = x$ (and $i(\kappa) > \lambda$)—is sufficent to characterize each type of Laver sequence. This observation will be important when we formulate a definition of Laver sequences for classes of embeddings. It also shows that whether we use transitive closure or rank in the definition is purely a matter of taste. (Having said this, it is perhaps interesting to note that the condition "$\lambda > \max(\kappa, rank(x))$" cannot be substituted for "$\lambda \geq \max(\kappa, |\mathrm{TC}(x)|)$" in the standard definition of Laver sequences (Definition 2.23): If κ is supercompact and $f : \kappa \to V_\kappa$, let $x = V_{\kappa+3}$ and let $\lambda = \kappa + 4$. If U is a normal ultrafilter over $P_\kappa\lambda$ and $i_U : V \to M$ is the canonical embedding, then one has

$$|M \cap V_\lambda| \leq 2^{\lambda^{<\kappa}} < |x|,$$

and so $x \notin M$. Thus, $i_U(f)(\kappa) \neq x$.)

We can now describe the classes of embeddings that will concern us.

4.12 Definition (*Regular Classes*) Let $\theta(x, y, z, w)$ be a first-order formula (in the language $\{\in\}$) with all free variables displayed. We will call θ a *suitable formula* if the following sentence is provable in ZFC:

$$\forall x, y, z, w\, [\theta(x, y, z, w) \Longrightarrow \text{"}w \text{ is a transitive set"} \wedge z \in \mathrm{ON} \wedge \\ \wedge \text{"}x : V_z \to w \text{ is an elementary embedding with critical point } y\text{"}].$$

For each cardinal κ and each suitable $\theta(x, y, z, w)$, let

$$\mathcal{E}_\kappa^\theta = \{(i, M) : \exists\beta\, \theta(i, \kappa, \beta, M)\}.$$

The definition of $\mathcal{E}_\kappa^\theta$ requires some explanation. The reason that the codomain of an elementary embedding i needs to be explicitly associated with i arises from the fact that, unlike most properties of functions, the elementarity of i depends on its codomain, but the dependence is not built into the definition of i as a function (i.e. as a set of ordered pairs). Thus (as the referee points out), it is possible for an elementary embedding $i : V_\beta \to M$ to be contained in some V_ρ and yet $M \notin V_\rho$; in this case, i generally loses its elementarity from the point of view of V_ρ. Therefore, if we were to define $\mathcal{E}_\kappa^\theta$ to be the class $\{i : \exists\beta\,\exists M\, \theta(i, \kappa, \beta, M)\}$, then for most of the suitable formulas θ that will concern us, the classes $\mathcal{E}_\kappa^\theta \cap V_\rho$ and $\left(\mathcal{E}_\kappa^\theta\right)^{V_\rho}$ would be unequal—even if ρ is inaccessible—and this would create numerous technical inconveniences.

For readability, however, we will treat elements of $\mathcal{E}_\kappa^\theta$ as if they were elementary embeddings rather than ordered pairs, whenever there is no possibility of ambiguous interpretation. Thus, in such cases, we will write "$i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$" when we mean "$(i : V_\beta \to M, M) \in \mathcal{E}_\kappa^\theta$" and "$i \in \mathcal{E}_\kappa^\theta$" when we mean "$(i, M) \in \mathcal{E}_\kappa^\theta$ for some M." In cases where this approach would lead to ambiguity, we will treat the elements of $\mathcal{E}_\kappa^\theta$ explicitly as ordered pairs.

In the sequel, we will often declare, or attempt to demonstrate, that a class $\mathcal{E}$ is *definable from a suitable formula*. The obvious meaning here is that for some suitable formula θ, $\mathcal{E} = \mathcal{E}_\kappa^\theta$.

However, because of the definition of suitability, the only free parameter that is allowed in such a defining formula is κ; in this paper, we do not allow the notion "definable by a suitable formula with extra parameters."

Continuing with our definition:

$$\text{Dom } \mathcal{E}^\theta_\kappa = \{\beta : \exists i \in \mathcal{E}^\theta_\kappa \ (\text{dom } i = V_\beta)\};$$
$$\mathcal{E}^\theta_\kappa(\lambda) = \{i \in \mathcal{E}^\theta_\kappa : i(\kappa) > \lambda\}.$$

We shall call $\mathcal{E}^\theta_\kappa$ a *regular class of embeddings at* κ if

$$\forall \gamma > \kappa \, \exists \beta \geq \gamma \, \exists i \, \exists M \, [\theta(i, \kappa, \beta, M) \wedge i(\kappa) > \gamma \wedge V_\gamma \subset M].$$

In the above definition, if we remove the clause "$i(\kappa) > \gamma$," $\mathcal{E}^\theta_\kappa$ will be called *semi-regular*; if, instead, we change "$\exists \beta \geq \gamma$" to "$\exists \beta > \kappa$," $\mathcal{E}^\theta_\kappa$ will be called *weakly regular*; and, if we alter the above definition by making both of these changes, the resulting class will be called *weakly semi-regular*.

Our main results will be about regular classes or, when appropriate, arbitrary classes $\mathcal{E}$ defined by a suitable formula. Our results about regular classes can often be generalized to the broader classes described in the last paragraph. In Section 7 we examine the relationships between these classes. For now, we note that, of these four types of classes, only two imply the existence of a strong cardinal (though all four are equiconsistent with a strong cardinal):

4.13 Proposition. *Suppose κ is an infinite cardinal. Then the following are equivalent:*

(1) *κ is a strong cardinal;*
(2) *there is a suitable formula θ such that $\mathcal{E}^\theta_\kappa$ is regular;*
(3) *there is a suitable formula θ such that $\mathcal{E}^\theta_\kappa$ is weakly regular.*

Proof. For (1) $\Rightarrow$ (2), see Theorem 4.30 below; (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), suppose $\mathcal{E}^\theta_\kappa$ is weakly regular and $\lambda > \kappa$; we would like to show that κ is λ-strong by obtaining an extender with support V_λ using some $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa(\lambda)$ with $V_\lambda \subset M$. However, since it is possible that $\beta < \lambda$, such an extender can fail to be well defined from i, using the usual definition. This difficulty can be corrected by modifying the definition of extender so that normal measures are taken over sets of the form ${}^{lh(s)}V_\kappa$ (where for some n, $s : n \to V_\lambda$) rather than ${}^aV_\kappa$. The details of this modification are worked out in [24]; in particular, it is shown there that an extender $E = \langle E(s) : s \in {}^{<\omega}V_\lambda \rangle$ of this new variety can be obtained from an elementary embedding $j : V \to N$ by setting $E(a)(X) = 1$ iff $s \in j(X)$. Because the sets X lie in $V_{\kappa+1}$, we can define E in the same way using any $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa(\lambda)$ with $V_\lambda \subset M$. It follows (see [24]) that κ is λ-strong.∎

We should point out that Proposition 4.13 has not been properly stated in ZFC since we have apparently quantified over the formula θ. To state Proposition 4.13 formally, we can, for example, express (1) $\Rightarrow$ (2) by saying "if κ is a strong cardinal, then $\mathcal{E}^{\theta_{str}}_\kappa$ is a regular class," where θ_{str}

is defined below, after Definition 4.28. On the other hand, (2) ⇒ (1) should be thought of as a schema, so that for each θ the formal version of the following statement, dependent on θ, can be proven in ZFC:

$$\text{“}\theta \text{ is a suitable formula } \wedge \mathcal{E}_\kappa^\theta \text{ regular } \Longrightarrow \kappa \text{ is strong”}.$$

These ideas can be applied to restate the entire theorem (schema) in ZFC.

4.14 Definition (*$\mathcal{E}$-Laver Sequences*) Suppose $\mathcal{E}_\kappa^\theta$ is a class of embeddings, where θ is a suitable formula. A function $g : \kappa \to V_\kappa$ is said to be *$\mathcal{E}_\kappa^\theta$-Laver at κ* if for each set x and each $\lambda > \max(\kappa, rank(x))$ there are $\beta > \lambda$, and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$.

For a given suitable formula θ, we define a formula $\phi(g, x, \lambda)$, depending on θ, as follows:

$$\begin{array}{c}\text{“there exists a cardinal } \alpha \text{ with } g : \alpha \to V_\alpha, \text{ such that } \lambda > \max(\alpha, rank(x)) \text{ and} \\ \text{for all } \beta > \lambda \text{ and all } i : V_\beta \to M \in \mathcal{E}_\alpha^\theta, \text{ if } i(\alpha) > \lambda \text{ then } i(g)(\alpha) \neq x.\text{”}\end{array} \tag{4.4}$$

For the rest of the paper, we reserve the symbol 'ϕ' to refer to this formula exclusively.

It should be pointed out that our definition of $\mathcal{E}_\kappa^\theta$-Laver sequences makes sense even for non-regular classes. This latitude is necessary in order to correctly state results about classes that have no $\mathcal{E}_\kappa^\theta$-Laver function. This necessity will become apparent in our canonical construction of an $\mathcal{E}_\kappa^\theta$-Laver sequence below, and will be pervasive in Section 5. As the next proposition shows, however, if $\mathcal{E}_\kappa^\theta$ admits an $\mathcal{E}_\kappa^\theta$-Laver sequence, it must be regular.

4.15 Proposition. *Suppose θ is a suitable formula.*

(1) *A function $g : \kappa \to V_\kappa$ is $\mathcal{E}_\kappa^\theta$-Laver at κ if and only if the following holds:*

$$\forall x\, \forall \lambda\, \neg\phi(g, x, \lambda).$$

(2) *If, in Definition 4.14, the variable λ is restricted to limit ordinals only, we obtain an equivalent definition.*

(3) *If $\mathcal{E}_\kappa^\theta$ admits an $\mathcal{E}_\kappa^\theta$-Laver sequence, then $\mathcal{E}_\kappa^\theta$ is a regular class.*

Proof. Parts (1) and (2) are immediate. For (3), let $g : \kappa \to V_\kappa$ be $\mathcal{E}_\kappa^\theta$-Laver. Let $\gamma > \kappa$ and, as in Definition 4.14, obtain $\beta > \gamma$ and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $i(\kappa) > \lambda$ and $i(g)(\kappa) = V_\gamma$. Since M is transitive, $V_\gamma \subseteq M$. Thus, β, i, M meet the requirements in the definition of regularity. ■

In our definition of $\mathcal{E}$-Laver sequences, we have required that for a given x, there are embeddings i that capture x and for which $i(\kappa)$ is arbitrarily large. This second requirement is in the spirit of Laver's original definition; in some cases, however, it can be ignored. In particular, if there happens to be a certain kind of definable relationship between the domains of $i \in \mathcal{E}_\kappa^\theta$ and $i(\kappa)$, then if we can prove "$i(g)(\kappa) = x$," it will follow that $i(\kappa) > \lambda$. We make this idea precise in the following definition and proposition:

4.16 Definition. (*Correlated Classes*) A class $\mathcal{E}_\kappa^\theta$ defined by a suitable formula will be called *correlated* if there is a strictly increasing class function $\mathbf{F}$: Dom $\mathcal{E}_\kappa^\theta \to ON$ such that for each $\beta \in$ Dom $\mathcal{E}_\kappa^\theta$, $\mathbf{F}(\beta) \leq \beta$ and for each $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, $i(\kappa) \geq \mathbf{F}(\beta)$.

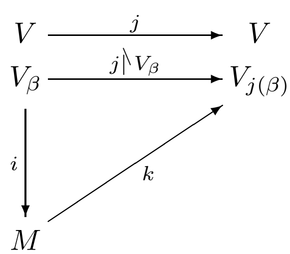
We leave it to the reader to verify that Definition 4.16 can be stated so that no quantification occurs over proper classes.

4.17 Proposition. *If $\mathcal{E}_\kappa^\theta$ is a correlated class of embeddings and $g : \kappa \to V_\kappa$ is such that for each set x and each $\lambda > \max(\kappa, rank(x))$, there are $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that $i(g)(\kappa) = x$, then g is $\mathcal{E}_\kappa^\theta$-Laver at κ.*

Proof. Suppose $\mathcal{E}_\kappa^\theta$ is a correlated class with class function $\mathbf{F}$: Dom $\mathcal{E}_\kappa^\theta \to ON$, and g satisfies the condition in the hypothesis; we show g is $\mathcal{E}_\kappa^\theta$-Laver. Suppose x is a set and $\lambda > \max(\kappa, rank(x))$. Use the fact that $\mathbf{F}$ is strictly increasing to pick $\beta \in$ Dom $\mathcal{E}_\kappa^\theta$ large enough so that $\mathbf{F}(\beta) > \lambda$ and such that for some $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, $i(g)(\kappa) = x$; then $\lambda < \mathbf{F}(\beta) \leq i(\kappa)$, as required.■

Theorem 4.22 shows that, assuming WA, every regular class—indeed, every class defined by a suitable formula—that is "compatible" with the WA-embedding $j : V \to V$ does have a Laver sequence. We begin with a definition of compatibility:

4.18 Definition (*Compatibility*) Suppose $\kappa < \lambda_0 < \beta \leq \gamma$, and $i_\beta : V_\beta \to N$, $i_\gamma : V_\gamma \to M_\gamma$ are elementary embeddings with critical point κ. Then i_β is *compatible with i_γ up to V_{λ_0}* if there is $k_\beta : N \to M_\beta = V_{i_\gamma(\beta)}^{M_\gamma}$ such that $k_\beta \circ i_\beta = i_\gamma \restriction V_\beta$ and $k_\beta \restriction V_{\lambda_0} \cap N = \mathrm{id}_{V_{\lambda_0} \cap N}$. Suppose $j : V \to \tilde{M}$ is an elementary embedding with critical point κ, and suppose θ is a suitable formula. We will say that $\mathcal{E}_\kappa^\theta$ is *compatible with* j if for each λ, $\kappa < \lambda < j(\kappa)$, there exist β, i such that $\beta \in$ Dom $\mathcal{E}_\kappa^\theta$, $\lambda < \beta < j(\kappa)$, $i : V_\beta \to M$, $(i, M) \in \mathcal{E}_\kappa^\theta$, and i is compatible with $j \restriction V_\beta$ up to V_λ.



In this section, our use of the notion of compatibility will be restricted to the the case in which the ambient elementary embedding is the WA-embedding $j : V \to V$; other codomains will be considered in the next section.

Although compatibility is a fairly natural concept, it is highly non-absolute. Our interest in this property lies in the fact that it has a handful of useful consequences. Our definition of weak compatibility below, though less intuitive, provides us with exactly what we need when some form of compatibility, and some degree of absoluteness, are required.

4.19 Definition. (*Weak Compatibility*) Suppose $j : V \to N$ is elementary with critical point κ and D is the normal ultrafilter over κ derived from j. Suppose θ is a suitable formula. Then $\mathcal{E}^\theta_\kappa$ is *weakly compatible* with j if for each λ, $g : \kappa \to V_\kappa$, $r : \kappa \to P(\kappa)$ for which $\kappa < \lambda < j(\kappa)$, $rank(j(g)(\kappa)) < \lambda$, and $r(\delta) \in D$ for all $\delta < \kappa$, there exist β, i, M such that $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$, $\lambda < \beta < j(\kappa)$, and

(1) $i(g)(\kappa) = j(g)(\kappa)$;

(2) $i(\kappa) > \lambda$;

(3) if D_i is the normal ultrafilter over κ derived from i, then $r(\delta) \in D_i$ for all $\delta < \kappa$.

4.20 Proposition. *Suppose $j : V \to N$ is elementary with critical point κ and θ is a suitable formula. Then if $\mathcal{E}^\theta_\kappa$ is compatible with j, $\mathcal{E}^\theta_\kappa$ is weakly compatible with j.*

Proof. Suppose $\mathcal{E}^\theta_\kappa$ is compatible with j. Let D be the normal ultrafilter derived from j. Given λ, g, and r as in the definition of weak compatibility, use compatibility to pick β and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ so that $\lambda + 1 < \beta < j(\kappa)$ and i is compatible with $j \restriction V_\beta$ up to $V_{\lambda+1}$. This immediately implies $i(\kappa) > \lambda$, and, by Theorem 2.26(II), since $rank(j(g)(\kappa)) < \lambda$, $i(g)(\kappa) = j(g)(\kappa)$. Finally, argue as in the proof of Theorem 4.4(3) to see that, if D_i is the normal ultrafilter derived from i, then $D_i = D$. ■

We are ready for the analogue to the construction CC(t):

4.21 Canonical Construction CC($t, \mathcal{E}^\theta_\kappa$). Suppose θ is a suitable formula and $t : \kappa \to V_\kappa$ is a κ-sequence. Define $f = f_{t,\theta,\kappa} : \kappa \to V_\kappa$ by

$$f(\alpha) = \begin{cases} t_\alpha & \text{if } f \restriction \alpha \text{ is a } \mathcal{E}^\theta_\alpha\text{-Laver sequence at } \alpha \\ x \in V_\kappa & \text{if } \alpha \text{ is a cardinal and } f \restriction \alpha \text{ is not } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha \\ & \text{where } \exists \lambda < \kappa\ \phi(f \restriction \alpha, x, \lambda); \\ \emptyset & \text{if } \alpha \text{ is not a cardinal.} \end{cases}$$

As in the Canonical Construction 4.3, it is quite possible that f is not defined at all $\alpha < \kappa$; this happens whenever there is a cardinal α for which $f \restriction \alpha$ is not $\mathcal{E}^\theta_\alpha$-Laver, but for which there is no counterexample in V_κ. Thus, verification that f is Laver for various classes will always require a proof that f is well-defined.

We now show that, assuming WA, if $\mathcal{E}$ is sufficiently compatible with the WA-embedding, then the f constructed above is $\mathcal{E}$-Laver at κ.

4.22 Theorem. *Assume* WA*, and κ is the critical point of the* WA*-embedding j. Suppose θ is a suitable formula and $\mathcal{E}^\theta_\kappa$ is weakly compatible with j. Then the function $f : \kappa \to V_\kappa$ defined in the Canonical Construction $CC(t, \mathcal{E}^\theta_\kappa)$ is $\mathcal{E}^\theta_\kappa$-Laver at κ.*

Proof. First we note that, in the construction of f, if α is a cardinal such that $f \restriction \alpha$ is not $\mathcal{E}^\theta_\kappa$-Laver at α, there is $(x, \lambda) \in V_\kappa$ such that $\phi(f \restriction \alpha, x, \lambda)$, since $f \restriction \alpha \in V_\kappa$ and $V_\kappa \prec V$; hence f is well-defined. Let D be the normal ultrafilter over κ derived from j. As in Theorem 4.4, it suffices to show that

$$\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\} \in D.$$

Assume this condition fails; then

$$\{\alpha < \kappa : \exists \lambda < \kappa \, \phi(f \restriction \alpha, f(\alpha), \lambda)\} \in D.$$

It follows that for some $\lambda < j(\kappa)$, $\phi(f, j(f)(\kappa), \lambda)$. Let $x = j(f)(\kappa)$. Since $\mathcal{E}^\theta_\kappa$ is weakly compatible with j, we can obtain β and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ with $\lambda < \beta < j(\kappa)$, $i(f)(\kappa) = j(f)(\kappa) = x$ and $i(\kappa) > \lambda$. If $M \notin V_{j(\kappa)}$, we can use elementarity to obtain an i' with the same properties, but which has its codomain in $V_{j(\kappa)}$. Thus, we have a contradiction. ■

One easy corollary to the theorem is that if $\mathcal{E}^\theta_\kappa$ is weakly compatible with the WA-embedding, then $\mathcal{E}^\theta_\kappa$ must be regular; this follows from the theorem and Proposition 4.15(3). A question that naturally arises here is whether there can be a regular class that does not admit its own brand of Laver sequence:

4.23 Open Question. Is it consistent[5] with the existence of a strong cardinal for there to be a regular class $\mathcal{E}^\theta_\kappa$ for which there is no $\mathcal{E}^\theta_\kappa$-Laver sequence? If yes, is this consistent[6] with WA? Is it consistent under any hypothesis (at least the existence of a strong cardinal) that all regular classes admit corresponding Laver sequences?

We pause here to consider an example that highlights an important caveat that must be respected in dealing with classes of embeddings in the presence of WA.

4.24 Example. (*A Paradoxical Class*) Assume WA, and let $j : V \to V$ be the WA-embedding. Let $\mathcal{E} = \{j^m \restriction V_{j^n(\kappa)} : m, n \geq 1\}$. It is clear that for each $\lambda > \kappa$ there is $\beta > \lambda$ and $i : V_\beta \to M \in \mathcal{E}$ such that $i(\kappa) > \lambda$ and $V_\lambda \subset M$. It is also clear that $\mathcal{E}$ is compatible with j. However, $\mathcal{E}$ does not admit an $\mathcal{E}$-Laver sequence; indeed, for any $g : \kappa \to V_\kappa$, if $x = j(g)(\kappa)$, then, as one easily verifies, for all $i \in \mathcal{E}$, $i(g)(\kappa) = x$. This result apparently contradicts Theorem 4.22.

The paradox is resolved by observing that there is no suitable formula θ which defines $\mathcal{E}$ (for, if there were such a $\theta(x, y, z, w)$, we could let $\psi(x, y)$ be the $\in$-formula

"y is of least rank such that, for some β, i, M, $\theta(i, \kappa, \beta, M) \wedge (x, y) \in V_\beta \wedge i(x) = y$".

Then if $j : V \to V$ is the WA-embedding, for all x, y, $j(x) = y$ iff $\psi(x, y)$). The moral is that, in devising examples of regular classes, we must exercise care not to use **j**-formulas. More generally,

[5] In [9] it is shown that, assuming there is a strong cardinal κ with an inaccessible above, there is a forcing extension in which κ is still strong and there is a regular class of embeddings with no corresponding Laver sequnce.

[6] In [8] it is shown that, assuming the consistency of an I_1 embedding, there is a forcing extension in which WA holds and there is a regular class with no corresponding Laver sequence.

the example illustrates the importance of checking that a class $\mathcal{E}$ can be defined by a suitable formula. A less contrived example of the need for care in this regard comes up in Example 7.3(3).

Returning to ramifications of Theorem 4.22, we can generalize the equivalence relation mentioned in the remarks following Proposition 4.8 to the present context: Define $\sim_L^\theta$ over $\mathcal{E}_\kappa^\theta$-Laver sequences as in (4.3) by $f \sim_L^\theta g$ iff for all $i \in \mathcal{E}_\kappa^\theta$, $i(f)(\kappa) = i(g)(\kappa)$; our arguments in the remarks following Proposition 4.8 still go through, *mutatis mutandis*, in this general case, showing that there must be 2^κ $\sim_L^\theta$-equivalence classes, each containing 2^κ elements.

We note also that, as in Theorem 4.4, we can ensure that the functions $\alpha \mapsto |f(\alpha)|$ and $\alpha \mapsto rank(f(\alpha))$, where f is a $\mathcal{E}_\kappa^\theta$-Laver sequence obtained in $\mathbf{CC}(t, \mathcal{E}_\kappa^\theta)$, dominate, on D-measure 1 sets, the functions $\kappa \to \kappa$ definable in V_κ. We can also generalize the notion of special Laver sequences and prove their existence under the hypotheses of Theorem 4.22. Since we are using ranks instead of transitive closures in our general notion of Laver sequences, we will actually generalize the notion of special* Laver sequences, as defined immediately after Definition 4.5.

4.25 Definition. (*Special $\mathcal{E}_\kappa^\theta$-Laver sequences.*) An $\mathcal{E}_\kappa^\theta$-Laver sequence $g : \kappa \to V_\kappa$ is *special* if there exist $\lambda, \beta, i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, such that $\kappa < \lambda < \beta$, λ is a beth fixed point, and

(1) M is λ-closed;

(2) $rank(i(g)(\kappa)) < \lambda < i(\kappa)$;

(3) if D is the normal ultrafilter over κ derived from i, then for each $h : \kappa \to \kappa$ definable in V_κ, the function $\alpha \mapsto rank(g(\alpha))$ dominates h on a set in D.

Proposition 4.32 below shows that this definition agrees with the definition of special* in the case that $\mathcal{E}_\kappa^\theta$ is the class of embeddings corresponding to a supercompact cardinal (for a definition of this class, see the discussion following Definition 4.28). In Section 7, we show, as in Theorem 4.9, that the existence of special $\mathcal{E}_\kappa^\theta$-Laver sequences is generally quite strong: If there exists a special $\mathcal{E}_\kappa^\theta$-Laver sequence, and if the embeddings in $\mathcal{E}_\kappa^\theta$ have "sufficiently many" extensions, it is consistent for κ to be the κth extendible cardinal (Theorem 7.17).

The next two results show that if a class $\mathcal{E}_\kappa^\theta$ is weakly compatible with the WA embedding and satisfies a reasonable closure condition, there must exist a special $\mathcal{E}_\kappa^\theta$-Laver sequence. This result is a generalization of part (3) of Theorem 4.4. In Corollary 4.35, we apply Corollary 4.27 to show that, assuming WA, whenever $\mathcal{E}_\kappa^\theta$ corresponds to one of the familiar globally defined large cardinals, there must exist a special $\mathcal{E}_\kappa^\theta$-Laver sequence.

4.26 Proposition. *Suppose θ is suitable. Suppose that for each $t : \kappa \to V_\kappa$, if $f = f_t$ is the function obtained from $\mathbf{CC}(t, \mathcal{E}_\kappa^\theta)$, there are $\lambda, D, i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ such that λ is a beth fixed point, D is the normal ultrafilter over κ derived from i, and*

(1) M is λ-closed;

(2) $rank(i(f)(\kappa)) < \lambda < i(\kappa)$;

(3) for each $h : \kappa \to \kappa$ definable in V_κ, if $X_h = \{\alpha < \kappa : rank(f(\alpha)) > h(\alpha)\}$, $X_h \in D$;

(4) f is $\mathcal{E}_\kappa^\theta$-Laver at κ.

Then t can be chosen so that f is a special $\mathcal{E}_\kappa^\theta$-Laver sequence.

Proof. Immediate. ▪

Suppose λ is a cardinal and θ is suitable. We will say that $\mathcal{E}_\kappa^\theta$ is *upward λ-closed* if for each $\beta \in \mathrm{Dom}\ \mathcal{E}_\kappa^\theta$ for which $\beta > \lambda$ and each $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, we have $i(\kappa) \geq \lambda$ and M is λ-closed (recall Definition 2.25).

4.27 Corollary. *Assume* WA *and let j denote the* WA*-embedding and κ its critical point. Suppose θ is suitable and $\mathcal{E}_\kappa^\theta$ is weakly compatible with j. Assume either of the following:*

(A) *The set $\{\beta < j(\kappa) : j \restriction V_\beta \in \mathcal{E}_\kappa^\theta\}$ is unbounded in $j(\kappa)$*

(B) *$\mathcal{E}_\kappa^\theta$ is upward λ-closed for every cardinal $\lambda < j(\kappa)$.*

Then the parameter t in the construction $\mathbf{CC}(t, \mathcal{E}_\kappa^\theta)$ can be chosen so that the constructed function f is a special $\mathcal{E}_\kappa^\theta$-Laver sequence.

Proof. We apply Proposition 4.26. Let $t : \kappa \to V_\kappa$. By Theorem 4.22 and the fact that $\mathcal{E}_\kappa^\theta$ is weakly compatible with j, the function f obtained from $\mathbf{CC}(t, \mathcal{E}_\kappa^\theta)$ must be $\mathcal{E}_\kappa^\theta$-Laver; this establishes (4) of Proposition 4.26. Moreover, by the proof of Theorem 4.22, the set $\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}_\kappa^\theta\text{-Laver at } \alpha\}$ is in D, where D is the normal ultrafilter over κ derived from j. Let $\langle h_\delta : \delta < \kappa\rangle$ be an enumeration of the functions definable (with parameters) in V_κ, and for each δ, let $X_\delta = \{\alpha < \kappa : rank(f(\alpha)) > h(\alpha)\}$; then $X_\delta \in D$. Let $r : \kappa \to P(\kappa) : \delta \mapsto X_\delta$. Let λ be a beth fixed point such that $rank(j(f)(\kappa)) < \lambda < j(\kappa)$. To complete the proof, we obtain $i \in \mathcal{E}_\kappa^\theta$ so that

(a) the codomain of i is λ-closed;

(b) $i(f)(\kappa) = j(f)(\kappa)$;

(c) $i(\kappa) > \lambda$;

(d) if D_i is the normal ultrafilter derived from i, then $r(\delta) \in D_i$ for each $\delta < \kappa$.

Assuming (A), pick β so that $\lambda < \beta < j(\kappa)$ and $j \restriction V_\beta : V_\beta \to V_{j(\beta)} \in \mathcal{E}_\kappa^\theta$. We let $i = j \restriction V_\beta$. Clearly (a) - (d) are satisfied.

Assuming (B), pick witnesses β and $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ to weak compatibility with j, where β and i are obtained on the basis of the parameters f, λ, and r. Parts (b) - (d) follow immediately from the properties of i guaranteed by weak compatibility. Finally, part (a) follows from the fact that $\mathcal{E}_\kappa^\theta$ is upward λ-closed. ▪

We turn to the task of verifying that each of the most familiar globally defined large cardinals can be defined in terms of a regular class of embeddings that is compatible (or weakly compatible) with the WA-embedding j. As a first step, it will be convenient to isolate a property that tells us that a regular class $\mathcal{E}_\kappa^\theta$ is "derived from" some large cardinal property $A(\kappa)$. In the definition, by a "large cardinal property", we mean a property $A(x)$ such that the sentence $\forall\alpha$ ("α is an ordinal" $\wedge$ $A(\alpha) \Longrightarrow V_\alpha \models \mathrm{ZFC}$) holds in some transitive model of set theory

4.28 Definition. Suppose $A(x)$ is a large cardinal property and θ is a suitable formula. We shall call $A(x)$ a *normal property with suitable formula* θ if for each ordinal α we have the following:

(1) $A(\alpha) \Longleftrightarrow \forall\gamma > \alpha\,\exists\beta \geq \gamma\,\exists i\,\exists M\theta(i, \alpha, \beta, M)$; and

(2) $A(\alpha) \Longrightarrow \mathcal{E}^\theta_\alpha$ is a regular class.

If, for each ordinal α, we have

$$A(\alpha) \Longrightarrow \text{“there is an } \mathcal{E}^\theta_\alpha\text{-Laver sequence at } \alpha\text{”},$$

we will call $A(x)$ *Laver-generating.*

If $A(x)$ is normal with suitable formula θ, we may informally call $\mathcal{E}^\theta_\kappa$ a normal class, or state that $\mathcal{E}^\theta_\kappa$ is Laver-generating. The first of these should be understood to mean “$A(x)$ is normal;” the second means “$A(x)$ is Laver-generating.”

4.29 Proposition. *Suppose $A(x)$ is a normal large cardinal property with suitable formula θ. Then for each ordinal α, if $\mathcal{E}^\theta_\alpha$ is regular, then $A(\alpha)$ holds.*

Proof. Notice that regularity of $\mathcal{E}^\theta_\alpha$ implies the right-hand side of Definition 4.28(1); by normality of $A(x)$, the result follows. ∎

We proceed to show that a number of large cardinals are normal. We begin by defining the corresponding suitable formulas:

$\theta_{sc}(i, \kappa, \beta, M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\, \beta = \lambda + \omega\ \wedge i : V_\beta \to M$ is an elementary embedding with critical point $\kappa \wedge i(\kappa) > \lambda \wedge \forall f : \lambda \to M\ [\exists x \in M\ (\text{range}(f) \subseteq x) \Longrightarrow f \in M]]$.

$\theta_{str}(i, \kappa, \beta, M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\, \beta = \lambda + \omega\ \wedge i : V_\beta \to M$ is an elementary embedding with critical point $\kappa \wedge i(\kappa) > \lambda \wedge V_\lambda \subseteq M]$.

$\theta_{ext}(i, \kappa, \beta, M)$: $\exists\delta > 0\,\exists\zeta\ [\beta = \kappa + \delta \wedge M = V_\zeta \wedge i : V_\beta \to M$ is elementary with critical point $\kappa \wedge \beta < i(\kappa) < \zeta]$.

$\theta_{sh}(i, \kappa, \beta, M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\,\lambda$ is inaccessible $\wedge\,\beta = \lambda + \omega \wedge i : V_\beta \to M$ is elementary with critical point $\kappa \wedge i(\kappa) = \lambda \wedge \forall f : \lambda \to M\,\forall x \in M\ (\text{range}(f) \subseteq x \Longrightarrow f \in M)]$.

$\theta_{sah}(i, \kappa, \beta, M)$: $\exists\lambda > \kappa\ [M$ is transitive $\wedge\,\lambda$ is inaccessible $\wedge\,\beta = \lambda + \omega \wedge i : V_\beta \to M$ is elementary with critical point $\kappa \wedge i(\kappa) = \lambda \wedge \forall\mu(\kappa \leq \mu < \lambda)\forall f : \mu \to M\,\forall x \in M\ (\text{range}(f) \subseteq x \Longrightarrow f \in M)]$.

It is easy to verify that the formulas given above are all suitable; Theorem 4.30 shows that they witness the normality of the large cardinals under consideration. We adopt the following notational convention: we shall write $\mathcal{E}^z_\kappa$ instead of $\mathcal{E}^{\theta_z}_\kappa$ for each $z \in \{sc, ext, sh, sah, str\}$.

4.30 Theorem. *Each of the large cardinal properties “supercompact”, “strong”, “extendible”, “superhuge”, and “super-almost-huge” is normal, and each corresponding class $\mathcal{E}^\theta_\kappa$ is correlated.*

Proof. $\underline{\mathcal{E}^{sc}_\kappa}$: For part (1) of Definition 4.28, suppose κ is supercompact and $\gamma > \kappa$; let $\lambda \geq \gamma$ and let $\beta = \lambda + \omega$. Let $\hat{i} : V \to \hat{M}$ be such that $\text{cp}(\hat{i}) = \kappa, \hat{i}(\kappa) > \lambda$, and ${}^\lambda\hat{M} \subseteq \hat{M}$. Let $i = \hat{i} \restriction V_\beta$

and $M = V^{\hat{M}}_{\hat{i}(\beta)}$. Now, given $f : \lambda \to M$ and $x \in M$ such that range$(f) \subseteq x$, note that $f \in \hat{M}$ and $f \subseteq \lambda \times x \in V_{\hat{i}(\beta)}$; thus $f \in M$. Conversely, assume the condition holds, and let $\lambda > \kappa, \beta = \lambda + \omega$, and $i : V_\beta \to M$ satisfy $\theta_{sc}(i, \kappa, \beta, M)$. Define U over $P_\kappa\lambda$ by putting $X \in U$ iff $i''\lambda \in i(X)$. Note that the definition makes sense since $i''\lambda \subseteq i(\lambda) \in M, i(\kappa) > \lambda$, and $PP_\kappa\lambda \subseteq V_\beta$; thus, U witnesses λ-supercompactness of κ.

For part (2), we show that $\mathcal{E}^{sc}_\kappa$ is a regular class. Given $\gamma > \kappa$, let $\hat{i} : V \to \hat{M}$ be a supercompact embedding such that $V_\gamma \subset \hat{M}$ and $\hat{i}(\kappa) > \gamma$. Let $\lambda = \gamma, \beta = \lambda + \omega$, and $i = \hat{i} \restriction V_\beta : V_\beta \to V^{\hat{M}}_{\hat{i}(\beta)} = M$. Then $(i, M) \in \mathcal{E}^{sc}_\kappa$, $i(\kappa) > \gamma$, and and $V_\gamma \subset M$.

Finally, to see that $\mathcal{E}^{sc}_\kappa$ is correlated, define $\mathbf{F}$ on Dom $\mathcal{E}^{sc}_\kappa$ by letting $\mathbf{F}(\beta) =$ the largest limit ordinal $< \beta$; since each $\beta \in$ Dom $\mathcal{E}^{sc}_\kappa$ is of the form $\lambda + \omega$ for some λ, the definition makes sense; it is easy to check that $\mathbf{F}$ has the required properties.

$\underline{\mathcal{E}^{str}_\kappa}$: For part (1), suppose κ is strong and let $\gamma > \kappa, \lambda > \gamma$, and $\beta = \lambda + \omega$. Let $\hat{i} : V \to \hat{M}$ be an embedding with $i(\kappa) > \lambda$ and $V_\lambda \subseteq M$. If we let $i = \hat{i} \restriction V_\beta$ and $M = V^{\hat{M}}_{\hat{i}(\beta)}$, then i, κ, β, M satisfy $\theta_{str}(i, \kappa, \beta, M)$. Conversely, assuming the condition and given $\lambda > \kappa$, we show that κ is λ-strong. Let $\gamma \geq \lambda + \omega + \omega$. Let $\beta \geq \gamma$ and let i be such that $\theta(i, \kappa, \beta)$; in particular, for some limit ordinal $\lambda' > \lambda$, $\beta = \lambda' + \omega$, $i : V_\beta \to M$ has critical point κ, $V_{\lambda'} \subset M$, and $i(\kappa) > \lambda'$. Let E be the extender with support $V_{\lambda'}$ derived from i. Note that for each $a \in {}^{<\omega}[V_{\lambda'}]$, $i^{-1} \restriction i(a) \in M$, and since λ' is a limit, $P({}^aV_\kappa) \in V_{\lambda'}$. Thus, for each a, $E_a \in V_{\lambda'}$, and E is well-defined. But now $i_E(\kappa) \geq \lambda' > \lambda$ and so i_E is the required λ-strong embedding. Finally, reason as in the supercompact case to show that $\mathcal{E}^{str}_\kappa$ is regular and correlated.

$\underline{\mathcal{E}^{ext}_\kappa}$: Normality is obvious; note that in this case, β is an extraneous variable in θ_{ext}. To show $\mathcal{E}^{ext}_\kappa$ is correlated, let $\mathbf{F}$ be the identity.

$\underline{\mathcal{E}^{sh}_\kappa}$: For part (1), suppose κ is superhuge. Let $\gamma > \kappa$ and let $\hat{i} : V \to \hat{M}$ be a huge embedding with critical point κ and $\hat{i}(\kappa) > \gamma$. Let $\lambda = \hat{i}(\kappa), \beta = \lambda + \omega, i = \hat{i} \restriction V_\beta$ and $M = V^{\hat{M}}_{\hat{i}(\beta)}$. Again, if $f : \lambda \to M$ and $x \in M$ with range$(f) \subseteq x$, then $f \in \hat{M}$ by $\hat{M}$'s closure property, and as $f \subseteq \lambda \times x$, $f \in V_{i(\beta)}$; thus $f \in M$. Conversely, assume the condition holds and let $\gamma > \kappa$. Let β and $i : V_\beta \to M$ satisfy $\theta_{sh}(i, \kappa, \beta, M)$ with witness $\lambda > \gamma$ (so that $\beta = \lambda + \omega$). Define an ultrafilter U over $P(\lambda)$ by $X \in U$ iff $i''\lambda \in i(X)$. The definition makes sense since $i''\lambda \in M$ and $PP(\lambda) \subseteq V_\beta$. U is a huge ultrafilter over κ since $\{X \subseteq P(\lambda) : \text{ ot}(X) = \kappa\} \in U$. Since λ is an arbitrarily large target, κ is superhuge. Finally, one proves that $\mathcal{E}^{sh}_\kappa$ is regular and correlated as in the supercompact case.

$\underline{\mathcal{E}^{sah}_\kappa}$: The forward direction for part (1) of the definition is like the proof for superhuge cardinals. For the other direction, let $\gamma > \kappa$, and let $\beta, i : V_\beta \to M$ satisfy $\theta(i, \kappa, \beta, M)$ with witness $\lambda > \gamma$. For each $\eta, \kappa \leq \eta < \lambda$, let $U_\eta = \{X \subseteq P_\kappa\eta : i''\eta \in i(X)\}$. Now $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ is coherent and satisfies $\mathcal{B}(\kappa, \lambda)$ (see Section 2). Thus, κ is almost huge with target λ; since λ was chosen arbitrarily large, we conclude that κ is super-almost-huge. Finally, prove that $\mathcal{E}^{sah}_\kappa$ is regular and correlated as in the supercompact case. ∎

As an application of these techniques, one can verify that our new notion of $\mathcal{E}^\theta_\kappa$-Laver sequences coincides with Laver's original notion in the case of a supercompact:

4.31 Proposition. *Suppose κ is an infinite cardinal and $g : \kappa \to V_\kappa$ is a function. Then g is a Laver sequence at κ if and only if g is $\mathcal{E}_\kappa^{sc}$-Laver at κ.* ■

We can now show, as promised earlier, that the notion of a special* Laver sequence (for supercompact cardinals) coincides with the notion of a special $\mathcal{E}_\kappa^{sc}$-Laver sequence:

4.32 Proposition. *Suppose $g : \kappa \to V_\kappa$. Then g is a special* Laver sequence if and only if g is a special $\mathcal{E}_\kappa^{sc}$-Laver sequence.*

Proof. Suppose g is special* with witnesses λ, U, D. Let $\beta = \lambda + \omega$ and let $i : V_\beta \to M$ be $i_U \upharpoonright V_\beta$. Properties (1) - (3) of Definition 4.25 are easily verified.

Conversely, assume g is a special $\mathcal{E}_\kappa^{sc}$-Laver sequence with witnesses $\lambda, \beta, i : V_\beta \to M$, and D. Since M is λ-closed, $i''\lambda \in M$. Thus we can define U over $P_\kappa\lambda$ from i in the usual way. By Lemma 2.26(II), $i_U(g)(\kappa) = i(g)(\kappa)$; also, $rank(i_U(g)(\kappa)) < \lambda$. Finally, by the usual arguments (see Theorem 4.4(3)), one shows that D is the normal ultrafilter derived from i_U. ■

To conclude this section, we observe that the five large cardinal notions treated in Theorem 4.30 are weakly compatible with the WA-embedding j. As the referee points out, because of the way we defined the classes $\mathcal{E}_\kappa^\theta$, for $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{ext}\}$, they happen to contain $j \upharpoonright V_\beta$ whenever $\beta \in \text{Dom } \mathcal{E}_\kappa^\theta$; thus, it is easy to show that these classes are compatible (not just weakly compatible) with j: Given $\lambda, \kappa < \lambda < j(\kappa)$, we can find β such that $\lambda < \beta < j(\kappa)$ for which $\beta \in \text{Dom } \mathcal{E}_\kappa^\theta$. Then $j \upharpoonright V_\beta \in \mathcal{E}_\kappa^\theta$ witnesses compatiblity.

However, neither $\mathcal{E}_\kappa^{sah}$ nor $\mathcal{E}_\kappa^{sh}$ has this property; indeed, the only restrictions of j that are in these classes are of the form $j^n \upharpoonright V_{\lambda+\omega}$ where $\lambda = j^n(\kappa)$, and such restrictions cannot be used to verify compatibility (though they can be used for weak compatibility). At this time we do not know whether $\mathcal{E}_\kappa^{sh}$ is compatible with j, but we can show that it is weakly compatible. We reserve the somewhat complicated proof that $\mathcal{E}_\kappa^{sah}$ is compatible with j for the next section (Theorem 5.20); for now, since $\mathcal{E}_\kappa^{sh} \subset \mathcal{E}_\kappa^{sah}$, $\mathcal{E}_\kappa^{sah}$ inherits weak compatibility. We turn to the proof of weak compatibility of $\mathcal{E}_\kappa^{sh}$:

Given $\kappa < \lambda < j(\kappa)$, $g : \kappa \to V_\kappa$ with $rank(j(g)(\kappa)) < \lambda$, $x = j(g)(\kappa)$, and $r : \kappa \to P(\kappa)$, let D be the normal ultrafilter over κ derived from j and let $\beta = j(\kappa) + \omega$. Clearly, $j \upharpoonright V_\beta \to V_{j(\beta)} \in \mathcal{E}_\kappa^{sh}$. Thus,

$$V_{j^3(\kappa)} \models \exists i \, \exists M \, \big[i : V_\beta \to M \in \mathcal{E}_\kappa^{sh} \,\wedge\, i(\kappa) > \lambda \,\wedge\, i(g)(\kappa) = x \,\wedge\, \forall \delta < \kappa \, (r(\delta) \in D) \,\wedge\, \text{“}D \text{ is the normal measure on } \kappa \text{ derived from } i\text{”} \big]$$

Note that the parameters $\beta, D, \lambda, g, \kappa, x, r$ of the displayed formula all lie in $V_{j^2(\kappa)}$. Thus, since $V_{j^2(\kappa)} \prec V_{j^3(\kappa)}$ the same formula holds in $V_{j^2(\kappa)}$. It follows that

$$V_{j^2(\kappa)} \models \exists i \, \exists M \, \exists \beta \, \exists D \, \big[i : V_\beta \to M \in \mathcal{E}_\kappa^{sh} \,\wedge\, i(\kappa) > \lambda \,\wedge\, i(g)(\kappa) = x \,\wedge\, \forall \delta < \kappa \, (r(\delta) \in D) \,\wedge\, \text{“}D \text{ is the normal measure on } \kappa \text{ derived from } i\text{”} \big]$$

Since the parameters λ, g, κ, x, r of the displayed formula all lie in $V_{j(\kappa)}$, the same formula holds in $V_{j(\kappa)}$; let $i : V_\beta \to M \in \left(\mathcal{E}_\kappa^{sh}\right)^{V_{j(\kappa)}}$ and D_i be witnesses. It is easy to verify that $\left(\mathcal{E}_\kappa^{sh}\right)^{V_{j(\kappa)}} = \mathcal{E}_\kappa^{sh} \cap V_{j(\kappa)}$. Properties (1) - (3) of weak compatibility follow immediately.

The next theorem sums up the work in the previous paragraphs:

4.33 Theorem. *Assume* WA*, and let j be the* WA*-embedding with critical point κ.*

(1) *If $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{ext}\}$, then for each $\beta \in \mathrm{Dom}\, \mathcal{E}^\theta_\kappa$, $j \upharpoonright V_\beta \in \mathcal{E}^\theta_\kappa$. Thus, for such θ, $\mathcal{E}^\theta_\kappa$ is compatible with j.*

(2) *Both $\mathcal{E}^{sah}_\kappa$ and $\mathcal{E}^{sh}_\kappa$ are weakly compatible with j.* ∎

4.34 Open Question. Assuming WA, is $\mathcal{E}^{sh}_\kappa$ compatible with the WA-embedding?

4.35 Corollary. *Assume* WA *and let j be the* WA*-embedding with critical point κ. For $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{ext}, \theta_{sah}, \theta_{sh}\}$, the parameter t in the construction $\mathbf{CC}(t, \mathcal{E}^\theta_\kappa)$ can be chosen so that the constructed function f is a special $\mathcal{E}^\theta_\kappa$-Laver sequence.*

Proof. For $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{ext}\}$, this follows from Theorem 4.33(1) and Corollary 4.27(A). For $\theta \in \{\theta_{sah}, \theta_{sh}\}$, the result follows from Theorem 4.33(2), Corollary 4.27(B), and the fact that $\mathcal{E}^\theta_\kappa$ is upward λ-closed for every cardinal $\lambda < j(\kappa)$. ∎

Though we will not need to do so in this paper, it is sometimes useful to use a different suitable formula in defining a regular class that corresponds to a particular large cardinal, especially when the large cardinal is definable in terms of ultrafilters or extenders. Using the ultrafilter/extender definitions for supercompact, strong, super-almost-huge, and huge cardinals, one can obtain suitable formulas $\theta'_{sc}, \theta'_{str}, \theta'_{sah}$, and θ'_{sh} so that normality of these large cardinal properties is witnessed by these formulas, and the resulting regular classes $\mathcal{E}^{\theta'}_\kappa$ are (at least) weakly compatible with the WA-embedding j; moreover, none of these classes contains a restriction of j.

We give an example for the supercompact case. Define θ'_{sc} by:

$\theta'_{sc}(i, \kappa, \beta, M)$: $\exists \lambda > \kappa\, \exists U\, \exists \gamma\, \exists j\, \exists N\, [\beta = \lambda + \omega \wedge U$ is a normal ultrafilter over $P_\kappa \lambda \wedge \gamma$ is a regular cardinal $> 2^{\lambda^{<\kappa}} \wedge N$ is the transitive collapse of the ultrapower $V_\gamma^{P_\kappa \lambda}/U \wedge j : V_\gamma \to N$ is the canonical embedding $\wedge\, i = j \upharpoonright V_\beta \wedge M = V^N_{j(\beta)}]$.

Now if κ is supercompact, $\mathcal{E}^{sc'}_\kappa$ is a regular subclass of $\mathcal{E}^{sc}_\kappa$, and if WA holds, $\mathcal{E}^{sc'}_\kappa$ is compatible with the WA-embedding j: Given $\kappa < \lambda < j(\kappa)$, let $\lambda' = |V_\lambda|$ and $\beta = \lambda' + \omega$. Let U be the normal ultrafilter over $P_\kappa \lambda'$ derived from j. Let $\hat{i} : V \to \hat{M}$ be the canonical embedding, and let $\hat{k} : M \to V$ be the usual embedding for which $\hat{k} \circ \hat{i} = j$. As usual, $\hat{i}(\kappa) > \lambda$, $V_\lambda \subseteq \hat{M}$, and $\hat{k} \upharpoonright V_\lambda = \mathrm{id}_{V_\lambda}$. Thus the required maps are given by $i = \hat{i} \upharpoonright V_\beta : V_\beta \to M$ and $k = \hat{k} \upharpoonright M : M \to V_{j(\beta)}$, where $M = V^{\hat{M}}_{\hat{i}(\beta)}$.

§5. Laver Sequences Under Weaker Hypotheses

In this section, we wish to obtain results as in the last section concerning the existence of $\mathcal{E}^\theta_\kappa$-Laver sequences for various θ, but with weaker hypotheses than WA. It is reasonable to try replacing the WA-embedding with some weaker kind of embedding $j : V \to N$. When we do this, we are immediately faced with several hurdles.

Hurdle #1. We would like to guarantee that in the canonical construction of the function f, for any $\alpha < \kappa$, if $f \upharpoonright \alpha$ fails to be $\mathcal{E}^\theta_\alpha$-Laver, then there is a witness $x \in V_\kappa$. When we assumed WA, this was easily accomplished by using the fact that $V_\kappa \prec V$.

Hurdle #2. Let D be the normal ultrafilter over κ derived from j. We wish to guarantee that $\{\alpha < \kappa : f \upharpoonright \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver}\} \in D$; in other words, we need to obtain a contradiction from the assumption that, in N, $j(f)(\kappa)$ is a witness to the failure of f to be $\mathcal{E}^\theta_\kappa$-Laver. In the case of WA, this was done by showing that there was an $i \in \mathcal{E}^\theta_\kappa$ weakly compatible with j and concluding that $i(f)(\kappa) = j(f)(\kappa)$.

Hurdle #3. If we can show that $\{\alpha < \kappa : f \upharpoonright \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver}\} \in D$, we will be able to conclude only that f is $\mathcal{E}^\theta_\kappa$-Laver in N, and since Laverness is not absolute, some other technique will be needed to conclude that f is really $\mathcal{E}^\theta_\kappa$-Laver. Of course, none of this is a problem under WA since $N = V$.

We attempt to surmount these hurdles in a straightforward way. We will approach *Hurdle* #1 by trying to prove directly that witnesses to Laver failures can always be found in V_κ if we start with a strong enough embedding $j : V \to N$. We will attack *Hurdle* #2 by trying to find in N, directly, an $i \in \mathcal{E}^\theta_\kappa$ that is weakly compatible with $j : V \to N$. To handle *Hurdle* #3, we try a less direct approach: We try to show that, given a superstrong embedding $j : V \to N$, if f is $\mathcal{E}^\theta_\kappa$-Laver in N, then this fact holds in $V_{j(\kappa)} = V^N_{j(\kappa)}$. Thus, if we assume that f fails to be $\mathcal{E}^\theta_\kappa$-Laver with witness x, and we are able to pick $j : V \to N$ with target $> rank(x)$, then we can use this reflection property to obtain a contradiction inside $V_{j(\kappa)}$.

Theorem 5.13 shows that if we surmount the three hurdles as we have suggested, then f is indeed $\mathcal{E}^\theta_\kappa$-Laver. Theorem 5.38 shows that each of these hurdles can be overcome for each of the five large cardinal classes we have been considering, all under reasonable hypotheses. Theorem 5.37 provides sufficient abstract conditions on a class $\mathcal{E}^\theta$ for these hurdles to be overcome.

Section 6 will show that it isn't necessary to handle all the hurdles mentioned here in order to obtain Laver sequences for each of the five large cardinal classes; however, to arrive at this result, we will need to modify our construction somewhat.

To conclude this preliminary discussion, we remark that our commitment to handle Hurdle #2 in the way we have described — by ensuring that

$$\{\alpha < \kappa : f \upharpoonright \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver}\} \in D, \tag{5.1}$$

where D is derived from the ambient embedding — accomplishes two things: On the one hand, it ensures that $\mathcal{E}^\theta_\kappa$-Laver sequence f that is obtained must agree with the parameter $t : \kappa \to V_\kappa$ on a normal measure 1 set (and in most cases, guarantees that f is special). It also limits our ability to obtain optimal hypotheses under which Laver sequences for various classes $\mathcal{E}^\theta_\kappa$ can be proven to exist. This latter remark follows because (5.1) implies that the $\alpha < \kappa$ for which $\mathcal{E}^\theta_\alpha$ is regular have normal measure 1, a consequence that is much stronger than simply the fact that $\mathcal{E}^\theta_\kappa$ is regular. The contrast is more apparent in the case in which $A(x)$ is a normal large cardinal property with suitable formula θ; in that case, by Proposition 4.29, the statement (5.1) implies that the $\alpha < \kappa$ for which $A(\alpha)$ holds forms a normal measure 1 set. See the discussion at the end of Section 6.

We turn to some definitions en route to a precise formulation of the three conditions mentioned above. We begin with observations about the construction of a class $\mathcal{E}_\kappa^\theta$ inside a rank V_ρ. If $\rho > \kappa$ is a limit ordinal, it is easy to verify that for $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{ext}, \theta_{sah}, \theta_{sh}\}$, $\theta^{V_\rho}(i, \kappa, \beta, M) \iff \theta(i, \kappa, \beta, M)$, and hence $\left(\mathcal{E}_\kappa^\theta\right)^{V_\rho} = \mathcal{E}_\kappa^\theta \cap V_\rho$. (Note that for this observation to hold true, it is not enough for members of the classes $\mathcal{E}_\kappa^\theta$ to be simply elementary embeddings; their codomains must be included too (see the remarks following Definition 4.12).) For general θ, however, there is no guarantee of absoluteness.

5.1 Definition. *(Adequate Absoluteness)* Suppose θ is a suitable formula. We will call θ *adequately absolute* if for all beth fixed points ρ and all $i, \kappa, \beta, M \in V_\rho$, we have

$$\theta^{V_\rho}(i, \kappa, \beta, M) \iff \theta(i, \kappa, \beta, M).$$

For each θ, the sentence that asserts that θ is adequately absolute can be expressed formally as follows:

$$\mathfrak{a}_\theta : \forall \rho \in \mathbf{BF}\, \forall i, \kappa, \beta, M \in V_\rho \left(\theta(i, \kappa, \beta, M) \iff \theta^{V_\rho}(i, \kappa, \beta, M)\right),$$

where **BF** is the class of all beth fixed points.

The next lemma describes two useful consequences of adequate absoluteness:

5.2 Lemma. *Suppose θ is adequately absolute.*

(1) *If $j : V \to N$ is an elementary embedding, then, in N, θ is adequately absolute.*
(2) *θ is $\Sigma_2^{\mathrm{ZFC}+\mathfrak{a}_\theta}$.*

Proof of (1). Because j is elementary, V and N are elementarily equivalent. ∎

Proof of (2). It suffices to show—in the theory ZFC $+\, \mathfrak{a}_\theta$—that θ is a local property. Let σ be a beth fixed point sentence (see the beginning of Section 2), and let $\psi(x, y, z, w) \equiv \theta(x, y, z, w) \wedge \sigma$. We prove $\theta(x, y, z, w) \iff \exists \delta\, V_\delta \models \psi(x, y, z, w)$.

For one direction, if $\theta(i, \kappa, \beta, M)$ holds and δ is a beth fixed point larger than the ranks of θ's parameters, then $V_\delta \models \sigma$, and by adequate absoluteness, $V_\delta \models \theta[i, \kappa, \beta, M]$. For the other direction, given a δ such that $V_\delta \models \psi[i, \kappa, \beta, M]$, δ must be a beth fixed point and so, by adequate absoluteness again, $\theta(i, \kappa, \beta, M)$ must be true (in V). ∎

Next we consider the notion of Laver-closure, which will prove useful in establishing absoluteness properties in some cases. We begin with two preliminary definitions. Suppose $\mathcal{E}_\kappa^\theta$ is regular and $\lambda < \rho$ are ordinals. We will say that $\mathcal{E}_\kappa^\theta$ *has a representative β in (λ, ρ)* if $\lambda < \beta < \rho$ and $\beta \in \mathrm{Dom}\ \mathcal{E}_\kappa^\theta$. Also, for any $i \in \mathcal{E}_\kappa^\theta$, any function $g : \kappa \to V_\kappa$, any ordinal $\lambda > \kappa$ and any $x \in V_\lambda$, we will say that *i has Laver-like values with respect to κ, λ, x, g* if $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$.

5.3 Definition. (*Laver-closed Classes*) Suppose $\mathbf{C}$ is a class of cardinals. Suppose θ is a suitable formula. The class $\mathcal{E}_\kappa^\theta$ will be called *Laver-closed over* $\mathbf{C}$ if for all $g : \kappa \to V_\kappa$ and all λ, ρ, x where λ is a limit ordinal, $\rho \in \mathbf{C}, x \in V_\lambda$, and $\lambda < \rho$, if $\mathcal{E}_\kappa^\theta$ has a representative in (λ, ρ), then $\mathcal{E}_\kappa^\theta$ has a representative β_0 in (λ,ρ) such that whenever $\beta_0 \leq \beta < \rho$ and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ has Laver-like values with respect to κ, λ, x, g, then there is $j : V_\beta \to N \in \mathcal{E}^\theta_\kappa$ such that j has Laver-like values with respect to κ, λ, x, g and $(j,N) \in V_\rho$. The ordinal β_0 will be called a *witness* to Laver-closure (over **C**) relative to λ, ρ. If $\mathcal{E}^\theta_\kappa$ is Laver-closed over **C**, we define a partial function ℓc on $\{(\lambda,\rho) : \lambda < \rho, \lambda \text{ is a limit ordinal, and } \rho \in \mathbf{C}\}$ by

$$\ell c(\lambda,\rho) = \begin{cases} \beta_0 & \text{if there is a witness to Laver-closure relative to } \lambda, \rho \\ & \text{and } \beta_0 \text{ is the least such} \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

The ordinal $\ell c(\lambda,\rho)$ (if defined) is called the *Laver-closure index for* $\mathcal{E}^\theta_\kappa$ *at* (λ,ρ) *over* **C.** $\mathcal{E}^\theta_\kappa$ will be called *simply Laver-closed* if, whenever $\ell c(\lambda,\rho)$ is defined, $\ell c(\lambda,\rho)$ is the least $\beta > \lambda$ lying in Dom $\mathcal{E}^\theta_\kappa$.

The classes **C** that will concern us in this paper in applications of Definition 5.3 will be the beth fixed points and the inaccessibles; if $\mathcal{E}^\theta_\kappa$ is Laver-closed over one of these classes, we will say that it is Laver-closed *at beth fixed points* or Laver-closed *at inaccessibles.* Note that the criterion given in the previous definition involves *all* functions $\kappa \to V_\kappa$, and not just those that happen to be Laver functions.

In the definition, we have required λ to be a limit ordinal in order to avoid certain technical inconveniences. However, by Proposition 4.15(2), restricting to limit ordinals does not restrict the applicability of Laver-closure.

As we shall see shortly, our main examples of Laver-closed classes will be simply Laver-closed, with the exception of $\mathcal{E}^{sc}_\kappa$; because this class does *not* have the convenient feature that for $\beta = \lambda + \omega \in$ Dom $\mathcal{E}^\theta_\kappa$, V_λ is a subset of the codomain of embeddings with domain V_β, we have allowed, in the definition, some flexibility in the possible values for the Laver-closure index.

One consequence of Laver-closure at beth fixed points that we will not need in this paper, but that is quite useful is the following: Suppose $\alpha \leq \kappa$, and $j : V \to N$ is a superstrong embedding with critical point κ. Suppose θ is adequately absolute and $\mathcal{E}^\theta_\alpha$ is Laver-closed at beth fixed points. Then Dom $\mathcal{E}^\theta_\alpha \cap j(\kappa) \subseteq \left(\text{Dom } \mathcal{E}^\theta_\alpha\right)^N \cap j(\kappa)$.

The next proposition puts together the last two definitions in a way that will be useful for a number of results in this section; the proof is easy and we omit it.

5.4 Proposition. *Suppose θ is a suitable formula that is adequately absolute. Suppose $\kappa < \lambda < \rho$ and ρ is a beth fixed point. Suppose $g : \kappa \to V_\kappa$ is a function, $x \in V_\lambda$, and $\beta < \rho$. Then* (1) $\Rightarrow$ (2), *where:*

(1) $\exists i : V_\beta \to M\, [(i,M) \in V_\rho \wedge V_\rho \models (i,M) \in \mathcal{E}^\theta_\kappa \wedge i(\kappa) > \lambda \wedge i(g)(\kappa) = x]$;

(2) $\exists i : V_\beta \to M\, [(i,M) \in \mathcal{E}^\theta_\kappa \wedge i(\kappa) > \lambda \wedge i(g)(\kappa) = x]$.

Moreover, if $\mathcal{E}^\theta_\kappa$ is Laver-closed over **C** $\subseteq \{\alpha : \alpha$ *is a beth fixed point*$\}$, $\rho \in$ **C**, λ *is a limit, and $\ell c(\lambda,\rho) \leq \beta$, then (2) implies (1) as well.* ∎

As the next proposition shows, most of the classes of embeddings we are considering are Laver-closed at beth fixed points:

5.5 Proposition. *For all $\theta \in \{\theta_{sc}, \theta_{str}, \theta_{sah}, \theta_{sh}\}$, $\mathcal{E}_\kappa^\theta$ is Laver-closed at beth fixed points; indeed, with the exception of $\mathcal{E}_\kappa^{sc}$, each of these classes is simply Laver-closed at beth fixed points.*

Proof. Let $\kappa < \lambda < \rho$ and assume that ρ is a beth fixed point. If $\theta = \theta_{sc}$, let $\beta_0 = |V_\lambda| + \omega$. For $\theta = \theta_{str}, \theta_{sah}$, and θ_{sh}, we let β_0 be least such that $\beta_0 > \lambda$ and $\beta_0 \in \text{Dom } \mathcal{E}_\kappa^\theta$. Let $g : \kappa \to V_\kappa$ and $x \in V_\lambda$ be given. Let β be such that $\beta_0 \le \beta < \rho$ and $\beta \in \text{Dom } \mathcal{E}_\kappa^\theta$, and assume that for some $i : V_\beta \to M$, with $(i, M) \in \mathcal{E}_\kappa^\theta$, i has Laver-like values with respect to κ, λ, x, g. For each case we obtain $j : V_\beta \to N$ with $(j, N) \in \mathcal{E}_\kappa^\theta \cap V_\rho$ for which $j(\kappa) > \lambda$ and $j(g)(\kappa) = x$.

For $\theta = \theta_{sc}$, let U be the normal ultrafilter over $P_\kappa |V_\lambda|$ derived from i and let $j = i_U \restriction V_\beta : V_\beta \to N$. Now $(j, N) \in \mathcal{E}_\kappa^{sc}$, $j(\kappa) > \lambda$, and by Lemma 2.26, $j(g)(\kappa) = x$. By Proposition 2.28(A,3), $(j, N) \in V_\rho$.

For $\theta = \theta_{str}$, let E be the extender with critical point κ and support V_λ. If $i_E(\kappa) > \lambda$, then let $j = i_E \restriction V_\beta : V_\beta \to N$. If $i_E(\kappa) = \lambda$, then let $j = \hat{i} \restriction V_\beta : V_\beta \to N$, where $\hat{i} = (i_E \cdot i_E) \circ i_E$ (see the remarks following Definition 2.16). In either case, $(j, N) \in \mathcal{E}_\kappa^{str}$ and $j(\kappa) > \lambda$. In the first case, we can use Lemma 2.26(II) to conclude that $j(g)(\kappa) = x$ and Proposition 2.28(B,3) to conclude $(j, N) \in V_\rho$. A bit more work is required to get these last two results for the second case. To see $j(g)(\kappa) = x$, we perform a computation:

$$\begin{aligned} j(g)(\kappa) &= [(i_E \cdot i_E)(i_E(g))](\kappa) \\ &= [(i_E \cdot i_E)(i_E(g))][(i_E \cdot i_E)(\kappa)] \\ &= (i_E \cdot i_E)[i_E(g)(\kappa)] \\ &= (i_E \cdot i_E)(x) \\ &= x. \end{aligned}$$

The last equality follows since $x \in V_\lambda$ and $\text{cp}(i_E \cdot i_E) = \lambda$ in this case.

To see that $(j, N) \in V_\rho$, we perform a computation in the codomain $\tilde{M}$ of i_E that is similar to the one in Proposition 2.28(B). Write $i_E : V \to \tilde{M}$ and $i_E \cdot i_E : \tilde{M} \to \tilde{M}_1$. Then $i_E \cdot i_E$ is the canonical embedding of $\tilde{M}$ into its ultrapower $\tilde{M}_1$ formed by the $\tilde{M}$-extender $\tilde{E} = i_E(E) \in \tilde{M}$. Reasoning as in Proposition 2.28(B), we can show in $\tilde{M}$ that

$$i_E \cdot i_E(i_E(\beta)) < \big(\big|\sum_{|V_{i_E(\lambda)}|} i_E(\beta)^{i_E(\kappa)}\big|\big)^+.$$

Evaluating the sizes of $V_{i_E(\lambda)}$, $i_E(\beta)$, $i_E(\kappa)$ in V (again reasoning as in Proposition 2.28) shows that each has (real) cardinality $< \rho$. It follows that

$$\begin{aligned} \max(rank(j \restriction V_\beta), rank(N)) &< [(i_E \cdot i_E) \circ i_E](\beta) + \omega \\ &< \rho, \end{aligned}$$

whence $(j, N) \in V_\rho$.

For $\theta = \theta_{sah}$, let $\hat{\lambda} = i(\kappa)$. Then $\lambda < \hat{\lambda} < \hat{\lambda} + \omega = \beta$. Obtain from i a coherent sequence $\langle U_\eta : \kappa \le \eta < \hat{\lambda} \rangle$ of normal ultrafilters satisfying $\mathcal{B}(\kappa, \hat{\lambda})$. Let $\hat{i}$ be the a.h. embedding with target

$\hat{\lambda}$ obtained from this sequence. Let $j = \hat{i} \upharpoonright V_\beta : V_\beta \to N$. Now $(j, N) \in \mathcal{E}_\kappa^{sah}$, $j(\kappa) = \hat{\lambda} > \lambda$, and $j(g)(\kappa) = x$, by Lemma 2.26(II). The fact that $(j, N) \in V_\rho$ follows from Proposition 2.28(C).

For $\theta = \theta_{sh}$, obtain $j : V_\beta \to N$, as in the supercompact case, from the normal ultrafilter (in this case over $P(\lambda)$) derived from i. Now, as in the other cases, $(j, N) \in \mathcal{E}_\kappa^{sh}$, and $j(\kappa) > \lambda$; by Lemma 2.26(II), $j(g)(\kappa) = x$; by Proposition 2.28(D,3), $(j, N) \in V_\rho$. ∎

Whether it is consistent for $\mathcal{E}_\kappa^{ext}$ to be Laver-closed at inaccessibles (assuming κ is extendible) is open.

5.6 Open Question. Is there a model[7] of ZFC + "κ is extendible" in which $\mathcal{E}_\kappa^{ext}$ is Laver-closed at inaccessibles?

Another abstract property of classes of embeddings that we will often make use of is *coherence:*

5.7 Definition. (*Coherence*) Suppose $\beta, \gamma \in \text{Dom } \mathcal{E}_\kappa^\theta$ and $\kappa < \beta < \gamma$. Then $\mathcal{E}_\kappa^\theta$ is *coherent at* (β, γ) if, for each $i : V_\gamma \to N \in \mathcal{E}_\kappa^\theta$, the embedding $i \upharpoonright V_\beta$ is also in $\mathcal{E}_\kappa^\theta$. $\mathcal{E}_\kappa^\theta$ is *coherent* if, for all $\beta, \gamma \in \text{Dom } \mathcal{E}_\kappa^\theta$ such that $\kappa < \beta < \gamma$, $\mathcal{E}_\kappa^\theta$ is coherent at (β, γ).

5.8 Proposition. *The classes $\mathcal{E}_\kappa^{sc}, \mathcal{E}_\kappa^{str}$ and $\mathcal{E}_\kappa^{ext}$ are coherent.*

Proof. For $\mathcal{E}_\kappa^{ext}$, this is obvious. We give the proof for $\mathcal{E}_\kappa^{sc}$; the proof for $\mathcal{E}_\kappa^{str}$ is similar. Given $i : V_\gamma \to N \in \mathcal{E}_\kappa^\theta$ and $\beta \in \text{Dom } \mathcal{E}_\kappa^{sc}$ with $\beta < \gamma$, we need to verify that $i \upharpoonright V_\beta \in \mathcal{E}_\kappa^{sc}$. Write $\beta = \lambda + \omega$. Note that $i \upharpoonright V_\beta : V_\beta \to V_{i(\beta)}^N = M$. It suffices to show that for each $x \in M$ and each $g : \lambda \to x$, we have $g \in M$. But since $i \in \mathcal{E}_\kappa^{sc}$, we have $g \in N$, and clearly $g \in V_{i(\beta)}$; the result follows. ∎

Note that neither $\mathcal{E}_\kappa^{sah}$ nor $\mathcal{E}_\kappa^{sh}$ is coherent. One of the useful features of coherent classes $\mathcal{E}_\kappa^\theta$ is that one can show a $g : \kappa \to V_\kappa$ is *not* $\mathcal{E}_\kappa^\theta$-Laver by exhibiting a witness x and a *single* $\beta > \max(\kappa, rank(x))$:

5.9 Proposition. *Suppose θ is a suitable formula and $\mathcal{E}_\kappa^\theta$ is a coherent class of embeddings. Suppose that $g : \kappa \to V_\kappa$ is a function and that there are x, λ such that for some β, one of the following holds:*

$$\lambda > \max(\kappa, rank(x)) \wedge \beta > \lambda \ \wedge \ \beta \in \text{Dom } \mathcal{E}_\kappa^\theta \wedge \\ \forall i : V_\beta \to M \in \mathcal{E}_\kappa^\theta \, [i(\kappa) > \lambda \Longrightarrow i(g)(\kappa) \neq x]; \tag{5.2}$$

$$\lambda > \max(\kappa, rank(x)) \ \wedge \ \beta > \lambda \ \wedge \ \forall \beta' \geq \beta \, (\beta' \notin \text{Dom } \mathcal{E}_\kappa^\theta). \tag{5.3}$$

Let $\lambda' \geq \beta$. Then x, λ' witness that g is not $\mathcal{E}_\kappa^\theta$-Laver at κ. Moreoever, if (5.2) holds for $\beta = \min\{\gamma : \gamma > \lambda \wedge \gamma \in \text{Dom } \mathcal{E}_\kappa^\theta\}$, or if (5.3) holds for $\beta = \lambda + 1$, then x, λ witness that g is not $\mathcal{E}_\kappa^\theta$-Laver at κ.

[7] In [9] it is shown that the answer is no.

Remark. When either (5.2) or (5.3) holds for x, λ, β, we will say that x, λ, β satisfy the *coherence criterion for Laver failure*. If the "moreover" clause holds, we will say that x, λ, β satisfy the *strong coherence criterion for Laver failure*.

Proof. Let $\lambda' \geq \beta$. Now $\lambda' > \max(\kappa, rank(x))$. If condition (5.3) holds, then the result follows vacuously. Assume condition (5.2). Let $\beta' > \lambda'$ with $\beta' \in \text{Dom } \mathcal{E}_\kappa^\theta$. Suppose $i : V_{\beta'} \to M \in \mathcal{E}_\kappa^\theta$ and $i(\kappa) > \lambda'$. By coherence, $i \upharpoonright V_\beta \in \mathcal{E}_\kappa^\theta$, and clearly $i(\kappa) > \lambda$. By (5.2), $i(g)(\kappa) \neq x$. Thus g is not $\mathcal{E}_\kappa^\theta$-Laver at κ, with witnesses x, λ'. Similar reasoning can be used to prove the "moreover" clause. ∎

The case in Proposition 5.9 in which (5.3) holds clearly does not require the hypothesis that $\mathcal{E}_\kappa^\theta$ is coherent; we have included this case here because our applications of the proposition will typically require consideration of both cases, (5.2) and (5.3).

We turn to a description of two concepts that are central to our proof of the existence of $\mathcal{E}_\kappa^\theta$-Laver sequences under weaker hypotheses.

5.10 Definition. (*Reflecting Laver Sequences*) Suppose θ is a suitable formula, $\mathcal{E}_\kappa^\theta$ is a class of embeddings, and $\rho > \kappa$. Then $\mathcal{E}_\kappa^\theta$ is *Laver reflecting in* V_ρ if, whenever $g : \kappa \to V_\kappa$ is $\mathcal{E}_\kappa^\theta$-Laver at κ, we have

$$V_\rho \models \text{"}g \text{ is } \mathcal{E}_\kappa^\theta\text{-Laver at } \kappa.\text{"}$$

In other words, $\mathcal{E}_\kappa^\theta$ is Laver reflecting in V_ρ if and only if for each $g : \kappa \to V_\kappa$, the statement "g is $\mathcal{E}_\kappa^\theta$-Laver at κ" is downward absolute for V_ρ. We have avoided this terminology because the conditions that we need in order to show that the canonically constructed f is $\mathcal{E}_\kappa^\theta$-Laver are close to, but different from, the absoluteness of $\mathcal{E}_\kappa^\theta$-Laverness of f.

5.11 Definition. (*Localized Laver Failures*) Suppose θ is a suitable formula, $\mathcal{E}_\kappa^\theta$ is a class of embeddings, and $\rho > \kappa$. Then *$\mathcal{E}_\kappa^\theta$-Laver failures are localized below ρ* if for each $g : \kappa \to V_\kappa$,

$$g \text{ is } \textit{not } \mathcal{E}_\kappa^\theta\text{-Laver at } \kappa \Longleftrightarrow \exists x \in V_\rho \, \exists \lambda < \rho \, \phi(g, x, \lambda).$$

Note that the only "localization" required by the above definition is with respect to the pair (x, λ); in particular, the assertion that "g is not $\mathcal{E}_\kappa^\theta$-Laver at κ" is downward absolute for V_ρ is not enough to show that $\mathcal{E}_\kappa^\theta$-Laver failures are localized below ρ. To see the problem, suppose "g is not $\mathcal{E}_\kappa^\theta$-Laver at κ" is downward absolute for V_ρ and g is not in fact $\mathcal{E}_\kappa^\theta$-Laver. While it is true that there are $x, \lambda \in V_\rho$ such that $\phi^{V_\rho}(g, x, \lambda)$, ϕ may not be absolute for V_ρ, even if θ is adequately absolute; indeed, there may be $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, not in V_ρ, for which $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$. The required ingredients for ensuring the equivalence of these two concepts are adequate absoluteness, Laver-closure, coherence, and unboundedness of Dom $\mathcal{E}_\kappa^\theta$ in ρ:

5.12 Proposition. *Suppose θ is adequately absolute and $\mathcal{E}_\kappa^\theta$ is coherent and Laver-closed at inaccessibles. Suppose $\rho > \kappa$ is inaccessible and* Dom $\mathcal{E}_\kappa^\theta$ *is cofinal in ρ. Then TFAE:*

(1) *$\mathcal{E}_\kappa^\theta$-failures are localized below ρ;*

(2) *for all $g : \kappa \to V_\kappa$, the statement "g is not $\mathcal{E}^\theta_\kappa$-Laver at κ" is downward absolute for V_ρ.*

Proof. (1) $\Rightarrow$ (2) follows from the fact that θ is adequately absolute. For (2) $\Rightarrow$ (1), assume that for all g, the statement "g is not $\mathcal{E}^\theta_\kappa$-Laver at κ" is downward absolute for V_ρ and that g is not $\mathcal{E}^\theta_\kappa$-Laver (in V). By downward absoluteness (and adequate absoluteness), there are $x, \lambda \in V_\rho$ with $\lambda > \max(\kappa, rank(x))$ such that for all β, i for which $\lambda < \beta < \rho$ and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa \cap V_\rho$, if $i(\kappa) > \lambda$ then $i(g)(\kappa) \neq x$. Without loss of generality, assume λ is a limit ordinal. Assume, by way of contradiction, that for some $j : V_\gamma \to N \in \mathcal{E}^\theta_\kappa$, we have $i(\kappa) > \lambda$ and $i(g)(\kappa) = x$. Let $\beta \in \mathrm{Dom}\ \mathcal{E}^\theta_\kappa$ be such that $\lambda < \ell c(\lambda, \rho) \leq \beta < \rho$, where $\ell c(\lambda, \rho)$ is the Laver-closure index for $\mathcal{E}^\theta_\kappa$ at (λ, ρ); we can find such a β because $\mathrm{Dom}\ \mathcal{E}^\theta_\kappa$ is cofinal in ρ. By coherence, $i = j \restriction V_\beta : V_\beta \to M \in \mathcal{E}^\theta_\kappa$. However, (i, M) may not be in V_ρ. Since $\mathcal{E}^\theta_\kappa$ is Laver-closed at inaccessibles, ρ is inaccessible, and $\beta \geq \ell c(\lambda, \rho)$, we can find $i' : V_\beta \to M' \in \mathcal{E}^\theta_\kappa$ such that $(i', M') \in V_\rho$, $i'(\kappa) > \lambda$, and $i'(g)(\kappa) = x$, and we have a contradiction. ∎

We are now ready to formulate the three conditions mentioned at the beginning of this section; Theorem 5.13 shows that these three are sufficient to prove that f is $\mathcal{E}^\theta_\kappa$-Laver.

5.13 Theorem. *Suppose $\mathcal{E}^\theta_\kappa$ is a regular class, θ is adequately absolute, and κ is globally superstrong. Let $f : \kappa \to V_\kappa$ be defined as in $CC(t, \mathcal{E}^\theta_\kappa)$. Assume*

(1) *for all $\alpha < \kappa$, $\mathcal{E}^\theta_\alpha$-Laver failures are localized below κ.*

Also, assume that, for each γ, there is a superstrong embedding $j : V \to N$ with critical point κ such that $j(\kappa) > \gamma$ and the following statements hold in N:

(2) $\forall \lambda < j(\kappa) \neg\phi(f, j(f)(\kappa), \lambda)$;

(3) *$\mathcal{E}^\theta_\kappa$ is Laver-reflecting in $V_{j(\kappa)}$.*

Then f is $\mathcal{E}^\theta_\kappa$-Laver at κ. Moreover, for each such superstrong j, if D is the normal ultrafilter over κ derived from j, then $\{\alpha : f \restriction \alpha$ is $\mathcal{E}^\theta_\alpha$-Laver at $\alpha\} \in D$.

Proof. To begin, note that (1) guarantees that f is well-defined. Suppose $j : V \to N$ is any superstrong embedding with critical point κ, satisfying properties (2) and (3) above, and let D be the normal ultrafilter over κ derived from j. Observe that, by the definition of f in Construction 4.21, one of the following sets must be in D:

$$S_1 = \{\alpha < \kappa : \exists \lambda < \kappa\, \phi(f \restriction \alpha, f(\alpha), \lambda)\};$$
$$S_2 = \{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\}.$$

Since f is well-defined (and since the non-cardinals below κ form a nonstationary set), S_1 represents the only way that $f \restriction \alpha$ could fail to be $\mathcal{E}^\theta_\alpha$-Laver at α for all α in a set in D. However, by (2), $S_1 \notin D$. It follows that $S_2 \in D$ (establishing the "moreover" clause). It follows that

$$N \models \text{"}f \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa.\text{"}$$

However, "f is $\mathcal{E}^\theta_\kappa$-Laver at κ" is not generally absolute for such N. We "simulate" absoluteness by making use of the fact that we can choose j so that it has arbitrarily large targets and still satisfies (2) and (3). Thus, suppose f is not $\mathcal{E}^\theta_\kappa$-Laver at κ, and let x, λ be such that $\phi(f, x, \lambda)$. Now let $j : V \to N$ be superstrong with critical point κ so that $j(\kappa) > \lambda$. As before,

$$N \models \text{"}f \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa.\text{"}$$

By (3),

$$V_{j(\kappa)} = V^N_{j(\kappa)} \models \text{"}f \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa.\text{"}$$

But now, since θ is adequately absolute and $j(\kappa)$ is a beth fixed point, we have, by Proposition 5.4, that there are β, i such that $\lambda < \beta < j(\kappa)$, $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa \cap V_{j(\kappa)}$, $i(\kappa) > \lambda$, and $i(f)(\kappa) = x$. But this contradicts the fact that x, λ have been chosen so that $\phi(f, x, \lambda)$. ■

In Theorem 5.13, the fact that $\{\alpha : f \upharpoonright \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\} \in D$ whenever D is derived from one of the superstrong embeddings satisfying (2) and (3) is significant: it will guarantee (as we shall show) that for "typical" classes, the parameter t in the construction of f can be chosen so that f is special.

We now describe conditions on classes $\mathcal{E}^\theta_\kappa$ and embeddings $j : V \to N$ for which properties (1)-(3) hold. We will use two strategies. One of these will use the fact that adequately absolute formulas are $\Sigma_2^{\mathrm{ZFC}+\mathfrak{a}_\theta}$; in the presence of a superstrong embedding, or an extendible cardinal κ, there will be enough reflection to establish our results concerning these properties; Theorem 5.22 is an example of this approach. The other strategy will consist of isolating abstract properties of classes $\mathcal{E}^\theta_\kappa$ that suffice to establish the required property. The former approach provides more general results and locks us into certain large cardinal hypotheses (e.g., existence of an extendible cardinal or superstrong embedding), whereas the abstract approach leaves open the possibility of obtaining the desired results for certain classes of embeddings under weaker hypotheses; Theorem 5.18 is an example of this phenomenon.

We begin with two lemmas that provide tools for reflection arguments.

5.14 Lemma. *Suppose θ is an adequately absolute suitable formula, $\alpha < \kappa$, κ is a strong cardinal, and $g : \alpha \to V_\alpha$ and $r : \alpha \to P(\alpha)$ are functions. Suppose x, λ are such that $x \in V_\lambda \in V_\kappa$. Suppose there is $i : V_\beta \to M \in \mathcal{E}^\theta_\alpha$ having Laver-like values with respect to α, λ, x, g, and such that, if D is the normal ultrafilter derived from i, then $r(\delta) \in D$ whenever $\delta < \alpha$. Then there is $\tilde{i} : V_{\tilde{\beta}} \to \tilde{M} \in \mathcal{E}^\theta_\alpha \cap V_\kappa$ having Laver-like values with respect to α, λ, x, g, and such that, if $\tilde{D}$ is the normal ultrafilter derived from $\tilde{i}$, then $r(\delta) \in \tilde{D}$ whenever $\delta < \alpha$.*

Remark. Note that the lemma does not assert (or imply) that $\mathcal{E}^\theta_\alpha$ is Laver-closed over any $\mathbf{C}$.

Proof. Assuming the hypotheses, the following holds (in V):

$$\begin{aligned}(5.4)\qquad &\exists i\, \exists M\, \exists \beta\, \exists P\, \exists D\, \big[i : V_\beta \to M \in \mathcal{E}^\theta_\alpha \wedge i(g)(\alpha) = x \wedge i(\alpha) > \lambda \wedge P = P(\alpha) \wedge \\ &\qquad \forall \delta < \alpha\, (r(\delta) \in D) \wedge \forall X (X \in D \Longleftrightarrow X \in P \wedge \alpha \in i(X))\big].\end{aligned}$$

By Lemma 5.2(2) and the fact that the formula $P = P(\alpha)$ and the last '$\forall$'-clause in (5.4) are Π_1^{ZFC}, it follows that (5.4) is $\Sigma_2^{\mathrm{ZFC}+\mathfrak{a}_\theta}$. Since κ is strong, and the parameters α, g, λ, r of (5.4) all lie in V_κ, we can conclude as in Theorem 2.18(1) that (5.4) holds in V_κ; since θ is adequately absolute, the result follows. ∎

5.15 Lemma. *If θ is adequately absolute and $g : \kappa \to V_\kappa$ is a function, then the statement "g is $\mathcal{E}_\kappa^\theta$-Laver at κ" is $\Pi_3^{\mathrm{ZFC}+\mathfrak{a}_\theta}$.*

Remark. We proved this result directly in the strong and supercompact cases in Corollary 2.33; for the super-almost-huge and superhuge cases, a direct proof could also be obtained using the observations in Remark 2.34.

Proof. Using Lemma 5.2(2) and the fact that $\theta = \theta(x, y, z, w)$ is adequately absolute, we observe that the formula $\psi_1(y, \lambda, u, g)$ defined below is $\Sigma_2^{\mathrm{ZFC}+\mathfrak{a}_\theta}$:

$$\psi_1(y, \lambda, u, g) \equiv \exists x \exists z \exists w \left[\theta(x, y, z, w) \wedge x(y) > \lambda \wedge x(g)(y) = u\right].$$

Thus the formula $\psi_2(y, g)$ below is $\Pi_3^{\mathrm{ZFC}+\mathfrak{a}_\theta}$:

$$\psi_2(y, g) \equiv \forall u \forall \lambda \, (\lambda > \max(y, rank(u)) \Longrightarrow \psi_1(y, \lambda, u, g)),$$

and it is easy to verify that $\psi_2(\kappa, g)$ is equivalent (in ZFC) to the statement "g is $\mathcal{E}_\kappa^\theta$-Laver at κ". ∎

The next theorem gives conditions for property (1) of Theorem 5.13 to hold. Theorem 5.18 arrives at the same result under weaker hypotheses, in special cases.

5.16 Theorem. *Suppose κ is extendible or globally superstrong and θ is adequately absolute. Then $\mathcal{E}_\alpha^\theta$-Laver failures are localized below κ for all $\alpha < \kappa$.*

Proof. We begin with the case in which we assume κ is extendible. Suppose $g : \alpha \to V_\alpha$ is not $\mathcal{E}_\alpha^\theta$-Laver. Since θ is adequately absolute, it follows from Lemma 5.15 that the statement

(5.5) $$\text{"}g \text{ is not } \mathcal{E}_\alpha^\theta\text{-Laver"}$$

is $\Sigma_3^{\mathrm{ZFC}+\mathfrak{a}_\theta}$. Since κ is extendible, we have by Theorem 2.18(2),

(5.6) $$V_\kappa \models \text{"}g \text{ is not } \mathcal{E}_\alpha^\theta\text{-Laver"}.$$

Let $(x, \lambda) \in V_\kappa$ be a witness for (5.6). But now Lemma 5.14 implies that (x, λ) is also a witness for (5.5), and we are done.

Finally, assume κ is globally superstrong. Suppose $g : \alpha \to V_\alpha$ is not $\mathcal{E}_\alpha^\theta$-Laver, with witnesses x, λ. Let $j : V \to N$ be a superstrong embedding with critical point κ and with $j(\kappa) > \lambda$.

Claim. $N \models \phi(g, x, \lambda)$.

Proof of Claim. Suppose that for some $\beta > \lambda$ and some $i : V_\beta \to M \in \left(\mathcal{E}_\alpha^\theta\right)^N$, i has Laver-like values with respect to α, λ, x, g. Since θ is adequately absolute in N, by Lemma 5.14 some $\tilde{i} : V_{\tilde{\beta}} \to \tilde{M} \in \left(\mathcal{E}_\alpha^\theta \cap V_{j(\kappa)}\right)^N = \left(\mathcal{E}_\alpha^\theta\right)^{V_{j(\kappa)}}$ also has Laver-like values with respect to α, λ, x, g. By adequate absolutness in V, $(\tilde{i}, \tilde{M}) \in \mathcal{E}_\alpha^\theta$, contradicting $\phi(g, x, \lambda)$. ∎

Continuation of Proof of Theorem. To complete the proof for this case, we begin by observing that, by the Claim,

$$N \models \exists x, \lambda \in V_{j(\kappa)}\, \phi(g, x, \lambda).$$

Pulling back with j (and noting that $j(\alpha) = \alpha$ and $j(g) = g$), we obtain

$$V \models \exists x, \lambda \in V_\kappa\, \phi(g, x, \lambda).$$

Thus, since g was arbitrary, we have shown that $\mathcal{E}_\alpha^\theta$-Laver failures are localized below κ. ∎

5.17 Corollary. *Suppose $\theta \in \{\theta_{ext}, \theta_{sah}, \theta_{sh}\}$. Suppose κ is globally superstrong or extendible. Then $\mathcal{E}_\alpha^\theta$-Laver failures are localized below κ for all $\alpha < \kappa$.* ∎

We can obtain the result for $\theta = \theta_{str}$ and θ_{sc} under weaker hypotheses:

5.18 Theorem. *Suppose $\theta \in \{\theta_{str}, \theta_{sc}\}$ and assume κ is a strong cardinal. Then $\mathcal{E}_\alpha^\theta$-Laver failures are localized below κ for all $\alpha < \kappa$.*

Proof. Suppose $\alpha < \kappa$ and $g : \alpha \to V_\alpha$ is not $\mathcal{E}_\alpha^\theta$-Laver at α. Then by Proposition 2.29 and the remark following, we can find $x \in V_\kappa$ and $\lambda < \kappa$ with $\lambda > \max(\alpha, rank(x))$ such that for all F, $i_F(g)(\alpha) \neq x$, where F is either a normal ultrafilter over $P_\alpha\lambda$ or an extender with critical point α and support V_λ. In particular, we use the remark following Proposition 2.29 to pick $\lambda > \max(\alpha, rank(x))$ in the supercompact case and to ensure λ is a successor ordinal in the strong case. Let $\beta = \lambda + \omega$.

Claim. *x, λ, β satisfy the strong coherence criterion for Laver failure (as in Proposition 5.9).*

Proof of Claim. Since α may have no special large cardinal properties, it is possible that $\beta \notin \text{Dom}\ \mathcal{E}_\alpha^\theta$. In that case, by coherence, (5.3) must hold—and by the particular definitions of $\mathcal{E}_\alpha^{sc}$ and $\mathcal{E}_\alpha^{str}$, it must hold when $\beta = \lambda + 1$ as well. On the other hand, if $\beta \in \text{Dom}\ \mathcal{E}_\alpha^\theta$, then $\beta = \min\{\gamma : \gamma > \lambda \wedge \beta \in \text{Dom}\ \mathcal{E}_\alpha^\theta\}$, and no $i : V_\beta \to M \in \mathcal{E}_\alpha^\theta$ has Laver-like values with respect to κ, λ, x, g. In other words, x, λ, β satisfy (5.2), as required. ∎

Continuation of Proof of Theorem. By Proposition 5.9, $x, \lambda \in V_\kappa$ witness that g is not $\mathcal{E}_\alpha^\theta$-Laver at α. Thus we have shown that for all $\alpha < \kappa$,

$$\forall g : \alpha \to V_\alpha\, [\text{“}g \text{ is not } \mathcal{E}_\alpha^\theta\text{-Laver at } \alpha\text{”} \implies \exists x, \lambda \in V_\kappa\, \phi(g, x, \lambda)]. \blacksquare$$

We were able to obtain the result in Theorem 5.18 assuming only a strong cardinal because of some of the special properties of the classes $\mathcal{E}_\kappa^{str}$ and $\mathcal{E}_\kappa^{sc}$. One such property is coherence. Another is the property shared by those $\mathcal{E}_\kappa^\theta$ that have some sort of ultrafilter definition. This property makes it possible (as in Proposition 2.29) to form a *set* of all possible counterexamples, and then pick a ν-strong embedding $j : V \to N$ whose codomain contains a V_ν large enough to contain this set. The fact that the possible counterexamples form a set, rather than a proper class, can be formulated as an abstract property of classes $\mathcal{E}_\kappa^\theta$—that of being *set-based*. We have been able to obtain the results of Theorem 5.16(1) using coherence and the notion of set bases, but only under the assumption that κ is globally superstrong (and Theorem 5.16(1) shows that under such a strong assumption, coherence and set bases aren't needed). Since the notion of set bases is of independent interest, we give the definition and some basic results in Section 7.

The next block of results address property (2) from Theorem 5.13. Theorem 5.19 shows that property (2) can be proven to hold in N if $\mathcal{E}_\kappa^\theta$ is weakly compatible with j (and is well enough behaved in other ways), just as in the case in which j is the WA-embedding (Theorem 4.22); Theorem 5.24 provides conditions under which the result holds for each of the specific classes we have been studying.

5.19 Theorem. *Suppose $j : V \to N$ is a superstrong embedding with critical point κ. Suppose θ is adequately absolute and $\mathcal{E}_\kappa^\theta$ is Laver-closed at beth fixed points, and weakly compatible with j. Then both of the following hold for any function $g : \kappa \to V_\kappa$:*
(A) $N \models \forall\lambda < j(\kappa)\, \neg\phi(g, j(g)(\kappa), \lambda)$
(B) $V_{j(\kappa)} \models \forall\lambda\, \neg\phi(g, j(g)(\kappa), \lambda)$.

Proof. First note that if θ is adequately absolute and j is a superstrong embedding, then (B) $\Rightarrow$ (A). We prove (B). Assume $\lambda < j(\kappa)$ satisfies $V_{j(\kappa)} \models \phi(g, j(g)(\kappa), \lambda)$, and assume λ is a limit. Let $x = j(g)(\kappa)$. Since $\mathcal{E}_\kappa^\theta$ is weakly compatible with j, there is $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ that has Laver-like values with respect to κ, λ, x, g, and $\lambda < \beta < j(\kappa)$. Since $\mathcal{E}_\kappa^\theta$ is Laver-closed at beth fixed points and $\mathcal{E}_\kappa^\theta$ has a representative in $(\lambda, j(\kappa))$, the Laver-closure index ℓc is defined at $\lambda, j(\kappa)$. To ensure that β is large enough, use weak compatibility again to obtain $i_1 : V_{\beta_1} \to M_1$ for which i_1 has Laver-like values with respect to κ, λ, x, g, and $\ell c(\lambda, j(\kappa)) \le \beta_1 < j(\kappa)$. We are not done, howewver, since (i_1, M_1) may not be in $V_{j(\kappa)}$. Because j is superstrong, $j(\kappa)$ is a beth fixed point; thus we can use Proposition 5.4 to obtain $i_1' : V_{\beta_1} \to M_1'$ for which

$$(i_1', M_1') \in \left(\mathcal{E}_\kappa^\theta\right)^{V_{j(\kappa)}}, i_1'(\kappa) > \lambda, \text{ and } i_1'(g)(\kappa) = x,$$

and this is a contradiction. ∎

In the proof of Theorem 5.19, we have spelled out in detail a typical application of weak compatibility and Laver-closure. Since the same reasoning works every time, we will abbreviate such arguments in later proofs.

In Theorem 5.24, we state conditions under which property (2) from Theorem 5.13 holds (both in N and in $V_{j(\kappa)}$) for the specific classes of embeddings we have been studying. As Theorem 5.19 shows, it suffices to show that each of these classes is weakly compatible with a sufficiently strong ambient embedding. We give some preliminary results about compatibility and weak compatibility of these classes.

5.20 Theorem. *Suppose $j : V \to N$ is elementary with critical point κ.*

(1) *$\mathcal{E}^{str}$ is compatible with j if j is a superstrong embedding.*
(2) *$\mathcal{E}^{sc}$ is compatible with j if j is an almost huge embedding.*
(3) *$\mathcal{E}^{sah}$ is compatible with j if j is a huge embedding.*

Proof of (1). Suppose $\kappa < \lambda < j(\kappa)$. We may assume that λ is a successor ordinal. Use the fact that j is a superstrong embedding to obtain the extender having critical point κ and support V_λ that is derived from j. From this extender we can define $i : V_\beta \to M \in \mathcal{E}_\kappa^{str}$ for which $\lambda < \beta < j(\kappa)$, and there is $k : M \to V_{j(\beta)}^N$ such that $k \circ i = j \restriction V_\beta$ and $k \restriction V_\lambda = \mathrm{id}_{V_\lambda}$. ∎

Proof of (2). Suppose $\kappa < \lambda < j(\kappa)$. Using the fact that j is an almost huge embedding , we may obtain the normal ultrafilter over $P_\kappa|V_\lambda|$ derived from j and then, as in part (1), define the required i and k. (Note that a superstrong embedding is not enough here: We may not be able to define the required normal ultrafilter over $P_\kappa|V_\lambda|$, since $j''|V_\lambda|$ may not be in N.) ∎

Proof of (3). We wish to apply Theorem 2.11; to do so, we first show that whenever $j : V \to N$ is a huge embedding with critical point κ and $\langle U_\eta : \kappa \le \eta < j(\kappa)\rangle$ is the derived coherent sequence of normal ultrafilters over $P_\kappa\eta$, $\kappa \le \eta < j(\kappa)$ (which necessarily satisfies $\mathcal{B}(\kappa, j(\kappa))$), the set

$$\{\alpha < j(\kappa) : \langle U_\eta : \kappa \le \eta < \alpha\rangle \text{ satisfies } \mathcal{B}(\kappa, \alpha)\}$$

is unbounded in $j(\kappa)$. (Note that this claim asserts more than just that the almost huge cardinals are unbounded below $j(\kappa)$.)

To prove the claim, we fix such a $j : V \to N$ and $\langle U_\eta : \kappa \le \eta < j(\kappa)\rangle$ and set $h = \langle U_\eta : \kappa \le \eta < j(\kappa)\rangle$. By hugeness of κ, $h \in N$, and by absoluteness,

$$N \models \text{“}h \text{ is coherent and satisfies } \mathcal{B}(\kappa, j(\kappa))\text{”}. \tag{5.7}$$

Assume there is $\alpha_0 < j(\kappa)$ such that $h \restriction \alpha$ does not satisfy $\mathcal{B}(\kappa, \alpha)$ whenever $\alpha_0 \le \alpha < j(\kappa)$. By absoluteness again,

$$N \models \forall\alpha\, (\alpha_0 \le \alpha < j(\kappa) \Longrightarrow \text{“}h \restriction \alpha \text{ does not satisfy } \mathcal{B}(\kappa, \alpha)\text{”}).$$

Applying jj and noting that $jj(\alpha_0) = \alpha_0$,

$$N_1 \models \forall\alpha\, (\alpha_0 \le \alpha < j^2(\kappa) \Longrightarrow \text{“}jj(h) \restriction \alpha \text{ does not satisfy } \mathcal{B}(\kappa, \alpha)\text{”}).$$

In particular, setting $\alpha = j(\kappa)$ and noting that $jj(h) \restriction j(\kappa) = h$,

$$N_1 \models \text{“}h \text{ does not satisfy } \mathcal{B}(\kappa, j(\kappa))\text{”}. \tag{5.8}$$

By absoluteness, (5.8) contradicts (5.7), and our claim is proved.

For the proof of (3), suppose $\kappa < \lambda_0 < j(\kappa)$. Let $\langle U_\eta : \kappa \leq \eta < j(\kappa)\rangle$ be the coherent sequence derived from j. By the claim just proved, there is λ such that $\lambda_0 < \lambda < j(\kappa)$ and $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ satisfies $\mathcal{B}(\kappa, \lambda)$. Let $i : V \to M$ denote the canonical embedding into the direct limit of the ultrapowers, obtained from $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ in the usual way. By Theorem 2.11, there is $e : M \to N$ such that $e \restriction V_{\lambda_0} = \mathrm{id}_{V_{\lambda_0}}$ and $j = e \circ i$. Now $i \restriction V_{\lambda+\omega} \in \mathcal{E}_\kappa^{sah}$ and a suitable restriction of e satisfy the requirements for compatibility with $j \restriction V_\beta$ up to V_{λ_0}. Since λ_0 was arbitrary (below $j(\kappa)$), it follows that $\mathcal{E}_\kappa^{sah}$ is compatible with j. ∎

An easy argument shows that $\mathcal{E}_\kappa^{ext}$ is compatible with a 2-huge embedding with critical point κ, but this is not an optimal bound for obtaining the results of Theorem 5.13 for this class. Whether this bound can be improved is open:

5.21 Open Question. Is $\mathcal{E}_\kappa^{ext}$ compatible with an almost huge embedding having critical point κ?

As we mentioned at the end of Section 4, we are unable to prove that $\mathcal{E}_\kappa^{sh}$ is compatible even with the strongest embeddings. We can, however, prove that $\mathcal{E}_\kappa^{ext}$ and $\mathcal{E}_\kappa^{sh}$ are weakly compatible with almost huge and 2-huge embeddings, respectively, as an application of a fairly general result about weak compatibility:

5.22 Theorem. *Suppose $j : V \to N$ is a superstrong embedding having a critical point κ that is a strong cardinal. Suppose θ is a suitable, adequately absolute formula. Assume also that for each $\lambda < j(\kappa)$, there is a $\beta \in \left(\mathrm{Dom}\ \mathcal{E}_\kappa^\theta\right)^N$ such that $\beta > \lambda$ and if $i = j \restriction V_\beta : V_\beta \to M$, where $M = V_{j(\beta)}^N$, then $(i, M) \in \left(\mathcal{E}_\kappa^\theta\right)^N$. Then $\mathcal{E}_\kappa^\theta$ is weakly compatible with j.*

Proof. Let $j : V \to N$ be as in the hypothesis. Let $\lambda, g : \kappa \to V_\kappa, r : \kappa \to P(\kappa)$ satisfy

(a) $\kappa < \lambda < j(\kappa)$;
(b) $rank(j(g)(\kappa)) < \lambda$;
(c) if D is the normal ultrafilter derived from j, then for all $\delta < \kappa$, $r(\delta) \in D$.

Let $x = j(g)(\kappa)$. Let $\beta \in \left(\mathrm{Dom}\ \mathcal{E}_\kappa^\theta\right)^N$ be such that $\beta > \lambda$ and, if $i = j \restriction V_\beta : V_\beta \to M \in \mathcal{E}_\kappa^\theta$, where $M = V_{j(\beta)}^N$, then $(i, M) \in \left(\mathcal{E}_\kappa^\theta\right)^N$. Clearly, in N, i has Laver-like values with respect to κ, λ, x, g and satisfies (c) (with j replaced by i). Since $N \models$ "$j(\kappa)$ is strong", we can apply Lemma 5.14 in N to obtain another embedding $\tilde{i} : V_{\tilde{\beta}} \to \tilde{M} \in \left(\mathcal{E}_\kappa^\theta \cap V_{j(\kappa)}\right)^N$ that has the same properties (in N). By adequate absoluteness and the fact that $V_{j(\kappa)}^N = V_{j(\kappa)}$,

$$\left(\mathcal{E}_\kappa^\theta \cap V_{j(\kappa)}\right)^N = \left(\left(\mathcal{E}_\kappa^\theta\right)^{V_{j(\kappa)}}\right)^N = \left(\mathcal{E}_\kappa^\theta\right)^{V_{j(\kappa)}} = \mathcal{E}_\kappa^\theta \cap V_{j(\kappa)}.$$

By a simple absoluteness argument, it follows that $\tilde{i}$ witnesses weak compatibility, as required. ∎

5.23 Corollary. *Suppose $j : V \to N$ is elementary with critical point κ.*

(1) *$\mathcal{E}_\kappa^{ext}$ is weakly compatible with j if κ is extendible and j is an almost huge embedding.*

(2) $\mathcal{E}_\kappa^{sh}$ *is weakly compatible with* j *if* κ *is superhuge and* j *is a 2-huge embedding.*

Proof of (1). Theorem 5.22. Let $j : V \to N$ be an almost huge embedding with critical point κ. Given $\lambda < j(\kappa)$, let $\beta = \lambda + 1$ and let $i = j \upharpoonright V_{\beta+1}$. By almost-hugeness of i and adequate absoluteness, $N \models i \in \mathcal{E}_\kappa^{ext}$. The result follows.∎

Proof of (2). We apply Theorem 5.22 again. Let $j : V \to N$ be a 2-huge embedding with critical point κ. Let $\beta = j(\kappa) + \omega$. By 2-hugeness of j and adequate absoluteness, $j \upharpoonright V_\beta \in \left(\mathcal{E}_\kappa^{sh}\right)^N$, as required. ∎

Unlike Theorem 5.20, the proof of Corollary 5.23 requires the classes $\mathcal{E}_\kappa^\theta$ to be regular — this could be improved by simply requiring κ to be a strong cardinal, but doing so still introduces a global requirement that is absent from the hypotheses of Theorem 5.20. In the case of $\mathcal{E}_\kappa^{ext}$, using a different argument (see Proposition 7.19), we can drop this requirement; we do not know how to do this for $\mathcal{E}_\kappa^{sh}$.

As a corollary, we can now state conditions under which property (2) of Theorem 5.13 holds in both N and in $V_{j(\kappa)}$ for our five regular classes. The usefulness of proving the result for $V_{j(\kappa)}$ will become clear in the next section where we consider a modified construction for $\mathcal{E}_\kappa^\theta$-Laver sequences.

5.24 Theorem. *Suppose* $j : V \to N$ *is an elementary embedding with critical point* κ *and* $g : \kappa \to V_\kappa$ *is a function. Then both of the following hold:*

(A) $N \models \forall \lambda < j(\kappa) \neg\phi(g, j(g)(\kappa), \lambda)$

(B) $V_{j(\kappa)} \models \forall \lambda \neg\phi(g, j(g)(\kappa), \lambda)$,

assuming any one of the following conditions:

(1) $\theta = \theta_{str}$ *and* $j : V \to N$ *is a superstrong embedding;*

(2) $\theta = \theta_{sc}$ *and* $j : V \to N$ *is an almost huge embedding;*

(3) $\theta = \theta_{ext}$, $j : V \to N$ *is an almost huge embedding, and* κ *is extendible;*

(4) $\theta = \theta_{sah}$ *and* $j : V \to N$ *is a huge embedding;*

(5) $\theta = \theta_{sh}$, $j : V \to N$ *is a 2-huge embedding, and* κ *is superhuge.*

Proof. Parts (1), (2), and (4) follow from Theorems 5.19 and 5.20. Parts (3) and (5) follow from Theorem 5.19 and Corollary 5.23. ∎

Next, we consider the Laver-reflecting property, corresponding to (3) of Theorem 5.13.

5.25 Theorem. *Suppose* θ *is adequately absolute and* $g : \kappa \to V_\kappa$ *is a function. Suppose* $j : V \to N$ *is an elementary embedding with critical point* κ. *Then, assuming either of the conditions below,* $N \models$ *"*$\mathcal{E}_\kappa^\theta$ *is Laver-reflecting in* $V_{j(\kappa)}$*":*

(A) κ *is extendible;*

(B) κ *is globally superstrong and* j *is a superstrong embedding.*

Proof. The result under condition (A) follows from the fact that Laver-ness is $\Pi_3^{\mathrm{ZFC}+\mathfrak{a}_\theta}$ and from Theorem 2.18(2) applied in N. To prove the result assuming (B), let j be superstrong with critical point κ. Suppose that, in N, $g:\kappa\to V_\kappa$ is $\mathcal{E}_\kappa^\theta$-Laver; we show this is true inside $V_{j(\kappa)}$. Let x,λ be such that $x\in V_\lambda\in V_{j(\kappa)}$. Since θ is adequately absolute, the formula

$$\psi(g,x,\lambda)\equiv \exists i\exists\beta\exists M\,(\theta(i,\kappa,\beta,M)\ \wedge\ i(\kappa)>\lambda\ \wedge\ i(g)(\kappa)=x)$$

is $\Sigma_2^{\mathrm{ZFC}+\mathfrak{a}_\theta}$, and holds in N. Since $j(\kappa)$ is strong in N (by Theorem 2.22(5)), we can apply Theorem 2.18(1) to conclude that

$$V_{j(\kappa)}\models\psi(g,x,\lambda),$$

as required. ∎

5.26 Corollary. *If $\theta\in\{\theta_{ext},\theta_{sah},\theta_{sh}\}$, $j:V\to N$ is an elementary embedding with critical point κ, and either*

(A) *κ is extendible, or*

(B) *κ is globally superstrong and j is a superstrong embedding,*

then, in N, $\mathcal{E}_\kappa^\theta$ is Laver-reflecting in $V_{j(\kappa)}$.

Proof. This follows immediately from Theorem 5.25. ∎

We note that Corollary 5.26 also applies to $\theta=\theta_{str}$ and $\theta=\theta_{sc}$, but the hypotheses are stronger than necessary (at least when $\theta=\theta_{str}$). We give other general conditions for a class $\mathcal{E}_\kappa^\theta$ to be Laver-reflecting that make use of some of the special properties of these two classes:

5.27 Theorem. *Suppose θ is a suitable, adequately absolute formula.*

(1) *Suppose that $\mathcal{E}_\kappa^\theta$ is coherent and Laver-closed at inaccessibles. Suppose $\rho>\kappa$ is inaccessible and that $\mathrm{Dom}\ \mathcal{E}_\kappa^\theta\cap\rho$ is cofinal in ρ. Then $\mathcal{E}_\kappa^\theta$ is Laver-reflecting in V_ρ.*

(2) *Suppose $j:V\to N$ is an elementary embedding with critical point κ. Suppose that each of the following is true in N: $\mathcal{E}_\kappa^\theta$ is coherent and Laver-closed at inaccessibles, and* $\mathrm{Dom}\ \mathcal{E}_\kappa^\theta\cap j(\kappa)$ *is cofinal in $j(\kappa)$. Then, in N, $\mathcal{E}_\kappa^\theta$ is Laver-reflecting in $V_{j(\kappa)}$.*

Proof. Part (2) is proved by applying part (1) in N, using the fact that adequate absoluteness of θ holds in N. For (1), suppose $g:\kappa\to V_\kappa$ is $\mathcal{E}_\kappa^\theta$-Laver at κ. By Proposition 4.15(3), $\mathcal{E}_\kappa^\theta$ must be regular. Let $x\in V_\rho$ and let λ be such that $\max(\kappa,rank(x))<\lambda<\rho$; without loss of generality, assume λ is a limit. Using Laver-ness, let $j:V_\gamma\to M\in\mathcal{E}_\kappa^\theta$ be such that $\lambda<\ell c(\lambda,\rho)<\gamma$, $j(\kappa)>\lambda$, and $j(g)(\kappa)=x$. If $\gamma<\rho$, we would be done because, by Laver-closure at inaccessibles, Proposition 5.4 would give us a $j':V_\gamma\to M'$ with $(j',M')\in\left(\mathcal{E}_\kappa^\theta\right)^{V_\rho}$, having Laver-like values with respect to κ,λ,x,g. Thus, assume instead that $\gamma\geq\rho$. Let $\beta\in\mathrm{Dom}\ \mathcal{E}_\kappa^\theta$ with $\lambda<\ell c(\lambda,\rho)<\beta<\rho$ (such a β can be found since $\mathrm{Dom}\ \mathcal{E}_\kappa^\theta\cap\rho$ is cofinal in ρ). By coherence, $i=j\restriction V_\beta\in\mathcal{E}_\kappa^\theta$; clearly,

$i(\kappa) > \lambda$ and $i(g)(\kappa) = x$. Now we can use Proposition 5.4 again to obtain another embedding $i' : V_\beta \to N$ with $(i', N) \in \left(\mathcal{E}_\kappa^\theta\right)^{V_\rho}$. Thus, $V_\rho \models$ "g is $\mathcal{E}_\kappa^\theta$-Laver at κ". ∎

5.28 Corollary. *Suppose $j : V \to N$ is an elementary embedding with critical point κ, and suppose θ is adequately absolute. Let D be the normal ultrafilter derived from j. Then, in N, $\mathcal{E}_\kappa^\theta$ is Laver-reflecting in $V_{j(\kappa)}$, assuming either of the following conditions:*

(A) *$\theta = \theta_{str}$ and for some $X \in D$ and all $\alpha \in X$, α is γ-strong for all $\gamma < \kappa$.*

(B) *$\theta = \theta_{sc}$ and for some $X \in D$ and all $\alpha \in X$, α is γ-supercompact for all $\gamma < \kappa$.*

Proof. To apply Theorem 5.27, the only verification that is not immediate is the fact that Dom $\mathcal{E}_\kappa^\theta$ is, in N, cofinal in $j(\kappa)$. However, notice that (A) implies that in N,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is strong”}$$

and (B) implies that in N,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is supercompact”},$$

as required. ∎

We can put Theorems 5.27(1) and 5.16(1) together to obtain a general condition under which Laver-ness is absolute for ranks V_ρ, for ρ inaccessible:

5.29 Proposition. *Suppose θ is adequately absolute and $\mathcal{E}_\kappa^\theta$ is a coherent class of embeddings that is Laver-closed at inaccessibles. Suppose $\rho > \kappa$ is a globally superstrong cardinal and Dom $\mathcal{E}_\kappa^\theta \cap \rho$ is cofinal in ρ. Then for all $g : \kappa \to V_\kappa$, the statement "g is $\mathcal{E}_\kappa^\theta$-Laver at κ" is absolute for V_ρ.*

Proof. Notice that the conditions in Theorem 5.27(1) are satisfied, so for all g, "g is $\mathcal{E}_\kappa^\theta$-Laver" is downward absolute for V_ρ. The conditions in Theorem 5.16 are also satisfied, so $\mathcal{E}_\kappa^\theta$-Laver failures are localized below ρ. The latter statement is equivalent to the assertion that for all g, "g is $\mathcal{E}_\kappa^\theta$-Laver" is upward absolute for V_ρ, by Proposition 5.12, and we are done. ∎

We pause here to make some observations about Question #5, raised in Section 4, concerning absoluteness of Laver sequences relative to inner models N. One strategy for proving absoluteness of an $\mathcal{E}_\kappa^\theta$-Laver sequence g in N, given a superstrong embedding $j : V \to N$, is to show that in both N and V, $\mathcal{E}_\kappa^\theta$ is Laver reflecting in $V_{j(\kappa)}$ and $\mathcal{E}_\kappa^\theta$-Laver failures are localized below $j(\kappa)$; to do this, we could use the hypotheses in Proposition 5.29. Then by a simple combination of downward and upward absoluteness arguments, we could obtain the absoluteness result for N. However, the large cardinal strength of the hypotheses from Proposition 5.29 is greater than necessary to obtain the result; indeed, even the single hypothesis that there is a superstrong cardinal above κ is far more than is needed. Thus, in addressing Question #5, we will content ourselves with a couple of minor observations.

5.30 Proposition. *Suppose θ is a suitable formula and $\mathcal{E}_\kappa^\theta$ is a class of embeddings. Suppose there is an elementary embedding $j : V \to N$ with $\mathrm{cp}(j) > \kappa$. Then for all $g : \kappa \to V_\kappa$, the statement "g is $\mathcal{E}_\kappa^\theta$-Laver at κ" is absolute for N.*

Proof. Use elementarity and the fact that $j(\kappa) = \kappa$ and $j(g) = g$. ∎

Proposition 5.30 gives us the following information:

5.31 Corollary. *Suppose κ is extendible (superhuge). Then for each $\alpha > \kappa$, there is an inner model N_α of* ZFC *such that for all $g : \kappa \to V_\kappa$, the statement "g is $\mathcal{E}_\kappa^{ext}$-Laver at κ" ("g is $\mathcal{E}_\kappa^{sh}$-Laver at κ") is absolute for N_α and $V_\alpha \subset N_\alpha$.*

Proof. If κ is extendible or superhuge, there is a proper class of measurables above κ. ∎

The measurable cardinals mentioned in the previous proof arise as the targets of the embeddings that define extendibility or superhugeness of κ; one can ask whether the targets of a super-almost-huge cardinal are also measurable. We observe here that they are not, in general; indeed, we will show that if κ is almost huge and λ is the least a.h. target for κ, then λ is not even weakly compact. Our proof will be brief and assumes familiarity with Π_n^m-describability (see [17,Chapter 32] for background material). To obtain the result, we show that λ is Π_1^1-describable. Let $\mathcal{B}_\kappa^-$ be the same as Barbanel's criterion $\mathcal{B}(\kappa, \lambda)$, without the clause "$\lambda$ is inaccessible", and with all mention of λ suppressed. We let X denote a second-order variable in the language $\{\in, \mathbf{U}\}$, where $\mathbf{U}$ is a unary relation symbol. Consider the following formulas:

$$\sigma_1(X) \equiv \big[\text{“}X \text{ is a function”} \wedge \text{“dom } X \text{ is an ordinal } \alpha\text{”} \wedge X''\alpha \subset ON\big] \Longrightarrow \text{“}X \text{ is bounded”};$$

$$\sigma_2 \equiv \exists\kappa\,\big[\mathbf{U} \text{ is a coherent sequence of normal ultrafilters and satisfies } \mathcal{B}_\kappa^-\big];$$

$$\sigma_3 \equiv \forall\alpha\,\exists\beta\,\exists g : \beta \to V_\alpha\,[\text{“}g \text{ is a surjection”}].$$

Let $\tau = \forall X\,[\sigma_1(X) \wedge \sigma_2 \wedge \sigma_3]$. Then under the usual second-order interpretation, and letting U be a coherent sequence $\langle U_\eta : \kappa \le \eta < \lambda\rangle$ satisfying $\mathcal{B}(\kappa, \lambda)$,

$$\langle V_\lambda, \in, U\rangle \models \tau.$$

($\forall X\,\sigma_1(X)$ and σ_3 together imply that λ is inaccessible, and these along with σ_2 imply that λ is an a.h. target via Barbanel's Criterion $\mathcal{B}(\kappa, \lambda)$.) However, for all $\alpha < \lambda$,

$$\langle V_\alpha, \in, U \cap V_\alpha\rangle \not\models \tau$$

since λ has been chosen to be the least a.h. target for κ. Thus λ is not weakly compact.

We note that these observations are not really about Laver sequences at all; they are equally true for any defined notion whose parameters lie below the critical point of one of the elementary embeddings described above.

5.32 Open Question. Describe conditions on classes $\mathcal{E}_\kappa^\theta$ and inner models N under which "g is $\mathcal{E}_\kappa^\theta$-Laver" is absolute for N.

A curious application of absoluteness of $\mathcal{E}^{sc}_\kappa$ is the following:

5.33 Proposition. *Suppose κ is both strong and almost huge. Moreover, suppose there exists an almost huge embedding $j : V \to N$ such that $\mathrm{cp}(j) = \kappa$ and for all $g : \kappa \to V_\kappa$, the statement "g is $\mathcal{E}^{sc}_\kappa$-Laver at κ" is absolute for N. Then κ is supercompact.*

Proof. Since j is an a.h. embedding, $V_{j(\kappa)} \models$ "κ is supercompact;" this property relativizes up (using Theorem 2.18(1)), so that in N, κ is supercompact and admits a Laver sequence. By absoluteness relative to N, there is a Laver sequence at κ in V, and so (by Proposition 4.15) κ is really supercompact. ∎

Returning to the main thread of ideas, we sum up our sufficient conditions for properties (1) - (3) of Theorem 5.13 to hold:

5.34 Theorem. *Suppose that κ is globally superstrong and suppose $j : V \to N$ is a superstrong embedding with critical point κ. Suppose θ is adequately absolute and $\mathcal{E}^\theta_\kappa$ is Laver-closed at beth-fixed points and weakly compatible with j. Then properties* (1)-(3) *of Theorem 5.13 are satisfied.*

Proof. Theorems 5.16(2), 5.19, and 5.25(B) guarantee that the requirements (1), (2), and (3), respectively, are satisfied. ∎

5.35 Theorem. *Suppose $j : V \to N$ is elementary with critical point κ. Under any of the following conditions on θ, κ, and j, properties* (1)-(3) *of Theorem 5.13 are satisfied:*

(A) *$\theta = \theta_{str}$, κ is strong, and j is superstrong;*
(B) *$\theta = \theta_{sc}$, κ is strong, and j is almost huge;*
(C) *$\theta = \theta_{ext}$, κ is globally superstrong or extendible, and j is almost huge;*
(D) *$\theta = \theta_{sah}$, κ is globally superstrong or extendible, and j is huge;*
(E) *$\theta = \theta_{sh}$, κ is globally superstrong or extendible, and j is 2-huge.*

Proof. For $\theta \in \{\theta_{str}, \theta_{sc}\}$, we use Theorems 5.18 and 5.24, and Corollary 5.25. Certainly the conditions in Theorems 5.18 and 5.24 are satisfied for these two classes. For $\theta = \theta_{str}$, the fact that j is superstrong also guarantees that the condition in Corollary 5.25(A) holds. And for $\theta = \theta_{sc}$, almost hugeness of j guarantees the condition in Corollary 5.25(B).

For $\theta \in \{\theta_{ext}, \theta_{sah}, \theta_{sh}\}$, it is easy to verify that the conditions in Theorem 5.24 and Corollaries 5.17 and 5.26 are satisfied. ∎

We conclude with our results from this section about $\mathcal{E}^\theta_\kappa$-Laver sequences. We will combine these with our work in Section 4 to show that, under the hypotheses we have used in this section, if a class $\mathcal{E}^\theta_\kappa$ admits an $\mathcal{E}^\theta_\kappa$-Laver sequence at all, it typically admits a special $\mathcal{E}^\theta_\kappa$-Laver sequence. As the next lemma shows, this phenomenon can be attributed largely to the fact that the ambient embeddings $j : V \to N$ that we have required in the hypotheses typically have the property that $\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver}\} \in D$, where D is the normal ultrafilter over j.

5.36 Lemma. *Suppose the suitable formula θ and the elementary embedding $j : V \to N$, with critical point κ, satisfy the following:*

(1) *for each $t : \kappa \to V_\kappa$, the function $f = f_t$ obtained in the construction $\mathbf{CC}(t, \mathcal{E}^\theta_\kappa)$ is $\mathcal{E}^\theta_\kappa$-Laver;*

(2) *$\{\alpha < \kappa : f \restriction \alpha$ is $\mathcal{E}^\theta_\alpha$-Laver$\} \in D$, where D is the normal ultrafilter over κ derived from j;*

(3) *$\mathcal{E}^\theta_\kappa$ is weakly compatible with j;*

(4) *$\mathcal{E}^\theta_\kappa$ is upward λ-closed for arbitrarily large λ below $j(\kappa)$;*

(5) *$j(\kappa)$ is a limit of beth fixed points.*

Then t may be chosen so that f is a special $\mathcal{E}^\theta_\kappa$-Laver sequence.

Proof. Let $x = j(f)(\kappa)$, and use (4) and (5) to pick a beth fixed point λ_0 and a cardinal λ_1 such that $rank(x) < \lambda_0 \leq \lambda_1 < j(\kappa)$ and $\mathcal{E}^\theta_\kappa$ is upward λ_1-closed. Let $\langle h_\alpha : \alpha < \kappa\rangle$ be an enumeration of the functions $\kappa \to \kappa$ that are definable in V_κ. By (2), if the parameter $t : \kappa \to V_\kappa$ is defined as in (4.2) (in Theorem 4.4) we can obtain sets $\langle X_\alpha : \alpha < \kappa\rangle$ with the property that, for each α, $X_\alpha \in D$ and $\forall \xi \in X_\alpha, rank(f(\xi)) > h_\alpha(\xi)$. Let $r : \kappa \to P(\kappa)$ be defined by $r(\alpha) = X_\alpha$. By (3), we can find $\beta, i : V_\beta \to M$ such that $\lambda_1 < \beta < i(\kappa), (i, M) \in \mathcal{E}^\theta_\kappa$, $i(f)(\kappa) = x$, and if D_i is the normal ultrafilter derived from i, then $r(\delta) \in D_i$ for each $\delta < \kappa$. It follows immediately that f is special. ∎

5.37 Theorem. *Suppose θ is a suitable, adequately absolute formula, f is the function constructed in $\mathbf{CC}(t, \mathcal{E}^\theta_\kappa)$, κ is globally superstrong, and $\mathcal{E}^\theta_\kappa$ is Laver-closed at beth-fixed points. Suppose that for each γ, there is a superstrong embedding $j : V \to N$ with critical point κ such that $j(\kappa) > \gamma$ and $\mathcal{E}^\theta_\kappa$ is weakly compatible with j. Then f is an $\mathcal{E}^\theta_\kappa$-Laver sequence. Moreover, if for one such j, $\mathcal{E}^\theta_\kappa$ is upward λ-closed for all $\lambda < j(\kappa)$, then the parameter t in the construction of f can be defined so that f is special.*

Proof. The main result follows from Theorems 5.13 and 5.34. We prove the "moreover" clause: Let $j : V \to N$ be a superstrong embedding for which $\mathcal{E}^\theta_\kappa$ is upward λ-closed for all $\lambda < j(\kappa)$. We apply Lemma 5.36. Parts (1) and (2) of the lemma hold by Theorem 5.13; parts (3) and (4) hold by hypothesis; and part (5) holds because j is a superstrong embedding. ∎

5.38 Theorem. *Suppose θ is a suitable formula, κ is a cardinal, and f is the function constructed in $\mathbf{CC}(t, \mathcal{E}^\theta_\kappa)$. Then, assuming any one of the following conditions, f is $\mathcal{E}^\theta_\kappa$-Laver at κ:*

(A) *$\theta = \theta_{str}$ and κ is globally superstrong;*

(B) *$\theta = \theta_{sc}$ and κ is super-almost-huge;*

(C) *$\theta = \theta_{ext}$ and κ is super-almost-huge;*

(D) *$\theta = \theta_{sah}$ and κ is superhuge;*

(E) *$\theta = \theta_{sh}$ and κ is super-2-huge.*

Moreover, the parameter t in the construction of f can be defined so that f is special if $\theta \in \{\theta_{sc}, \theta_{ext}, \theta_{sah}, \theta_{sh}\}$.

Proof. The main result follows from Theorems 5.13 and 5.35. We use Lemma 5.36 to prove the "moreover" clause: Parts (1) and (2) hold because of Theorems 5.13 and 5.35; part (3) follows from

Theorems 5.20 and 5.23; part (4) is easy to verify; and part (5) holds because in each case $j(\kappa)$ is inaccessible. ∎

§6. A Modified Construction

In this section, we show how to obtain $\mathcal{E}_\kappa^\theta$-Laver sequences as Section 5, but with less work. Our efforts in Section 5 were directed toward proving that properties (1)-(3) of Theorem 5.13 hold for $\mathcal{E}_\kappa^\theta$ for various θ, assuming the proper kind of embeddings $j : V \to N$; with these established, we could use Theorem 5.13 to show that the canoncially constructed function is $\mathcal{E}_\kappa^\theta$-Laver. However, examining these properties and the reasons for proving them suggests a way to obtain the desired results even without most of these properties.

Property (1) ("Laver failures are localized below κ") is used to ensure that the f we constructed was well-defined (if $f \restriction \alpha$ is not $\mathcal{E}_\alpha^\theta$-Laver, we wanted to be sure that some x, λ in V_κ witness this fact). Property (3) ("$\mathcal{E}_\kappa^\theta$ is Laver-reflecting in $V_{j(\kappa)}$") is used to ensure that if $\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}_\alpha^\theta\text{-Laver}\}$ is in the ultrafilter derived from $j : V \to N$, then we may conclude not only that f is $\mathcal{E}_\kappa^\theta$-Laver in N, but in V as well. Both these properties show how to push concerns about Laver-ness down to the model $V_{j(\kappa)}$. On the other hand, Property (2) ("$j(f)(\kappa)$ is not a witness to Laver failure of f") is already essentially an assertion about sets in $V_{j(\kappa)}$, as Theorems 5.19 and 5.24 show.

These observations suggest that we could save ourselves the effort of establishing properties (1) and (3) if we work inside a $V_{j(\kappa)}$ in the first place. To carry out the plan, we will use the same definition of f as before, except that now it will take place inside a structure of type $\langle V_\kappa, \in, R\rangle$, where R is a well-ordering of V_κ. With this approach, our concern about whether f is well-defined evaporates. And the reason f still ends up being Laver in V is because verification of Laver-ness always takes place within some $V_{j(\kappa)}$ and never involves all of V. With this lightened load, our task becomes simply one of verifying Property (2), now inside a structure of type $\langle V_{j(\kappa)}, \in, j(R)\rangle$. However, this was essentially accomplished in Theorems 5.19 and 5.24. We devote the rest of this section to filling in the details.

6.1 Canonical Construction $\mathbf{CC}^R(t, \mathcal{E}_\kappa^\theta)$. *Suppose $\mathcal{E}_\kappa^\theta$ is a regular class and let R be a well-ordering of V_κ. Let $t : \kappa \to V_\kappa$ be definable in the structure $\langle V_\kappa, \in, R\rangle$. In $\langle V_\kappa, \in, R\rangle$, define $f^R : \kappa \to V_\kappa$ by*

$$f^R(\alpha) = \begin{cases} t_\alpha & \text{if } f^R \restriction \alpha \text{ is a } \mathcal{E}_\alpha^\theta\text{-Laver sequence at } \alpha \\ x \in V_\kappa & \text{if } \alpha \text{ is a cardinal and } f^R \restriction \alpha \text{ is not } \mathcal{E}_\alpha^\theta\text{-Laver at } \alpha, \\ & \text{where } x \text{ is } R\text{-least such that } \exists\lambda\, \phi(f^R \restriction \alpha, x, \lambda) \\ \emptyset & \text{if } \alpha \text{ is not a cardinal.} \end{cases}$$

First note that, unlike the construction $\mathrm{CC}(t, \mathcal{E}_\kappa^\theta)$, f^R is always well-defined. Also, although f^R is definable in the structure $\langle V_\kappa, \in, R\rangle$, the definition of f^R is not absolute (since, for any α, $f^R \restriction \alpha$ may be $\mathcal{E}_\alpha^\theta$-Laver at α in $\langle V_\kappa, \in, R\rangle$ but not in V).

Joel Hamkins pointed out to the author that, as in the case of our earlier construction $\mathbf{CC}(t, \mathcal{E}_\kappa^\theta)$, it is possible to use an indirect argument, more reminiscent of Laver's original proof, to show that, assuming only supercompactness (strongness) of κ, the above construction yields a $\mathcal{E}^{sc}_\kappa$-Laver ($\mathcal{E}^{str}_\kappa$-Laver) sequence. As before, the advantage to this approach is that in some cases we can minimize the large cardinal hypotheses required; the price we pay for this is that $\{\alpha : \langle V_\kappa, \in, R\rangle \models \text{“}f^R \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\text{”}\}$ is not guaranteed to be of normal measure 1, and so we lose the luxury of forcing f^R to equal some $t : \kappa \to V_\kappa$ on a large set. We outline the proof below:

Assume κ is supercompact. For the proof, we will revert to the original definition of Laver sequences and use the definition of ϕ given in Construction 4.3. Suppose the claim is false. Let x, λ be such that $\phi(f^R, x, \lambda)$. Let ν be a beth fixed point greater than $\gamma = (2^{\lambda^{<\kappa}})^+$. Let $j : V \to M_\nu$ be any ν-supercompact embedding and let D be the normal ultrafilter over κ derived from j.

Note that each normal ultrafilter over $P_\kappa\lambda$ lies in M_ν. For each such U, let

$$i_U : V \to N \cong V^{P_\kappa\lambda}/U;$$
$$\bar{i}_U : V_\gamma \to \bar{N} \cong V_\gamma^{P_\kappa\lambda}/U,$$

denote the resulting canonical embeddings for the ultrapowers relative to V, V_γ, respectively. By Proposition 2.28(A),

$$i_U \restriction V_{\kappa+1} = \bar{i}_U \restriction V_{\kappa+1} \text{ and } (\bar{i}_U, \bar{N}) \in V_\nu \in V^{M_\nu}_{j(\kappa)}. \tag{6.1}$$

Suppose first that $\{\alpha : \langle V_\kappa, \in, R\rangle \models \text{“}f^R \restriction \alpha \text{ is Laver at } \alpha\text{”}\} \in D$. Then there is a normal ultrafilter U over $P_\kappa\lambda$ such that in $\langle V_{j(\kappa)}, \in, j(R)\rangle^{M_\nu}$, $i_U(f^R)(\kappa) = x$. It follows that (in V) $x = \bar{i}_U(f^R)(\kappa) = i_U(f^R)(\kappa)$, and this is impossible.

Thus, $\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \exists\lambda\phi(f^R \restriction \alpha, f^R(\alpha), \lambda)\} \in D$, and so $\langle V_{j(\kappa)}, \in, j(R)\rangle^{M_\nu} \models \exists\lambda\, \phi(f^R, j(f^R)(\kappa), \lambda)$. Let $\mathcal{M} = \langle V_{j(\kappa)}, \in, j(R)\rangle^{M_\nu}$ and let U be the normal ultrafilter over $P_\kappa\lambda$ derived from j. We have, by (6.1) and Lemma 2.26,

$$j(f^R)(\kappa) = i_U(f^R)(\kappa) = \bar{i}_U(f^R)(\kappa) = \bar{i}^{\mathcal{M}}_U(f^R)(\kappa) = i^{\mathcal{M}}_U(f^R)(\kappa),$$

and we have a contradiction.

We now give a sufficient condition for the construction $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$ to yield an $\mathcal{E}^\theta_\kappa$-Laver sequence; this is the analogue to Theorem 5.13:

6.2 Theorem. *Suppose θ is adequately absolute and κ is globally superstrong. Let f^R be defined as in $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$. Assume that for each $\gamma > \kappa$ there is a superstrong embedding $j : V \to N$ such that $j(\kappa) > \gamma$ and*

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \forall\lambda\, \neg\phi(f^R, j(f^R)(\kappa), \lambda). \tag{6.2}$$

Then f^R is a $\mathcal{E}^\theta_\kappa$-Laver sequence at κ.

Remark. Note that by elementarity, $j(f^R)$ is a definable subclass of $\langle V_{j(\kappa)}, \in, j(R)\rangle$.

Proof of Theorem 6.2 We begin by noting that whenever j satisfies (6.2), we have

$$\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \forall\lambda\,\neg\phi(f^R \restriction \alpha, f^R(\alpha), \lambda)\} \in D, \tag{6.3}$$

where D is the normal ultrafilter derived from j. Thus, the first case in the definition of f^R (in $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$) must hold on a set in D:

$$\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \text{“}f^R \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver at } \alpha\text{”}\} \in D. \tag{6.4}$$

Applying j to (6.4), and noting that $R \in V_{j(\kappa)}$ and $j(f^R) \restriction \kappa = f$, we have

$$V_{j(\kappa)} \models \text{“}f^R \text{ is } \mathcal{E}^\theta_\kappa\text{-Laver at } \kappa\text{”}. \tag{6.5}$$

Thus, suppose there is a counterexample (x, λ) to $\mathcal{E}^\theta_\kappa$-Laverness of f. We pick $j : V \to N$ satisfying the hypotheses of the theorem—in particular, (6.2)—so that $j(\kappa) > \lambda$. Then (6.5) holds, so we pick $i : V_\beta \to M \in \left(\mathcal{E}^\theta_\kappa\right)^{V_{j(\kappa)}}$ that has Laver-like values with respect to κ, λ, x, f^R. By adequate absoluteness, $(i, M) \in \mathcal{E}^\theta_\kappa$, and we have the required contradiciton. ∎

Notice that since (6.4) holds, we should be able to reason as in Section 5 to show that the function f^R can be defined so that the functions $\alpha \mapsto |f^R(\alpha)|$ and $\alpha \mapsto rank(f^R(\alpha))$ dominate, on sets in D, the functions $\kappa \to \kappa$ that are definable in V_κ; the argument will work as long as we can obtain a dominating function $t : \kappa \to V_\kappa$ (as per the definition of f^R) that is definable in some $\langle V_\kappa, \in, R\rangle$. We show here (using the simplified argument suggested by the referee) that, in fact, *every* $t : \kappa \to V_\kappa$ is definable in such a structure.

Let Succ denote the successor ordinals below κ and Lim the limit ordinals below κ. Let $t : \kappa \to V_\kappa$ be a function. We define a bijection $h : \kappa \to V_\kappa$ from which t is definable. Let $q : \text{Lim} \to t = \{(\alpha, t(\alpha)) : \alpha < \kappa\}$ and $r : \text{Succ} \to V_\kappa \setminus t$ both be bijections. Define h by

$$h(\alpha) = \begin{cases} q(\alpha) & \text{if } \alpha \text{ is a limit} \\ r(\alpha) & \text{if } \alpha \text{ is a successor.} \end{cases}$$

Letting R_h denote the well-ordering of V_κ determined by h, we have that $x = t(\alpha)$ if and only if $\langle V_\kappa, \in, R_h\rangle \models \exists\beta\,[\text{“}\beta \text{ is a limit”} \wedge h(\beta) = (\alpha, x)]$.

The next theorem shows that, for “typical” classes $\mathcal{E}^\theta_\kappa$, weak compatibility with j suffices to establish (6.2); the result follows immediately from Theorem 5.19(B).

6.3 Theorem. *Suppose $j : V \to N$ is a superstrong embedding with critical point κ. Suppose θ is adequately absolute and $\mathcal{E}^\theta_\kappa$ is Laver-closed at beth fixed points and weakly compatible with j. Then for any function $g : \kappa \to V_\kappa$,*

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \forall\lambda\,\neg\phi(g, j(g)(\kappa), \lambda). \blacksquare \tag{6.6}$$

We now record the results of Theorem 5.24 in the present context.

6.4 Theorem. *Suppose $j : V \to N$ is an elementary embedding with critical point κ and $g : \kappa \to V_\kappa$ is a function. Then (6.6) holds, assuming any one of the following:*

(1) *$\theta = \theta_{str}$ and $j : V \to N$ is a superstrong embedding;*
(2) *$\theta = \theta_{sc}$ and $j : V \to N$ is an almost huge embedding;*
(3) *$\theta = \theta_{ext}$ and $j : V \to N$ is an almost huge embedding;*
(4) *$\theta = \theta_{sah}$ and $j : V \to N$ is a huge embedding;*
(5) *$\theta = \theta_{sh}$ and $j : V \to N$ is a 2-huge embedding.* ■

We conclude the section with results summarizing the work from this section, and combine these with our work in previous sections concerning special $\mathcal{E}^\theta_\kappa$-Laver sequences. As was the case for the construction $\mathbf{CC}(t, \mathcal{E}^\theta_\kappa)$, our methods typically allow us to choose t and R in $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$ so that f^R is special. We begin with the analogue of Lemma 5.36:

6.5 Lemma. *Suppose the suitable formula θ and the elementary embedding $j : V \to N$, with critical point κ satisfy the following:*

(1) *for each well-ordering R of V_κ*
    (a) *for each $t : \kappa \to V_\kappa$ definable in $\langle V_\kappa, \in, R\rangle$, the function $f^R = f_{R,t}$ obtained in the construction $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$ is $\mathcal{E}^\theta_\kappa$-Laver;*
    (b) *$\{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models$ "$f^R \upharpoonright \alpha$ is $\mathcal{E}^\theta_\alpha$-Laver"$\} \in D$, where D is the normal ultrafilter over κ derived from j;*
(2) *$\mathcal{E}^\theta_\kappa$ is weakly compatible with j;*
(3) *$\mathcal{E}^\theta_\kappa$ is upward λ-closed for arbitrarily large λ below $j(\kappa)$;*
(4) *$j(\kappa)$ is a limit of beth fixed points.*

Then R and t may be chosen so that f^R is a special $\mathcal{E}^\theta_\kappa$-Laver sequence.

Proof. The function t can be chosen just as in (4.2) (in Theorem 4.4), so that it dominates the functions $\kappa \to \kappa$ that are definable in V_κ on sets in D. By the discussion following Theorem 6.2, we can find a well-ordering R of V_κ for which t is definable in $\langle V_\kappa, \in, R\rangle$. The proof that this choice of R and t make f^R special is the same as the corresponding proof for Lemma 5.36. ■

Combining Theorems 6.2 and 6.3 with the previous lemma, we obtain:

6.6 Theorem. *Suppose R is a well-ordering of V_κ and f^R is constructed as in $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$. Suppose θ is an adequately absolute formula, κ is globally superstrong, and $\mathcal{E}^\theta_\kappa$ is Laver-closed at beth-fixed points. Suppose that for each γ, there is a superstrong embedding $j : V \to N$ with critical point κ such that $j(\kappa) > \gamma$ and $\mathcal{E}^\theta_\kappa$ is weakly compatible with j. Then f^R is an $\mathcal{E}^\theta_\kappa$-Laver sequence. Moreover, if for one such j, $\mathcal{E}^\theta_\kappa$ is upward λ-closed for all $\lambda < j(\kappa)$, then R and t can be chosen in the construction so that f^R is a special $\mathcal{E}^\theta_\kappa$-Laver sequence.*

Proof. The main result follows from Theorems 6.2 and 6.3. We prove the "moreover" clause: Let $j : V \to N$ be one of the superstrong embeddings described in the hypothesis for which $\mathcal{E}^\theta_\kappa$ is upward λ-closed for all $\lambda < j(\kappa)$. We apply Lemma 6.5. Part (1a) follows from Theorem 6.2. Part (1b) follows from the proof of Theorem 6.2 (note the display (6.4)). Parts (2) and (3) hold by hypothesis. And Part (4) holds because j is superstrong. ∎

Finally, the following results mirror Theorem 5.38.

6.7 Theorem. *Suppose θ is adequately absolute, κ is a cardinal, and f^R is defined as in* $\mathbf{CC}^R(t, \mathcal{E}^\theta_\kappa)$. *Then f^R is $\mathcal{E}^\theta_\kappa$-Laver assuming any one of the following conditions:*

(A) $\theta = \theta_{str}$ *and κ is globally superstrong;*
(B) $\theta = \theta_{sc}$ *and κ is super-almost-huge;*
(C) $\theta = \theta_{ext}$ *and κ is super-almost-huge;*
(D) $\theta = \theta_{sah}$ *and κ is superhuge;*
(E) $\theta = \theta_{sh}$ *and κ is super-2-huge.*

Moreover, in the construction, R and t can be chosen in the construction so that f^R is a special $\mathcal{E}^\theta_\kappa$-Laver sequence, whenever $\theta \in \{\theta_{sc}, \theta_{ext}, \theta_{sah}, \theta_{sh}\}$.

Proof. The main result follows from Theorems 6.2 and 6.4. We use Lemma 6.5 to prove the "moreover" clause: Part (1a) follows from Theorem 6.4. Part (1b) follows by the reasoning in the proof of Theorem 6.2 that leads to the relation (6.4). Part (2) follows from Theorems 5.20 and 5.23. Part (3) is easy to verify, and part (4) follows because, in each case, $j(\kappa)$ is inaccessible. ∎

Our efforts to obtain results about the Laver-ness of the functions constructed in this and the previous section have been guided by the strategy used in our WA-proof in Section 4. That proof guarantees that

$$\{\alpha : f \restriction \alpha \text{ is } \mathcal{E}^\theta_\alpha\text{-Laver}\} \in D, \tag{6.7}$$

where D is the normal ultrafilter over κ derived from the WA embedding j. Our arguments, under weakenings of WA, have, essentially, preserved the truth of (6.7). As we mentioned at the beginning of Section 5, this approach has the advantage of permitting us to construct a diverse range of $\mathcal{E}^\theta_\kappa$-Laver sequences, but limits our ability to obtain optimal hypotheses under which the bare existence of an $\mathcal{E}^\theta_\kappa$-Laver sequence could be proven. These concerns lead to an important open question:

6.8 Open Question. Is $\mathcal{E}^\theta_\kappa$ Laver-generating if θ is any[8] of the formulas $\theta_{sah}, \theta_{ext}, \theta_{sh}$?

§7. Related Results

In this section, we bring together a number of partial results and questions concerning the material of previous sections.

[8]The class $\mathcal{E}^{ext}_\kappa$ is now known to be Laver-generating; see [9], and also the Appendix of this paper for a correction.

Equivalent Forms of Barbanel's Criterion

We prove here that Barbanel's Criterion $\mathcal{B}(\kappa, \lambda)$, the criterion $\mathbf{SRK}(\kappa, \lambda)$, and the criterion actually used by Barbanel for almost huge cardinals in [3] are equivalent. We shall denote the last of these $\mathcal{B}'(\kappa, \lambda)$.

7.1 Proposition. *The following are equivalent:*

(1) *κ is almost huge with target λ.*

(2) *There is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ satisfying $\mathbf{SRK}(\kappa, \lambda)$, namely, λ is inaccessible and for all η, ρ for which $\kappa \leq \eta < \lambda$ and $\eta \leq \rho < j_\eta(\kappa)$ there is ζ such that $\eta \leq \zeta < \lambda$ and $k_{\eta\zeta}(\rho) = \zeta$.*

(3) *There is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ satisfying $\mathcal{B}(\kappa, \lambda)$, namely, λ is inaccessible and for all η for which $\kappa \leq \eta < \lambda$ and all $h : P_\kappa \eta \to ON$, if $\{x \in P_\kappa \eta : \mathrm{ot}(x) \leq h(x) < \kappa\} \in U_\eta$ then there is ζ such that $\eta \leq \zeta < \lambda$ and $\{x \in P_\kappa \zeta : \text{ ot } x = h(x \cap \eta)\} \in U_\zeta$.*

(4) *There is a coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ satisfying $\mathcal{B}'(\kappa, \lambda)$, namely, λ is inaccessible and for all η for which $\kappa \leq \eta < \lambda$, and all $h : P_\kappa \eta \to ON$, if $\{x \in P_\kappa \eta : \mathrm{ot}(x) \leq h(x) < \kappa\} \in U_\eta$, then there is ζ such that $\eta \leq \zeta < \lambda$ and $\{x \in P_\kappa \zeta : h(x \cap \eta) \leq |x|\} \in U_\zeta$.*

Proof. (1) $\Leftrightarrow$ (2) is proven in [27, Theorem 8].

To prove (1) $\Rightarrow$ (3), suppose κ is almost huge with target λ. Let $j : V \to M$ be an a.h. embedding with target λ. Let $\langle U_\eta : \kappa \leq \eta < \lambda \rangle$ be the coherent sequence derived from j (see the proof of [27, Theorem 8]). Given $h : P_\kappa \eta \to ON$, with $\kappa \leq \eta < \lambda$, assume $\{x \in P_\kappa \eta : \mathrm{ot}(x) \leq h(x) < \kappa\} \in U_\eta$. By definition of U_η, we have $\eta = \mathrm{ot}(j''\eta) \leq j(h)j''\eta < j(\kappa)$; thus, letting $\zeta = j(h)(j''\eta)$, we have $\eta \leq \zeta < j(\kappa)$. Also, since $\zeta = j(h)(j''\zeta \cap j(\eta))$, we have (as in [27])

$$\{x \in P_\kappa \zeta : \mathrm{ot}\ x = h(x \cap \eta)\} \in U_\zeta,$$

as required.

For (3) $\Rightarrow$ (4), under the hypotheses of either (3) or (4), simply note that

$$\{x \in P_\kappa \zeta : \mathrm{ot}\ x = |x|\} \in U_\zeta,$$

and

$$\{x \in P_\kappa \zeta : h(x \cap \eta) = |x|\} \in U_\zeta \text{ implies } \{x \in P_\kappa \zeta : h(x \cap \eta) \leq |x|\} \in U_\zeta.$$

Finally, (4) $\Rightarrow$ (1) was proven in the lemma of [3, p. 180]. ∎

Variations on Regularity

Let us recall the variations on the concept of a regular class, introduced in Section 4:

Given a suitable formula θ, we call $\mathcal{E}^\theta_\kappa$

regular if $\quad \forall \gamma > \kappa\, \exists \beta \geq \gamma\, \exists i \in \mathcal{E}^\theta_\kappa\, [i : V_\beta \to M \wedge i(\kappa) > \gamma \wedge V_\gamma \subset M]$;

weakly regular if $\quad \forall \gamma > \kappa\, \exists \beta > \kappa\, \exists i \in \mathcal{E}^\theta_\kappa\, [i : V_\beta \to M \wedge i(\kappa) > \gamma \wedge V_\gamma \subset M]$;

semi-regular if $\qquad \forall\gamma > \kappa\, \exists\beta \geq \gamma\, \exists i \in \mathcal{E}^\theta_\kappa\, [i : V_\beta \to M \ \wedge \ \wedge V_\gamma \subset M]$;

weakly semi-regular if $\qquad \forall\gamma > \kappa\, \exists\beta > \kappa\, \exists i \in \mathcal{E}^\theta_\kappa\, [i : V_\beta \to M \ \wedge \ \wedge V_\gamma \subset M]$.

Proposition 4.13 showed that the existence of a suitable θ for which $\mathcal{E}^\theta_\kappa$ is regular or weakly regular is equivalent to the statement that κ is a strong cardinal. As promised, we can obtain an equiconsistency result for the other two types of classes:

7.2 Proposition. *Suppose κ is an infinite cardinal. Then the following are equiconsistent:*

(1) *κ is a strong cardinal;*
(2) *for some suitable formula θ, $\mathcal{E}^\theta_\kappa$ is semi-regular;*
(3) *for some suitable formula θ, $\mathcal{E}^\theta_\kappa$ is weakly semi-regular.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are immediate. For (3) $\Rightarrow$ (1), assume $\mathcal{E}^\theta_\kappa$ is weakly semi-regular. Consider the following statement:

$$\exists\lambda > \kappa\, \exists i \in \mathcal{E}^\theta_\kappa\, [i : V_\beta \to M \ \wedge \ i(\kappa) \leq \lambda \ \wedge \ V_\lambda \subset M]. \tag{7.1}$$

If (7.1) is false, we can show that κ is a strong cardinal, as follows: Let $\lambda > \kappa$ and let $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ be such that $V_\lambda \subset M$. Since (7.1) fails, $i(\kappa) > \lambda$, as required.

On the other hand, if (7.1) is true, let $\lambda, i : V_\beta \to M$ satisfy the conditions in (7.1). Let γ be such that $\kappa < \gamma < i(\kappa)$. We can obtain from i, as in Proposition 2.5, an extender with critical point κ and support V_γ. Thus, $V_{i(\kappa)} \models$ "κ is a strong cardinal,", and we are done. ∎

Returning to our variations on the definition of regularity, it is clear that regular implies weakly regular and semi-regular, and each of these implies weakly semi-regular. We show that weakly semi-regular does not imply weakly regular or semi-regular (assuming something more than an extendible cardinal); that weakly regular does not imply semi-regular (assuming there is an extendible cardinal); and that, assuming $V = \text{HOD}$, semi-regular does not imply weakly regular (under very strong hypotheses). We also formulate fairly natural conditions under which semi-regular does imply weakly regular.

7.3 Examples.

(1) *A weakly semi-regular class that is neither weakly regular nor semi-regular.* We make use of the following hypothesis: There exists an extendible cardinal λ, ordinals κ, β, and an elementary embedding i with critical point κ such that $\kappa < \beta < \lambda$ and $i : V_\beta \to V_\lambda$. (The hypothesis holds under WA: let $\beta = j(\kappa)$ and $\lambda = j^2(\kappa)$.)

Given such $\kappa, \lambda, \beta, i$, recall that, since λ is extendible, there are arbitrarily large ν such that $V_\lambda \prec V_\nu$; for each $\gamma > \lambda$, let $\nu_\gamma > \gamma$ be least such that $V_\lambda \prec V_{\nu_\gamma}$. For each γ, let $i_\gamma : V_\beta \to V_{\nu_\gamma}$ be the composition $\text{incl}_\gamma \circ i$, where incl_γ is the inclusion map $V_\lambda \hookrightarrow V_{\nu_\gamma}$. Let $\mathcal{E} = \{i_\gamma : \gamma > \lambda\}$. Clearly, there is a suitable formula which defines $\mathcal{E}$ and $\mathcal{E}$ is weakly semi-regular. But $\mathcal{E}$ is neither semi-regular nor weakly regular because $\{i(\kappa) : i \in \mathcal{E}\} \cup \text{Dom}\ \mathcal{E} \subset V_\lambda$.

(2) *A weakly regular class that is not semi-regular.* Let $\mathcal{E} = \{i \in \mathcal{E}_\kappa^{ext} : \text{dom}\, i = V_{\kappa+1}\}$. $\mathcal{E}$ is clearly defined by some suitable formula; and it is weakly regular since for each $\lambda > \kappa$ there is $j \in \mathcal{E}_\kappa^{ext}$ such that $j(\kappa) > \lambda$, and $j \upharpoonright V_{\kappa+1} \in \mathcal{E}$. $\mathcal{E}$ is not semi-regular since the domains of its elements are uniformly bounded.

(3) *A semi-regular class that is not weakly regular.* This example is only a consistency result: We assume $V = \text{HOD}$. We also use the following strong hypothesis: There exist λ, U such that U is a huge ultrafilter over $P(\lambda)$ witnessing that κ is huge, and, for arbitrarily large β, the following holds:

$$\exists\zeta_\beta\, \exists k_\beta\, [k_\beta : V^N_{i_U(\beta)} \to V_{\zeta_\beta} \wedge \text{“}\zeta_\beta \text{ is extendible”} \wedge k_\beta(i_U(\kappa)) = \lambda)]. \tag{7.2}$$

(The hypothesis can be proven from WA as follows: If j is the WA-embedding with critical point κ, let U be the normal ultrafilter over $P(j(\kappa))$ derived from j and let $i : V \to N$ be the canonical embedding. As in Section 2 (see remarks following Proposition 2.13), there is $k : N \to V$ such that $k \circ i = j$. By WA, there are arbitrarily large extendibles β greater than $j(\kappa)$; for each such β, $N \models$ “$i(\beta)$ is extendible.”. Let $\zeta_\beta = j(\beta) = k(i(\beta))$ and let $k_\beta = k \upharpoonright V^N_{i(\beta)}$. Finally, let $\lambda = j(\kappa)$. (The main result in [8] shows that WA is consistent with the axiom $V = \text{HOD}$.))

Let $\mathbf{F}$ be a definable well-ordering of the universe (without parameters). We let λ be the least target of a canonical huge embedding having critical point κ for which the properties in the hypothesis hold. Use $\mathbf{F}$ to obtain the least huge ultrafilter U satisfying these properties with respect to κ and λ. Use $\mathbf{F}$ to obtain class functions $\mathbf{G}$ and $\mathbf{K}$ that select ζ_β and k_β for each β: $\mathbf{G}(\beta) = \zeta_\beta$, $\mathbf{K}(\beta) = k_\beta$. For convenience, we will make sure $\mathbf{G}$ and $\mathbf{K}$ are defined on all of ON by letting $\zeta_\beta = 0$ and $k_\beta = \emptyset$ whenever β does not satisfy (7.2). Note that these class functions can be defined with κ as their only parameter. Using the hypothesis, let $M_\beta = V^N_{i_U(\beta)}$ for each $\beta > j(\kappa)$. For each β satisfying (7.2), let $\langle \gamma^{(\beta)}_\alpha : \alpha \in ON\rangle$ be the increasing enumeration of the ordinals $\gamma > \zeta_\beta$ for which $V_{\zeta_\beta} \prec V_\gamma$. Let $\text{incl}_{\beta,\alpha}$ denote the inclusion map $V_{\zeta_\beta} \hookrightarrow V_{\gamma^{(\beta)}_\alpha}$. Finally, let $\mathcal{E} = \{\text{incl}_{\beta,\alpha} \circ k_\beta \circ i_U \upharpoonright V_\beta : \beta > j(\kappa) \text{ and } \zeta_\beta \text{ is extendible}\}$.

Because of our careful use of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{K}$, $\mathcal{E}$ can be defined by a suitable formula. Since we have ensured the existence of domains V_β for arbitrarily large β each having arbitrarily large codomains, $\mathcal{E}$ is semi-regular; but $\mathcal{E}$ is not weakly regular since $\{e(\kappa) : e \in \mathcal{E}\} = \{\lambda\}$. ∎

The next two definitions provide conditions under which semi-regular implies weakly regular (in fact, regular). The main idea is to mimic the proof that, if we remove the condition “$j(\kappa) > \lambda$” in the definition of λ-supercompact, there must be some $n \in \omega$ such that $j^n(\kappa) > \lambda$ (see [18, 23.15a]). We would like to show that if there is a uniform bound λ on the $i(\kappa)$ for $i \in \mathcal{E}^\theta_\kappa$ then for some $i \in \mathcal{E}^\theta_\kappa$ we would have $i^n(\kappa) < \lambda$ for all $n \in \omega$; we would then be able to obtain an embedding $V_{\gamma+2} \to V_{\gamma+2}$ with critical point κ and with $\gamma = \sup(\{i^n(\kappa) : n \in \omega\})$, which would contradict Kunen's theorem. However, because the codomains of embeddings in $\mathcal{E}^\theta_\kappa$ are generally larger than their corresponding domains, it is not generally possible to iterate embeddings. We could accomplish something roughly equivalent to one step of iteration if for a given $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$ we could find an extension $\hat{i} \in \mathcal{E}_\kappa^\theta$ with $M \in \text{dom } \hat{i}$; then $\hat{i}(i(\kappa))$ would essentially be $i^2(\kappa)$. Still, it's not necessarily true that $(\hat{i} \upharpoonright M) \circ i \in \mathcal{E}_\kappa^\theta$, so there is no guarantee that $\hat{i}(i(\kappa)) < \lambda$.

We will introduce conditions on a class $\mathcal{E}_\kappa^\theta$ that will eliminate these problems. We will say that a class *admits threads* if, roughly, every i has extensions with arbitrarily large domain. And a class will be *closed under powers* if "compositions," as described above, always remain in $\mathcal{E}_\kappa^\theta$.

7.4 Definition. (*Threads*) Suppose $\mathcal{E}_\kappa^\theta$ is a semi-regular class of embeddings. An element $i_\beta : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ *admits a thread* if there is a class function $\mathbf{T}$: Dom $\mathcal{E}_\kappa^\theta \setminus \beta \to \mathcal{E}_\kappa^\theta : \gamma \mapsto (i_\gamma, M_\gamma)$, called a *thread starting at* i_β, with the following properties:

(A) $\mathbf{T}(\beta) = (i_\beta, M)$ and dom $i_\beta = V_\beta$;

(B) whenever $i_\delta : V_\delta \to M_\delta$ and $i_\gamma : V_\gamma \to M_\gamma$ are both in $\mathcal{E}_\kappa^\theta$, where $\beta \le \delta < \gamma$, we have $i_\gamma \upharpoonright V_\delta = i_\delta$, and $i_\gamma(V_\delta) = M_\delta$.

Moreover, $\mathcal{E}_\kappa^\theta$ *admits threads* if for each $i : V_\beta \to M \in \mathcal{E}_\kappa^\theta$ and each $\lambda < |\beta|$ there is an $i_\beta : V_\beta \to N$ such that

(1) i_β is compatible with i up to V_λ;
(2) i_β admits a thread; and
(3) either
  (a) $V_\lambda \cap M = V_\lambda \cap N$ or
  (b) N is λ-closed.

If (3a) holds, we will say that $\mathcal{E}_\kappa^\theta$ *admits threads with rank closure*, whereas of (3b) holds we will say that $\mathcal{E}_\kappa^\theta$ *admits threads with sequential closure.* If instead of (3), the embedding i_β satisfies, in each case, property (3′) below, we will say that $\mathcal{E}_\kappa^\theta$ *admits threads in a strong sense.*

(3′) $V_\lambda \subset N$ and N is V_λ-closed.

Because of the peculiarities of our definitions of θ_{sc}, θ_{str}, etc., none of the particular classes we have been investigating admits threads. The reason is that each is correlated:

7.5 Proposition. *No member of a regular correlated class of embeddings admits a thread.*

Proof. Suppose $i_\beta : V_\beta \to M_\beta$ is a member of a regular correlated class $\mathcal{E}_\kappa^\theta$, and i_β admits a thread $\mathbf{T}$. Let $\mathbf{F}$ be the increasing class function that witnesses the fact that $\mathcal{E}_\kappa^\theta$ is correlated. Since $\mathbf{F}$ is strictly increasing and $\mathcal{E}_\kappa^\theta$ is regular, we can find $\gamma \in$ Dom $\mathcal{E}_\kappa^\theta$ such that $\mathbf{F}(\gamma) > i_\beta(\kappa)$. Let $\mathbf{T}(\gamma) = i_\gamma : V_\gamma \to M_\gamma$. By the definition of $\mathbf{F}$, we have $\mathbf{F}(\gamma) \le i_\gamma(\kappa)$. But this contradicts the fact that $i_\gamma(\kappa) = i_\beta(\kappa) < \mathbf{F}(\gamma)$. ∎

With a slight change in the definition of most of our classes, however, they can be transformed into classes that admit threads. We use $\mathcal{E}_\kappa^{sc}$ as a typical example. In the definition of θ_{sc}, replace the conjunct "$\beta = \lambda + \omega$" with "$\beta \ge \lambda + \omega$", and call the new formula θ''_{sc}. The proof given in Theorem 4.30 can be used to show that supercompactness is normal with respect to θ''_{sc}. We show that $\mathcal{E}_\kappa^{sc''}$ admits threads (with sequential closure) whenever κ is supercompact: Given $i : V_\beta \to M \in \mathcal{E}_\kappa^{sc}$ and $\lambda < |\beta|$, write $\beta = \hat{\lambda} + \omega$ (note $\hat{\lambda} > \lambda$), and let U be the normal ultrafilter over $P_\kappa\hat{\lambda}$.

Let $i_\beta = i_U \restriction V_\beta$. Then we can define a thread $\mathbf{T}$ starting at i_β by putting $\mathbf{T}(\gamma) = i_U \restriction V_\gamma$ for each $\gamma \in \text{Dom } \mathcal{E}_\kappa^{sc''}$. Performing similar modifications to the class definitions, one can show that if κ is strong, $\mathcal{E}_\kappa^{str''}$ admits threads (with rank closure) and if κ is super-almost-huge (superhuge) then $\mathcal{E}_\kappa^{sah''}$ ($\mathcal{E}_\kappa^{sh''}$) admits threads in a strong sense. Finally, note that no member of $\mathcal{E}_\kappa^{ext}$ admits a thread for a more fundamental reason: if $i : V_\beta \to V_\eta \in \mathcal{E}_\kappa^{ext}$, it is necessary that $\beta < i(\kappa)$, so no putative thread starting at i could have values at $i(\kappa)$ or greater.

7.6 Definition. (*n-power and Closure Under Powers*) Suppose $\mathcal{E}_\kappa^\theta$ is semi-regular and $i_1 : V_\beta \to M_1 \in \mathcal{E}_\kappa^\theta$. We define *n-power*$(i_1)$ inductively as follows: 1-power$(i_1) = \{i_1\}$. Assume n-power(i_1) has been defined. A function j_{n+1} is in $n+1$-power(i_1) if and only if j_{n+1} is an elementary embedding with domain V_β and there are $j_n : V_\beta \to N \in n$-power$(i_1)$ and $i_{n+1} : V_\gamma \to M_{n+1} \in \mathcal{E}_\kappa^\theta$ such that

(1) $N \in V_\gamma$;
(2) $i_{n+1} \restriction V_\beta = i_1$;
(3) $j_{n+1} = (i_{n+1} \restriction N) \circ j_n$.

We will say that $\mathcal{E}_\kappa^\theta$ is *closed under powers* if for each $m \geq 1$ and each $i \in \mathcal{E}_\kappa^\theta$, m-power$(i) \neq \emptyset$ and m-power$(i) \subset \mathcal{E}_\kappa^\theta$.

7.7 Theorem. *Suppose $\mathcal{E}_\kappa^\theta$ is semi-regular and coherent, admits threads with rank closure and is closed under powers. Then $\mathcal{E}_\kappa^\theta$ is regular.*

Proof. Assume $\mathcal{E}_\kappa^\theta$ satisfies the hypotheses but is not regular. Then there is $\gamma_0 > \kappa$ such that

$$\forall \beta \geq \gamma_0 \forall i : V_\beta \to M \in \mathcal{E}_\kappa^\theta \, [V_{\gamma_0} \subset M \Longrightarrow i(\kappa) < \gamma_0].$$

By semi-regularity, let $i : V_\gamma \to M \in \mathcal{E}_\kappa^\theta$ be such that $|\gamma| > \gamma_0$ and $V_{\gamma_0+2} \subset M$. Since $\mathcal{E}_\kappa^\theta$ admits threads with rank closure, there is $i_{\gamma_1} : V_\gamma \to M_1 \in \mathcal{E}_\kappa^\theta$ that is compatible with i up to V_{γ_0+2}, admits a thread $\mathbf{T}$ starting at i_{γ_1} and is such that $V_{\gamma_0+2} \subset M_1$. We define by induction $\langle j_n : V_\gamma \to N_n \mid n > 0 \rangle, \langle \gamma_n : n > 0 \rangle$, and $\langle i_{\gamma_n} : V_{\gamma_n} \to M_n \mid n > 0 \rangle$ so that for each $n \geq 1$

(a) $\gamma_1 = \gamma$, $j_1 = i_{\gamma_1}$ and $N_1 = M_1$;
(b) $\gamma_n \in \text{Dom } \mathcal{E}_\kappa^\theta$ and $\gamma \leq \gamma_n < \gamma_{n+1}$;
(c) $M_n \in V_{\gamma_{n+1}}$;
(d) $i_{\gamma_n} \in range(\mathbf{T})$;
(e) $j_{n+1} = \left(i_{\gamma_{n+1}} \restriction N_n\right) \circ j_n$.

To carry out the induction step of the construction, assume we have defined $j_n : V_\gamma \to N_n, \gamma_n$ and $i_{\gamma_n} : V_{\gamma_n} \to M_n$. By semi-regularity, let γ_{n+1} be large enough so that $M_n \in V_{\gamma_{n+1}}$ and $\gamma_{n+1} > \gamma_n$. Let $i_{\gamma_{n+1}} : V_{\gamma_{n+1}} \to M_{n+1}$ be $\mathbf{T}(\gamma_{n+1})$. Let $j_{n+1} = \left(i_{\gamma_{n+1}} \restriction N_n\right) \circ j_n$.

One proves by induction that for each $n > 0$, $V_{\gamma_0+2} \subset N_n \subseteq M_n$ and $j_n \in n$-power(i_{γ_1}). Since $\mathcal{E}_\kappa^\theta$ is closed under powers, for each n, $j_n \in \mathcal{E}_\kappa^\theta$. Thus, since $V_{\gamma_0} \subset N_n$, $j_n(\kappa) < \gamma_0$ for each n.

Pick $\gamma_\omega \in \text{Dom } \mathcal{E}_\kappa^\theta$ so that $\gamma_\omega \geq \sup(\{\gamma_n : n > 0\}) + \omega$. Let $i_{\gamma_\omega} : V_{\gamma_\omega} \to M_\omega = \mathbf{T}(\gamma_\omega)$. One shows by induction that for each n, $i_{\gamma_\omega}^n(\kappa) = j_n(\kappa)$ (where $i_{\gamma_\omega}^n$ denotes the nth iterate of i_{γ_ω}).

Now, let $\lambda = \sup(\{i^n_{\gamma_\omega}(\kappa) : n > 0\})$. Then $\lambda + 2 \le \gamma_0 + 2 < \gamma_\omega$; since $V_{\gamma_0+2} \subset M_\omega$, it follows that $i_{\gamma_\omega} \restriction V_{\lambda+2}$ is an embedding $V_{\lambda+2} \to V_{\lambda+2}$ with critical point κ, violating Kunen's Theorem (Theorems 2.14 and 2.15). ∎

Set Bases

We remarked in Section 5 that classes $\mathcal{E}^\theta_\kappa$ obtainable in the usual way from normal ultrafilters are "set-based"; in this subsection, we make this remark precise, indicate why it is true, and show that the class $\mathcal{E}^{ext}_\kappa$ may fail to have this property.

7.8 Definition. (*Bases*) Given a class $\mathcal{E}^\theta_\kappa$ and ordinals λ, β with $\kappa < \lambda < \beta$, a (λ, β)-*base for* $\mathcal{E}^\theta_\kappa$ is a set $B \subset \mathcal{E}^\theta_\kappa$ such that for each $j : V_\beta \to N \in \mathcal{E}^\theta_\kappa$ for which $j(\kappa) > \lambda$, there is $i : V_\beta \to M \in B$ such that $i(\kappa) > \lambda$ and i is compatible with j up to V_λ. Moreover, we will call $\mathcal{E}^\theta_\kappa$ *set-based* if for each $\lambda > \kappa$, if there is an element of Dom $\mathcal{E}^\theta_\kappa$ that is $> \lambda$, then there is such a β for which $\mathcal{E}^\theta_\kappa$ has a (λ, β)-base.

Suppose $\beta > \kappa$ and $\lambda_{\min} = \min\{j(\kappa) : j \in \mathcal{E}^\theta_\kappa\}$. Then if $\kappa < \lambda_1 < \lambda_2 < \min\{\lambda_{\min}, \beta\}$, every (λ_2, β)-base for $\mathcal{E}^\theta_\kappa$ is also a (λ_1, β)-base for $\mathcal{E}^\theta_\kappa$.

Also, notice that since a (λ, β)-base B is a subset of $\mathcal{E}^\theta_\kappa$, B actually consists of ordered pairs (i, M), where M is the codomain of i; we will observe the same conventions for bases in this regard as we have for regular classes in general (see comments after Definition 4.12).

The next proposition shows that bases are not generally trivial:

7.9 Proposition. *Suppose $\mathcal{E}^\theta_\kappa$ is regular and coherent. Then for each $\beta \in$ Dom $\mathcal{E}^\theta_\kappa$ and each λ with $\kappa < \lambda < \beta$, $\{(i, M) \in \mathcal{E}^\theta_\kappa : i(\kappa) > \lambda \wedge \mathrm{dom}\ i = V_\beta\}$ is a proper class.*

Proof. Let λ, β be as in the hypotheses and let Y be a set. By regularity, let $\gamma \ge \beta$ and $i : V_\gamma \to M$ be such that $(i, M) \in \mathcal{E}^\theta_\kappa$ and $i(\kappa) > \max(\lambda, rank(Y))$. Then by coherence,if $i \restriction V_\beta : V_\beta \to N$, then $(i \restriction V_\beta, N) \in \mathcal{E}^\theta_\kappa \setminus Y$.∎

The classes $\mathcal{E}^\theta_\kappa$, with $\theta \in \{\theta_{str}, \theta_{sc}, \theta_{sah}, \theta_{sh}\}$, are set-based. We outline the straightforward proofs for $\mathcal{E}^{sc}_\kappa$ and $\mathcal{E}^{str}_\kappa$. Given $\lambda > \kappa$, let $\beta = |V_\lambda| + \omega$, and let $B_\beta = \{i_U \restriction V_\beta :$ U is a normal ultrafilter over $P_\kappa|V_\lambda|\}$. Now, for every $j : V_\beta \to N \in \mathcal{E}^{sc}_\kappa$, there is $i \in B_\beta$ that is compatible with j up to V_λ. Similarly, $\mathcal{E}^{str}_\kappa$ is set-based: Given $\lambda > \kappa$, let $\beta = \lambda + \omega + \omega$ and let $B_\beta = \{i_E \restriction V_\beta :$ E is an extender having critical point κ and support $V_{\lambda+\omega}\}$. In a similar fashion, one can show that $\mathcal{E}^{sah}_\kappa$ and $\mathcal{E}^{sh}_\kappa$ are set-based. On the other hand, as we now show, it is consistent for κ to be extendible and $\mathcal{E}^{ext}_\kappa$ not to be set-based. We begin with a definition and some preliminary results; the main result is Theorem 7.13.

7.10 Definition. Suppose $\mathcal{E}$ is a class of embeddings all having critical point λ. Then λ is *locally gap extendible in* $\mathcal{E}$ if there is $\mu \ge \lambda$ such that $\{j(\mu) \mid j : V_{\mu+1} \to V_{j(\mu)+1} \in \mathcal{E}\}$ is unbounded (in ON).

If κ is extendible and $\mathcal{E} = \{j \in \mathcal{E}_\kappa^{ext} : \text{dom } j = V_{\kappa+1}\}$, then κ is locally gap extendible in $\mathcal{E}$. However, the kind of locally gap extendible cardinal that interests us here will not generally be extendible. Assuming it is consistent for $\mathcal{E}_\kappa^{ext}$ to be set-based, we will show that there must be a locally gap extendible cardinal λ above κ; the strong reflection guaranteed by the global definition of such a cardinal will provide a ZFC model of "κ is extendible", leading to a contradiction.

7.11 Theorem. *Suppose $\mathcal{E}$ is a class of embeddings all having critical point λ, and λ is locally gap extendible in $\mathcal{E}$. Then there is an inaccessible $\mu \geq \lambda$ such that for all Π_3 formulas $\phi(x)$ and all $x \in V_\lambda$,*

$$\phi[x] \Longleftrightarrow V_\mu \models \phi[x].$$

Remark. Notice that in the case $\mu = \lambda$, an argument like the one required for Theorem 2.18(2) could be used to prove the theorem.

Proof. Starting with $\mathcal{E}$ and λ as in the hypotheses, let μ be least such that $\{j(\mu) : j \in \mathcal{E}\}$ is unbounded. Define $K : \mu + 1 \to V$ by $K(\gamma) = \{j(\gamma) : j \in \mathcal{E}\}$; for $\gamma < \mu$, $K(\gamma)$ is bounded.

Claim. *The ordinal μ is inaccessible.*

Proof of Claim. For each $\gamma < \mu$, since $K(\gamma)$ is bounded, both $K(\gamma+1)$ and $K(2^\gamma)$ are bounded.∎

Continuation of Proof of Theorem. Because $\cup\{K(\gamma) : \gamma < \mu\}$ is bounded, there are $\mathcal{E}_0 \subseteq \mathcal{E}$ and $g : V_\mu \to V$ such that

(1) $\{j(\mu) : j \in \mathcal{E}_0\}$ is unbounded;
(2) for each $j \in \mathcal{E}_0$, $j \restriction V_\mu = g$.

Let $\phi(x)$ be Π_3 and write $\phi(x)$ as $\forall z \, \exists y \, \psi(x, y, z)$, where ψ is Π_1. For one direction, let $x_0 \in V_\lambda$ and assume $V \models \phi[x_0]$. Let z_0 be an arbitrary element of V_μ; it suffices to show that $V_\mu \models \exists y \, \psi[x_0, y, z_0]$. There must be a y_0 such that

$$V \models \psi[x_0, y_0, g(z_0)].$$

Choose $j \in \mathcal{E}_0$ such that $y_0, g(z_0) \in V_{j(\mu)}$ (by (1)). Using (2), the fact that μ is inaccessible, and the fact that ψ is Π_1, we obtain

$$V_{j(\mu)} \models \exists y \, \psi[x_0, y, j(z_0)].$$

Since $j(x_0) = x_0$, we have:

$$V_\mu \models \exists y \, \psi[x_0, y, z_0],$$

as required.

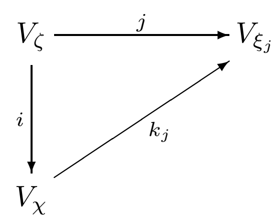
For the other direction, suppose $V_\mu \models \phi[x_0]$. Given an arbitrary z_0, it suffices to exhibit y_0 such that $V \models \psi(x_0, y_0, z_0)$. For such a z_0, let $j \in \mathcal{E}_0$ be such that $z_0 \in V_{j(\mu)}$. By elementarity and the fact that $j(x_0) = x_0$, we have

$$V_{j(\mu)} \models \exists y \, \psi[x_0, y, z_0].$$

Thus $V_{j(\mu)} \models \psi[x_0, y_0, z_0]$ for some y_0. Since ψ is Π_1, it holds in V as well (at x_0, y_0, z_0), and we are done. ∎

7.12 Lemma. *Suppose κ is extendible and $\mathcal{E}_\kappa^{ext}$ has a (δ,ζ)-base, for some δ,ζ. Then there is an inaccessible $\mu_0 > \kappa$ such that $V_{\mu_0} \models$ "κ is extendible".*

Proof. Let B denote a (δ,ζ)-base for $\mathcal{E}_\kappa^{ext}$. For this proof, we shall say that a subclass $\mathcal{F}$ of $\mathcal{E}_\kappa^{ext}$ is *unbounded* if the set $\{j(\kappa) : j \in \mathcal{F}\}$ is unbounded in ON. Let $\mathcal{E} = \{j \in \mathcal{E}_\kappa^{ext} : \mathrm{dom}\, j = V_\zeta\}$. Note that $\mathcal{E}$ is unbounded, since for any $\nu > \zeta$, we can find $\hat{j} \in \mathcal{E}_\kappa^{ext}$ having domain V_ν (whence $\hat{j}(\kappa) > \nu$), with $\hat{j} \upharpoonright V_\zeta \in \mathcal{E}$. For each $b \in B$, let $\mathcal{E}_b = \{j \in \mathcal{E} : b$ is compatible with j up to $V_\delta\}$. Since $\mathcal{E}$ is unbounded and B is a set, there is an $i : V_\zeta \to V_\chi \in B$ such that $\mathcal{E}_i$ is unbounded. Without loss of generality (using unboundedness of $\mathcal{E}_i$), we assume that for all $j \in \mathcal{E}_i$, $j(\kappa) > i(\kappa)$. By compatibility, for each $j : V_\zeta \to V_{\xi_j} \in \mathcal{E}_i$ there is $k_j : V_\chi \to V_{\xi_j}$ such that $k_j \upharpoonright V_\delta = \mathrm{id}_{V_\delta}$ and $k_j \circ i = j$.



Using $\mathcal{E}_i$ we will obtain λ, μ and a class $\mathcal{K}_1$ such that λ is locally gap extendible in $\mathcal{K}_1$ with witness μ: Let $\mu = i(\kappa)$. For each $j \in \mathcal{E}_i$, $k_j(\mu) > \mu$, and in fact $\{k_j(\mu) : j \in \mathcal{E}_i\}$ is unbounded in ON. Thus, each k_j has a critical point in the interval $(\kappa, \mu]$, whence there are $\lambda \in (\kappa, \mu]$ and a subclass $\mathcal{K}_0 \subseteq \{k_j : j \in \mathcal{E}_i\}$ such that $\{k(\mu) : k \in \mathcal{K}_0\}$ is unbounded, and for each $k \in \mathcal{K}_0$, $\mathrm{cp}(k) = \lambda$. Let $\mathcal{K}_1 = \{k \upharpoonright V_{\mu+1} : k \in \mathcal{K}_0\}$. Then λ is locally gap extendible in $\mathcal{K}_1$ with witness μ, and Theorem 7.11 applies. In particular, since "κ is extendible" is Π_3, this formula reflects to some V_{μ_0}, where μ_0 is inaccessible and $\lambda \leq \mu_0 \leq \mu$, as required. ∎

7.13 Theorem. *Con(ZFC+"κ is extendible") $\Longrightarrow$ Con(ZFC+"κ is extendible"+"$\mathcal{E}_\kappa^{ext}$ is not set-based").*

Proof. Given κ, assume, by way of contradiction, that ZFC $\vdash$ "κ is extendible" $\Longrightarrow$ "$\mathcal{E}_\kappa^{ext}$ is set-based", and that κ is extendible. By the previous lemma, pick an inaccessible $\mu_0 > \kappa$ such that $V_{\mu_0} \models$ "κ is extendible". Then, since $V_{\mu_0} \models$ ZFC, it follows that $V_{\mu_0} \models$ "$\mathcal{E}_\kappa^{ext}$ is set-based". Thus, we can again obtain, in V_{μ_0} an inaccessible $\mu_1 > \kappa$ such that $V_{\mu_1} \models$ "κ is extendible". Continuing this line of reasoning leads to an infinite descending chain $\mu_0 > \mu_1 > \mu_2 > \ldots$ of cardinals, which is impossible. This proves the theorem. ∎

Our work involving locally gap extendible cardinals in the present context raises some natural questions. If λ is locally gap extendible (in some class $\mathcal{E}$), let us call the least μ such that $\{j(\mu) : j \in \mathcal{E}\}$ is unbounded the *gap threshhold* for λ. Two natural questions are:

7.14 Open Question. *Locally gap extendible cardinals.*

(1) Under any large cardinal hypothesis, are there $\lambda, \mathcal{E}$ such that λ is locally gap extendible in $\mathcal{E}$ and such that the gap threshhold for λ is $> \lambda$?

(2) If the answer to (1) is "yes": Let $\mu, \lambda, \mathcal{E}$ be such that λ is locally gap extendible in $\mathcal{E}$, μ is the gap threshhold for λ, and $\mu > \lambda$. What is the large cardinal strength of μ?

For (2), we have seen in the proofs above that μ must be inaccessible. In fact, μ has to be at least λ-inaccessible as we now show: As in [18, p. 16-21], a cardinal κ is 1-inaccessible if κ is inaccessible; κ is $\alpha+1$-inaccessible if κ is the κth α-inaccessible; and κ is ζ-inaccessible, where ζ is a limit, if κ is α-inaccessible for each $\alpha < \zeta$.

Let $\mathcal{E}_0 \subset \mathcal{E}$ and $g : V_\mu \to V$ be as in (1) and (2) of the proof of Theorem 7.11. For the proof of λ-inaccessiblity we proceed by induction; the limit case is trivial. Let $\alpha < \lambda$ and assume that μ is α-inaccessible. For a contradiction, assume that $\exists \sigma < \mu \, \forall \xi < \mu \, (\text{ "}\xi \text{ is } \alpha\text{-extendible"} \Longrightarrow \xi < \sigma)$. Then clearly, there is $\sigma < \mu$ such that

$$V_\mu \models \forall \xi \, (\text{ "}\xi \text{ is } \alpha\text{-extendible"} \Longrightarrow \xi < \sigma). \tag{7.3}$$

Let $\rho = \sup(g''V_\mu)$. Let $i, j \in \mathcal{E}$ be such that $\rho < i(\mu) < j(\mu)$, and set $\nu = i(\mu)$. Then by (7.3) and the fact that $j(\alpha) = \alpha$,

$$V_{j(\mu)} \models \forall \xi \, (\text{"}\xi \text{ is } \alpha\text{-inaccessible"} \Longrightarrow \xi < j(\sigma) \leq \rho).$$

But this is impossible since

$$V_{j(\mu)} \models \text{"}\nu \text{ is } \alpha\text{-inaccessible"} \wedge \nu > \rho.$$

As a final remark in this vein, notice that (with the notation and hypotheses as in the last paragraph) if μ happens to be a closure point for some $j \in \mathcal{E}_0$ (that is, $j''\mu \subset \mu$) then μ must be Mahlo—indeed, λ-Mahlo—since, for any closed unbounded $C \subseteq \mu$, μ must be a limit point of (and hence a member of) $j(C)$.

Extensibility and Special $\mathcal{E}$-Laver Sequences

In Theorem 4.9 we showed that the existence of a special (or special*) Laver sequence at κ implies that it is consistent for κ to be the κth extendible cardinal. We show in this subsection that this strong consequence of specialness carries over into the general setting of special $\mathcal{E}^\theta_\kappa$-Laver sequences (recall Definition 4.25) for classes $\mathcal{E}^\theta_\kappa$ that admit sufficient *extensibility*:

7.15 Definition. Suppose λ is an ordinal and $i : V_\beta \to M$ is elementary with critical point κ. Then i is *extensible with λ-closure* if for each $\delta \geq \beta$ there are $\gamma > \delta$ and $\hat{i} : V_\gamma \to N$ such that

(1) $\hat{i} \restriction V_\beta = i$
(2) $\hat{i}(V_\beta) = M$
(3) N is λ-closed.

In this case, $\hat{i}$ is called a *λ-closed extension of i*. Moreover, a class $\mathcal{E}^\theta_\kappa$ *admits closed extensions* if, whenever $\lambda < |\beta|$ and $i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$, there is $i_\beta : V_\beta \to M_\beta \in \mathcal{E}^\theta_\kappa$ such that

(A) i_β is compatible with i up to $\lambda + 1$

(B) M_β is λ-closed
(C) i_β is extensible with λ-closure.

Clearly, we could have obtained a perfectly good definition of extensibility without including special closure requirements as we have in Definition 7.15; a more general definition would certainly have wider applicability. We use the more restricted definition because our only applications of extensibility always entail these closure requirements.

Notice that extensions are not required to be in $\mathcal{E}_\kappa^\theta$; without this laxity, none of the classes $\mathcal{E}_\kappa^\theta$ studied in this paper would admit closed extensions. Also notice that the "moreover" clause of the definition requires $\lambda < |\beta|$, but the first part of the definition does not. It is convenient to allow the possibility $\beta < \lambda$ for classes like $\mathcal{E}_\kappa^{ext}$ (see the proof of Proposition 7.16).

We can show that if $\theta \in \{\theta_{sc}, \theta_{sah}, \theta_{sh}\}$, then $\mathcal{E}_\kappa^\theta$ admits closed extensions; we prove this for $\mathcal{E}_\kappa^{sc}$ and leave the proofs of the other cases to the reader. Given $\lambda < |\beta|$ and $i : V_\beta \to M \in \mathcal{E}_\kappa^{sc}$, let U be the normal ultrafilter over $P_\kappa\lambda$ derived from i and let $i_\beta = i_U \restriction V_\beta$. Now i_β has the required properties.

On the other hand, we are unable to prove that $\mathcal{E}_\kappa^{ext}$ admits closed extensions. We have the following partial result:

7.16 Proposition. *Suppose κ is super-3-huge and $j : V \to N$ is a 3-huge embedding with critical point κ. Then the set $\{\lambda < j(\kappa) :$ for some $i : V_\beta \to V_\xi \in \mathcal{E}_\kappa^{ext},\ i$ is extensible with λ-closure$\}$ has normal measure 1.*

Proof. Let $S = \{\lambda < j(\kappa) : \lambda \text{ is a 2-huge target}\}$. Reasoning as in Theorem 2.22(1), one verifies that S has normal measure 1. Let $\lambda \in S$ and let $j_\lambda : V \to M_\lambda$ be a 2-huge embedding with critical point κ and target λ. Pick β with $\kappa < \beta < \lambda$. Note that by 2-hugeness, the codomain of $j_\lambda \restriction V_\beta$ is some V_ξ; it follows that $j_\lambda \restriction V_\beta \in \mathcal{E}_\kappa^{ext}$. Clearly, $j_\lambda \restriction V_\beta$ is extensible with λ-closure. ∎

Next we show that the existence of a special $\mathcal{E}_\kappa^\theta$-Laver sequence, is, as in the supercompact case, strong enough for κ to be (consistently) the κth extendible cardinal, as long as $\mathcal{E}_\kappa^\theta$ has sufficient extensibility to allow the arguments of Theorem 4.9 to go through.

7.17 Theorem. *Suppose θ is a suitable formula.*

(1) *If $\mathcal{E}_\kappa^\theta$ admits closed extensions and there is a special $\mathcal{E}_\kappa^\theta$-Laver sequence at κ, it is consistent for κ to be the κth extendible.*
(2) *If there is a special $\mathcal{E}_\kappa^\theta$-Laver sequence at κ with witnesses i, β, λ such that i is extensible with λ-closure, then it is consistent for κ to be the κth extendible.*

Proof. The proofs of (1) and (2) are nearly the same, but (2) is easier; we prove (1). We argue as in Theorem 4.9. The idea is to replace the embedding i_U in (the special* version of) Theorem 4.9 with some $i_\beta : V_\beta \to M_\beta \in \mathcal{E}_\kappa^\theta$ that witnesses specialness. The argument goes through as long as there is $i_\gamma : V_\gamma \to M_\gamma$ satisfying the following:

(a) $i_\gamma \restriction V_\beta = i_\beta$;
(b) $\langle \mathcal{M}'_\xi : \xi < i_\gamma(\kappa) \rangle \in V_\gamma$;
(c) $i_\gamma \restriction \mathcal{M}'_\kappa \in M_\gamma$,

where

$$\langle \mathcal{M}'_\xi : \xi < i_\gamma(\kappa) \rangle = i_\gamma(\langle \mathcal{M}_\xi : \xi < \kappa \rangle).$$

(See the proof of Theorem 4.9 for the definition of $\langle \mathcal{M}_\xi : \xi < \kappa \rangle$.)

Let g be a special $\mathcal{E}^\theta_\kappa$-Laver with witnesses $\lambda, \beta, i : V_\beta \to M \in \mathcal{E}^\theta_\kappa$, and D. Since $\mathcal{E}^\theta_\kappa$ admits closed extensions, we obtain $i_\beta : V_\beta \to M_\beta \in \mathcal{E}^\theta_\kappa$ satisfying (A) - (C) of Definition 7.15. It is straightforward to verify that D is the normal ultrafilter over κ derived from i_β and that $\lambda, \beta, i_\beta, D$ witness the specialness of g. Pick $\gamma > \beta$ so that $i_\beta(\langle \mathcal{M}_\xi : \xi < \kappa \rangle) \in V_\gamma$ and there is a λ-closed extension $i_\gamma : V_\gamma \to M_\gamma$ of i_β. Now i_γ clearly satisfies (a), and (b) holds since

$$i_\gamma(\langle \mathcal{M}_\xi : \xi < \kappa \rangle) = i_\beta(\langle \mathcal{M}_\xi : \xi < \kappa \rangle).$$

To prove (c), we argue as in Theorem 4.9: define $h : \kappa \to \kappa$ by $h(\xi) = rank(\mathcal{M}_\xi)$. Observe

$$i_\gamma(h)(\kappa) < rank(i_\gamma(g)(\kappa)) < \lambda.$$

Since $rank(\mathcal{M}'_\kappa) = i_\gamma(h)(\kappa)$ and λ is a beth fixed point, $|\mathcal{M}'_\kappa| < \lambda$. Hence, as M_γ is λ-closed, $i_\gamma \restriction \mathcal{M}'_\kappa \in M_\gamma$, and we are done. ∎

Although Theorem 7.17 is a more general result than Theorem 4.9, it provides us with no new information concerning the five classes we have been considering in this paper. By Theorem 7.17 and our observations following Definition 7.15, the existence of a special $\mathcal{E}^\theta_\kappa$-Laver sequence implies the consistency of a proper class of extendibles whenever $\theta \in \{\theta_{sc}, \theta_{sah}, \theta_{sh}\}$; but Theorem 4.9 gave us this result for the case $\theta = \theta_{sc}$; for $\theta = \theta_{sah}$ or θ_{sh}, the result follows from the existence of an ordinary $\mathcal{E}^\theta_\kappa$-Laver sequence. Using Theorem 7.17, we can't draw any conclusions about the strength of special $\mathcal{E}^{str}_\kappa$-Laver sequences, or special $\mathcal{E}^{ext}_\kappa$-Laver sequences. We are left with the following questions:

7.18 Open Question.

(1) Is the existence of a special $\mathcal{E}^{str}_\kappa$-Laver sequence (special $\mathcal{E}^{ext}_\kappa$-Laver sequence) stronger than the existence of a strong (extendible) cardinal?
(2) Under any hypothesis, does $\mathcal{E}^{ext}_\kappa$ admit closed extensions?

Weak Compatibility of $\mathcal{E}^{ext}_\kappa$

In this subsection, we improve Theorem 5.23(1) slightly by eliminating "κ is extendible" from the hypothesis:

7.19 Proposition. *Suppose $j : V \to N$ is an almost huge embedding with critical point κ. Then $\mathcal{E}^{ext}_\kappa$ is weakly compatible with j.*

Proof. Suppose $j : V \to N$ is almost huge with critical point κ. Given $\lambda, g : \kappa \to V_\kappa, r : \kappa \to P(\kappa)$ such that $\kappa < \lambda < j(\kappa)$, $rank(j(g)(\kappa)) < \lambda$ and for all $\delta < \kappa, r(\delta) \in D$, let $x = j(g)(\kappa)$, let U^j be defined as in Definition 2.16, and let $N_1 = (j \cdot j)(N)$. Observe that

$$\begin{aligned} V^{N_1}_{j^2(\kappa)} \models \forall\beta\,\exists\eta\,\big(\kappa < \beta < j(\kappa) \Longrightarrow \\ &\exists i : V_\beta \to V_\eta\,\exists D_i\,[\mathrm{cp}(i) = \kappa \,\wedge\, i(\kappa) > \beta \,\wedge\, i(g)(\kappa) = x \,\wedge \\ &\text{“}D_i \text{ is the normal ultrafilter derived from } i\text{”} \,\wedge\, \forall\delta < \kappa(r(\delta) \in D_i)]\big).\end{aligned}$$

(For each β, let $\eta = j(\beta)$, $i = j \restriction V_\beta$, and let D_i be the normal ultrafilter over κ derived from j.) Thus, $S \in U^j$, where

$$\begin{aligned} S = \{\beta < j(\kappa) : &V_{j(\kappa)} \models \forall\alpha\exists\eta\,\big(\kappa < \alpha < \beta \Longrightarrow \\ &\exists i : V_\alpha \to V_\eta\,\exists D_i[\mathrm{cp}(i) = \kappa \,\wedge\, i(\kappa) > \alpha \,\wedge\, i(g)(\kappa) = x \,\wedge \\ &\text{“}D_i \text{ is the normal ultrafilter over } \kappa \text{ derived from } i\text{”} \,\wedge\, \forall\delta < \kappa(r(\delta) \in D_i)]\big)\}.\end{aligned}$$

S may not be stationary (in V), but by absoluteness, it contains arbitrarily large limits below $j(\kappa)$. We can therefore pick a limit β such that $\lambda < \beta < j(\kappa)$ and $\beta \in S$, and pick α so that $\lambda < \alpha < \beta$. The corresponding embedding $i : V_\alpha \to V_\eta$ is in $\mathcal{E}^{ext}_\kappa$ and properties (1)-(3) of weak compatibility are satisfied. ∎

§8. Appendix

The purpose of this section is to correct several errors that were published in [9]. These errors became apparent during the period that the present paper was being reviewed and modified. The first error is an erroneous statement about the existence of $\mathcal{E}^{sah}_\kappa$-Laver sequences that was based on an earlier (incorrect) version of the present paper. The other two errors are incorrect proofs of correctly stated theorems about the existence of $\mathcal{E}^{ext}_\kappa$-Laver sequences. We present an outline of the results that are in error and follow these with corrections.

Error #1: [9, Theorem 5.1(5)]. This theorem refers to a result that had been stated in an earlier version of this paper which we formulated in [9] as an axiom $\mathrm{SAH}_4(\kappa)$. We formulate an alternative version of this axiom and show that it has all the features that the version in [9] was supposed to have.

Error #2: [9, Theorem 4.1]. The theorem states that, assuming only that κ is extendible, the construction $\mathbf{CC}^R(t, \mathcal{E}^{ext}_\kappa)$ produces an $\mathcal{E}^{ext}_\kappa$-Laver sequence. The result is true and, though the ideas for a correct proof do appear in [9], the proof given there is not correct; we give a correct proof here.

Error #3: [9, Theorems 4.4 and 4.5]. These theorems are used in [9] to demonstrate that the construction $\mathbf{CC}(t, \mathcal{E}^{ext}_\kappa)$ also produces an $\mathcal{E}^{ext}_\kappa$-Laver sequence assuming only that κ is extendible. This result is also basically true, as we show below, but the statements of the theorems mentioned are incorrect.

Correction to Error #1

Our original approach to the proof that the functions obtained from either $\mathbf{CC}(t, \mathcal{E}_\kappa^{sah})$ or $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ are $\mathcal{E}_\kappa^{sah}$-Laver, and our original proof of Theorem 5.20(3) in particular, were somewhat different from our current version. In [9], an axiom that we called $\mathrm{SAH}_4(\kappa)$ was extracted from the original paper, and we argued in [9, Theorem 5.3] that $\mathrm{SAH}_4(\kappa)$ is sufficient to prove that f^R obtained in $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ is $\mathcal{E}_\kappa^{sah}$-Laver; in [9, Theorem 5.1(5)] that "κ is superhuge" strongly implies $\mathrm{SAH}_4(\kappa)$; and in [9, Theorem 5.1(4)] that $\mathrm{SAH}_4(\kappa)$ is strictly consistency-wise stronger than another axiom $\mathrm{SAH}_2(\kappa)$ that is also concerned with super-almost-huge cardinals. The error occurs in [9, Theorem 5.1(5)]: our proof, which first appeared in the original version of this paper, is incorrect; we don't know at this time the consistency strength of $\mathrm{SAH}_4(\kappa)$, or whether it is consistent with any known large cardinal axiom. The other results are correct but not particularly meaningful in the absence of a reasonable upper bound on $\mathrm{SAH}_4(\kappa)$.

Our plan here is to replace $\mathrm{SAH}_4(\kappa)$ with a different axiom, and show that the results originally obtained for $\mathrm{SAH}_4(\kappa)$ go through for our new axiom. Since we have not given the details of $\mathrm{SAH}_4(\kappa)$ as it appears in [9], there should be no confusion if we use the same name for our new version of this axiom. We begin by giving the statements of the axioms $\mathrm{SAH}_0(\kappa)$-$\mathrm{SAH}_3(\kappa)$ from [9], and then stating our new $\mathrm{SAH}_4(\kappa)$. We then provide some other background material and conclude with the relevant proofs.

If κ is super-almost-huge, let $\Lambda = \{\lambda : \lambda \text{ is an a.h. target for } \kappa\}$, and, for any class $\mathbf{C}$, $\mathbf{C}' = \{\nu : \nu \text{ is a limit point of } \mathbf{C}\}$.

$\mathrm{SAH}_0(\kappa)$: κ is super-almost-huge.

$\mathrm{SAH}_1(\kappa)$: κ is super-almost-huge and the class $\Lambda \cap \Lambda'$ is bounded.

$\mathrm{SAH}_2(\kappa)$: κ is super-almost-huge, and the class $\Lambda \cap \Lambda'$ is unbounded, and there is μ such that for all regular $\rho > \mu$ the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is nonstationary in ρ.

$\mathrm{SAH}_3(\kappa)$: κ is super-almost-huge and for arbitrarily large regular ρ, the set $\{\gamma < \rho : \gamma \text{ is an a.h. target of } \kappa\}$ is stationary in ρ.

Our new version of $\mathrm{SAH}_4(\kappa)$ is the following:

$\mathrm{SAH}_4(\kappa)$: κ is super almost huge and there are unboundedly many λ such that λ is an a.h. target, and for some coherent sequence $\langle U_\eta : \kappa \leq \eta < \lambda\rangle$ that satisfies $\mathcal{B}(\kappa, \lambda)$, the set $\{\alpha < \lambda : \langle U_\eta : \kappa \leq \eta < \alpha\rangle \text{ satisfies } \mathcal{B}(\kappa, \alpha)\}$ is unbounded in λ.

To compare relative strengths of these hypotheses, we introduced in [9] the following notation: Given properties $A(\kappa), B(\kappa)$ that depend on an infinite cardinal κ, we write:

$$\begin{array}{lll} A(\kappa) \overset{\text{ZFC}}{\longrightarrow} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ implies } B(\kappa)\text{”} \\ & \text{iff} & \text{ZFC} \vdash A \longrightarrow B; \\ A(\kappa) \overset{con}{\longrightarrow} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ is consistency-wise at least as strong as } B(\kappa)\text{”} \\ & \text{iff} & Con(\text{ZFC} + A(\kappa)) \overset{\text{ZFC}}{\longrightarrow} Con(\text{ZFC} + B(\kappa)); \\ A(\kappa) \overset{s}{\longrightarrow} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ is strictly consistency-wise stronger than } B(\kappa)\text{”} \\ & \text{iff} & A(\kappa) \overset{con}{\longrightarrow} B(\kappa) \text{ and } A(\kappa) \overset{\text{ZFC}}{\longrightarrow} Con(\text{ZFC} + B(\kappa)); \\ A(\kappa) \overset{si}{\longrightarrow} B(\kappa) & \text{iff} & \text{“}A(\kappa) \text{ strongly implies } B(\kappa)\text{”} \\ & \text{iff} & A(\kappa) \overset{\text{ZFC}}{\longrightarrow} B(\kappa) \text{ and } \{\alpha < \kappa : B(\alpha)\} \text{ has normal measure 1.} \end{array}$$

The terminology *strongly implies* was introduced in [27].

8.1 Theorem. *$SAH_4(\kappa) \overset{con}{\longrightarrow} SAH_2(\kappa)$.*

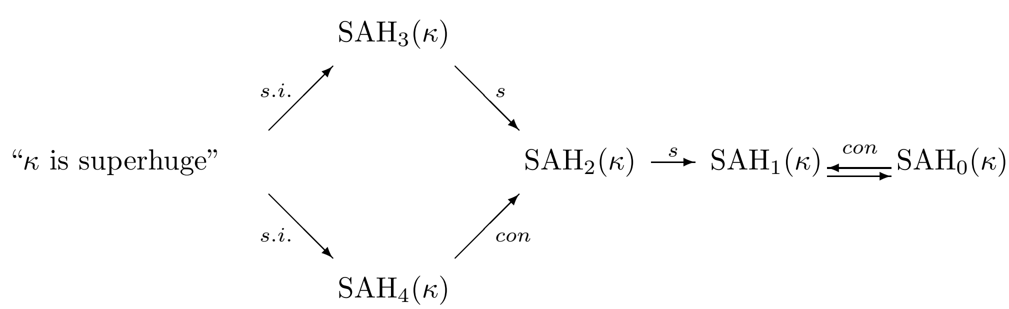
Proof. It is clear that $\text{SAH}_4(\kappa)$ implies that $\Lambda \cap \Lambda'$ is unbounded. If there is a regular cardinal in which Λ is stationary, let ρ be the least such. Then $\Lambda \cap \Lambda' \cap \Lambda''$ is also stationary in ρ. Then if $\lambda = \min \Lambda \cap \Lambda' \cap \Lambda''$, it follows that $V_\lambda \models \text{SAH}_2(\kappa)$. ∎

8.2 Theorem. *“κ is superhuge” $\overset{s.i.}{\longrightarrow}$ $SAH_4(\kappa)$.*

Proof. Let $j : V \to N$ be a huge embedding with critical point κ. It suffices to show that

$$N \models \text{SAH}_4(\kappa).$$

The proof of Theorem 5.20(3) shows that $V_{j(\kappa)} \models \text{SAH}_4(\kappa)$. It is easy to see that the statement $\text{SAH}_4(\kappa)$ is globalized local and hence Π_3^{ZFC}. Thus, if $N \models \neg\text{SAH}_4(\kappa)$, this fact would reflect down to $V_{j(\kappa)}$ (since, in N, $j(\kappa)$ is superhuge, whence $V_{j(\kappa)} \prec_3 V$), giving a contradiction. ∎



8.3 Theorem. *$SAH_4(\kappa)$ implies that for any well-ordering R of V_κ, the function f^R obtained in the construction $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{sah})$ is $\mathcal{E}_\kappa^{sah}$-Laver.*

Proof. This follows from Theorem 5.20(3) and Theorem 6.2. ■

Correction to Error #2.

The second error in [9] that we address is the proof of [9, Theorem 4.1], which asserts that, assuming only that κ is extendible, there is a $\mathcal{E}_\kappa^{ext}$-Laver sequence. The theorem is true; we provide a correct proof below. We begin with a definition:

8.4 Definition. A well-ordering R of V_δ is *rank-preserving* if, for all $x, y \in V_\delta$, if $rank(x) < rank(y)$ then $(x, y) \in R$.

8.5 Theorem. *If κ is extendible, there is an $\mathcal{E}_\kappa^{ext}$-Laver sequence at κ. In particular, assuming κ is extendible, for every rank-preserving well-ordering R and every $t : \kappa \to V_\kappa$ definable in $\langle V_\kappa, \in, R\rangle$, the function f^R constructed in $\mathbf{CC}^R(t, \mathcal{E}_\kappa^{ext})$ is $\mathcal{E}_\kappa^{ext}$-Laver.*

Proof. Suppose x_0, λ_0 witness that f^R is not $\mathcal{E}_\kappa^{ext}$-Laver at κ. Let α be such that $V_\kappa \prec V_\alpha$ and $x, \lambda \in V_\alpha$. Let $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$. Let D denote the normal ultrafilter derived from j. As usual, the definition of f^R implies that exactly one of the following sets is in D:

$$S_1 = \{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \text{“}f^R \restriction \alpha \text{ is } \mathcal{E}_\alpha^{ext}\text{-Laver at } \alpha\text{”}\};$$
$$S_2 = \{\alpha < \kappa : \langle V_\kappa, \in, R\rangle \models \exists\lambda\, \phi(f^R \restriction \alpha, f^R(\alpha), \lambda)\}.$$

If $S_1 \in D$, then, $\langle V_{j(\kappa)}, \in, j(R)\rangle \models$ “f^R is $\mathcal{E}_\kappa^{ext}$-Laver”. By adequate absoluteness of θ_{ext}, we can find $i : V_\xi \to V_\gamma \in \mathcal{E}_\kappa^{ext} \cap V_{j(\kappa)}$ for which $i(f^R)(\kappa) = x_0$ and $i(\kappa) > \lambda_0$, which contradicts our assumptions about x_0, λ_0.

Thus, $S_2 \in D$, whence

$$\langle V_{j(\kappa)}, \in, j(R)\rangle \models \exists\lambda\, \phi(f^R, j(f^R)(\kappa), \lambda). \tag{8.1}$$

Let $x_1 = j(f^R)(\kappa)$ and $\lambda_1 < j(\kappa)$ be witnesses to this formula. By the formulation of the second case in the definition of f^R, x_1 must be the R-least set for which f^R fails to be $\mathcal{E}_\kappa^{ext}$-Laver. Since R is rank-preserving, this means that, since x_0 is another set witnessing Laver-failure of f^R,

$$rank(x_1) \le rank(x_0) < \alpha.$$

Since $V_\kappa \prec V_\alpha$, α must be a limit; thus, let β be such that $rank(x_1) < \beta < \alpha$. Then if $i = j \restriction V_\beta : V_\beta \to V_{j(\beta)}$, $(i, V_{j(\beta)}) \in \mathcal{E}_\kappa^{ext}$, $i(\kappa) > \lambda_1$ and $i(f^R)(\kappa) = x_1$. Since $\beta < \alpha$, it follows that $j(\beta) < \eta$ and $(i, V_{j(\beta)}) \in V_\eta$. Thus

$$V_\eta \models \exists e\, \exists\beta\, \exists\zeta\, \big[(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \text{cp}(e) = \kappa \wedge$$
$$\beta < \alpha \wedge e(\kappa) > \lambda_1 \wedge e(f^R)(\kappa) = x_1\big].$$

Because $V_\kappa \prec V_\alpha$, it follows that $V_{j(\kappa)} \prec V_\eta$. Since $\kappa, f^R, \alpha, \lambda_1, x_1 \in V_{j(\kappa)}$, we have:

$$V_{j(\kappa)} \models \exists e \, \exists \beta \, \exists \zeta \, \big[(e, V_\zeta) \in \mathcal{E}_\kappa^{ext} \wedge \text{dom } e = V_\beta \wedge \text{cp}(e) = \kappa \wedge \\ \beta < \alpha \wedge e(\kappa) > \lambda_1 \wedge e(f^R)(\kappa) = x_1\big].$$

But now any witness $(e, V_\zeta) \in \left(\mathcal{E}_\kappa^{ext}\right)^{V_{j(\kappa)}} = \mathcal{E}_\kappa^{ext} \cap V_{j(\kappa)}$ contradicts (8.1), and we have a contradiction. ∎

Correction to Error #3. The goal of [9, Theorems 4.4, 4.5] was to show that the function defined in $\mathbf{CC}(t, \mathcal{E}_\kappa^{ext})$ is $\mathcal{E}_\kappa^{ext}$-Laver, assuming only that κ is extendible. This result is basically true, but the two theorems cited are incorrect.

Theorem 4.4 in [9] claimed that, if in Theorem 5.13 of the present paper, we replace "superstrong embeddings having arbitrarily large targets" with embeddings $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$ for which α is an arbitrarily large inaccessible, then the conclusion of Theorem 5.13 will hold for $\mathcal{E}_\kappa^{ext}$. However, this replacement is not sufficient for the proof to go through. Indeed, conditions (1) and (2) would guarantee that for each such $j : V_\alpha \to V_\eta$,

$$\{\alpha < \kappa : f \restriction \alpha \text{ is } \mathcal{E}_\alpha^{ext}\text{-Laver}\} \in D, \tag{8.2}$$

where D is the normal ultrafilter over κ derived from j; clearly this would imply that κ is the κth extendible cardinal, a conclusion that is too strong to be obtained from the hypothesis.

Theorem 4.5 in [9] asserted that properties (1) - (3) of Theorem 5.13 must hold in V_η whenever $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$ is such that $V_\kappa \prec V_\alpha$ and α is inaccessible. Again this is impossible because properties (1) and (2) would again imply (8.2) where D is the normal ultrafilter derived from j, and the consequences are too strong for the hypothesis.

Nonetheless, the f obtained from $\mathbf{CC}(t, \mathcal{E}_\kappa^{ext})$ can be shown to be $\mathcal{E}_\kappa^{ext}$-Laver, assuming only that κ is extendible, if we make the following slight modification in the definition of f: In the second clause, we now require that if α is a cardinal but $f \restriction \alpha$ is not $\mathcal{E}_\alpha^{ext}$-Laver, then $f(\alpha)$ is defined to be a set $x \in V_\kappa$ *of least possible rank* that witnesses this failure. Let us call the function defined in this way $f' = f'_t$. Then the following is true:

8.6 Theorem. *Suppose κ is extendible. Then for any choice of the parameter t, the function f' is $\mathcal{E}_\kappa^{ext}$-Laver at κ.*

Proof. The idea of the proof is basically the same as the proof of Theorem 8.5; we give an outline. Assume x_0, λ_0 witness that f' is not $\mathcal{E}_\kappa^{ext}$-Laver. Pick an inaccessible α so that $x_0, \lambda_0 \in V_\alpha$ and $V_\kappa \prec V_\alpha$, and pick any $j : V_\alpha \to V_\eta \in \mathcal{E}_\kappa^{ext}$. If D is the normal ultrafilter over κ derived from j, either $S_1 \in D$ or $S_2 \in D$ where

$$S_1 = \{\alpha < \kappa : \text{“}f' \restriction \alpha \text{ is } \mathcal{E}_\alpha^{ext}\text{-Laver at } \alpha\text{”}\};$$
$$S_2 = \{\alpha < \kappa : \exists \lambda \, \phi(f' \restriction \alpha, f'(\alpha), \lambda)\}.$$

Reasoning as in Theorem 8.5, one shows $S_1 \notin D$. Assuming $S_2 \in D$, we have that for some $\lambda_1 < j(\kappa)$,

$$V_\eta \models \phi(f', j(f')(\kappa), \lambda_1). \tag{8.3}$$

Let $x_1 = j(f')(\kappa)$. As in Theorem 8.5, $rank(x_1) < \alpha$, so, as in that proof, we can use $j \restriction V_\beta$ as a counterexample to (8.3), giving the desired contradiction. ∎

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